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The automorphism group of the random lattice is not amenable

by

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Abstract. We prove that the automorphism group of the random lattice is not amenable, and we identify the universal minimal flow for the automorphism group of the random distributive lattice.

1. Introduction. In this note we answer a question posed by A. Kechris and M. Sokić [6], who asked whether the automorphism group $\mathrm{Aut}(\boldsymbol{L}_\infty)$ of the random lattice $\boldsymbol{L}_\infty$ is amenable. Recall that a topological group G is *amenable* if every G-flow, that is, a continuous action of G on a non-empty compact Hausdorff space, has an invariant Borel probability measure. We show that $\mathrm{Aut}(\boldsymbol{L}_\infty)$ is not amenable. We also identify the universal minimal flow for the automorphism group $\mathrm{Aut}(\boldsymbol{D}_\infty)$ of the random distributive lattice $\boldsymbol{D}_\infty$. This has been first done in [6, Theorem 5.1] using a different approach.

Let L be a countable first-order language. A class $\mathcal{K}$ of finite L-structures is called a *Fraïssé class* if it contains structures of arbitrarily large finite size, is countable (in the sense that it contains only countably many isomorphism types) and satisfies the following:

(1) *Hereditary Property* (HP): If $\boldsymbol{A}_1 \in \mathcal{K}$ and $\boldsymbol{A}_0$ can be embedded in $\boldsymbol{A}_1$, then $\boldsymbol{A}_0 \in \mathcal{K}$.
(2) *Joint Embedding Property* (JEP): If $\boldsymbol{A}_0, \boldsymbol{A}_1 \in \mathcal{K}$, there is $\boldsymbol{A}_2 \in \mathcal{K}$ such that $\boldsymbol{A}_0, \boldsymbol{A}_1$ can be embedded in $\boldsymbol{A}_2$.
(3) *Amalgamation Property* (AP): If $\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2 \in \mathcal{K}$ and $f : \boldsymbol{A}_0 \to \boldsymbol{A}_1$ and $g : \boldsymbol{A}_0 \to \boldsymbol{A}_2$ are embeddings, there is $\boldsymbol{A}_3 \in \mathcal{K}$ and embeddings $r : \boldsymbol{A}_1 \to \boldsymbol{A}_3$ and $s : \boldsymbol{A}_2 \to \boldsymbol{A}_3$ such that $r \circ f = s \circ g$.

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If (3) holds with the extra property that $r[\boldsymbol{A}_1] \cap s[\boldsymbol{A}_2] = r \circ f[\boldsymbol{A}_0]$, we say that $\mathcal{K}$ has the *Strong Amalgamation Property* (SAP).

For every Fraïssé class $\mathcal{K}$ there is a unique, up to isomorphism, countably infinite structure $\boldsymbol{K}_\infty$ which is locally finite (i.e., finitely generated substructures are finite), ultrahomogeneous (i.e., isomorphisms between finite substructures can be extended to automorphisms of the structure) and is such that, up to isomorphism, its finite substructures are exactly those in $\mathcal{K}$. We call this the *Fraïssé limit* of $\mathcal{K}$.

For a language L and a class $\mathcal{K}$ of finite L-structures, a class $\mathcal{K}^*$ of structures in the language $L \cup \{\prec\}$ is called an *order expansion* of $\mathcal{K}$ if every $\boldsymbol{A}^* \in \mathcal{K}^*$ is of the form $\langle \boldsymbol{A}, \prec \rangle$, where $\boldsymbol{A} \in \mathcal{K}$ and $\prec$ is a linear ordering of $\boldsymbol{A}$. A linear ordering $\prec$ on $\boldsymbol{A} \in \mathcal{K}$ is called *admissible* if $\langle \boldsymbol{A}, \prec \rangle \in \mathcal{K}^*$. Similarly, if $\mathcal{K}$ is a Fraïssé class and $\boldsymbol{K}_\infty$ is its Fraïssé limit, then a linear ordering $\prec$ on $\boldsymbol{K}_\infty$ is admissible if $\langle \boldsymbol{A}, \prec \restriction \boldsymbol{A} \rangle \in \mathcal{K}^*$ for every finite substructure $\boldsymbol{A} \subseteq \boldsymbol{K}_\infty$. We denote the compact space of all admissible orderings on $\boldsymbol{K}_\infty$ by $X_{\mathcal{K}^*}$.

We will say that an order expansion $\mathcal{K}^*$ of $\mathcal{K}$ is *reasonable* if for every $\langle \boldsymbol{A}_0, \prec_0 \rangle \in \mathcal{K}^*$ and $\boldsymbol{A}_1 \in \mathcal{K}$, and for every embedding $i : \boldsymbol{A}_0 \to \boldsymbol{A}_1$, there exists an ordering $\prec_1$ of $\boldsymbol{A}_1$ such that $\prec_1$ extends the image of $\prec_0$ in $\boldsymbol{A}_1$, and $\langle \boldsymbol{A}_1, \prec_1 \rangle \in \mathcal{K}^*$. A crucial property of reasonable order expansions is that if $\mathcal{K}^*$ is reasonable then $X_{\mathcal{K}^*}$ is non-empty (see [6, Section 2]).

A *lattice* is a partially ordered set $\langle L, < \rangle$ with unique greatest lower bounds and least upper bounds. However, in this paper we will view lattices as structures $\boldsymbol{L} = \langle L, \wedge, \vee \rangle$, where $\wedge$ is the greatest lower bound operation and $\vee$ is the least upper bound operation. Then the ordering $<$ can be reconstructed from $\boldsymbol{L}$ because

$$x < y \;\Leftrightarrow\; x \wedge y = x \;\Leftrightarrow\; x \vee y = y$$

for every $x, y \in \boldsymbol{L}$ with $x \neq y$.

A *semilattice* is a partially ordereded set with unique least upper bounds. We will view semilattices as structures $\boldsymbol{S} = \langle S, \vee \rangle$ with the least upper bound operation $\vee$.

A *distributive lattice* is a lattice $\boldsymbol{D}$ satisfying

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for every $x, y, z \in \boldsymbol{D}$. Finally, a *Boolean lattice* is a complemented distributive lattice, that is, a distributive lattice $\boldsymbol{B}$ which has the smallest element $\mathbf{0}$, the largest element $\mathbf{1}$, and for every $x \in \boldsymbol{B}$ there exists a unique $y \in \boldsymbol{B}$ such that

$$x \wedge y = \mathbf{0}, \qquad x \vee y = \mathbf{1}.$$

It is known (see [4]) that the class $\mathcal{L}$ of all finite lattices, the class $\mathcal{S}$ of all finite semilattices, and the class $\mathcal{D}$ of all finite distributive lattices, are all Fraïssé classes, and that the classes $\mathcal{L}$ and $\mathcal{S}$ satisfy SAP. Let $\mathcal{L}^*$ be an order expansion of $\mathcal{L}$ defined as the set of all structures of the form $\langle \boldsymbol{L}, \prec \rangle$, where $\boldsymbol{L} \in \mathcal{L}$ and $\prec$ is a linear ordering extending the lattice ordering of $\boldsymbol{L}$, that is, the ordering induced by the meet operation (equivalently, the join operation) in $\boldsymbol{L}$.

We denote by $\boldsymbol{L}_\infty$ the Fraïssé limit of $\mathcal{L}$, by $\boldsymbol{D}_\infty$ the Fraïssé limit of $\mathcal{D}$, and by $\boldsymbol{S}_\infty$ the Fraïssé limit of $\mathcal{S}$. The lattice $\boldsymbol{L}_\infty$ is often called the *random lattice*, $\boldsymbol{D}_\infty$ is the *random distributive lattice*, and $\boldsymbol{S}_\infty$ is the *random semilattice.*

2. The automorphism group of the random lattice

LEMMA 2.1. *The class $\mathcal{L}^*$ is a reasonable Fraïssé expansion of $\mathcal{L}$.*

Proof. Fix $\langle \boldsymbol{A}_0, \prec_0 \rangle \in \mathcal{L}^*$, $\boldsymbol{A}_1 \in \mathcal{L}$, and an embedding $i : \boldsymbol{A}_0 \to \boldsymbol{A}_1$. Then the image of $\prec_0$ under i is a linear ordering on $i[\boldsymbol{A}_0]$ which extends the lattice ordering on $i[\boldsymbol{A}_0]$. By [7, Lemma 14] (see the algorithm presented in the proof of the lemma), there exists a linear ordering $\prec_1$ on $\boldsymbol{A}_1$ which extends the image of $\prec_0$ and the lattice ordering on $\boldsymbol{A}_1$. Thus $\mathcal{L}^*$ is reasonable.

In a similar way we can prove that $\mathcal{L}^*$ is a Fraïssé class. The property HP is obvious. In order to show AP, fix $\langle \boldsymbol{A}_0, \prec_0 \rangle, \langle \boldsymbol{A}_1, \prec_1 \rangle, \langle \boldsymbol{A}_2, \prec_2 \rangle \in \mathcal{L}^*$ and embeddings $i_1 : \langle \boldsymbol{A}_0, \prec_0 \rangle \to \langle \boldsymbol{A}_1, \prec_1 \rangle$ and $i_2 : \langle \boldsymbol{A}_0, \prec_0 \rangle \to \langle \boldsymbol{A}_2, \prec_2 \rangle$. Without loss of generality we can assume that $\boldsymbol{A}_1 \cap \boldsymbol{A}_2 = \boldsymbol{A}_0$, and i_1, i_2 are the identity embeddings. Using SAP for the class of finite lattices, find a lattice $\boldsymbol{A}_3$ such that

$$\boldsymbol{A}_1 \cup \boldsymbol{A}_2 \subseteq \boldsymbol{A}_3.$$

Now extend the orderings $\prec_1$, $\prec_2$ on $\boldsymbol{A}_1, \boldsymbol{A}_2$, respectively, to a linear ordering $\prec'$ on $\boldsymbol{A}_1 \cup \boldsymbol{A}_2$ using the well-known fact that the class of finite linear orderings has SAP. Finally, extend the ordering $\prec'$ to a linear ordering $\prec_3$ which extends the lattice ordering on $\boldsymbol{A}_3$, using [7, Lemma 14].

The same argument with $\boldsymbol{A}_0 = \emptyset$ shows that $\mathcal{L}^*$ has JEP. Thus, $\mathcal{L}^*$ is a Fraïssé class which is a reasonable expansion of $\mathcal{L}$. ∎

THEOREM 2.2. *The group* $\mathrm{Aut}(\boldsymbol{L}_\infty)$ *is not amenable.*

Proof. Since $\mathcal{L}^*$ is reasonable, the compact space $X_{\mathcal{L}^*}$ is non-empty. For $\langle \boldsymbol{A}, \prec \rangle \in \mathcal{L}^*$, let $N_{\langle \boldsymbol{A}, \prec \rangle}$ denote the non-empty basic clopen set in $X_{\mathcal{L}^*}$, consisting of $\prec^* \in X_{\mathcal{L}^*}$ such that $\prec^* \restriction \boldsymbol{A} = \prec$.

Suppose that μ is a Borel probability measure on $X_{\mathcal{L}^*}$ which is invariant with respect to the natural action of $\mathrm{Aut}(\boldsymbol{L}_\infty)$ on $X_{\mathcal{L}^*}$. Define

$$\mathcal{O} = \{V \subseteq X_{\mathcal{L}^*} : V \text{ is open and } \mu(V) = 0\}.$$

Then $\mathrm{Aut}(\boldsymbol{L}_\infty)$ permutes the family $\mathcal{O}$, so $O = \bigcup \mathcal{O}$ is an open invariant subset of $X_{\mathcal{L}^*}$. Moreover, $\mu(O) = 0$, and the restriction of μ to the invariant set $C = X_{\mathcal{L}^*} \setminus O$ has full support, that is, $\mu(V \cap C) > 0$ for every open $V \subseteq X_{\mathcal{L}^*}$ such that $V \cap C$ is non-empty.

Fix a finite Booelan lattice $\boldsymbol{A} \in \mathcal{L}$ with two atoms $x, y \in \boldsymbol{A}$. Let $\prec_0$ be an $\mathcal{L}^*$-admissible ordering for $\boldsymbol{A}$ such that $x \prec_0 y$, and let $\prec_1$ be an $\mathcal{L}^*$-admissible ordering for $\boldsymbol{A}$ such that $y \prec_1 x$.

Now fix a finite Boolean lattice $\boldsymbol{B} \in \mathcal{L}$ with three atoms $a, b, c \in \boldsymbol{B}$. Denote by $\boldsymbol{A}_0$ the Boolean lattice generated by $b \vee c$, a, and by $\boldsymbol{A}_1$ the Boolean lattice generated by $a \vee c$, b. Because $b \vee c$ and a are incomparable, there exists an $\mathcal{L}^*$-admissible ordering $\prec_0'$ for $\boldsymbol{A}_0$ such that $b \vee c \prec_0' a$. Similarly, there exists an $\mathcal{L}^*$-admissible ordering $\prec_1'$ for $\boldsymbol{A}_1$ such that $a \vee c \prec_1' b$.

Clearly, the unique lattice isomorphism $\pi_0 : \boldsymbol{A} \to \boldsymbol{A}_0$ that maps x to $b \vee c$ and y to a, is also an isomorphism of the orderings $\prec_0$ and $\prec_0'$. Similarly, the unique lattice isomorphism $\pi_1 : \boldsymbol{A} \to \boldsymbol{A}_1$ that maps y to $a \vee c$ and x to b, is an isomorphism of the orderings $\prec_1$ and $\prec_1'$.

Since $a \vee b$, c are incomparable and C is non-empty and invariant under the action of $\mathrm{Aut}(\boldsymbol{L}_\infty)$, there exists a linear ordering $\prec_2$ on $\boldsymbol{B}$ that can be extended to an element of C, and is such that c is the maximal atom in $\boldsymbol{B}$ with respect to $\prec_2$. As a is the maximal atom for any ordering of $\boldsymbol{B}$ extending $\prec_0'$, no such ordering extends $\prec_2$. Similarly, no ordering extending $\prec_1'$ extends $\prec_2$.

Also, if an ordering of $\boldsymbol{B}$ extends $\prec_0'$, then a is the maximal atom, so it cannot extend $\prec_1'$ which forces b to be the maximal atom. Analogously, no ordering extending $\prec_1'$ can extend $\prec_0'$.

The above observations imply that

$$N_{\langle \boldsymbol{B}, \prec_2 \rangle} \cap C \neq \emptyset, \tag{1}$$

$$N_{\langle \boldsymbol{B}, \prec_2 \rangle} \subseteq X_{\mathcal{L}^*} \setminus (N_{\langle \boldsymbol{A}_0, \prec_0' \rangle} \cup N_{\langle \boldsymbol{A}_1, \prec_1' \rangle}), \tag{2}$$

$$N_{\langle \boldsymbol{A}_0, \prec_0' \rangle} \cap N_{\langle \boldsymbol{A}_1, \prec_1' \rangle} = \emptyset. \tag{3}$$

In order to complete the proof, we will follow the main argument of [6, Proposition 2.1]. As already mentioned, $\mathrm{Aut}(\boldsymbol{L}_\infty)$ naturally acts on $X_{\mathcal{L}^*}$: for $\varphi \in \mathrm{Aut}(\boldsymbol{L}_\infty)$, $\prec \in X_{\mathcal{L}^*}$, and $x, y \in \boldsymbol{L}_\infty$,

$$x \prec_\varphi y \iff \varphi^{-1}(x) \prec \varphi^{-1}(y).$$

Fix $\varphi_0, \varphi_1 \in \mathrm{Aut}(\boldsymbol{L}_\infty)$ extending π_0, π_1, respectively. Then

$$\varphi_0[N_{\langle \boldsymbol{A}, \prec_0 \rangle}] = N_{\langle \boldsymbol{A}_0, \prec_0' \rangle}, \qquad \varphi_1[N_{\langle \boldsymbol{A}, \prec_1 \rangle}] = N_{\langle \boldsymbol{A}_1, \prec_1' \rangle},$$

so

$$\mu(N_{\langle \boldsymbol{A}, \prec_0 \rangle}) = \mu(N_{\langle \boldsymbol{A}_0, \prec_0' \rangle}), \qquad \mu(N_{\langle \boldsymbol{A}, \prec_1 \rangle}) = \mu(N_{\langle \boldsymbol{A}_1, \prec_1' \rangle}). \tag{4}$$

Moreover, by (1),

$$\mu(N_{\langle B,\prec_2\rangle}) > 0.$$

By (2) and (3), this implies that

$$\mu(N_{\langle \boldsymbol{A}_0,\prec_0'\rangle}) + \mu(N_{\langle \boldsymbol{A}_1,\prec_1'\rangle}) = \mu(N_{\langle \boldsymbol{A}_0,\prec_0'\rangle} \cup N_{\langle \boldsymbol{A}_1,\prec_1'\rangle}) < 1,$$

which contradicts (4) because, obviously,

$$N_{\langle \boldsymbol{A},\prec_0\rangle} \cup N_{\langle \boldsymbol{A},\prec_1\rangle} = X_{\mathcal{L}^*}, \qquad N_{\langle \boldsymbol{A},\prec_0\rangle} \cap N_{\langle \boldsymbol{A},\prec_1\rangle} = \emptyset,$$

so

$$\mu(N_{\langle \boldsymbol{A},\prec_0\rangle}) + \mu(N_{\langle \boldsymbol{A},\prec_1\rangle}) = \mu(N_{\langle \boldsymbol{A},\prec_0\rangle} \cup N_{\langle \boldsymbol{A},\prec_1\rangle}) = 1. \ \blacksquare$$

It has been pointed out to us by Miodrag Sokić and by the anonymous referee that exactly the same proof works for the class $\mathcal{S}$ of all finite semilattices, and the order expansion $\mathcal{S}^*$ of $\mathcal{S}$ such that $\langle S,\prec\rangle \in \mathcal{S}^*$ if $S \in \mathcal{S}$ and $\prec$ is a linear ordering extending the semilattice ordering of S. Thus we have the following corollary.

Corollary 2.3. *Let $\mathcal{S}$ be the Fraïssé class of all finite semilattices, and let $\boldsymbol{S}_\infty$ be the Fraïssé limit of $\mathcal{S}$. Then* $\mathrm{Aut}(\boldsymbol{S}_\infty)$ *is not amenable.*

3. The automorphism group of the random distributive lattice. In this section we present an alternative method of identifying the universal minimal flow for the automorphism group $\mathrm{Aut}(\boldsymbol{D}_\infty)$ of the random distributive lattice $\boldsymbol{D}_\infty$. The universal minimal flow for this group has been first found in [6, Theorem 5.1]. Let us start by defining the notions involved.

Recall that a G-flow X is called *minimal* if all its orbits are dense. A minimal G-flow X is called the *universal minimal flow* for G if any minimal G-flow Y is a factor of X, i.e., there is a continuous surjection $\pi : X \to Y$ which is a G-map: $\pi(g\cdot x) = g\cdot \pi(x)$ for all $g \in G$ and $x \in X$.

For a compact space X, a *chain* in X is a collection of compact subsets of X that is linearly ordered by inclusion. The space $\mathcal{C}(X)$ of all maximal chains in X can be naturally viewed as a compact subspace of the space $\mathcal{K}(\mathcal{K}(X))$ endowed with the Vietoris topology (here, $\mathcal{K}(X)$ is the space of all compact subsets of X). Then $\mathrm{Hom}(X)$ continuously acts on $\mathcal{C}(X)$ in a natural way, that is, $\mathcal{C}(X)$ is a $\mathrm{Hom}(X)$-flow.

A compact space X endowed with an ordering $\le$ with the least element 0 and the greatest element 1 is called a *bounded Priestley space* if $0 \ne 1$ and it is totally order-disconnected: for any $x, y \in X$ with $x \not\le y$ there exists a clopen downset containing y but not containing x.

Theorem 3.1. *The random distributive lattice $\boldsymbol{D}_\infty$ is isomorphic to the lattice of all clopen subsets of the Cantor space X that contain a distinguished point $x_0 \in X$, and do not contain a distinguished point $x_1 \in X$, where $x_0 \ne x_1$.*

In particular, $\mathrm{Aut}(\boldsymbol{D}_\infty)$ *is isomorphic to the group* $\mathrm{Hom}(X, x_0, x_1)$ *of all homeomorphisms of* X *fixing* x_0 *and* x_1*, and the universal minimal flow for* $\mathrm{Aut}(\boldsymbol{D}_\infty)$ *is the space of all maximal chains in* X *containing the sets* $\{x_0\}$ *and* $\{x_0, x_1\}$*.*

Proof. It is well known that every distributive lattice $\boldsymbol{D}$ is isomorphic to the lattice of proper, non-empty clopen downsets of the bounded Priestley space X of all prime ideals in $\boldsymbol{D}$ ordered by inclusion (see [2, Section 1.2]). A subbasis in X is defined by clopen sets of the form

$$U(a) = \{x \in X : a \in x\}, \qquad V(a) = \{x \in X : a \notin x\},$$

where $a \in \boldsymbol{D}$. Moreover, automorphisms of $\boldsymbol{D}$ correspond to homeomorphisms of X that preserve the ordering.

Let X be the Priestley space for $\boldsymbol{D}_\infty$. Obviously, the least element of the ordering of X is $x_0 = \emptyset$, and the largest element is $x_1 = \boldsymbol{D}_\infty$. Moreover, all non-trivial prime ideals in $\boldsymbol{D}_\infty$ are maximal, that is, any two distinct, non-trivial elements in X are incomparable. This is because $\boldsymbol{D}_\infty$ is relatively complemented (see below), and it is known that in relatively complemented, distributive lattices all non-trivial prime ideals are maximal (see [5, Theorem 4.2]).

Recall that a lattice $\boldsymbol{D}$ is *relatively complemented* if for all $x, y, z \in \boldsymbol{D}$ with $x < y < z$ there exists $w \in \boldsymbol{D}$ such that

$$y \wedge w = x, \qquad y \vee w = z.$$

To see that $\boldsymbol{D}_\infty$ is relatively complemented, fix $a, b, c, d \in \boldsymbol{D}_\infty$ such that

$$a = b \wedge c, \qquad d = b \vee c.$$

By the ultrahomogeneity of $\boldsymbol{D}_\infty$, for arbitrary $x, y, z \in \boldsymbol{D}_\infty$ such that $x < y < z$, there exists an automorphism $\varphi \in \mathrm{Aut}(\boldsymbol{D}_\infty)$ such that

$$\varphi(a) = x, \qquad \varphi(b) = y, \qquad \varphi(d) = z.$$

Then $\varphi(c)$ must be the complement of y relative to x and z:

$$y \wedge \varphi(c) = x, \qquad y \vee \varphi(c) = z.$$

We show that there are no isolated points in X. Fix a prime ideal $x \in X$. Suppose first that x is non-trivial, and fix $a_0, \ldots, a_n \in x$, $b_0, \ldots, b_m \notin x$. We will find $y \in X$ such that $y \neq x$, and

$$y \in U(a_0) \cap \cdots \cap U(a_n) \cap V(b_0) \cap \cdots \cap V(b_m).$$

Put $a = a_0 \vee \cdots \vee a_n$, $b = b_0 \wedge \cdots \wedge b_m$. Because x is prime, we have $a \in x$ and $b \notin x$. Suppose that $a < b$. By the ultrahomogeneity of $\boldsymbol{D}_\infty$, there exist $c, d \in \boldsymbol{D}_\infty$ such that $c \wedge d = a$ and $c \vee d = b$. Since x is prime, exactly one of the elements c, d is in x; say it is c. By [3, Theorem 115], there exists a prime ideal $y \in X$ such that $d \in y$ and $b \notin y$. Then $c \notin y$, so $y \neq x$, and y is as required.

On the other hand, if a, b are incomparable, using the ultrahomogeneity of $\boldsymbol{D}_\infty$, we find $c \in \boldsymbol{D}_\infty$ such that each of the sets $\{a, b, c\}$, $\{a \vee c, b\}$, $\{a, b \wedge c\}$ is pairwise incomparable. If $c \notin x$, then $a \vee c$ generates an ideal which does not contain b, and, by [3, Theorem 115], there exists $y \in X$ such that $a, c \in y$ and $b \notin y$. Thus, $y \neq x$, and y is as required. If $c \in x$, then $b \wedge c$ generates a filter which does not contain a, and we can use the same argument as in the case that $c \notin x$.

In order to show that the trivial ideals $\emptyset$ and $\boldsymbol{D}_\infty$ are not isolated, we proceed analogously, using the fact that there is no smallest nor largest element in $\boldsymbol{D}_\infty$.

It is well known that every compact space with a clopen countable basis and no isolated points is isomorphic to the Cantor space. Thus, X is the Cantor space with two distinct, distinguished points $x_0, x_1 \in X$ corresponding to the ideals $\emptyset$ and $\boldsymbol{D}_\infty$. Moreover, any two distinct, non-trivial elements in X are incomparable with respect to the inclusion ordering. It follows that $\mathrm{Aut}(\boldsymbol{D}_\infty)$ is isomorphic to the group $\mathrm{Hom}(X, x_0, x_1)$ of all homeomorphisms of X fixing x_0 and x_1.

Observe now that $\mathrm{Hom}(X, x_0, x_1)$ is equal to the group of all homeomorphisms of X fixing setwise the sets $\{x_0\}$ and $\{x_0, x_1\}$. But it is proved in [1, Theorem 12] that the universal minimal flow for this group is the space of all maximal chains in X containing these sets. ∎

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