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Locally Σ_1-definable well-orders of $\mathrm{H}(\kappa^+)$

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Abstract. Given an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ and 2^κ regular, we show that there is a forcing that preserves cofinalities less than or equal to 2^κ and forces the existence of a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Σ_1-formula with parameters. This shows that, in contrast to the case "$\kappa = \omega$", the existence of a locally definable well-order of $\mathrm{H}(\kappa^+)$ of low complexity is consistent with failures of the GCH at κ. We also show that the forcing mentioned above introduces a Bernstein subset of ${}^\kappa\kappa$ that is definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$ by a Δ_1-formula with parameters.

1. Introduction. A classical theorem of Mansfield (see [Man75] and [Kec78]) says that the existence of a well-ordering of $\mathbb{R}$ that is a $\boldsymbol{\Sigma}^1_2$-subset of $\mathbb{R} \times \mathbb{R}$ is equivalent to the statement that there is a real number x such that all reals are contained in $\mathrm{L}[x]$. Since a set of reals is a $\boldsymbol{\Sigma}^1_2$-subset of $\mathbb{R}$ if and only if it is definable over the structure $\langle \mathrm{H}(\omega_1), \in\rangle$ by a Σ_1-formula with parameters (see [Jec03, Lemma 25.25]), Mansfield's theorem has the following corollary: *if there is a well-ordering of* $\mathrm{H}(\omega_1)$ *that is definable over the structure* $\langle \mathrm{H}(\omega_1), \in\rangle$ *by a* Σ_1*-formula with parameters, then CH holds.* Note that such well-orders of $\mathrm{H}(\omega_1)$ exist in $\mathrm{L}[x]$ whenever $x \in \mathbb{R}$.

It is natural to ask whether the above corollary generalizes to higher cardinalities: *if* κ *is an uncountable cardinal, does the existence of a well-ordering of* $\mathrm{H}(\kappa^+)$ *that is definable over the structure* $\langle \mathrm{H}(\kappa^+), \in\rangle$ *by a* Σ_1*-formula* (1) *with parameters imply that the GCH holds at* κ*?* In this paper, we provide a negative answer to this question by proving the following result.

Theorem 1.1. *Let* κ *be an uncountable cardinal such that* $\kappa = \kappa^{<\kappa}$ *and* 2^κ *is regular* (2)*. Then there is a partial order* $\mathbb{P}$ *with the following properties:*

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(1) Note that every Σ_n-definable well-order $\prec$ is automatically Δ_n-definable, because $x \prec y$ holds if and only if $x \neq y$ and $y \not\prec x$.

(2) Note that every uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ is regular.

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(i) $\mathbb{P}$ *is* $<\kappa$*-closed, and forcing with* $\mathbb{P}$ *preserves cofinalities less than or equal to* 2^κ *and the value of* 2^κ*.*
(ii) *If* G *is* $\mathbb{P}$*-generic over the ground model* V*, then there is a well-ordering of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *that is definable over* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ *by a* Σ_1*-formula with parameters.*

In order to motivate our construction of a forcing with the above properties, we give a brief history of results that allow us to obtain definable well-orders of $\mathrm{H}(\kappa^+)$ of low complexity by forcing when κ is an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. The following theorem is due to the second author. Note that we use ${}^\kappa\kappa$ to denote the set of all functions from κ to κ, and ${}^\kappa 2$ to denote the set of all such functions whose range is a subset of $\{0, 1\}$.

Theorem 1.2 ([Lüc12, Theorem 1.5]). *If* κ *is an uncountable cardinal with* $\kappa = \kappa^{<\kappa}$ *and* A *is a subset of* ${}^\kappa\kappa$*, then there is a* $<\kappa$*-closed partial order* $\mathbb{P}(A)$ *such that* $\mathbb{P}(A) \subseteq \mathrm{H}(\kappa^+)$*,* $\mathbb{P}(A)$ *satisfies the* κ^+*-chain condition and the subset* A *is definable over* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ *by a* Σ_1*-formula with parameters whenever* G *is* $\mathbb{P}(A)$*-generic over* V*.*

This result can then be used to prove the following statement.

Theorem 1.3 ([Lüc12, Theorem 1.9]). *If* κ *is an uncountable cardinal with* $\kappa = \kappa^{<\kappa}$*, then there is a* $<\kappa$*-closed partial order* $\mathbb{P}$ *such that* $\mathbb{P}$ *satisfies the* κ^+*-chain condition and there is a well-ordering of* $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ *definable over* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ *by a* Σ_2*-formula with parameters whenever* G *is* $\mathbb{P}$*-generic over* V*.*

The basic idea of the proof of the latter theorem is to choose (in the ground model) an arbitrary well-order $\prec$ of $\mathrm{H}(\kappa^+)$, code it into a subset of ${}^\kappa\kappa$ and force to make this subset definable using Theorem 1.2 (3). Since this forcing satisfies the κ^+-chain condition and is contained in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$, every element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ is represented by a name in $\mathrm{H}(\kappa^+)^{\mathrm{V}}$. Moreover, it can be shown that $\mathbb{P}$, the generic filter G and its complement relative to $\mathbb{P}$ are all definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters. In this situation, we obtain a Σ_2-definable well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ by setting $x \prec_* y$ if and only if some name $\dot{x}$ that is evaluated to x by the generic filter is $\prec$-less than any name $\dot{y}$ that is evaluated to y.

Now, if the GCH holds at κ, then every initial segment of $\prec$ is an element of $\mathrm{H}(\kappa^+)$ and we can instead code the set of all these initial segments. This allows us to spare one quantifier and obtain the following result, which has independently been obtained by Sy Friedman and the first author in [FH11] using different techniques.

(3) Note that the forcing $\mathbb{P}(A)$ introduces new subsets of κ. Hence the relation $\prec$ does not well-order $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.

THEOREM 1.4. *If κ is an uncountable cardinal satisfying $\kappa = \kappa^{<\kappa}$ and $2^\kappa = \kappa^+$, then there is a $<\kappa$-closed partial order $\mathbb{P}$ such that $\mathbb{P}$ satisfies the κ^+-chain condition and there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters whenever G is $\mathbb{P}$-generic over* V.

The forcing used in [FH11] to prove the above theorem is an iteration of length κ^+ that satisfies the κ^+-chain condition and adds new subsets of κ at cofinally many stages of the iteration. These properties can be used to show that Mansfield's theorem itself does not generalize to higher cardinalities, in the sense that the existence of a locally Σ_1-definable well-ordering of $\mathrm{H}(\kappa^+)$ does not imply that all subsets of κ are contained in $\mathrm{L}[x]$ for some $x \subseteq \kappa$: Assume this were the case for some $x \subseteq \kappa$ in the model obtained by forcing with the partial order $\mathbb{P}$ constructed in [FH11] in the proof of the above theorem. Then x is added by some initial segment of that iteration (by the κ^+-chain condition) and there is a subset $y \subseteq \kappa$ which is added at a later stage and hence cannot be an element of $\mathrm{L}[x]$. However the question remained open whether the existence of such a well-ordering of $\mathrm{H}(\kappa^+)$ implies that the GCH holds at κ (see [Lüc12, Question 10.4]).

If the GCH does not hold at κ, the above approaches can no longer be used and a totally different strategy is needed to force the existence of a Σ_1-definable well-order of $\mathrm{H}(\kappa^+)$ while preserving failures of the GCH at κ. We will recursively define a forcing $\mathbb{P}$ that preserves all cofinalities less than or equal to 2^κ while simultaneously performing the following two tasks:

- Generically add a sequence $\vec{A} = \langle A_\delta \mid \delta < 2^\kappa\rangle$ of subsets of κ in the $\mathbb{P}$-generic extension $\mathrm{V}[G]$ such that every element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ is coded (in a sense made precise later on) by exactly one A_δ.
- Generically code $\vec{A}$ to ensure that it is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters.

In this situation, we can well-order $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ in the desired way by identifying each element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ with the unique A_δ coding it. The generic coding used in this construction will be a variation of the *almost disjoint coding forcing* (see [JS70] for the original) introduced in [AHL, Section 2]. The recursive definition of our forcing heavily uses ideas from [AF12].

The coding techniques developed in the proof of Theorem 1.1 are in fact quite a general way of generically adding subsets of $\mathrm{H}(\kappa^+)$ with certain properties while simultaneously making them Σ_1-definable (or even Δ_1-definable) over $\langle \mathrm{H}(\kappa^+), \in\rangle$. An example of another application of these techniques can be found in [LS, Remark 5.2]. In the following, we discuss yet another example: Equip the set ${}^\kappa\kappa$ with the topology whose basic open subsets are of the form $N_s = \{x \in {}^\kappa\kappa \mid s \subseteq x\}$ for some function $s : \alpha \to \kappa$ with $\alpha < \kappa$.

A closed subset of ${}^{\kappa}\kappa$ is *perfect* if it is homeomorphic to ${}^{\kappa}2$ equipped with the subspace topology. Finally, a subset X of ${}^{\kappa}\kappa$ is a *Bernstein subset of* ${}^{\kappa}\kappa$ if neither X nor its complement contain a perfect subset of ${}^{\kappa}\kappa$.

With a slight modification of the construction presented in Section 2, one could obtain a $<\kappa$-closed forcing that preserves cofinalities less than or equal to 2^{κ} and the value of 2^{κ} and introduces a Bernstein subset of ${}^{\kappa}\kappa$ that is Δ_1-definable over $\langle \mathrm{H}(\kappa^+), \in\rangle$. Instead of presenting this construction, we will show that such a subset can already be found in any generic extension obtained by forcing with the partial order $\mathbb{P}$ that witnesses Theorem 1.1 ([4]).

COROLLARY 1.5. *Forcing with the partial order* $\mathbb{P}$ *that witnesses Theorem* 1.1 *introduces a Bernstein subset of* ${}^{\kappa}\kappa$ *that is* Δ_1*-definable with parameters over* $\langle \mathrm{H}(\kappa^+), \in\rangle$.

Note that this result again contrasts with the case when "$\kappa = \omega$", because [BL99, Theorem 7.1] shows that the existence of a Bernstein subset of ${}^{\omega}\omega$ that is Δ_1-definable with parameters over $\langle \mathrm{H}(\omega_1), \in\rangle$ is equivalent to the existence of an $x \in \mathbb{R}$ with $\mathbb{R} \subseteq \mathrm{L}[x]$.

2. The forcing. For the remainder of this paper, we fix an uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$ and $\lambda = 2^{\kappa}$ regular. We use ${}^{<\kappa}2$ to denote the set of all functions $s : \alpha \to 2$ with $\alpha < \kappa$. Moreover, we let $\prec\cdot,\cdot\succ : \mathrm{On} \times \mathrm{On} \to \mathrm{On}$ denote the *Gödel pairing function*.

We say that *a subset* A *of* κ *codes an element* z *of* $\mathrm{H}(\kappa^+)$ if there is a bijection $b : \kappa \to \mathrm{tc}(\{z\})$ such that

$$A = \{\prec 0, \prec\alpha, \beta\succ\succ \mid \alpha, \beta < \kappa,\, b(\alpha) \in b(\beta)\} \cup \{\prec 1, \alpha\succ \mid \alpha < \kappa,\, b(\alpha) \in z\}.$$

Note that z and b are uniquely determined by A.

Given $x, y \in {}^{\kappa}\kappa$, we define $x \oplus y \in {}^{\kappa}\kappa$ by setting

$$(x \oplus y)(\alpha) := \begin{cases} x(\beta) & \text{if } \alpha = \prec 0, \beta\succ, \\ y(\beta) & \text{if } \alpha = \prec 1, \beta\succ, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\alpha < \kappa$. In addition, if $\alpha, \beta < \kappa$, then we define $c(\alpha, \beta) \in {}^{\kappa}2$ by setting

$$c(\alpha, \beta)(\gamma) := \begin{cases} 1 & \text{if } \gamma \in \{\prec 0, \alpha\succ, \prec 1, \beta\succ\}, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\gamma < \kappa$.

Fix a sequence $\vec{w} = \langle w_{\gamma} \mid \gamma < \lambda\rangle$ consisting of pairwise distinct elements of ${}^{\kappa}2$. We inductively construct a sequence $\vec{\mathbb{P}}_{\vec{w}} = \langle \mathbb{P}_{\gamma} \mid \gamma \leq \lambda\rangle$ of partial

([4]) Note that, in general, one can construct a locally Δ_2-definable Bernstein subset of ${}^{\kappa}\kappa$ from a locally Δ_1-definable well-order of $\mathrm{H}(\kappa^+)$. The existence of a Bernstein set of lower complexity follows from specific properties of our definable well-order.

orders with the property that $\mathbb{P}_\delta$ is a complete subforcing of $\mathbb{P}_\gamma$ whenever $\delta \leq \gamma \leq \lambda$. Fix $\gamma \leq \lambda$ and assume that we constructed $\mathbb{P}_\delta$ with the above property for every $\delta < \gamma$.

DEFINITION 2.1. We call a tuple

$$p = \langle s_p, t_p, \vec{c}_p, \vec{A}_p \rangle$$

a $\mathbb{P}_\gamma$-*candidate* if the following statements hold for some ordinals $\beta_p < \kappa$ and $\gamma_p < \min\{\gamma + 1, \lambda\}$:

(i) $s_p : \beta_p + 1 \to {}^{<\kappa}2$.
(ii) $t_p : \beta_p + 1 \to 2$.
(iii) $\vec{c}_p = \langle c_{p,x} \mid x \in a_p \rangle$ is a sequence with the following properties:
  (a) a_p is a subset of $\{w_\delta \oplus c(\alpha, i) \mid \delta < \gamma_p,\ \alpha < \kappa,\ i < 2\}$ of cardinality less than κ.
  (b) If $x \in a_p$, then $c_{p,x}$ is a closed subset of β_p+1 and the implication

$$s_p(\alpha) \subseteq x \to t_p(\alpha) = 1$$

  holds for every $\alpha \in c_{p,x}$.
(iv) $\vec{A}_p = \langle \dot{A}_{p,\delta} \mid \delta < \gamma_p \rangle$ is a sequence such that:
  (a) If $\delta < \gamma_p$, then $\dot{A}_{p,\delta}$ is a $\mathbb{P}_\delta$-nice name for a subset of κ (and, by our assumptions, also a $\mathbb{P}_{\tilde{\delta}}$-nice name for a subset of κ for every $\delta \leq \tilde{\delta} < \gamma$).
  (b) If $\bar{\gamma} < \gamma_p$ and G is $\mathbb{P}_{\bar{\gamma}}$-generic over the ground model V, then either $|\lambda|^{\mathrm{V}[G]} = |\bar{\gamma}|^{\mathrm{V}[G]}$ holds (5), or in $\mathrm{V}[G]$ there is a sequence $\langle y_\delta \mid \delta \leq \bar{\gamma} \rangle$ of pairwise distinct elements of $\mathrm{H}(\kappa^+)$ such that $\dot{A}^G_{p,\delta}$ codes y_δ for every δ less than or equal to $\bar{\gamma}$.

Given a $\mathbb{P}_\gamma$-candidate p and $\delta \leq \gamma$, we define $p{\restriction}\delta$ to be the tuple

$$\langle s_p, t_p, \langle c_{p,x} \mid x \in a_p{\restriction}\delta \rangle, \vec{A}_p{\restriction}\min\{\gamma_p, \delta\} \rangle,$$

where $a_p{\restriction}\delta = a_p \cap \{w_{\bar{\delta}} \oplus c(\alpha, i) \mid \bar{\delta} < \delta,\ \alpha < \kappa,\ i < 2\}$.

DEFINITION 2.2. A $\mathbb{P}_\gamma$-candidate p is a *condition* in $\mathbb{P}_\gamma$ if the following statement holds for all $\delta < \gamma_p$, $\alpha < \kappa$ and $i < 2$ with $w_\delta \oplus c(\alpha, i) \in a_p$:

(v) If $p{\restriction}\delta$ is a condition in $\mathbb{P}_\delta$, then

$$p{\restriction}\delta \Vdash_{\mathbb{P}_\delta} \text{“} i = 1 \leftrightarrow \check{\alpha} \in \dot{A}_{p,\delta} \text{”} \ (^6).$$

(5) We will show later that this case never occurs (see Corollary 2.11).

(6) The idea behind this construction is that the set a_p collects information about the interpretations of names in $\vec{A}_p$ that is already decided by the condition p. This will allow us to use the almost disjoint coding part of the forcing (see clause (iii)(b)) to add a subset of κ that in the end codes $\bigcup_{p \in G} a_p$ and thus also $\bigcup_{p \in G} \vec{A}_p$ whenever G is $\mathbb{P}_\lambda$-generic.

Given conditions p and q in $\mathbb{P}_\gamma$, we define $p \leq_{\mathbb{P}_\gamma} q$ to hold if $s_q = s_p\restriction(\beta_q+1)$, $t_q = t_p\restriction(\beta_q + 1)$, $a_q \subseteq a_p$, $\vec{A}_q = \vec{A}_p\restriction\gamma_q$ and $c_{q,x} = c_{p,x}\restriction(\beta_q + 1)$ for every $x \in a_q$.

Proposition 2.3. *If p is a condition in $\mathbb{P}_\gamma$ and $\delta < \gamma$, then $p\restriction\delta$ is a condition in $\mathbb{P}_\delta$. In particular, every condition p in $\mathbb{P}_\gamma$ is also a condition in $\mathbb{P}_{\gamma_p}$.*

Proof. Let $\delta < \gamma$ and assume that $p\restriction\bar{\delta}$ is a condition in $\mathbb{P}_{\bar{\delta}}$ for every $\bar{\delta} < \delta$. Then it is easy to see that $p\restriction\delta$ is a $\mathbb{P}_\delta$-candidate. Fix $\bar{\delta} < \delta$, $\alpha < \kappa$ and $i < 2$ with $w_{\bar{\delta}} \oplus c(\alpha, i) \in a_{p\restriction\delta}$. Then $(p\restriction\delta)\restriction\bar{\delta} = p\restriction\bar{\delta}$ is a condition in $\mathbb{P}_{\bar{\delta}}$ and $a_{p\restriction\delta} = a_p\restriction\delta \subseteq a_p$. Since p is a condition in $\mathbb{P}_\gamma$, this implies $\bar{\delta} < \gamma_p$ and

$$(p\restriction\delta)\restriction\bar{\delta} \Vdash_{\mathbb{P}_\delta} \text{“} i = 1 \leftrightarrow \check{\alpha} \in \dot{A}_{p,\bar{\delta}} \text{”}.$$

We can conclude that $p\restriction\delta$ is a condition in $\mathbb{P}_\delta$. ∎

The following statement is a direct consequence of the above definition.

Proposition 2.4. *If p is a condition in $\mathbb{P}_\gamma$ and $\vec{A}$ is a sequence of length smaller than* $\min\{\gamma + 1, \lambda\}$ *such that $\vec{A}_p \subseteq \vec{A}$ and $\vec{A}$ satisfies the statements listed in Definition* 2.1(iv)*, then the tuple $\langle s_p, t_p, \vec{c}_p, \vec{A}\rangle$ is a condition in $\mathbb{P}_\gamma$ that is stronger than p.* ∎

Proposition 2.5. *If $\bar{\gamma} < \min\{\gamma + 1, \lambda\}$, then the set of all conditions p in $\mathbb{P}_\gamma$ with $\gamma_p \geq \bar{\gamma}$ is dense in $\mathbb{P}_\gamma$.*

Proof. Fix a condition p in $\mathbb{P}_\gamma$ with $\gamma_p < \bar{\gamma}$. Since $\bar{\gamma} < \lambda = 2^\kappa$, we can recursively construct a sequence $\vec{A}$ of length $\bar{\gamma}$ that satisfies the statements listed in Definition 2.1(iv). By Proposition 2.4, the resulting tuple $\langle s_p, t_p, \vec{c}_p, \vec{A}\rangle$ is a condition in $\mathbb{P}_\gamma$ that is stronger than p. ∎

Lemma 2.6. *If $\delta < \gamma$, then $\mathbb{P}_\delta$ is a complete subforcing of $\mathbb{P}_\gamma$.*

Proof. Every condition in $\mathbb{P}_\delta$ is a condition in $\mathbb{P}_\gamma$, $\leq_{\mathbb{P}_\delta} = \leq_{\mathbb{P}_\gamma}\restriction(\mathbb{P}_\delta \times \mathbb{P}_\delta)$ and, if q is a condition in $\mathbb{P}_\delta$ and p is a condition in $\mathbb{P}_\gamma$ with $p \leq_{\mathbb{P}_\gamma} q$, then Proposition 2.3 shows that $p\restriction\delta$ is a condition in $\mathbb{P}_\delta$ and it is easy to check that $p\restriction\delta \leq_{\mathbb{P}_\delta} q$. Hence it suffices to show that every maximal antichain in $\mathbb{P}_\delta$ is maximal in $\mathbb{P}_\gamma$.

Fix a maximal antichain $\mathcal{A}$ of $\mathbb{P}_\delta$ and a condition p_0 in $\mathbb{P}_\gamma$. By Proposition 2.5, there is a condition p with $p \leq_{\mathbb{P}_\gamma} p_0$ and $\gamma_p \geq \delta$. Proposition 2.3 implies that $p\restriction\delta$ is a condition in $\mathbb{P}_\delta$. Hence we find a condition q in $\mathbb{P}_\delta$ and $r \in \mathcal{A}$ with $q \leq_{\mathbb{P}_\delta} p\restriction\delta, r$. Then $\gamma_q = \delta$. Define p^* to be the tuple

$$\langle s_q, t_q, \langle c_{p,x} \mid x \in a_p \setminus a_q\rangle \cup \langle c_{q,x} \mid x \in a_q\rangle, \vec{A}_p\rangle.$$

Then p^* is a $\mathbb{P}_\gamma$-candidate with $\gamma_{p^*} = \gamma_p$. Fix $\bar{\delta} < \gamma_p$, $\alpha < \kappa$ and $i < 2$ such that $p^*\restriction\bar{\delta}$ is a condition in $\mathbb{P}_{\bar{\delta}}$ and $x = w_{\bar{\delta}} \oplus c(\alpha, i) \in a_p \cup a_q$. If $x \in a_q$, then

$\bar{\delta} < \delta \leq \gamma_{p^*}$ and $\vec{A}_q = \vec{A}_p \restriction \delta$ implies that $p^* \restriction \bar{\delta} \leq_{\mathbb{P}_{\bar{\delta}}} q \restriction \bar{\delta}$. Hence

$$p^* \restriction \bar{\delta} \Vdash_{\mathbb{P}_{\bar{\delta}}} \text{“}i = 1 \leftrightarrow \check{\alpha} \in \dot{A}_{p,\bar{\delta}}\text{”}$$

holds in this case. Now assume that $x \in a_p \setminus a_q$. Since $q \leq_{\mathbb{P}_\delta} p \restriction \delta$, we have $p^* \restriction \bar{\delta} \leq_{\mathbb{P}_{\bar{\delta}}} p \restriction \bar{\delta}$ and this implies that the above forcing statement also holds in this case. Therefore p^* is a condition in $\mathbb{P}_\gamma$ and our construction ensures that $p^* \leq_{\mathbb{P}_\gamma} p, q$. Hence $\mathcal{A}$ is a maximal antichain in $\mathbb{P}_\gamma$. ∎

This completes the construction of the sequence $\vec{\mathbb{P}}_{\vec{w}}$ of partial orders. In the remainder of this section, we prove some basic properties of these forcings.

Proposition 2.7. *Let $\gamma \leq \lambda$, $\bar{\lambda} < \lambda$ and $\langle p_\alpha \mid \alpha < \bar{\lambda} \rangle$ be a sequence of conditions in $\mathbb{P}_\gamma$ such that $\vec{A}_{p_\alpha} \subseteq \vec{A}_{p_\beta}$ for all $\alpha < \beta < \bar{\lambda}$. Then $\vec{A} = \bigcup\{\vec{A}_{p_\alpha} \mid \alpha < \bar{\lambda}\}$ satisfies the statements listed in Definition* 2.1(iv). ∎

Lemma 2.8. *If $\gamma \leq \lambda$, then $\mathbb{P}_\gamma$ is $<\kappa$-closed.*

Proof. Let $\bar{\kappa} \in \mathrm{Lim} \cap \kappa$ and $\langle p_\alpha \mid \alpha < \bar{\kappa} \rangle$ be a descending sequence of conditions in $\mathbb{P}_\gamma$. Define $\vec{A} = \bigcup\{\vec{A}_{p_\alpha} \mid \alpha < \bar{\kappa}\}$, $a = \bigcup\{a_{p_\alpha} \mid \alpha < \bar{\kappa}\}$ and $c_x = \bigcup\{c_{p_\alpha, x} \mid x \in a_{p_\alpha}\}$ for each $x \in a$. By Proposition 2.7, $\vec{A}$ satisfies the statements listed in Definition 2.1(iv).

First assume that there is $\bar{\alpha} < \bar{\kappa}$ such that $\beta_{p_\alpha} = \beta_{p_{\bar{\alpha}}}$ for all $\bar{\alpha} \leq \alpha < \bar{\kappa}$. Then the tuple $p_* = \langle s_{p_{\bar{\alpha}}}, t_{p_{\bar{\alpha}}}, \langle c_x \mid x \in a \rangle, \vec{A} \rangle$ is a $\mathbb{P}_\gamma$-candidate. To show that p_* is a condition in $\mathbb{P}_\gamma$, fix $\delta < \gamma$, $\beta < \kappa$ and $i < 2$ with $x = w_\delta \oplus c(\beta, i) \in a$. Then there is $\bar{\alpha} \leq \alpha < \bar{\kappa}$ with $x \in a_{p_\alpha}$ and hence $\delta < \gamma_{p_\alpha} \leq \gamma_{p_*}$. If $p_* \restriction \delta$ is a condition in $\mathbb{P}_\delta$, then it is stronger than $p_\alpha \restriction \delta$ and hence it forces the desired statement (v) in Definition 2.2. This shows that p_* is a condition in $\mathbb{P}_\gamma$ and our construction ensures that $p_* \leq_{\mathbb{P}_\gamma} p_\alpha$ for every $\alpha < \bar{\kappa}$.

Now assume that for every $\bar{\alpha} < \bar{\kappa}$ there is $\bar{\alpha} < \alpha < \bar{\kappa}$ with $\beta_{p_{\bar{\alpha}}} < \beta_{p_\alpha}$. Define

$$\begin{aligned} \beta &= \sup_{\alpha < \bar{\kappa}} \beta_{p_\alpha}, \\ s &= \{\langle \beta, \emptyset \rangle\} \cup \bigcup\{s_{p_\alpha} \mid \alpha < \bar{\kappa}\}, \\ t &= \{\langle \beta, 1 \rangle\} \cup \bigcup\{t_{p_\alpha} \mid \alpha < \bar{\kappa}\}, \\ p_* &= \big\langle s, t, \langle c_x \cup \{\beta\} \mid x \in a \rangle, \vec{A} \big\rangle. \end{aligned}$$

This construction ensures that p_* is a $\mathbb{P}_\gamma$-candidate and the same argument as above shows that p_* is actually a condition in $\mathbb{P}_\gamma$ with $p_* \leq_{\mathbb{P}_\gamma} p_\alpha$ for all $\alpha < \bar{\kappa}$. ∎

Proposition 2.9. *If $\gamma < \lambda$ and p is a condition in $\mathbb{P}_\gamma$ with $\gamma = \gamma_p$, then $\mathbb{P}_\gamma$ satisfies the κ^+-chain condition below p.*

Proof. Let $\mathcal{A}$ be a set of conditions below p in $\mathbb{P}_\gamma$ of cardinality κ^+. Then $\vec{A}_p = \vec{A}_q$ for all $q \in \mathcal{A}$. By our assumptions and the Δ-system lemma, there are $q_0, q_1 \in \mathcal{A}$ such that $q_0 \neq q_1$, $s_{q_0} = s_{q_1}$, $t_{q_0} = t_{q_1}$ and $c_{q_0,x} = c_{q_1,x}$ for all $x \in a_{q_0} \cap a_{q_1}$. Then the tuple

$$r = \langle s_{q_0}, t_{q_0}, \langle c_{q_0,x} \mid x \in a_{q_0}\rangle \cup \langle c_{q_1,x} \mid x \in a_{q_1}\rangle, \vec{A}_p\rangle$$

is a $\mathbb{P}_\gamma$-candidate. If $\delta < \gamma$ is such that $r\restriction\delta$ is a condition in $\mathbb{P}_\delta$, then $r\restriction\delta \leq_{\mathbb{P}_\delta} q_i\restriction\delta$ for $i < 2$. This shows that r is a condition in $\mathbb{P}_\gamma$ witnessing that the conditions q_0 and q_1 are compatible in $\mathbb{P}_\gamma$. ■

LEMMA 2.10. *If q is a condition in $\mathbb{P}_\lambda$ and $\mathcal{D}$ is a collection of less than λ-many dense open subsets of $\mathbb{P}_\lambda$, then there is a condition p in $\mathbb{P}_\lambda$ such that $p \leq_{\mathbb{P}_\lambda} q$ and the set $D \cap \mathbb{P}_{\gamma_p}$ is dense below p in $\mathbb{P}_{\gamma_p}$ for every $D \in \mathcal{D}$.*

Proof. We start by proving the following claim. An iterated application of this claim will yield the statement of the lemma.

CLAIM. *Let q_0 be a condition in $\mathbb{P}_\lambda$ and D be a dense open subset of $\mathbb{P}_\lambda$. Then there is a condition q_0^* in $\mathbb{P}_\lambda$ such that $q_0^* = \langle s_{q_0}, t_{q_0}, \vec{c}_{q_0}, \vec{A}_{q_0^*}\rangle \leq_{\mathbb{P}_\lambda} q_0$ and $D \cap \mathbb{P}_{\gamma_{q_0^*}}$ is dense below q_0^* in $\mathbb{P}_{\gamma_{q_0^*}}$.*

Proof. We inductively construct a sequence $\langle q_\alpha \mid 0 < \alpha < \theta\rangle$ of incompatible conditions below q_0 in $\mathbb{P}_\lambda$ with $0 < \theta \leq \kappa^+$ and $\vec{A}_{q_{\bar\alpha}} \subseteq \vec{A}_{q_\alpha}$ for all $\bar\alpha < \alpha < \theta$: Assume that the sequence $\langle q_{\bar\alpha} \mid 0 < \bar\alpha < \alpha\rangle$ is already constructed. If there is a $p_\alpha \in D$ such that $p_\alpha \leq_{\mathbb{P}_\lambda} \langle s_{q_0}, t_{q_0}, \vec{c}_{q_0}, \bigcup_{\bar\alpha<\alpha} \vec{A}_{p_{\bar\alpha}}\rangle$ and the conditions p_α and $q_{\bar\alpha}$ are incompatible in $\mathbb{P}_\lambda$ for all $0 < \bar\alpha < \alpha$, then we set $q_\alpha = p_\alpha$ and we continue our construction. Otherwise, we stop our construction and set $\theta = \alpha$.

Define $\vec{A} = \bigcup_{\alpha<\theta} \vec{A}_{q_\alpha}$ and $q_\alpha^* = \langle s_{q_\alpha}, t_{q_\alpha}, \vec{c}_{q_\alpha}, \vec{A}\rangle$ for all $\alpha < \theta$. Given $\alpha < \theta$, Proposition 2.7 shows that q_α^* is a condition in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^* and q_α. In particular, the set $\mathcal{A} = \{q_\alpha^* \mid 0 < \alpha < \theta\}$ is an antichain in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^*. By Proposition 2.9, $\mathbb{P}_{\gamma_{q_0^*}}$ satisfies the κ^+-chain condition below q_0^* and therefore $\theta < \kappa^+$. This means that the above construction has stopped at stage $\theta < \kappa^+$, because no suitable condition p_θ could be found. This implies that $\mathcal{A}$ is a *maximal* antichain in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^*. Pick a condition p in $\mathbb{P}_{\gamma_{q_0^*}}$ below q_0^*. Then there is $0 < \alpha < \theta$ and a condition r in $\mathbb{P}_{\gamma_{q_0^*}}$ with $r \leq_{\mathbb{P}_{\gamma_{q_0^*}}} p, q_\alpha^*$. Since q_α^* is an element of D, we get $r \in D$. This shows that the condition q_0^* has the desired properties. ■

Let $\langle D_\alpha \mid \alpha < \bar\lambda\rangle$ be an enumeration of $\mathcal{D}$ such that $\bar\lambda < \lambda$ is a limit ordinal. By the above Claim and Proposition 2.7, we can construct a decreasing sequence $\langle q_\alpha \mid \alpha \leq \bar\lambda\rangle$ of conditions in $\mathbb{P}_\lambda$ with the property that

$q = q_0$, $q_\alpha = \langle s_q, t_q, \vec{c}_q, \vec{A}_{q_\alpha}\rangle$ for all $\alpha \leq \bar{\lambda}$ and $D_\alpha \cap \mathbb{P}_{\gamma_{q_{\alpha+1}}}$ is dense below $q_{\alpha+1}$ in $\mathbb{P}_{\gamma_{q_{\alpha+1}}}$ for all $\alpha < \bar{\lambda}$.

Pick a condition r in $\mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$ below $q_{\bar{\lambda}}$ and $\alpha < \bar{\lambda}$. Then $\vec{A}_r = \vec{A}_{q_{\bar{\lambda}}}$ and $r\restriction\gamma_{q_{\alpha+1}} \leq q_{\bar{\lambda}}\restriction\gamma_{q_{\alpha+1}} = q_{\alpha+1}$. So we can find $\bar{r}_\alpha \leq_{\mathbb{P}_{\gamma_{q_{\alpha+1}}}} r\restriction\gamma_{q_{\alpha+1}}$ such that $\bar{r}_\alpha \in D_\alpha$. Define $\vec{c} = \langle c_x \mid x \in a_r \cup a_{\bar{r}_\alpha}\rangle$ by letting $c_x = c_{\bar{r}_\alpha,x}$ if $x \in a_{\bar{r}_\alpha}$ and letting $c_x = c_{r,x}$ otherwise. Then $r_\alpha = \langle s_{\bar{r}_\alpha}, t_{\bar{r}_\alpha}, \vec{c}, \vec{A}_r\rangle$ is a $\mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$-candidate with $\bar{r}_\alpha = r_\alpha\restriction\gamma_{q_{\alpha+1}}$. Moreover, if $\delta < \gamma_{q_{\bar{\lambda}}}$ and $r_\alpha\restriction\delta$ is a condition in $\mathbb{P}_\delta$, then this condition is stronger than $r\restriction\delta$. We can conclude that r_α is actually a condition in $\mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$ that is a common extension of r and $\bar{r}_\alpha$ contained in $D_\alpha \cap \mathbb{P}_{\gamma_{q_{\bar{\lambda}}}}$. This shows that $p = q_{\bar{\lambda}}$ has the desired properties. ∎

Corollary 2.11. *Forcing with $\mathbb{P}_\lambda$ preserves all cofinalities less than or equal to λ.*

Proof. By Lemma 2.8, forcing with $\mathbb{P}_\lambda$ preserves cofinalities less than or equal to κ. Let $\gamma \leq \lambda$ be a limit ordinal with $\mathrm{cof}(\gamma) > \kappa$ and let $\kappa \leq \nu < \mathrm{cof}(\gamma)$ be a regular cardinal. Assume, towards a contradiction, that there is a condition q in $\mathbb{P}_\lambda$ and a $\mathbb{P}_\lambda$-name $\dot{c}$ with $q \Vdash_{\mathbb{P}_\lambda}$ "$\dot{c} : \check{\nu} \to \check{\gamma}$ is cofinal". Given $\alpha < \nu$, define

$$D_\alpha = \{p \in \mathbb{P}_\lambda \mid \exists\beta < \gamma\ p \Vdash_{\mathbb{P}_\lambda} \text{“}\dot{c}(\check{\alpha}) = \check{\beta}\text{”}\}.$$

Let G be P_λ-generic over V. By Lemma 2.10, there is a $p \in G$ with the property that the set $D_\alpha \cap \mathbb{P}_{\gamma_p}$ is dense below p in $\mathbb{P}_{\gamma_p}$ for every $\alpha < \nu$. By Proposition 2.9, $\mathbb{P}_{\gamma_p}$ satisfies the κ^+-chain condition below p. Therefore we can define $c : \nu \to \gamma$ in V by setting

$$c(\alpha) = \mathrm{lub}\{\beta < \gamma \mid \exists r \in \mathbb{P}_{\gamma_p}\ [r \leq_{\mathbb{P}_{\gamma_p}} p \wedge r \Vdash_{\mathbb{P}_\lambda} \text{“}\dot{c}(\check{\alpha}) = \check{\beta}\text{”}]\}$$

for all $\alpha < \nu$. Pick $\alpha < \nu$. By Lemma 2.6, $\bar{G} = G \cap \mathbb{P}_{\gamma_p}$ is $\mathbb{P}_{\gamma_p}$-generic over V. Since $p \in \bar{G}$, the above computations show that there is an $r \in D_\alpha \cap \bar{G}$. If $\beta < \gamma$ witnesses that r is an element of D_α, then $\dot{c}^G(\alpha) = \beta < c(\alpha)$. This shows that the range of c is unbounded in γ, a contradiction. ∎

Corollary 2.12. *Let G be $\mathbb{P}_\lambda$-generic over* V, *let $\bar{\lambda} < \lambda$ and let A be a subset of $\bar{\lambda}$ in* V$[G]$. *Then there is a $\gamma < \lambda$ such that $A = \dot{A}^{G\cap\mathbb{P}_\gamma}$ for some $\mathbb{P}_\gamma$-name $\dot{A}$ for a subset of $\bar{\lambda}$.*

Proof. Let $\dot{A}_0$ be a $\mathbb{P}_\lambda$-name for a subset of $\bar{\lambda}$ with $A = \dot{A}_0^G$ and, given $\alpha < \bar{\lambda}$, let D_α be the dense open subset of $\mathbb{P}_\lambda$ consisting of all conditions in $\mathbb{P}_\lambda$ that decide the statement "$\check{\alpha} \in \dot{A}_0$". By Lemma 2.10, there is a $p \in G$ such that the set $D_\alpha \cap \mathbb{P}_{\gamma_p}$ is dense below p for every $\alpha < \bar{\lambda}$. Define

$$\dot{A} \ = \ \{\langle\check{\alpha}, r\rangle \mid \alpha < \bar{\lambda},\ r \in D_\alpha \cap \mathbb{P}_{\gamma_p},\ r \leq_{\mathbb{P}_\lambda} p,\ r \Vdash_{\mathbb{P}_\lambda} \text{“}\check{\alpha} \in \dot{A}_0\text{”}\}.$$

Then $\dot{A}$ is a $\mathbb{P}_{\gamma_p}$-name for a subset of $\bar{\lambda}$ and we can use Lemma 2.6 to conclude that $A = \dot{A}^G = \dot{A}^{G\cap\mathbb{P}_{\gamma_p}}$. ∎

We use this corollary to show that forcing with $\mathbb{P}_\lambda$ can collapse cardinals.

PROPOSITION 2.13. *Forcing with $\mathbb{P}_\lambda$ collapses $2^{<\lambda}$ to λ.*

Proof. Let G be $\mathbb{P}_\lambda$-generic over V. Given $\gamma < \lambda$, we define A_γ to be the unique set that is equal to $\dot{A}^G_{p,\gamma}$ for all $p \in G$ with $\gamma < \gamma_p$. A standard density argument using Proposition 2.4 and Corollary 2.12 shows that for every ordinal $\bar{\lambda} < \lambda$ and every subset a of $\bar{\lambda}$ there is a $\gamma < \lambda$ such that a is equal to the set $\{\delta < \bar{\lambda} \mid 0 \in A_{\gamma+\delta}\}$. This yields the assertion. ∎

3. Proof of Theorem 1.1. We are now ready to show how the forcing constructed in the last section can be used to produce a locally Σ_1-definable well-order of $\mathrm{H}(\kappa^+)$.

LEMMA 3.1. *If G is $\mathbb{P}_\lambda$-generic over* V *and $y \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, then there is a unique ordinal $\delta < \lambda$ such that $\delta < \gamma_p$ and $\dot{A}^G_{p,\delta}$ codes y for some condition $p \in G$.*

Proof. By Corollary 2.12, there is a $\gamma < \lambda$ and a $\mathbb{P}_\gamma$-name $\dot{y}$ such that $y = \dot{y}^{G \cap \mathbb{P}_\gamma}$. Fix a condition p in $\mathbb{P}_\lambda$ with $\gamma_p \geq \gamma$. Let $\dot{A}$ be a $\mathbb{P}_{\gamma_p}$-name for a subset of κ such that the following statements hold whenever H is $\mathbb{P}_{\gamma_p}$-generic over V with $p \in H$ and $\dot{y}^H \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$:

- If there is no $\delta < \gamma_p$ such that $\dot{A}^H_{p,\delta}$ codes $\dot{y}^H$, then $\dot{A}^H$ codes $\dot{y}^H$.
- Otherwise, $\dot{A}^H$ codes an element of $\mathrm{H}(\kappa^+)^{\mathrm{V}}$ that is not coded by some $\dot{A}^H_{p,\delta}$ with $\delta < \gamma_p$ (note that Corollary 2.11 implies that such an element always exists).

Define $\vec{A} = \vec{A}_p \cup \{\langle \gamma_p, \dot{A} \rangle\}$. Then $\vec{A}$ satisfies the statements listed in Definition 2.1(iv) and $\langle s_p, t_p, \vec{c}_p, \vec{A} \rangle$ is a condition in $\mathbb{P}_\lambda$ below p. The above computations show that there is a condition q in G and a $\delta < \gamma_q$ such that $\gamma_q > \gamma$ and $\dot{A}^{G \cap \mathbb{P}_{\gamma_q}}_{q,\delta} = \dot{A}^G_{q,\delta}$ codes $\dot{y}^{G \cap \mathbb{P}_{\gamma_q}} = \dot{y}^G$.

Now, assume, towards a contradiction, that there are $\delta_0 < \delta_1 < \lambda$ and $p_0, p_1 \in G$ such that both $\dot{A}^G_{p_0,\delta_0}$ and $\dot{A}^G_{p_1,\delta_1}$ code y. Pick $p \in G$ with $p \leq_{\mathbb{P}_\lambda} p_0, p_1$. Then $\bar{G} = G \cap \mathbb{P}_{\delta_1}$ is $\mathbb{P}_{\delta_1}$-generic over V and Corollary 2.11 implies $|\delta_1|^{\mathrm{V}[\bar{G}]} < |\lambda|^{\mathrm{V}[\bar{G}]}$. The above assumption now implies that the subsets $\dot{A}^{\bar{G}}_{p,\delta_0} = \dot{A}^G_{p_0,\delta_0}$ and $\dot{A}^{\bar{G}}_{p,\delta_1} = \dot{A}^G_{p_1,\delta_1}$ code the same element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[\bar{G}]}$, contradicting Definition 2.1(iv) for the condition p. ∎

COROLLARY 3.2. *Forcing with $\mathbb{P}_\lambda$ preserves the value of 2^κ.* ∎

LEMMA 3.3. *If G is $\mathbb{P}_\lambda$-generic over* V*, then the set*

$$D(G) = \{w_\delta \oplus c(\alpha, i) \mid i < 2,\ \exists p \in G\ [\delta < \gamma_p \wedge (i = 1 \leftrightarrow \alpha \in \dot{A}^G_{p,\delta})]\}$$

is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters.

Proof. Let G be $\mathbb{P}_\lambda$-generic over V. We prove a number of claims whose combination will imply the statement of the lemma.

CLAIM 1. *If $x = w_\delta \oplus c(\alpha, i) \in D(G)$, then there is a $p \in G$ with $x \in a_p$.*

Proof. There is a $q \in G$ witnessing that x is an element of $D(G)$ and $q{\restriction}\delta \Vdash_{\mathbb{P}_\delta}$ "$i = 1 \leftrightarrow \check{\alpha} \in \dot{A}_{q,\delta}$". We may assume that $x \notin a_q$. Fix $p_0 \in \mathbb{P}_\lambda$ with $p_0 \leq_{\mathbb{P}_\lambda} q$ and $x \notin a_{p_0}$. If we define

$$p = \langle s_{p_0}, t_{p_0}, \{\langle x, \emptyset\rangle\} \cup \langle c_{p_0,y} \mid y \in a_{p_0}\rangle, \vec{A}_{p_0}\rangle,$$

then the above assumptions imply that p is a condition in $\mathbb{P}_\lambda$ that is stronger than p_0. Hence the set of all conditions p in $\mathbb{P}_\lambda$ with $x \in a_p$ is dense below $q \in G$. ∎

CLAIM 2. *$\kappa = \sup\{\beta_p \mid p \in G\}$ and $\kappa = \sup\{\sup c_{p,x} \mid p \in G,\, x \in a_p\}$ whenever $x \in D(G)$.*

Proof. Fix a condition q in $\mathbb{P}_\lambda$ with $x \in a_q$ and fix $\beta_q < \beta < \kappa$. Define

$$\begin{aligned} s &= s_q \cup \{\langle \alpha, \emptyset\rangle \mid \beta_q < \alpha \leq \beta\},\\ t &= t_q \cup \{\langle \alpha, 1\rangle \mid \beta_q < \alpha \leq \beta\},\\ p &= \langle s, t, \langle c_{q,x} \cup (\beta_q, \beta] \mid x \in a_q\rangle, \vec{A}_q\rangle. \end{aligned}$$

Then p is a condition in $\mathbb{P}_\lambda$ with $p \leq_{\mathbb{P}_\lambda} q$, $\beta_p = \beta$ and $\sup c_{p,x} = \beta$. ∎

We fix $\mathbb{P}_\lambda$-names $\dot{s}$ and $\dot{t}$ in V such that $\dot{s}^H = \bigcup\{s_p \mid p \in H\} : \kappa \to {}^{<\kappa}2$ and $\dot{t}^H = \bigcup\{t_p \mid p \in H\} : \kappa \to 2$ whenever H is $\mathbb{P}_\lambda$-generic over V. The following claim is a direct consequence of the definition of $\mathbb{P}_\lambda$ and of Claim 2.

CLAIM 3. *If $x \in D(G)$, then $C^x_G = \bigcup\{c_{p,x} \mid p \in G,\, x \in a_p\}$ is a club subset of κ such that the implication*

$$\dot{s}^G(\alpha) \subseteq x \to \dot{t}^G(\alpha) = 1 \tag{1}$$

holds for all $\alpha \in C^x_G$. ∎

CLAIM 4. *Assume that $x \in ({}^\kappa 2)^{\mathrm{V}[G]}$ is such that* (1) *holds for every element α of some club subset C of κ. Then x is an element of $D(G)$.*

Proof. Let $\dot{a}$ be the canonical $\mathbb{P}_\lambda$-name with the property that $\dot{a}^H = \bigcup\{a_p \mid p \in H\}$ whenever H is $\mathbb{P}_\lambda$-generic over V. Assume, towards a contradiction, that x is not an element of $\dot{a}^G$. Then we can find $q \in G$ and $\mathbb{P}_\lambda$-names $\dot{C}$ and $\dot{x}$ such that $x = \dot{x}^G$ and

$$q \Vdash_{\mathbb{P}_\lambda} \text{“}\dot{x} \in {}^{\check{\kappa}}2 \setminus \dot{a} \wedge \dot{C} \subseteq \check{\kappa} \text{ club} \wedge \forall \alpha \in \dot{C}\ [\dot{s}(\alpha) \subseteq \dot{x} \to \dot{t}(\alpha) = 1]\text{”}.$$

Fix a condition p_0 in $\mathbb{P}_\lambda$ that is stronger than q. By using the above assumptions, we can recursively construct:

- a descending sequence $\langle p_n \mid n < \omega\rangle$ of conditions in $\mathbb{P}_\lambda$,
- strictly increasing sequences $\langle \alpha_n \mid n < \omega\rangle$ and $\langle \beta_n \mid n < \omega\rangle$ of ordinals less than κ, and

- a sequence $\langle s_n \mid n < \omega\rangle$ of elements of ${}^{<\kappa}2$

that satisfy the following statements for all $n < \omega$:

(i) $\beta_{p_n} < \alpha_n \leq \beta_n < \beta_{p_{n+1}}$.
(ii) $s_n \neq y\restriction\alpha_n$ for all $y \in a_{p_n}$.
(iii) $p_{n+1} \Vdash_{\mathbb{P}_\lambda}$ "$\dot{x}\restriction\check{\alpha}_n = \check{s}_n \wedge \check{\beta}_n = \min(\dot{C} \setminus \check{\alpha}_n)$".

Next, we define

$$\begin{aligned} \beta &= \sup_{n<\omega} \alpha_n = \sup_{n<\omega} \beta_n, \\ s_\omega &= \bigcup\{s_n \mid n < \omega\}, \\ s &= \{\langle \beta, s_\omega\rangle\} \cup \bigcup\{s_{p_n} \mid n < \omega\}, \\ t &= \{\langle \beta, 0\rangle\} \cup \bigcup\{t_{p_n} \mid n < \omega\}, \\ a &= \bigcup\{a_{p_n} \mid n < \omega\}, \\ c_y &= \{\beta\} \cup \bigcup\{c_{p_n,y} \mid n < \omega,\, y \in a_{p_n}\} \quad \text{for every element } y \text{ of } a, \\ \vec{A} &= \bigcup\{\vec{A}_{p_n} \mid n < \omega\}. \end{aligned}$$

Since $s_\omega \not\subseteq y$ for every $y \in a$, the tuple $p = \langle s, t, \langle c_y \mid y \in a\rangle, \vec{A}\rangle$ is a condition in $\mathbb{P}_\lambda$ that is stronger than p_0. Our construction ensures

$$p \Vdash_{\mathbb{P}_\lambda} \text{"}\check{\beta} \in \dot{C} \wedge \dot{s}(\check{\beta}) = \check{s} \subseteq \dot{x} \wedge \dot{t}(\check{\beta}) = 0\text{"},$$

a contradiction. Hence we can conclude that $x \in \dot{a}^G$.

The above computations show that there are $p \in G$, $\delta < \gamma_p$, $\alpha < \kappa$ and $i < 2$ with $x = w_\delta \oplus c(\alpha, i) \in a_p$. Since $p\restriction\delta \in G \cap \mathbb{P}_\delta$, Definition 2.2 implies that we have "$i = 1$" if and only if $\alpha \in \dot{A}_{p,\delta}^{G\cap\mathbb{P}_\delta} = \dot{A}_{p,\delta}^G$. Hence p witnesses that x is an element of $D(G)$. ∎

Claims 1–4 allow us to conclude that

$$D(G) = \{x \in ({}^\kappa 2)^{\mathrm{V}[G]} \mid \exists C \subseteq \kappa \text{ club } \forall \alpha \in C\ [\dot{s}^G(\alpha) \subseteq x \rightarrow \dot{t}^G(\alpha) = 1]\},$$

and this equality yields a Σ_1-definition of $D(G)$ over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ using the parameters $\dot{s}^G$ and $\dot{t}^G$. ∎

Lemma 3.4. *Let G be $\mathbb{P}_\lambda$-generic over* V *and assume that the set*

$$\prec_{\vec{w}} = \{\langle w_{\bar{\delta}}, w_\delta\rangle \mid \bar{\delta} < \delta < \lambda\} \tag{2}$$

is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters. Then there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters.

Proof. Define $W = \{w_\delta \mid \delta < \lambda\}$. Then our assumptions imply that W is also definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters.

CLAIM 1. *If $p \in G$ and $\delta < \gamma_p$, then*

$$\dot{A}^G_{p,\delta} = \{\alpha < \kappa \mid w_\delta \oplus c(\alpha,1) \in D(G)\} = \{\alpha < \kappa \mid w_\delta \oplus c(\alpha,0) \notin D(G)\}.$$

Proof. By the definition of $D(G)$, we have

$$\alpha \in \dot{A}^G_{p,\delta} \;\Leftrightarrow\; \exists q \in G\ [\delta < \gamma_q \wedge \alpha \in \dot{A}^G_{q,\delta}] \;\Leftrightarrow\; w_\delta \oplus c(\alpha,1) \in D(G)$$

and

$$\alpha \notin \dot{A}^G_{p,\delta} \;\Leftrightarrow\; \exists q \in G\ [\delta < \gamma_q \wedge \alpha \notin \dot{A}^G_{q,\delta}] \;\Leftrightarrow\; w_\delta \oplus c(\alpha,0) \in D(G).$$

These equivalences imply the assertion. ∎

We define P to be the set of all pairs $\langle z, w\rangle$ such that $z \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$, $w \in W$ and there is a subset A of κ coding z and satisfying

(3) $\qquad [\alpha \in A \to w \oplus c(\alpha,1) \in D(G)] \wedge [\alpha \notin A \to w \oplus c(\alpha,0) \in D(G)].$

Lemma 3.3 implies that P is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters.

CLAIM 2. *Let $z \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and let δ_z be the unique ordinal* (*given by Lemma* 3.1) *such that $\delta_z < \gamma_p$ and $\dot{A}^G_{p,\delta_z}$ codes z for some $p \in G$. Then w_{δ_z} is the unique element of W with $\langle z, w_{\delta_z}\rangle \in P$.*

Proof. By Claim 1, the subset $\dot{A}^G_{p,\delta_z}$ of κ witnesses that the pair $\langle z, w_{\delta_z}\rangle$ is an element of P. Now assume, towards a contradiction, that there is a $\delta < \lambda$ with $\delta \neq \delta_z$ and $\langle z, w_\delta\rangle \in P$. Let $A \subseteq \kappa$ satisfy (3). Then these implications together with Claim 1 show that $A = \dot{A}^G_{q,\delta}$ for some $q \in G$ with $\bar{\gamma} = \max\{\delta, \delta_z\} < \gamma_q$. If we set $\bar{G} = G \cap \mathbb{P}_{\bar{\gamma}}$, then Corollary 2.11 implies $|\bar{\gamma}|^{\mathrm{V}[\bar{G}]} < |\lambda|^{\mathrm{V}[\bar{G}]}$ and the subsets $\dot{A}^{\bar{G}}_{q,\delta} = \dot{A}^G_{q,\delta}$ and $\dot{A}^{\bar{G}}_{q,\delta_z} = \dot{A}^G_{q,\delta_z}$ code the same element of $\mathrm{H}(\kappa^+)^{\mathrm{V}[\bar{G}]}$. This contradicts Definition 2.1 (iv). ∎

Define $\prec_*$ to be the set of all pairs $\langle z, \bar{z}\rangle$ in $\mathrm{H}(\kappa^+)$ such that

$$\exists w, \bar{w} \in W\ [\langle z, w\rangle \in P \wedge \langle \bar{z}, \bar{w}\rangle \in P \wedge w \prec_{\vec{w}} \bar{w}].$$

Then our assumptions and the above remarks imply that this relation is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters. Given $z_0, z_1 \in \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ and $\delta_0, \delta_1 < \lambda$ such that δ_i is the unique ordinal with the property that $\delta_i < \gamma_p$ and $\dot{A}^G_{p,\delta_i}$ codes z_i for some $p \in G$, we have $z_0 \prec_* z_1$ if and only if $\delta_0 < \delta_1$. This shows that $\prec_*$ is a well-ordering of $\mathrm{H}(\kappa^+)$. ∎

The following *absoluteness version* of Theorem 1.2 proven in [Lüc12] will allow us to show that the hypotheses of Lemma 3.4 can be forced to hold by a forcing that preserves our assumptions on κ and λ.

THEOREM 3.5 ([Lüc12, Theorem 1.5]). *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$. Given a subset A of ${}^\kappa\kappa$, there is a partial order $\mathbb{P}(A)$ with the following properties:*

(i) $\mathbb{P}(A)$ *is* $<\kappa$*-closed, satisfies the* κ^+*-chain condition and has cardinality* 2^κ.
(ii) *If* $\dot{\mathbb{Q}}$ *is a* $\mathbb{P}(A)$*-name for a* σ*-strategically closed partial order that preserves the regularity of* κ *and* $G * H$ *is* $(\mathbb{P}(A) * \dot{\mathbb{Q}})$*-generic over* V, *then* A *is definable over the structure* $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G*H]}, \in\rangle$ *by a* Σ_1*-formula with parameters.*

Proof of Theorem 1.1. Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$ and $\lambda = 2^\kappa$ is regular. Fix an injective sequence $\vec{w} = \langle w_\gamma \mid \gamma < \lambda\rangle$ of elements of ${}^\kappa 2$ and define $A = \{w_\delta \oplus w_\gamma \mid \delta < \gamma < \lambda\}$. Let $\mathbb{P}(A)$ be the notion of forcing corresponding to A that is given by Theorem 3.5. Since forcing with $\mathbb{P}(A)$ preserves the above assumptions on κ and λ, there is a canonical $\mathbb{P}(A)$-name $\dot{\mathbb{Q}}$ with the property that $\dot{\mathbb{Q}}^G = \mathbb{P}_\lambda^{\mathrm{V}[G]}$ whenever G is $\mathbb{P}(A)$-generic over V and $\vec{\mathbb{P}}_{\vec{w}}^{\mathrm{V}[G]} = \langle \mathbb{P}_\gamma^{\mathrm{V}[G]} \mid \gamma \leq \lambda\rangle$. Then the combination of Lemma 2.8, Corollary 2.11, Corollary 3.2 and Theorem 3.5 implies that $\mathbb{P} = \mathbb{P}(A) * \dot{\mathbb{Q}}$ is $<\kappa$-closed and forcing with $\mathbb{P}(A) * \dot{\mathbb{Q}}$ preserves all cofinalities less than or equal to λ and the value of 2^κ.

Let $G * H$ be $(\mathbb{P}(A) * \dot{\mathbb{Q}})$-generic over V. By Theorem 3.5, the set A is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G*H]}, \in\rangle$ by a Σ_1-formula with parameters and this implies that the relation $\prec_{\vec{w}}$ defined by (2) is definable in the same way. In this situation, Lemma 3.4 implies that there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G*H]}$ that is definable over the structure $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G*H]}, \in\rangle$ by a Σ_1-formula with parameters. ■

4. Definable Bernstein sets. In this short section, we prove Corollary 1.5. We start by introducing some vocabulary needed in this proof. A subset T of ${}^{<\kappa}\kappa$ is a *subtree* of ${}^{<\kappa}\kappa$ if T is closed under initial segments. Given such a subtree T, we define $[T] = \{x \in {}^\kappa\kappa \mid \forall \alpha < \kappa \;\; x\restriction\alpha \in T\}$. Note that a subset of ${}^\kappa\kappa$ is closed with respect to the topology introduced at the end of the first section if and only if it is equal to the set $[T]$ for some subtree T of ${}^{<\kappa}\kappa$. An easy argument shows that a closed subset $[T]$ of ${}^\kappa\kappa$ contains a perfect subset if and only if there is an order-preserving injection $e : {}^{<\kappa}2 \to T$.

Proof of Corollary 1.5. We work in the setting of the proof of Theorem 1.1 and show that forcing with $\mathbb{P} = \mathbb{P}(A) * \dot{\mathbb{Q}}$ adds a subset X of ${}^\kappa\kappa$ such that both X and its complement intersect every perfect subset of ${}^\kappa\kappa$ and such that X is Δ_1-definable with parameters over $\langle \mathrm{H}(\kappa^+), \in\rangle$. If $x \in {}^\kappa\kappa$ and $\alpha < \kappa$, we define $y = \alpha^\frown x$ if $y(0) = \alpha$ and $y(1+\beta) = x(\beta)$ for every $\beta < \kappa$. We work in a $\mathbb{P}$-generic extension $\mathrm{V}[G, H]$ of V. Assume that $\prec^*$ is the locally Δ_1-definable well-order of $\mathrm{H}(\kappa^+)$ constructed in Section 3. Define

$$X = \{x \in {}^\kappa\kappa \mid 0^\frown x \prec^* 1^\frown x\}.$$

Let $[T]$ be a perfect subset of ${}^{\kappa}\kappa$ and $e : {}^{<\kappa}2 \to T$ be an order-preserving injection. Now work in $\mathrm{V}[G]$ and pick a condition p in $\mathbb{P}_\lambda = \dot{\mathbb{Q}}^G$. By Corollary 2.12, there is a $\gamma < \lambda$ with $p \in \mathbb{P}_\gamma$, a $\mathbb{P}_\gamma$-name $\dot{T}$ for a subtree of ${}^{<\kappa}\kappa$ with $T = \dot{T}^{H\cap\mathbb{P}_\gamma}$ and a $\mathbb{P}_\gamma$-name $\dot{e}$ for an order-preserving injection of ${}^{<\kappa}2$ into $\dot{T}$. Note that this implies that $[\dot{T}]$ has cardinality 2^κ in every $\mathbb{P}_\gamma$-generic extension of $\mathrm{V}[G]$. Hence we can find a $\mathbb{P}_\gamma$-name $\dot{x}$ for an element of $[\dot{T}]$ such that whenever $\bar{H}$ is $\mathbb{P}_\gamma$-generic over $\mathrm{V}[G]$ and $\dot{y}_i$ is the canonical $\mathbb{P}_\gamma$-name for $i^\frown\dot{x}$, then neither $\dot{y}_0^{\bar{H}}$ nor $\dot{y}_1^{\bar{H}}$ are coded by any $\dot{A}_{p,\delta}^{\bar{H}}$ for $\delta < \gamma_p$. Given $i < 2$, we can extend $p = \langle s_p, t_p, \vec{c}_p, \vec{A}_p\rangle$ to a condition $q_i = \langle s_p, t_p, \vec{c}_p, \vec{A}_{q_i}\rangle$ in $\mathbb{P}_\lambda$ such that $\gamma_{q_i} = \gamma + 2$, $\dot{A}_{q_i,\gamma}$ is a $\mathbb{P}_\gamma$-nice name for a subset of κ coding $\dot{y}_i$, and $\dot{A}_{q_i,\gamma+1}$ is a $\mathbb{P}_\gamma$-nice name for a subset of κ coding $\dot{y}_{1-i}$. Let $\dot{X}$ be the canonical $\mathbb{P}_\lambda$-name for the set X. The above construction ensures that $q_0 \Vdash_{\mathbb{P}_\lambda}$ "$\dot{x} \in [\dot{T}] \cap \dot{X}$" and $q_1 \Vdash_{\mathbb{P}_\lambda}$ "$\dot{x} \in [\dot{T}] \setminus \dot{X}$". We can conclude that, in $\mathrm{V}[G, H]$, the perfect subset $[T]$ is contained neither in X nor in the complement of X. ∎

5. Open questions. We close this paper with questions induced by the above results.

The parameter in the Σ_1-definition of the well-order constructed above is a subset of κ that is added by forcing and therefore is, in a certain sense, a very complicated object. It is natural to ask if it is possible to force Σ_1-definable well-orderings of $\mathrm{H}(\kappa^+)$ that use *simpler* parameters which are contained in some prescribed set P.

QUESTION 5.1. *Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and let P be a subset of $\mathrm{H}(\kappa^+)$. Is there a partial order $\mathbb{P}$ with the following properties?*

- (i) *Forcing with $\mathbb{P}$ preserves cofinalities less than or equal to 2^κ and the value of 2^κ.*
- (ii) *If G is $\mathbb{P}$-generic over the ground model V, then there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters contained in P.*

Interesting examples of such restricted parameter sets would be $P = \{\kappa\}$ or $P = \mathrm{H}(\kappa^+)^{\mathrm{V}}$.

QUESTION 5.2. *Is it possible to iterate forcings of the form $\mathbb{P}_\lambda$ to add locally Σ_1-definable well-orderings of $\mathrm{H}(\kappa^+)$ for many different κ simultaneously while preserving certain structural properties of the ground model?*

Interesting examples of such structural properties would be the cardinal structure, the continuum function and the existence of large cardinals.

A completely satisfactory positive answer to the above question would probably depend on a positive answer to the following.

QUESTION 5.3. *Is it possible to obtain a result as in Theorem* 1.1*, however witnessed by a* cofinality-preserving *forcing* $\mathbb{P}$*?*

Note that by Proposition 2.13, the forcing $\mathbb{P}$ constructed in the proof of Theorem 1.1 changes the cardinality of $(2^{<2^\kappa})^V$ if this cardinal is larger than $(2^\kappa)^V$.

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References

[AF12] D. Asperó and S.-D. Friedman, *Definable well-orders of* $H(\omega_2)$ *and GCH*, J. Symbolic Logic 77 (2012), 1101–1121.

[AHL] D. Asperó, P. Holy and P. Lücke, *Forcing lightface definable well-orders without the GCH*, submitted.

[BL99] J. Brendle and B. Löwe, *Solovay-type characterizations for forcing-algebras*, J. Symbolic Logic 64 (1999), 1307–1323.

[FH11] S.-D. Friedman and P. Holy, *Condensation and large cardinals*, Fund. Math. 215 (2011), 133–166.

[Jec03] T. Jech, *Set Theory. The Third Millennium Edition, Revised and Expanded*, Springer Monogr. Math., Springer, Berlin, 2003.

[JS70] R. B. Jensen and R. M. Solovay, *Some applications of almost disjoint sets*, in: Mathematical Logic and Foundations of Set Theory (Jerusalem, 1968), North-Holland, Amsterdam, 1970, 84–104.

[Kec78] A. S. Kechris, *The perfect set theorem and definable wellorderings of the continuum*, J. Symbolic Logic 43 (1978), 630–634.

[Lüc12] P. Lücke, Σ^1_1*-definability at uncountable regular cardinals*, J. Symbolic Logic 77 (2012), 1011–1046.

[LS] P. Lücke and P. Schlicht, *Continuous images of closed sets in generalized Baire spaces*, submitted.

[Man75] R. Mansfield, *The non-existence of* Σ^1_2 *well-orderings of the Cantor set*, Fund. Math. 86 (1975), 279–282.

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