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The sizes of the classes of $H^{(N)}$-sets

by

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Abstract. The class of $H^{(N)}$-sets forms an important subclass of the class of sets of uniqueness for trigonometric series. We investigate the size of this class which is reflected by the family of measures (called polar) annihilating all sets from the class. The main aim of this paper is to answer in the negative a question stated by Lyons, whether the polars of the classes of $H^{(N)}$-sets are the same for all $N \in \mathbb{N}$. To prove our result we also present a new description of $H^{(N)}$-sets.

1. Introduction. Let M be a collection of closed subsets of $[0,1]$, and $\mathcal{M}([0,1])$ be the set of all Radon measures on the interval $[0,1]$. Then the *polar* $M^{\perp} \subset \mathcal{M}([0,1])$ is defined by

$$M^{\perp} = \{\nu \in \mathcal{M}([0,1]);\ \forall B \in M : \nu(B) = 0\}.$$

We say that $\mu \in \mathcal{M}([0,1])$ is *Rajchman* if $\lim_{|n|\to\infty} \widehat{\mu}(n) = 0$. The family of all Rajchman measures is denoted by $\mathcal{R}$. Let us recall that closed sets of *extended uniqueness* (U_0 sets) are those closed sets which are annihilated by every Rajchman measure. Thus by definition we have $\mathcal{R} \subset U_0^{\perp}$.

Rajchman [9] investigated classes A with the property $A^{\perp} = \mathcal{R}$. He introduced an important subclass of U sets, called H-sets (or $H^{(1)}$-sets) (see the next section or [4] for the definitions of U and $H^{(1)}$) and investigated whether $H^{\perp} = \mathcal{R}$. Lyons [5] showed that $\mathcal{R} = U_0^{\perp}$. On the other hand Kaufman [3] proved that $U^{\perp} \neq U_0^{\perp} = \mathcal{R}$. Thus U_0 can be considered much larger than U in the sense of polars. More generally, one can consider two families of closed sets $A \subset B$ and may ask whether $B^{\perp} \subsetneq A^{\perp}$. If this is the case then B can be considered much larger than A.

Rajchman conjectured that every set of uniqueness was a countable union of H-sets. This was disproved by Pyatetskiĭ–Shapiro [7] (see also [8]), who also introduced the classes of $H^{(N)}$-sets for $N \in \mathbb{N}$. Further he showed that $H^{(N)} \subset H^{(N+1)} \subset U \subset U_0$ and that there is an $H^{(N+1)}$-set which

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cannot be written as a countable union of $H^{(N)}$-sets. Lyons [6] showed that $\mathcal{R} \subsetneq (\bigcup_{N\in\mathbb{N}} H^{(N)})^{\perp}$. Thus, the classes $H^{(N)}$ are "small" in U_0 in the sense given above. Lyons [6] asked whether $(H^{(N+1)})^{\perp} = (H^{(N)})^{\perp}$. The aim of this paper is to prove the next theorem which answers Lyons' question in the negative for every $N \in \mathbb{N}$.

Theorem 1.1. *Let $N \in \mathbb{N}$. Then $(H^{(N+1)})^{\perp} \neq (H^{(N)})^{\perp}$.*

We will prove Theorem 1.1 using a description of $H^{(N)}$-sets in Theorem 2.5. This result can be used to reprove Šleich's result that each $H^{(N)}$-set is σ-porous ([12]).

The case $N = 1$ of Theorem 1.1, which is much simpler, was presented without proof in [11].

The question also arises whether $(\bigcup_{N\in\mathbb{N}} H^{(N)})^{\perp} \supsetneq U^{\perp}$. Zelený and Pelant [13] show that there is a non-σ-porous closed set of uniqueness. Thus this set is a set of uniqueness which cannot be written as a countable union of elements of $\bigcup_{N\in\mathbb{N}} H^{(N)}$.

2. Proof of Theorem 1.1

Notation 2.1.

- We denote the Lebesgue measure on $\mathbb{R}$ by λ and the number of elements of a finite set A by $\sharp A$.
- The symbol $\langle x\rangle$ stands for the fractional part of $x \in \mathbb{R}$, i.e., $\langle x\rangle = x - [x]$, where $[x]$ is the integer part of x. Further, for $B \subset \mathbb{R}$ we denote $\langle B\rangle = \{\langle x\rangle; x \in B\}$.
- For $N \in \mathbb{N}$ and $\boldsymbol{a} \in (\mathbb{R}^N)^{\mathbb{N}}$, we write $\boldsymbol{a} = \{a_j\}_{j\in\mathbb{N}}$ and $a_j = (a_j^1, \dots, a_j^N) \in \mathbb{R}^N$.
- By an *open interval* $J \subset \mathbb{R}^N$ we mean any product of nonempty open intervals $J^i \subset \mathbb{R}$, $i = 1, \dots, N$.
- Let $x \in \mathbb{R}$ and $r > 0$. We denote the interval $(x - r, x + r)$ by $B(x, r)$.

Definition 2.2. Let $N \in \mathbb{N}$, $L \in \mathbb{R}$, and $P \subset \mathbb{R}$.

- A sequence of vectors $\boldsymbol{a} \in (\mathbb{R}^N)^{\mathbb{N}}$ is *quasi-independent* if for every nonzero $\alpha \in \mathbb{Z}^N$ we have $\lim_j |(\alpha, a_j)| = \infty$, where (u, v) denotes the scalar product of vectors $u, v \in \mathbb{R}^N$. The set of all quasi-independent sequences of vectors from P^N is denoted by $\mathcal{Q}(P^N)$.
- A closed set $A \subset [0, 1]$ is in $H^{(N)}(P)$ if there exist $\boldsymbol{a} \in \mathcal{Q}(P^N)$ and an open interval $J \subset [0, 1]^N$ such that for every $x \in A$ and every $j \in \mathbb{N}$ we have $\langle xa_j\rangle := (\langle xa_j^1\rangle, \dots, \langle xa_j^N\rangle) \notin J$. We will write just $H^{(N)}$ instead of $H^{(N)}(\mathbb{N})$, and $H^{(N)*}$ instead of $H^{(N)}(\mathbb{R} \setminus \{0\})$. Subsets of elements of $H^{(N)}$ are called *$H^{(N)}$-sets*.

- A closed set $A \subset [0,1]$ is in $H_L^{(N)*}$ if there exist $\boldsymbol{a} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$ and an open interval $J = \prod_{i=1}^N J^i \subset [0,1]^N$ witnessing $A \in H^{(N)*}$ and satisfying
$$\left|\frac{a_j^{i+1}\lambda(J^i)}{a_j^i}\right| \geq L$$
for every $i \in \{1,\dots,N-1\}$ and $j \in \mathbb{N}$.

The notion of $H^{(N)*}$ is well known but $H_L^{(N)*}$ is a new notion.

Remark 2.3. (i) Let $N, M \in \mathbb{N}$, $N \leq M$, and $L, K \in \mathbb{R}$, $L \leq K$. Then clearly $H_K^{(N)*} \subset H_L^{(M)*}$, $H^{(N)*} = H_0^{(N)*}$ and $H^{(N)} \subset H^{(N)*}$. Further, the family $H^{(N)}$ is hereditary, i.e., if $A \in H^{(N)}$, $A \supset B$ and B is closed then $B \in H^{(N)}$. Similarly, the families $H^{(N)*}$ and $H_L^{(N)*}$ are also hereditary.

(ii) Bari [1] denotes $H^{(N)*}$ by $H^{(N)}(\mathbb{R})$. We use $\mathbb{R} \setminus \{0\}$ instead of $\mathbb{R}$ to avoid dividing by zero. It is easy to see that $H^{(N)}(\mathbb{R}) = H^{(N)}(\mathbb{R} \setminus \{0\})$. Thus, both of these definitions define the same object. Note that each set from $H^{(N)*}$ is a finite union of elements of $H^{(N)}$ (see [1, pp. 919–921]). Consequently, $(H^{(N)*})^\perp = (H^{(N)})^\perp$.

(iii) Let $N \in \mathbb{N}$. Then the collection $H^{(N)}$ consists of closed $H^{(N)}$-sets.

The proof of the main result is based on the following two results which will be proved in the next sections.

Lemma 2.4. *Let $N \in \mathbb{N}$. Then* $(H^{(N+1)})^\perp \subsetneq (H_{10}^{(N)*})^\perp$.

Theorem 2.5. *Let $N, L \in \mathbb{N}$. Then* $H_L^{(N)*} = H^{(N)*}$.

Granting these results the proof goes as follows.

Proof of Theorem 1.1. By Lemma 2.4, Theorem 2.5, and Remark 2.3(ii) we get
$$(H^{(N+1)})^\perp \subsetneq (H_{10}^{(N)*})^\perp = (H^{(N)*})^\perp = (H^{(N)})^\perp. \blacksquare$$

3. Proof of Lemma 2.4. Throughout this section $N \in \mathbb{N}$ will be fixed. We will construct a measure $\mu \in (H_{10}^{(N)*})^\perp \setminus (H^{(N+1)})^\perp$.

3.1. Construction of the measure μ

Notation 3.1. We fix $\boldsymbol{x} \in (\mathbb{N}^{N+1})^{\mathbb{N}}$ such that for every $n \in \mathbb{N}$ and $j = 1,\dots,N$ both $x_n^{j+1}/(2x_n^j)$ and $x_{n+1}^1/(2x_n^{N+1})$ are natural numbers greater than n^2.

For $n \in \mathbb{N}$ and $j = 1, \dots, N+1$ we set

$$P_n = \{x \in [0,1];\ \langle x \cdot x_i \rangle \notin (1/2,1)^{N+1},\ i = 1, \dots, n\}, \tag{3.1}$$

$$\mathcal{P}_{n,j} = \left\{ \left[\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j} \right] \subset [0,1];\ i \in \mathbb{N},\ \left(\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j} \right) \subset P_n \right\},$$

$$\|\mathcal{P}_{n,j}\| = 1/(2x_n^j).$$

Notation 3.2. Let $\mathcal{A}$ be a collection of subsets of $\mathbb{R}$, and let $S \subset \mathbb{R}$. We denote

$$\mathcal{A}^S = \{V \in \mathcal{A};\ V \subset S\}.$$

Notation 3.3. Let $V \subset [0,1]$ and $x \in \mathbb{R} \setminus \{0\}$. We set

$$\mathcal{T}(x, V) = \left\{ \frac{1}{x}(V+n);\ n \in \mathbb{Z} \right\}.$$

The following remark explains the notions P_n and $\mathcal{P}_{n,j}$. I hope this clarifies these notions and the important Remark 3.5 below.

Remark 3.4. Fix some $n \in \mathbb{N}$. If $n = 1$ then set $I = [0,1]$, otherwise fix some $I \in \mathcal{P}_{n-1,N+1}$. Let $0 < j \le N+1$. Define

$$\mathcal{M}_j = \left\{ \left[\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j} \right] \subset J;\ i \in \mathbb{N} \right\},$$

$$\widetilde{\mathcal{M}}_j = \left\{ \left[\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j} \right] \subset J;\ i \text{ is an odd natural number} \right\}.$$

Clearly, $\mathcal{M}_j^J = \mathcal{T}^J(2x_n^j, [0,1])$ and $\widetilde{\mathcal{M}}_j^J = \mathcal{T}^J(x_n^j, [0,1/2]) = \{x \in [0,1];\ x \cdot x_n^j \notin (1/2,1)\}$. It is easy to see that $\mathcal{P}_{n,1}^I = \widetilde{\mathcal{M}}_1^I$. Let $0 < j \le N$. Since $x_n^{j+1}/(2x_n^j)$ is a natural number we have

$$\begin{aligned} \mathcal{P}_{n,j+1}^I &= \{V \in \mathcal{M}_{j+1}^I;\ (\exists J \in \mathcal{P}_{n,j}^I : V \subset J) \vee (V \in \widetilde{\mathcal{M}}_{j+1}^I)\} \\ &= \widetilde{\mathcal{M}}_{j+1}^I \cup \bigcup_{J \in \mathcal{P}_{n,j}^I} \mathcal{M}_{j+1}^J. \end{aligned}$$

Remark 3.5 and Lemma 3.6 below will explain some basic facts concerning the collections $\mathcal{P}_{n,j}^I$.

Remark 3.5. Let $n \in \mathbb{N}$. Since $x_n^{j+1}/(2x_n^j)$ and $x_{n+1}^1/(2x_n^{N+1})$ are natural numbers we can easily obtain the following three statements:

- $\bigcup \mathcal{P}_{n,N+1} = P_n$.
- $\mathcal{P}_{n+1,j} = \bigcup_{I \in \mathcal{P}_{n,N+1}} \mathcal{P}_{n+1,j}^I$.
- If $j \in \{1, \dots, N+1\}$, $i \in \mathbb{N}$, $I \in \mathcal{P}_{n,N+1}$ and $\left[\frac{i-1}{2x_{n+1}^j}, \frac{i+1}{2x_{n+1}^j}\right] \subset I$ then

$$\left[\frac{i-1}{2x_{n+1}^j}, \frac{i}{2x_{n+1}^j} \right] \in \mathcal{P}_{n+1,j} \quad \text{or} \quad \left[\frac{i}{2x_{n+1}^j}, \frac{i+1}{2x_{n+1}^j} \right] \in \mathcal{P}_{n+1,j}.$$

LEMMA 3.6.

(i) *If $V \in \mathcal{P}_{n,j}$, then $\|\mathcal{P}_{n,j}\| = \lambda(V)$.*
(ii) *Let $k \geq n$ and $i, j \leq N+1$ be such that $k > n$ or $j \geq i$. Let $I, J \in \mathcal{P}_{n,i}$. Then $\sharp\,\mathcal{P}^I_{k,j} = \sharp\,\mathcal{P}^J_{k,j}$.*
(iii) *Let $n > 1$, $I \in \mathcal{P}_{n-1,N+1}$ and $1 \leq j \leq i \leq N+1$. Then*

$$\sharp\,\mathcal{P}^I_{n,i} \leq 2 \sum_{R \in \mathcal{P}^I_{n,j}} \sharp\,\mathcal{P}^R_{n,i}.$$

(iv) *Let $n_1, n_2, n_3 \in \mathbb{N}$, $n_1 < n_2 \leq n_3$, $j_1, j_2, j_3 \in \{1, \ldots, N+1\}$ and $I \in \mathcal{P}_{n_1,j_1}$ be such that $n_2 < n_3$ or $j_2 \leq j_3$. Then*

$$\sharp\,\mathcal{P}^I_{n_3,j_3} \leq 2 \sum_{R \in \mathcal{P}^I_{n_2,j_2}} \sharp\,\mathcal{P}^R_{n_3,j_3}.$$

(v) *Let $n \in \mathbb{N}$ and $1 \leq j \leq N$. Then $\|\mathcal{P}_{n,j}\| \geq n^2 \|\mathcal{P}_{n,j+1}\|$.*

Proof. (i) Let $V \in \mathcal{P}_{n,j}$. Then there exists $i \in \mathbb{N}$ such that $V = \left[\frac{i-1}{2x_n^j}, \frac{i}{2x_n^j}\right]$. Thus, $\lambda(V) = 1/(2x_n^j) = \|\mathcal{P}_{n,j}\|$.

(ii) Let $x = \min(I)$ and $y = \min(J)$. It is easy to verify that $\mathcal{P}^J_{k,j} = \mathcal{P}^I_{k,j} + y - x$.

(iii) By Remark 3.5 we can easily obtain

$$\sharp\,\mathcal{P}^I_{n,i} \leq 2x_n^i \lambda(I) \leq 2 \sum_{R \in \mathcal{P}^I_{n,j}} \sharp\,\mathcal{P}^R_{n,i}.$$

(iv) Assume $n_2 < n_3$. Then

$$\sharp\,\mathcal{P}^I_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sum_{W \in \mathcal{P}^V_{n_2,N+1}} \sharp\,\mathcal{P}^W_{n_3,j_3},$$

$$\sum_{R \in \mathcal{P}^I_{n_2,j_2}} \sharp\,\mathcal{P}^R_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sum_{R \in \mathcal{P}^V_{n_2,j_2}} \sum_{W \in \mathcal{P}^R_{n_2,N+1}} \sharp\,\mathcal{P}^W_{n_3,j_3}.$$

Using (ii) and (iii) we obtain the desired inequality.

Assume $n_2 = n_3$ and $j_2 \leq j_3$. Then

$$\sharp\,\mathcal{P}^I_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sharp\,\mathcal{P}^V_{n_3,j_3},$$

$$\sum_{R \in \mathcal{P}^I_{n_2,j_2}} \sharp\,\mathcal{P}^R_{n_3,j_3} = \sum_{V \in \mathcal{P}^I_{n_2-1,N+1}} \sum_{R \in \mathcal{P}^V_{n_2,j_2}} \sharp\,\mathcal{P}^R_{n_3,j_3}.$$

Using (ii) and (iii) we obtain the desired inequality.

(v) Clearly, $\|\mathcal{P}_{n,j}\| = (x_n^{j+1}/x_n^j)\|\mathcal{P}_{n,j+1}\| \geq 2n^2 \|\mathcal{P}_{n,j+1}\|$. ∎

Lemma 3.7. *Let $W, S \subset [0,1]$ be intervals, $x \in \mathbb{R}\setminus\{0\}$ and $\lambda(S) \geq 4/|x|$. Then $\lambda(\bigcup \mathcal{T}(x,W)^S) \geq \frac{1}{2}\lambda(S)\lambda(W)$.*

Proof. Clearly, $\sharp\, \mathcal{T}(x,W)^S \geq \lambda(S)\cdot |x| - 2$. Thus,

$$\lambda\left(\bigcup \mathcal{T}(x,W)^S\right) = \frac{\lambda(W)}{|x|}\cdot \sharp\, \mathcal{T}(x,W)^S \geq \lambda(S)\lambda(W) - \frac{2\lambda(W)}{|x|}.$$

Since $\lambda(S) \geq 4/|x|$ we have

$$\lambda(S)\lambda(W) - \frac{2\lambda(W)}{|x|} \geq \frac{1}{2}\lambda(S)\lambda(W). \ \blacksquare$$

Lemma 3.8. *Let $n, s, j \in \mathbb{N}$, $n > 1$, $s, j \leq N+1$, $I \in \mathcal{P}_{n-1,s}$ and let $S \subset I$ be an interval with $\lambda(S) \geq 8\|\mathcal{P}_{n,j}\|$. Then $\lambda(\bigcup \mathcal{P}_{n,j}^S) \geq \frac{1}{4}\lambda(S)$.*

Proof. It is easy to verify that $I = \bigcup \mathcal{P}^I_{n-1,N+1}$ and $\mathcal{P}^V_{n,j} \supset \mathcal{T}(x_n^j, [0,1/2])^V$ for every $V \in \mathcal{P}^I_{n-1,N+1}$. Consequently, $\mathcal{P}^I_{n,j} \supset \mathcal{T}(x_n^j,[0,1/2])^I$. Thus $\mathcal{P}^S_{n,j} \supset \mathcal{T}(x_n^j,[0,1/2])^S$. Hence

$$\lambda\Big(\bigcup \mathcal{P}^S_{n,j}\Big) \geq \lambda\Big(\bigcup \mathcal{T}(x_n^j,[0,1/2])^S\Big).$$

We know that $\lambda(S) \geq 8\|\mathcal{P}_{n,j}\| = 4/x_n^j$. Thus Lemma 3.7 yields

$$\lambda\Big(\bigcup \mathcal{T}(x_n^j,[0,1/2])^S\Big) \geq \frac{1}{4}\lambda(S). \ \blacksquare$$

Construction 3.9. For $I = \left[\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}\right]$, where $n \in \mathbb{N}$ and $i \in \{1,\dots,2x_n^{N+1}\}$, we define

$$\mu(I) = \begin{cases} 1/\sharp\, \mathcal{P}_{n,N+1} & \text{whenever } I \in \mathcal{P}_{n,N+1}, \\ 0 & \text{whenever } I \notin \mathcal{P}_{n,N+1}. \end{cases} \tag{3.2}$$

Now we use the standard mass distribution principle (see e.g. [2, Proposition 1.7]) to extend μ to the desired measure.

We also set

$$P = \left\{x \in [0,1];\ \forall i \in \mathbb{N} : \langle x\cdot x_i\rangle \notin (1/2,1)^{N+1}\right\}. \tag{3.3}$$

We can easily obtain the following properties of the measure μ.

Lemma 3.10. *The measure μ is a continuous Radon probability measure and the support of μ is a subset of P.*

Proof. Let $x \in [0,1]$ and $n \in \mathbb{N}$. Then there exists $1 \leq i \leq 2x_n^{N+1}$ such that $x \in \left[\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}\right]$. By (3.2) we have

$$\mu(\{x\}) \leq \mu\left(\left[\frac{i-1}{2x_n^{N+1}}, \frac{i}{2x_n^{N+1}}\right]\right) \leq \frac{1}{\sharp\, \mathcal{P}_{n,N+1}}.$$

Since $\lim_{n\to\infty} 1/\sharp\, \mathcal{P}_{n,N+1} = 0$ we have $\mu(\{x\}) = 0$.

By (3.2) and Remark 3.5 the support of μ is a subset of $\bigcup \mathcal{P}_{n,N+1} = P_n$ for every $n \in \mathbb{N}$. But by (3.1), $P = \bigcap_{n\in\mathbb{N}} P_n$. ∎

3.2. Verification of $\mu \notin (H^{(N+1)})^{\perp}$

LEMMA 3.11. *The set P is a closed $H^{(N+1)}$-set and $\mu(P) = 1$.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_{N+1}) \in \mathbb{Z}^{N+1} \setminus \{0\}$. We find the largest $i \le N+1$ such that $\alpha_i \neq 0$. Since $\lim_{n\to\infty} x_n^j/x_n^i = 0$ for every $1 \le j < i$, we have

$$\lim_{n\to\infty} |(x_n, \alpha)| = \lim_{n\to\infty} \Big|\sum_{j=1}^{i} x_n^j \alpha_j\Big| = \lim_{n\to\infty} x_n^i \Big|\sum_{j=1}^{i} \frac{x_n^j \alpha_j}{x_n^i}\Big| = |\alpha_i| \lim_{n\to\infty} x_n^i = \infty.$$

Thus $\{x_n\}_{n\in\mathbb{N}} \in \mathcal{Q}(\mathbb{N}^{N+1})$ and therefore $P \in H^{(N+1)}$. By Lemma 3.10 we have $\mu(P) = 1$. ∎

3.3. Verification of $\mu \in (H_{10}^{(N)*})^{\perp}$. Fix $X \in H_{10}^{(N)*}$. We find an open interval $W = \prod_{j=1}^{N} W_j \subset [0,1]^N$ and $\boldsymbol{z} \in \mathcal{Q}(\mathbb{R}\setminus\{0\}^N)$ witnessing $X \in H_{10}^{(N)*}$. Thus, we have

$$\Big|\frac{z_i^{j+1}\lambda(W_j)}{z_i^j}\Big| \ge 10 \quad \text{for all } i \in \mathbb{N},\ j \in \{1, \ldots, N-1\}. \tag{3.4}$$

Let $0 \le \sigma \le \rho \le N$ be integers. We set

$$\begin{aligned} A_{k,\sigma,\rho} &= \{x \in [0,1];\ \exists j \in \mathbb{N},\ \sigma < j \le \rho : \langle x \cdot z_k^j\rangle \notin W_j\},\\ A_k &= \{x \in [0,1];\ \forall i \le k : \langle x \cdot z_i\rangle \notin W\} = \bigcap_{i\le k} A_{i,0,N},\\ A &= \bigcap_{k\in\mathbb{N}} A_k = \bigcap_{k\in\mathbb{N}} A_{k,0,N}. \end{aligned}$$

We have $X \subset A$. We want to show that $\mu(X) = 0$, so it is sufficient to prove $\mu(A) = 0$.

Further in this section fix a constant $l \in \mathbb{N}$ such that

$$l > 100 \quad \text{and} \quad l > 1/\lambda(W_j), \quad j = 1, \ldots, N. \tag{3.5}$$

NOTATION 3.12. Let $n, k \in \mathbb{N}$, $S, T \subset [0,1]$ and $\mathcal{D}$ be a collection of subsets of $[0,1]$. We define

$$\mathcal{V}(\mathcal{D}, T) = \{V \in \mathcal{D};\ V \cap T = \emptyset\},$$

and if $\mathcal{P}_{n,N+1}^S \neq \emptyset$, then we set

$$\mu_{n,k}^S = 1 - \frac{\sharp\, \mathcal{V}(\mathcal{P}_{n,N+1}, A_k)^S}{\sharp\, \mathcal{P}_{n,N+1}^S} \quad \text{and} \quad \mu_{n,k} = \mu_{n,k}^{[0,1]}.$$

Lemma 3.13.

(i) *$\mu(A) \le \mu_{n,k}$ for all $n, k \in \mathbb{N}$.*
(ii) *If $n, s, k \in \mathbb{N}$ and $n \ge s$ then $\mu_{n,k} \le \sup\{\mu_{n,k}^V;\ V \in \mathcal{P}_{s,N+1}\} \cdot \mu_{s,k}$.*

Proof. (i) We have

$$A \cap P \subset A_k \cap P \subset A_k \cap P_n \subset \bigcup(\mathcal{P}_{n,N+1} \setminus \mathcal{V}(\mathcal{P}_{n,N+1}, A_k)). \tag{3.6}$$

Using Lemma 3.10, (3.6) and (3.2) we conclude that

$$\begin{aligned}\mu(A) &= \mu(A \cap P) \le \mu\Big(\bigcup(\mathcal{P}_{n,N+1} \setminus \mathcal{V}(\mathcal{P}_{n,N+1}, A_k))\Big)\\ &= \sum_{J \in \mathcal{P}_{n,N+1} \setminus \mathcal{V}(\mathcal{P}_{n,N+1}, A_k)} \mu(J) = \frac{\sharp(\mathcal{P}_{n,N+1} \setminus \mathcal{V}(\mathcal{P}_{n,N+1}, A_k))}{\sharp\,\mathcal{P}_{n,N+1}} = \mu_{n,k}.\end{aligned}$$

(ii) It is easy to verify that

$$\begin{aligned}\mu_{n,k} &= 1 - \frac{\sharp\,\mathcal{V}(\mathcal{P}_{n,N+1}, A_k)}{\sharp\,\mathcal{P}_{n,N+1}} = 1 - \sum_{V \in \mathcal{P}_{s,N+1}} \frac{\sharp\,\mathcal{V}(\mathcal{P}_{n,N+1}, A_k)^V}{\sharp\,\mathcal{P}_{s,N+1} \cdot \sharp\,\mathcal{P}_{n,N+1}^V}\\ &= \frac{1}{\sharp\,\mathcal{P}_{s,N+1}} \sum_{V \in \mathcal{P}_{s,N+1}} 1 - \frac{\sharp\,\mathcal{V}(\mathcal{P}_{n,N+1}, A_k)^V}{\sharp\,\mathcal{P}_{n,N+1}^V} = \frac{\sum_{V \in \mathcal{P}_{s,N+1}} \mu_{n,k}^V}{\sharp\,\mathcal{P}_{s,N+1}}\\ &= \frac{\sum_{V \in \mathcal{P}_{s,N+1} \setminus \mathcal{V}(\mathcal{P}_{s,N+1}, A_k)} \mu_{n,k}^V}{\sharp\,\mathcal{P}_{s,N+1}},\end{aligned}$$

where the last equality follows from the fact that $\mu_{n,k}^V = 0$ for all $V \in \mathcal{V}(\mathcal{P}_{s,N+1}, A_k)$. Thus, we have

$$\begin{aligned}\mu_{n,k} &\le \sup\{\mu_{n,k}^V;\ V \in \mathcal{P}_{s,N+1}\} \cdot \frac{\sharp(\mathcal{P}_{s,N+1} \setminus \mathcal{V}(\mathcal{P}_{s,N+1}, A_k))}{\sharp\,\mathcal{P}_{s,N+1}}\\ &= \sup\{\mu_{n,k}^V;\ V \in \mathcal{P}_{s,N+1}\} \cdot \mu_{s,k}. \ \blacksquare\end{aligned}$$

We assume that $k \in \mathbb{N}$ is fixed in the following definition and in Lemmas 3.15–3.17.

Definition 3.14. Let $S \subset [0,1]$ be an interval and $j \in \{0, \ldots, N-1\}$. We inductively define

$$\begin{aligned}\mathcal{K}_{j,j+1}(S) &= \mathcal{T}(z_k^{j+1}, W_{j+1})^S,\\ \mathcal{K}_{j,t}(S) &= \bigcup_{L \in \mathcal{K}_{j,t-1}(S)} \mathcal{T}(z_k^t, W_t)^L, \qquad t = j+2, \ldots, N.\end{aligned}$$

Lemma 3.15.

(i) *For every $Z \in \mathcal{K}_{j,t}(S)$ we have $\lambda(Z) = \lambda(W_t)/|z_k^t| \ge 1/(l|z_k^t|)$.*
(ii) *Let $K, L \subset [0,1]$ and $K \cap L = \emptyset$. Then $\mathcal{K}_{j,t}(K) \cap \mathcal{K}_{j,t}(L) = \emptyset$.*
(iii) *Let $K, L \in \mathcal{K}_{j,t}(S)$. Then $K = L$ or $K \cap L = \emptyset$.*
(iv) *$\bigcup \mathcal{K}_{j,t}(S) \cap A_{k,j,t} = \emptyset$.*

Proof. Statements (i)–(iii) are easy to verify.

(iv) It is straightforward to show that

$$\bigcup \mathcal{K}_{j,t}(S) \subset \bigcap_{i=j+1}^{t} \bigcup \mathcal{T}(z_k^i, W_i)^S,$$

$$A_{k,j,t} = \bigcup_{i=j+1}^{t} \Big([0,1] \setminus \bigcup \mathcal{T}(z_k^i, W_i)^{\mathbb{R}}\Big).$$

Since $\mathcal{T}(z_k^i, W_i)^S \subset \mathcal{T}(z_k^i, W_i)^{\mathbb{R}}$ for every $1 \le i \le N$, the right-hand sides above are disjoint. ∎

LEMMA 3.16. *Let $0 \le j < t \le N$ and let $S \subset [0,1]$ be an interval with $\lambda(S) \ge 4/|z_k^{j+1}|$. Then $\lambda(\bigcup \mathcal{K}_{j,t}(S)) \ge \lambda(S)(2l)^{j-t}$.*

Proof. We argue by induction. First, we assume that $t = j+1$. Then $\mathcal{K}_{j,t}(S) = \mathcal{T}(z_k^t, W_t)^S$ and $\lambda(S) \ge 4/|z_k^t|$. We have

$$\begin{aligned} \lambda\Big(\bigcup \mathcal{K}_{j,t}(S)\Big) &= \lambda\Big(\bigcup \mathcal{T}(z_k^t, W_t)^S\Big) \overset{\text{L.3.7}}{\ge} \frac{1}{2}\lambda(S)\lambda(W_t) \\ &\overset{(3.5)}{\ge} \lambda(S)(2l)^{-1} = \lambda(S)(2l)^{j-t}. \end{aligned}$$

Now, we assume that $t > j+1$ and that we have already proved

$$\lambda\Big(\bigcup \mathcal{K}_{j,t-1}(S)\Big) \ge \lambda(S)(2l)^{j-t+1}. \tag{3.7}$$

Let $L \in \mathcal{K}_{j,t-1}(S)$ be arbitrary. Then $\lambda(L) = \lambda(W_{t-1})/|z_k^{t-1}|$. By (3.4) we have $\lambda(W_{t-1})/|z_k^{t-1}| \ge 10/|z_k^t|$. Thus

$$\lambda(L) \ge 4/|z_k^t|. \tag{3.8}$$

We obtain

$$\begin{aligned} \lambda\Big(\bigcup \mathcal{K}_{j,t}(S)\Big) &= \lambda\Big(\bigcup \bigcup_{L \in \mathcal{K}_{j,t-1}(S)} \mathcal{T}(z_k^t, W_t)^L\Big) \\ &\overset{\text{L.3.15(ii),(iii)}}{=} \sum_{L \in \mathcal{K}_{j,t-1}(S)} \lambda\Big(\bigcup \mathcal{T}(z_k^t, W_t)^L\Big) \\ &\overset{\text{L.3.7}}{\ge} \sum_{L \in \mathcal{K}_{j,t-1}(S)} \frac{1}{2}\lambda(L)\lambda(W_t) \\ &\overset{(3.5)}{\ge} \sum_{L \in \mathcal{K}_{j,t-1}(S)} \lambda(L)(2l)^{-1} \\ &\overset{\text{L.3.15(iii)}}{=} (2l)^{-1}\lambda\Big(\bigcup \mathcal{K}_{j,t-1}(S)\Big) \\ &\overset{(3.7)}{\ge} \lambda(S)(2l)^{j-t}, \end{aligned}$$

where (3.8) was used to verify the condition of Lemma 3.7. ∎

Lemma 3.17. *Let $0 \le \sigma < \rho \le N$, $1 \le s \le N$, $1 \le j \le N+1$, n, k be natural numbers and $I \in \mathcal{P}_{n,s}$. Suppose that*

$$n \ge l^2, \tag{3.9}$$

$$n\|\mathcal{P}_{n,s+1}\| \ge \frac{1}{|z_k^i|} \ge (n+1)\|\mathcal{P}_{n+1,j}\|, \qquad \sigma < i \le \rho. \tag{3.10}$$

Then

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n+1,j}, A_{k,\sigma,\rho})}{\sharp\,\mathcal{P}^I_{n+1,j}} \ge \frac{1}{4}(2l)^{\sigma-\rho}. \tag{3.11}$$

Proof. By Lemma 3.15(iii),(iv) we have

$$\mathcal{V}(\mathcal{P}^I_{n+1,j}, A_{k,\sigma,\rho}) \supset \mathcal{P}^{\bigcup \mathcal{K}_{\sigma,\rho}(I)}_{n+1,j} \supset \bigcup_{K\in\mathcal{K}_{\sigma,\rho}(I)} \mathcal{P}^K_{n+1,j}. \tag{3.12}$$

By (3.12), Lemma 3.15(iii) and Lemma 3.6(i) we have

$$\begin{aligned}\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{n+1,j}, A_{k,\sigma,\rho})}{\sharp\,\mathcal{P}^I_{n+1,j}} &\ge \sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \frac{\sharp\,\mathcal{P}^K_{n+1,j}}{\sharp\,\mathcal{P}^I_{n+1,j}} = \sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \frac{\lambda(\bigcup \mathcal{P}^K_{n+1,j})}{\lambda(\bigcup \mathcal{P}^I_{n+1,j})}\\ &\ge \sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \frac{\lambda(\bigcup \mathcal{P}^K_{n+1,j})}{\lambda(I)}.\end{aligned}$$

Thus, it is enough to verify that

$$\sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \lambda\Big(\bigcup \mathcal{P}^K_{n+1,j}\Big) \ge \frac{1}{4}\lambda(I)(2l)^{\sigma-\rho}. \tag{3.13}$$

By (3.9) and (3.5) we have $n \ge l^2$ and $l > 4$. By Lemma 3.6(v) and (3.10),

$$\lambda(I) = \|\mathcal{P}_{n,s}\| \ge n^2\|\mathcal{P}_{n,s+1}\| \ge \frac{n}{z_k^{\sigma+1}} \ge \frac{4}{z_k^{\sigma+1}}.$$

Thus Lemma 3.16 yields

$$\lambda\Big(\bigcup \mathcal{K}_{\sigma,\rho}(I)\Big) \ge \lambda(I)(2l)^{\sigma-\rho}. \tag{3.14}$$

Let $K \in \mathcal{K}_{\sigma,\rho}(I)$. From Lemma 3.15(i), (3.10) and $n+1 > 8l$ we have

$$\lambda(K) \ge \frac{1}{l z_k^{\rho}} \ge 8\|\mathcal{P}_{n+1,j}\|.$$

Thus Lemma 3.8 implies

$$\lambda\Big(\bigcup \mathcal{P}^K_{n+1,j}\Big) \ge \frac{1}{4}\lambda(K). \tag{3.15}$$

By (3.15), Lemma 3.15(iii) and (3.14) we have

$$\sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \lambda\Big(\bigcup \mathcal{P}^K_{n+1,j}\Big) \ge \frac{1}{4}\sum_{K\in\mathcal{K}_{\sigma,\rho}(I)} \lambda(K) = \frac{1}{4}\lambda\Big(\bigcup \mathcal{K}_{\sigma,\rho}(I)\Big) \ge \frac{1}{4}\lambda(I)(2l)^{\sigma-\rho}.$$

So, we have verified (3.13). ■

LEMMA 3.18. *Let $n_0 \le n_1 < n_2 \in \mathbb{N}$, $1 \le j_1 < j_2 < j_3 \le N+1$ and $T_1, T_2 \subset [0,1]$. If there exist $\alpha_1, \alpha_2 > 0$ such that*

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I_1}_{n_1+1,j_2}, T_1)}{\sharp\,\mathcal{P}^{I_1}_{n_1+1,j_2}} \ge \alpha_1, \tag{3.16}$$

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I_2}_{n_2+1,j_3}, T_2)}{\sharp\,\mathcal{P}^{I_2}_{n_2+1,j_3}} \ge \alpha_2, \tag{3.17}$$

for every $I_1 \in \mathcal{P}_{n_0,j_1}$ and $I_2 \in \mathcal{P}_{n_2,j_2}$, then

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2+1,j_3}, T_1\cup T_2)}{\sharp\,\mathcal{P}^{I}_{n_2+1,j_3}} \ge \frac{1}{4}\alpha_1\alpha_2$$

for every $I \in \mathcal{P}_{n_0,j_1}$.

Proof. Let $I \in \mathcal{P}_{n_0,j_1}$. Clearly,

$$\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2,j_2}, T_1) \ge \sum_{V\in\mathcal{V}(\mathcal{P}^{I}_{n_1+1,j_2},T_1)} \sharp\,\mathcal{P}^{V}_{n_2,j_2}. \tag{3.18}$$

Hence

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2,j_2}, T_1)}{\sharp\,\mathcal{P}^{I}_{n_2,j_2}} \overset{(3.18),\text{L.3.6(iv)}}{\ge} \frac{\sum_{V\in\mathcal{V}(\mathcal{P}^{I}_{n_1+1,j_2},T_1)} \sharp\,\mathcal{P}^{V}_{n_2,j_2}}{2\sum_{W\in\mathcal{P}^{I}_{n_1+1,j_2}} \sharp\,\mathcal{P}^{W}_{n_2,j_2}} \overset{\text{L.3.6(ii)}}{\ge} \frac{\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_1+1,j_2}, T_1)}{2\,\sharp\,\mathcal{P}^{I}_{n_1+1,j_2}} \overset{(3.16)}{\ge} \frac{1}{2}\alpha_1. \tag{3.19}$$

Clearly,

$$\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2+1,j_3}, T_1\cup T_2) \ge \sum_{V\in\mathcal{V}(\mathcal{P}^{I}_{n_2,j_2},T_1)} \sharp\,\mathcal{V}(\mathcal{P}^{V}_{n_2+1,j_3}, T_2). \tag{3.20}$$

Therefore

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2+1,j_3}, T_1\cup T_2)}{\sharp\,\mathcal{P}^{I}_{n_2+1,j_3}} \overset{(3.20),\text{L.3.6(iv)}}{\ge} \frac{\sum_{V\in\mathcal{V}(\mathcal{P}^{I}_{n_2,j_2},T_1)} \sharp\,\mathcal{V}(\mathcal{P}^{V}_{n_2+1,j_3}, T_2)}{2\sum_{W\in\mathcal{P}^{I}_{n_2,j_2}} \sharp\,\mathcal{P}^{W}_{n_2+1,j_3}} \overset{(3.17)}{\ge} \alpha_2\frac{\sharp\,\mathcal{V}(\mathcal{P}^{I}_{n_2,j_2}, T_1)}{2\,\sharp\,\mathcal{P}^{I}_{n_2,j_2}} \overset{(3.19)}{\ge} \frac{1}{4}\alpha_2\alpha_1. \ ■$$

Lemma 3.19. *There exists $\varepsilon > 0$ such that for every $n, k \in \mathbb{N}$ there exist $\tilde{n} \in \mathbb{N}$ and $\tilde{k} \in \mathbb{N}$ such that $\tilde{n} > n$, $\tilde{k} > k$ and*

$$\frac{\sharp\, \mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1}, A_{\tilde{k},0,N})}{\sharp\, \mathcal{P}^I_{\tilde{n},N+1}} \geq \varepsilon$$

for every $I \in \mathcal{P}_{n,N+1}$.

Proof. Set $\varepsilon = 2(32l)^{-N}$. Let $n, k \in \mathbb{N}$. We set $n_0 = \max\{n+1, l^2\}$. We will construct $\tilde{k} > k$, $s \leq N$ and sequences $n_0 < n_1 < \cdots < n_s$ and $0 = v_0 < v_1 < \cdots < v_s = N$ such that

$$\forall 0 < i \leq s\ \forall v_{i-1} < j \leq v_i:\ n_i \|\mathcal{P}_{n_i, v_{i-1}+2}\| \geq \frac{1}{|z^j_{\tilde{k}}|} > (n_i + 1)\|\mathcal{P}_{n_i+1, v_i+1}\|.$$

Since $\boldsymbol{z} \in \mathcal{Q}(\mathbb{R}\setminus\{0\}^N)$ and (3.4) holds we have $\lim |z^1_i| = \infty$ and $|z^{j+1}_i| \geq 10|z^j_i|$ for every $i \in \mathbb{N}, j < N$. Thus, we can find $\tilde{k} > k$ such that $1/|z^1_{\tilde{k}}| \leq \|\mathcal{P}_{n_0+1,2}\|(n_0+1)$. We set $v_0 = 0$. Assume that we have already constructed $n_0, \ldots, n_i$ and $v_0, \ldots, v_i$ for some $i \geq 0$. If $v_i = N$ we set $s = i$ and we are done. If $v_i < N$ we find $n_{i+1} \in \mathbb{N}$ such that

$$n_{i+1}\|\mathcal{P}_{n_{i+1}, v_i+2}\| \geq \frac{1}{|z^{v_i+1}_{\tilde{k}}|} > (n_{i+1}+1)\|\mathcal{P}_{n_{i+1}+1, v_i+2}\|.$$

Further we find the largest $v_{i+1} \in \{v_i + 1, \ldots, N\}$ such that

$$\frac{1}{|z^{v_{i+1}}_{\tilde{k}}|} > (n_{i+1}+1)\|\mathcal{P}_{n_{i+1}+1, v_{i+1}+1}\|$$

and we are done. We set $\tilde{n} = n_s + 1$.

We use Lemma 3.17 replacing σ, ρ, s, j, n, k by v_{i-1}, v_i, $v_{i-1}+1$, v_i+1, n_i, $\tilde{k}$ respectively to obtain

$$\frac{\sharp\, \mathcal{V}(\mathcal{P}^V_{n_i+1, v_i+1}, A_{\tilde{k}, v_{i-1}, v_i})}{\sharp\, \mathcal{P}^V_{n_i+1, v_i+1}} \geq \frac{1}{4}(2l)^{v_{i-1}-v_i} \tag{3.21}$$

for every $V \in \mathcal{P}_{n_i, v_{i-1}+1}$ and $1 \leq i \leq s$.

We will prove by induction that

$$\frac{\sharp\, \mathcal{V}(\mathcal{P}^V_{n_j+1, v_j+1}, A_{\tilde{k}, v_0, v_j})}{\sharp\, \mathcal{P}^V_{n_j+1, v_j+1}} \geq 4^{-j}(2l)^{-v_j} \cdot 4^{-j+1} \tag{3.22}$$

for every $V \in \mathcal{P}_{n_1,1}$ and $1 \leq j \leq s$.

By (3.21) we have (3.22) for $j = 1$.

Suppose that $1 < j \leq s$ and (3.22) holds for $j - 1$. Thus, by (3.21) and Lemma 3.18 replacing n_0, n_1, n_2, j_1, j_2, j_3, T_1, T_2 by n_1, n_{j-1}, n_j, 1, $v_{j-1}+1$,

v_j+1, $A_{\tilde{k},v_0,v_{j-1}}$, $A_{\tilde{k},v_{j-1},v_j}$ respectively we have

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{n_j+1,v_j+1},A_{\tilde{k},v_0,v_j})}{\sharp\,\mathcal{P}^V_{n_j+1,v_j+1}} = \frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{n_j+1,v_j+1},A_{\tilde{k},v_0,v_{j-1}}\cup A_{\tilde{k},v_{j-1},v_j})}{\sharp\,\mathcal{P}^V_{n_j+1,v_j+1}}$$
$$\geq \frac{1}{4}(4^{-j+1}(2l)^{-v_{j-1}}\cdot 4^{-j+2})\left(\frac{1}{4}(2l)^{v_{j-1}-v_j}\right)$$
$$= 4^{-j}(2l)^{-v_j}\cdot 4^{-j+1}.$$

Thus we obtain (3.22).

Since $v_s=N$, $s\leq N$ and (3.22) holds, we have

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^V_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^V_{\tilde{n},N+1}} \geq 4^{-s}(2l)^{-N}\cdot 4^{-s+1}\geq 2\varepsilon \tag{3.23}$$

for every $V\in\mathcal{P}_{n_1,1}$. Fix $I\in\mathcal{P}_{n,N+1}$. Clearly,

$$\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})\geq \sum_{V\in\mathcal{P}^I_{n_1,1}}\sharp\,\mathcal{V}(\mathcal{P}^V_{\tilde{n},N+1},A_{\tilde{k},0,N}). \tag{3.24}$$

By (3.24), (3.23) and Lemma 3.6(iv),(ii) we have

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}} \geq \frac{\sum_{V\in\mathcal{P}^I_{n_1,1}}\sharp\,\mathcal{V}(\mathcal{P}^V_{\tilde{n},N+1},A_{\tilde{k},0,N})}{2\sum_{W\in\mathcal{P}^I_{n_1,1}}\sharp\,\mathcal{P}^W_{\tilde{n},N+1}}\geq\varepsilon. \ \blacksquare$$

Proof of Lemma 2.4. We need to show that $\mu(A)=0$. Set $\varepsilon=2(32l)^{-N}$. Let $n,k\in\mathbb{N}$. By Lemma 3.19 there exist $\tilde{n},\tilde{k}\in\mathbb{N}$ such that

$$\frac{\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}}\geq\varepsilon$$

for every $I\in\mathcal{P}_{n,N+1}$. Since $A_{\tilde{k}}\subset A_{\tilde{k},0,N}$ we have

$$\mu^I_{\tilde{n},\tilde{k}} = \frac{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}-\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k}})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}}$$
$$\leq \frac{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}-\sharp\,\mathcal{V}(\mathcal{P}^I_{\tilde{n},N+1},A_{\tilde{k},0,N})}{\sharp\,\mathcal{P}^I_{\tilde{n},N+1}}\leq 1-\varepsilon$$

for every $I\in\mathcal{P}_{n,N+1}$. Thus by Lemma 3.13(ii) we have $\mu_{\tilde{n},\tilde{k}}\leq(1-\varepsilon)\mu_{n,k}$. Hence $\inf\{\mu_{n,k};\ n,k\in\mathbb{N}\}=0$, and Lemma 3.13(i) yields

$$0\leq\mu(A)\leq\inf\{\mu_{n,k};\ n,k\in\mathbb{N}\}=0. \ \blacksquare$$

4. Proof of Theorem 2.5

NOTATION 4.1. Let $N, n \in \mathbb{N}$, $\boldsymbol{a} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$, $y \in \mathbb{R}$, and let $J \subset \mathbb{R}$ and $\mathcal{J} = \prod_{j=1}^N J^j \subset [0,1]^N$ be open intervals. We set

$$T(y, J) = \{x \in [0,1];\ \langle xy \rangle \in \langle J \rangle\},$$

$$H_n(\boldsymbol{a}, \mathcal{J}) = [0,1] \setminus \bigcap_{p=1}^{N} T(a_n^p, J^p), \qquad H(\boldsymbol{a}, \mathcal{J}) = \bigcap_{n\in\mathbb{N}} H_n(\boldsymbol{a}, \mathcal{J}).$$

NOTATION 4.2. Let $m \in \mathbb{N}$, $I \subset [0,1]^m$ be an interval and let $\boldsymbol{z} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^m)$. Then we define

$$H(\boldsymbol{z}, I) = \{x \in [0,1];\ \forall k \in \mathbb{N} : \langle x \cdot z_k \rangle \notin I\}.$$

REMARK 4.3. Let $m \in \mathbb{N}$.

(i) If $A \in H^{(m)*}$ then there exist $\boldsymbol{z} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^m)$ and an open interval $W \subset [0,1]^m$ such that $A \subset H(\boldsymbol{z}, W)$.

(ii) If $I \subset J \subset [0,1]^m$ are open intervals and $\boldsymbol{r} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^m)$, then $H(\boldsymbol{r}, J) \subset H(\boldsymbol{r}, I)$.

LEMMA 4.4. *Let $N \in \mathbb{N}$, $\boldsymbol{a} = \{a_j\} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$, $\{j_k\}$ be an increasing sequence of integers and $\mathcal{J} \subset \mathcal{U} \subset [0,1]^N$ be open intervals. Then:*

(i) $\{a_{j_k}\} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$.

(ii) $H(\boldsymbol{a}, \mathcal{U}) \subset H(\{a_{j_k}\}, \mathcal{U})$.

(iii) $H(\boldsymbol{a}, \mathcal{U}) = \bigcap_{n\in\mathbb{N}} H_n(\boldsymbol{a}, \mathcal{U})$.

(iv) *Let $L \in \mathbb{R}^{N\times N}$ be a nonsingular matrix. Then there exists a finite set $M \subset \mathbb{N}$ such that for every increasing sequence $\{v_k\}$ of elements from $\mathbb{N} \setminus M$ we have $\{L(a_{v_k})\} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$.*

(v) *Let $y \in \mathbb{R} \setminus \{0\}$ and $J \subset [0,1]$ be an open interval. Then $T(y, J) = \bigcup_{n\in\mathbb{Z}} \frac{1}{y}(J + n) \cap [0,1] = \bigcup \mathcal{T}(y, J) \cap [0,1]$.*

(vi) *Let $m \in \mathbb{Z} \setminus \{0\}$, $y \in \mathbb{R} \setminus \{0\}$ and $u, r \in \mathbb{R}$. Then $T(y, B(u, r)) \supset T(y/m, B(u/m, r/|m|))$, where $B(x, s) = (x - s, x + s)$ for $s > 0$.*

(vii) *Let $y \in \mathbb{R} \setminus \{0\}$, and let $J \subset \mathbb{R}$ and $V \subset \langle J \rangle$ be open intervals. Then $T(y, J) \supset T(y, V)$.*

Proof. (i)–(iii), (v) and (vii) are trivial.

(iv) We set $M = \{i \in \mathbb{N};\ \exists s \le N : (L(a_i))^s = 0\}$. Let $\{v_k\}$ be an increasing sequence of elements from $\mathbb{N} \setminus M$. Then $\{L(a_{v_k})\} \in ((\mathbb{R} \setminus \{0\})^N)^{\mathbb{N}}$. Let $\alpha \in \mathbb{Z}^N \setminus \{0\}$. Then $L^T(\alpha)$ is a nonzero vector, where L^T is the transpose of the matrix L. Thus we have

$$\lim_{n\to\infty} |(L(a_{v_k}), \alpha)| = \lim_{n\to\infty} |(a_{v_k}, L^T(\alpha))| = \infty.$$

Thus, $L(a_{v_k}) \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$.

(vi) Clearly, $\mathcal{T}(y, B(u,r)) \supset \mathcal{T}(y/m, B(u/m, r/|m|))$. Thus (vi) follows from (v). ∎

We will use the following well known approximation theorem.

Lemma 4.5 ([10, Dirichlet's Theorem on Simultaneous Approximations]). *Let $\alpha_1, \ldots, \alpha_n$ be real numbers and $Q > 1$ be an integer. Then there exist integers $q, p_1, \ldots, p_n$ with $1 \le q < Q^n$ and $|\alpha_i q - p_i| \le 1/Q$ for all $1 \le i \le n$.*

Lemma 4.6. *Let $N \in \mathbb{N}$, $\boldsymbol{a} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$ and let $\mathcal{U}_n = U^1 \times \cdots \times U^{N-1} \times U_n^N \subset [0,1]^N$ for $n \in \mathbb{N}$ be open intervals. If there exists $\alpha > 0$ such that $\lambda(U_n^N) \ge \alpha$ for all $n \in \mathbb{N}$ then there exist an increasing sequence $\{j_n\}$ of positive integers and an open interval $\mathcal{J} = U^1 \times \cdots \times U^{N-1} \times J^N \subset [0,1]^N$ such that for every $n \in \mathbb{N}$ we have*

(i) $4\lambda(J^N) \ge \lambda(U_{j_n}^N)$,
(ii) $H_n(\{a_{j_n}\}, \mathcal{U}_{j_n}) \subset H_n(\{a_{j_n}\}, \mathcal{J})$.

Proof. Since $\inf\{\lambda(U_n^N);\, n \in \mathbb{N}\} \ge \alpha > 0$ there exists an increasing sequence $\{v_n\}$ of positive integers such that

$$4 \inf\{\lambda(U_{v_n}^N);\, n \in \mathbb{N}\} > 3 \sup\{\lambda(U_{v_n}^N);\, n \in \mathbb{N}\}.$$

We find $l \in \mathbb{N}$ such that

$$2/l \le \inf\{\lambda(U_{v_n}^N);\, n \in \mathbb{N}\} < 3/l.$$

For all $j \in \mathbb{N}$ we find $b_j \in \mathbb{N}_0$ and an open interval $J_j^N = (b_j/l, (b_j+1)/l)$ such that $J_j^N \subset U_{v_j}^N$. Since the set $\{J_j^N;\, j \in \mathbb{N}\}$ is finite there exists an increasing sequence $\{p_n\}$ of positive integers and an open interval J^N such that $J_{p_n}^N = J^N$ for all $n \in \mathbb{N}$. We set $\mathcal{J} = U^1 \times \cdots \times U^{N-1} \times J^N$ and $j_n = v_{p_n}$. Thus,

$$H_n(\{a_{j_n}\}, \mathcal{U}_{j_n}) \subset H_n(\{a_{j_n}\}, \mathcal{J})$$

for every $n \in \mathbb{N}$. Clearly,

$$4\lambda(J^N) = \frac{4}{l} \ge \frac{4}{3} \inf\{\lambda(U_{v_n}^N);\, n \in \mathbb{N}\} > \sup\{\lambda(U_{v_n}^N);\, n \in \mathbb{N}\} \ge \lambda(U_{j_m}^N)$$

for all $m \in \mathbb{N}$. ∎

The following lemma was inspired by Zajíček [12].

Lemma 4.7. *Let $y, z \in \mathbb{R} \setminus \{0\}$, $y \ne z$, let $U = B(u, r_1)$ and $V = B(v, r_2)$ be subsets of $[0,1]$, and $\delta \le \min\{\lambda(V)/|y|, \lambda(U)/|z|\}$. If $4|y| > 3|z|$ then*

$$T(y,V) \cap T(z,U) \supset T(z, B(u, |z|\delta/4)) \cap T(y-z, B(v-u, r_2/4)).$$

Proof. Since $|z|\delta/4 \le r_1$ we have $B(u, |z|\delta/4) \subset U$. Thus

$$T(z,U) \supset T(z, B(u, |z|\delta/4)).$$

Let $x \in T(z, B(u, |z|\delta/4)) \cap T(y-z, B(v-u, r_2/4))$. Then there exist $\xi \in B(0, r_2/4)$, $\mu \in B(0, |z|\delta/4)$ and $m, n \in \mathbb{Z}$ such that

$$x = (\xi + v - u + n)\frac{1}{y-z}, \qquad x = (\mu + u + m)\frac{1}{z}.$$

Thus, $x = (\xi+\mu+v+m+n)\frac{1}{y}$. Since $|\xi+\mu| \le r_2/4+|z|\delta/4 < r_2/4+|y|\delta/3 < r_2/4 + 2r_2/3 < r_2$, we have $\xi + \mu + v \in V$. Thus, $x \in T(y, V)$. ∎

LEMMA 4.8. *Let $N \in \mathbb{N}$, $\boldsymbol{a} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$, let $\mathcal{U} = \prod_{i=1}^N U^i \subset [0,1]^N$ be an open interval, $L \in \mathbb{N}$ and $\delta_j = \min\{\lambda(U^i)/|a_j^i|;\ i = 1, \ldots, N\}$ for every $j \in \mathbb{N}$. Then there exist a nonsingular matrix $\mathcal{L} \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{v_n\}$ of positive integers and an open interval $\mathcal{J} = \prod_{i=1}^N J^i \subset [0,1]^N$ such that*

(a) $\boldsymbol{x} := \{\mathcal{L}(a_{v_n})\} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$,
(b) $\forall n \in \mathbb{N} : H_n(\{a_{v_n}\}, \mathcal{U}) \subset H_n(\boldsymbol{x}, \mathcal{J})$,
(c) $\forall n \in \mathbb{N}\ \forall i < N : |x_n^N \lambda(J^i)/x_n^i| \ge L$,
(d) $\lambda(J^N)/|x_n^N| \ge \delta_{v_n}/16$.

Proof. Passing to a subsequence and permuting indices if necessary, we can assume that $|a_n^i| < |a_n^{i+1}|$ for all $n \in \mathbb{N}$ and $i < N$. We find $Q \in \mathbb{N}$ such that $1/Q < \min\{\lambda(U^i);\ i = 1, \ldots, N\}/(8L)$. By Lemma 4.5 for every $j \in \mathbb{N}$ there exist $q_j, p_j^1, \ldots, p_j^{N-1} \in \mathbb{Z}$ such that

$$1 \le q_j \le Q^{N-1},$$

$$\left| q_j \frac{a_j^i}{a_j^N} - p_j^i \right| \le \frac{1}{Q}, \qquad i = 1, \ldots, N-1. \tag{4.1}$$

Since $|a_j^i|/|a_j^N| < 1$, we have $|p_j^i| \le Q^{N-1}$ for every $j \in \mathbb{N}$ and $i = 1, \ldots, N-1$. Passing to a subsequence if necessary, we can assume that there exist $q, p^1, \ldots, p^{N-1}$ such that $q = q_j$, $p^i = p_j^i$ for every $j \in \mathbb{N}$. Clearly, there exists $0 \le s < N$ such that $p^i = 0$ if and only if $i \le s$. Denote by u^i the center of the interval U^i and set

$$y_j^i = \begin{cases} a_j^i & \text{for } i \le s, \\ a_j^i/p^i - a_j^N/q & \text{for } s < i < N, \\ a_j^N/q & \text{for } i = N, \end{cases} \qquad j \in \mathbb{N}.$$

Further we define

- $J^i = U^i$ for $i \le s$,
- $\widetilde{J}^i = B(u^i/p^i - u^N/q, \lambda(U^i)/(8|p^i|))$ for $s < i < N$,
- $\widetilde{J}_j^N = B(u^N/q, \delta_j|y_j^N|/4)$ for $j \in \mathbb{N}$,
- $J_j^N = \widetilde{J}_j^N \cap (0,1)$.

Since $u^N/q \in (0,1)$ we have $\lambda(J_j^N) \ge \frac{1}{2}\lambda(\widetilde{J}_j^N)$. Passing to a subsequence if necessary and using Lemma 4.4(iv) we see that $\boldsymbol{y} := \big\{(y_j^1, \dots, y_j^N)\big\}_j$ is in $\mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$. For every $s < i < N$ we find an open interval $J^i \subset [0,1]$ such that $\lambda(J^i) \geq \lambda(\widetilde{J}^i)/2$ and $J^i \subset \langle \widetilde{J}^i \rangle$. By Lemma 4.4(vi) we have

$$T(a_j^i, U^i) \supset T\bigg(\frac{a_j^i}{p^i}, B\bigg(\frac{u^i}{p^i}, \frac{\lambda(U^i)}{2|p^i|}\bigg)\bigg),$$

$$T(a_j^N, U^N) \supset T\bigg(y_j^N, B\bigg(\frac{u^N}{q}, \frac{\lambda(U^N)}{2q}\bigg)\bigg). \tag{4.2}$$

Since

$$\bigg|q\frac{a_j^i}{a_j^N} - p^i\bigg| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\, i = 1, \dots, N\}}{8L} \leq \frac{1}{8},$$

we have $4|a_j^i/p^i| > 3|y_j^N|$. Since $a_j^i/p^i - y_j^N = y_j^i$ and $\boldsymbol{y} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$, it follows that $a_j^i/p^i \neq y_j^N$. We use Lemma 4.7 replacing y, v, r_2, δ, z, u, r_1 by a_j^i/p^i, u^i/p^i, $\lambda(U^i)/(2|p^i|)$, δ_j, y_j^N, u^N/q, $\lambda(U^N)/(2q)$ respectively to obtain

$$\begin{aligned} T\bigg(\frac{a_j^i}{p^i}, B\bigg(\frac{u^i}{p^i}, \frac{\lambda(U^i)}{2|p^i|}\bigg)\bigg) \cap T\bigg(y_j^N, B\bigg(\frac{u^N}{q}, \frac{\lambda(U^N)}{2q}\bigg)\bigg) \\ \supset T(y_j^N, \widetilde{J}_j^N) \cap T(y_j^i, \widetilde{J}^i). \end{aligned} \tag{4.3}$$

Recall that $y - z$ is replaced by $a_j^i/p^i - y_j^N = y_j^i$.

By Lemma 4.4(vii) and our choice of the sets J^i, J_j^N we have

$$T(y_j^N, \widetilde{J}_j^N) \cap T(y_j^i, \widetilde{J}^i) \supset T(y_j^N, J_j^N) \cap T(y_j^i, J^i). \tag{4.4}$$

By (4.2)–(4.4) we have

$$H_n(\boldsymbol{a}, \mathcal{U}) \subset H_n(\boldsymbol{y}, J^1 \times \dots \times J^{N-1} \times J_n^N). \tag{4.5}$$

Observe that

$$\begin{aligned} \lambda(J_j^N) &\geq \frac{1}{2}\lambda(\tilde{J}_j^N) = \frac{1}{4}\delta_j|y_j^N| = \frac{1}{4}\delta_j\frac{|a_j^N|}{q} \geq \frac{1}{4}\,\frac{\min\{\lambda(U^i); i = 1, \dots, N\}}{|a_j^N|}\,\frac{|a_j^N|}{q} \\ &= \frac{1}{4}\,\frac{\min\{\lambda(U^i);\, i = 1, \dots, N\}}{q}. \end{aligned}$$

Thus we can use Lemma 4.6 to get an open interval J^N and an increasing sequence v_n of positive integers such that for every $n \in \mathbb{N}$,

$$H_n(\{y_{v_n}\}, J^1 \times \dots \times J^{N-1} \times J_n^N) \subset H_n(\{y_{v_n}\}, J^1 \times \dots \times J^N), \tag{4.6}$$

and

$$4\lambda(J^N) \geq \lambda(J_{v_n}^N).$$

We set $x_n^i := y_{v_n}^i$ and $\mathcal{J} = J^1 \times \cdots \times J^N$. By the definition of $\boldsymbol{y}$ we easily see that $\mathcal{L}$ is a triangular matrix without any zero element on the diagonal. Thus we have (a). By (4.5) and (4.6) we get (b). Assume $i \leq s$. Since

$$\left|\frac{x_j^i}{x_j^N}\right| = \left|q\frac{a_{v_j}^i}{a_{v_j}^N} - p^i\right| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\, i = 1, \ldots, N\}}{8L},$$

we have

$$\left|\frac{x_j^N \lambda(J^i)}{x_j^i}\right| = \left|\frac{x_j^N \lambda(U^i)}{x_j^i}\right| \geq \left|\frac{x_j^N \cdot 8L}{x_j^i Q}\right| \geq 8L.$$

Let $s < i < N$. Since

$$\left|\frac{x_j^i p^i}{x_j^N}\right| = \left|q\frac{a_{v_j}^i}{a_{v_j}^N} - p^i\right| \leq \frac{1}{Q} < \frac{\min\{\lambda(U^i);\, i = 1, \ldots, N\}}{8L},$$

we deduce

$$\left|\frac{x_j^N \lambda(J^i)}{x_j^i}\right| \geq \left|\frac{x_j^N \lambda(\widetilde{J}^i)}{2x_j^i}\right| = \left|\frac{x_j^N \lambda(U^i)}{8x_j^i p^i}\right| \geq \left|\frac{x_j^N L}{x_j^i Q p^i}\right| \geq L.$$

Thus we have (c). Clearly,

$$16\lambda(J^N) \geq 4\lambda(J_{v_n}^N) \geq 2\lambda(\widetilde{J}_{v_n}^N) = \delta_{v_n}|x_n^N|$$

for all $n \in \mathbb{N}$. Thus we have (d). ∎

LEMMA 4.9. *Let $N \in \mathbb{N}$, $\boldsymbol{a} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$, let $\mathcal{U} = \prod_{i=1}^N U^i \subset [0,1]^N$ be an open interval, $L \in \mathbb{N}$ and $\delta_j = \min\{\lambda(U^i)/|a_j^i|;\, i = 1, \ldots, N\}$ for every $j \in \mathbb{N}$. Then there exist $\boldsymbol{x} \in (\mathbb{R}^N)^{\mathbb{N}}$, a nonsingular matrix $M \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{v_n\}$ of positive integers and an open interval $\mathcal{J} = \prod_{i=1}^N J^i \subset [0,1]^N$ such that*

(a) $\boldsymbol{x} := \{M(a_{v_n})\} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$,
(b) $\forall n \in \mathbb{N} : H_n(\{a_{v_n}\}, \mathcal{U}) \subset H_n(\boldsymbol{x}, \mathcal{J})$,
(c) $\forall n \in \mathbb{N}\ \forall i < N : |x_n^{i+1} \lambda(J^i)/x_n^i| \geq L$,
(d) $\lambda(J^N)/|x_n^N| \geq \delta_{v_n}/16$.

Proof. We use induction on N. The case $N = 1$ is trivial. Assume that our statement holds for some $N - 1 \in \mathbb{N}$; we show that it also holds for N. By Lemma 4.8 there exist a nonsingular matrix $\mathcal{L} \in \mathbb{Q}^{N\times N}$, an increasing sequence $\{p_n\}$ of positive integers and an open interval $\mathcal{V} = \prod_{i=1}^N V^i \subset [0,1]^N$ such that

(i) $\boldsymbol{y} := \{\mathcal{L}(a_{p_n})\} \in \mathcal{Q}(\mathbb{R} \setminus \{0\}^N)$,
(ii) $\forall n \in \mathbb{N} : H_n(\{a_{p_n}\}, \mathcal{U}) \subset H_n(\boldsymbol{y}, \mathcal{V})$,
(iii) $\forall n \in \mathbb{N}\ \forall i < N : |y_n^N \lambda(V^i)/y_n^i| \geq 16L$,
(iv) $\lambda(V^N)/|y_n^N| \geq \delta_{p_n}/16$.

Clearly, $\{y^1,\dots,y^{N-1}\}\in\mathcal{Q}(\mathbb{R}\setminus\{0\}^{N-1})$. By induction hypothesis there exist $\{x_n\}\in(\mathbb{Q}^{N-1})^{\mathbb{N}}$, a nonsingular matrix $\mathcal{Z}\in\mathbb{Q}^{(N-1)\times(N-1)}$, an increasing sequence $\{j_n\}$ of positive integers and open intervals $J^i\subset[0,1]$, $0<i<N$, such that

(1) $\{x_n^1,\dots,x_n^{N-1}\}:=\{\mathcal{Z}(y_{j_n}^1,\dots,y_{j_n}^{N-1})\}\in\mathcal{Q}(\mathbb{R}\setminus\{0\}^{N-1})$,
(2) $\forall n\in\mathbb{N}$:

$$H_n\Big(\{y_{j_n}^1,\dots,y_{j_n}^{N-1}\},\prod_{i=1}^{N-1}V^i\Big)\subset H_n\Big(\{x_n^1,\dots,x_n^{N-1}\},\prod_{i=1}^{N-1}J^i\Big),$$

(3) $\forall n\in\mathbb{N}\ \forall i<N-1:|x_n^{i+1}\lambda(J^i)/x_n^i|\geq L$,
(4) $\lambda(J^{N-1})/|x_n^{N-1}|\geq\frac{1}{16}\min\{\lambda(V^i)/|y_{j_n}^i|;\ i=1,\dots,N-1\}$.

We set $v_n=p_{j_n}$, $x_n^N=y_{j_n}^N$ and $J^N=V^N$. We define $\widetilde{\mathcal{Z}}\in\mathbb{Q}^{N\times N}$ by

$$\widetilde{\mathcal{Z}}_{i,j}=\begin{cases}\mathcal{Z}_{i,j} & \text{for } 0<i,j<N,\\ 1 & \text{for } i=j=N+1,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $\widetilde{\mathcal{Z}}$ is nonsingular. We set $M=\widetilde{\mathcal{Z}}\cdot\mathcal{L}$. Thus M is also nonsingular. Using (i) and (1) we easily obtain (a). By (2) we have

$$\forall n\in\mathbb{N}:H_n\big(\{y_{j_n}^1,\dots,y_{j_n}^N\},\mathcal{V}\big)\subset H_n(\boldsymbol{x},\mathcal{J}). \tag{4.7}$$

Using (4.7) and (ii) we get (b). Using (3) we obtain (c) for $i<N-1$. From (iii) we have $\min\{\lambda(V^i)/|y_{j_n}^i|;\ i=1,\dots,N-1\}=\lambda(V^{N-1})/|y_{j_n}^{N-1}|$. Using this, (4) and (iii) again we get the case $i=N-1$. Formula (iv) easily gives (d). ∎

Proof of Theorem 2.5. The inclusion $H^{(N)*}\supset H_L^{(N)*}$ is trivial.

Let $A\in H^{(N)*}$. Then there exists $\boldsymbol{a}\in\mathcal{Q}(\mathbb{R}\setminus\{0\}^N)$ and an open interval $\mathcal{U}\subset[0,1]^N$ such that $A\subset H(\boldsymbol{a},\mathcal{U})$. By Lemma 4.9 there exists $\boldsymbol{x}\in(\mathbb{Q}^N)^{\mathbb{N}}$ and an open interval $\mathcal{J}\subset[0,1]^N$ such that $H(\boldsymbol{a},\mathcal{U})\subset H(\boldsymbol{x},\mathcal{J})\in H_L^{(N)*}$. So, $A\in H_L^{(N)*}$. ∎

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