

Hereditarily indecomposable inverse limits of graphs

by

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Abstract. We prove the following theorem: Let G be a compact connected graph and let $f : G \rightarrow G$ be a piecewise linear surjection which satisfies the following condition: for each nondegenerate subcontinuum A of G , there is a positive integer n such that $f^n(A) = G$. Then, for each $\varepsilon > 0$, there is a map $f_\varepsilon : G \rightarrow G$ which is ε -close to f such that the inverse limit (G, f_ε) is hereditarily indecomposable.

1. Introduction. Throughout the paper, a *graph* means a compact, connected, one-dimensional polyhedron. A continuum (i.e. a compact, connected, metric space) is said to be *indecomposable* if it is not the union of two proper subcontinua. A continuum is said to be *hereditarily indecomposable* if each subcontinuum is indecomposable. For a continuous surjection $f : X \rightarrow X$ of a compact metric space X , (X, f) denotes the inverse limit of the inverse sequence

$$X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \dots$$

where all coordinate spaces are X 's and all bonding maps are f 's. In this paper, we are concerned with the following question:

What properties of a map $h : G \rightarrow G$ of a graph G imply that for each $\varepsilon > 0$, there is a map $h_\varepsilon : G \rightarrow G$ which is ε -close to h such that (G, h_ε) is hereditarily indecomposable?

Properties of maps on graphs which yield inverse limits *containing indecomposable subcontinua* have been investigated by several authors. In particular, Barge and Diamond [2] obtained necessary and sufficient conditions for piecewise monotone maps whose inverse limits contain indecomposable

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continua. However, the situation for maps of graphs with *hereditarily indecomposable* limits is slightly different. Block, Keesling and Uspenskij [4] have shown that the continuous surjections g of $[0, 1]$ such that $([0, 1], g)$ is hereditarily indecomposable form a nowhere dense set in the space of continuous surjections of $[0, 1]$. In [6] and [10], two very different surjections of $[0, 1]$ onto itself are constructed, both of which yield the pseudo-arc (the topologically unique hereditarily indecomposable arc-like continuum) as their inverse limits.

The map $f : [0, 1] \rightarrow [0, 1]$ constructed by Henderson in [6] is simple from the point of view of topological dynamics. It has exactly two fixed points 0 and 1, and every point x different from 1 satisfies $\lim_{n \rightarrow \infty} f^n(x) = 0$. In contrast, the map $f : [0, 1] \rightarrow [0, 1]$ constructed by Minc and Transue in [10] is topologically transitive, that is, there exists a point x such that the subset $\{f^n(x) \mid n \geq 0\}$ is dense in $[0, 1]$. Thus it seems to be difficult to obtain a characterization of the dynamics of maps which yield hereditarily indecomposable inverse limits (see also Kuykendall [8]). The purpose of the present paper is to prove the following theorem, generalizing the construction of the Minc–Transue map above.

MAIN THEOREM. *Let G be a compact connected graph and $f : G \rightarrow G$ a piecewise linear surjection which satisfies the following condition:*

- (*) *for each nondegenerate subcontinuum A of G , there is a positive integer n such that $f^n(A) = G$.*

Then for each $\varepsilon > 0$, there is a map $f_\varepsilon : G \rightarrow G$ which is ε -close to f such that (G, f_ε) is hereditarily indecomposable.

The proof of the Main Theorem follows that in [10], where G is assumed to be an arc. We construct inductively small “crooked perturbations” of a given map to obtain the desired map as the limit of a sequence of these perturbations. Our main result extends the Minc–Transue theorem [10] to maps of compact connected graphs satisfying condition (*). Another perturbation of the Minc–Transue map has been considered in [9] in a different context. Also a similar set of conditions has been considered by Williams in [12] and [13]. His local expansions on branched 1-manifolds satisfying Axioms 0-3 ([12], [13]) satisfy the conditions of our main theorem.

2. Preliminaries. In this section, we recall some definitions and results which are necessary for the proof of the main theorem.

DEFINITION 2.1. (1) A metric ϱ on a graph G is called a *convex* (or *arc length*) *metric* if, for each pair of points x and y of G , there is an isometry α of the interval $[0, \varrho(x, y)]$ in the real line into G such that $\alpha(0) = x$ and $\alpha(\varrho(x, y)) = y$.

- (2) Let G be a graph with a convex metric ϱ . For a subset A of G , $\text{diam}(A)$ denotes the diameter of A with respect to ϱ . The ε -neighborhood of A (with respect to ϱ) is denoted $N_\varepsilon(A)$. The interior and closure of A are denoted by $\text{int}(A)$ and $\text{cl}(A)$, respectively.

DEFINITION 2.2. Let $f : G \rightarrow G$ be a map of a graph G with metric ϱ and let $\varepsilon > 0$. A map $\omega : [0, 1] \rightarrow G$ is said to be (f, ε) -crooked if there exist s and t with $0 < s \leq t < 1$ such that $\varrho(f \circ \omega(s), f \circ \omega(1)) \leq \varepsilon$ and $\varrho(f \circ \omega(t), f \circ \omega(0)) \leq \varepsilon$.

REMARK. In Definition 2.2, notice that, if there exist intervals I_1 and I_2 in $[0, 1]$ disjoint except possibly for a common endpoint and such that $f(\omega([0, 1])) \subset \text{cl}(N_\varepsilon(f(\omega(I_i))))$ for $i = 1, 2$, then ω is (f, ε) -crooked.

First, we state three propositions without proof. They are analogues of Propositions 1–3 of [10] and the proofs are simple.

PROPOSITION 2.3. Let $f : G \rightarrow G$ be a map of a graph G . If $\alpha \geq \delta$, then every $\omega : [0, 1] \rightarrow G$ which is (f, δ) -crooked is also (f, α) -crooked.

PROPOSITION 2.4. Let G be a graph with metric ϱ and let $f, g : G \rightarrow G$ be two maps on G .

- (1) Suppose $\varrho(f(t), g(t)) < \varepsilon$ for each $t \in G$. If a map $\omega : [0, 1] \rightarrow G$ is (f, δ) -crooked, then ω is also $(g, \delta + 2\varepsilon)$ -crooked.
- (2) Suppose that, for each pair of points $x, y \in G$ with $\varrho(x, y) < \delta$, we have $\varrho(g(x), g(y)) < \varepsilon$. Then a map $\omega : [0, 1] \rightarrow G$ which is (f, δ) -crooked is also $(g \circ f, \varepsilon)$ -crooked.

PROPOSITION 2.5. Let G be a graph and $\delta > 0$. Let $\{f_i \mid i = 1, 2, \dots\}$ be a sequence of maps of G onto itself which converges uniformly to f . If a map $\omega : [0, 1] \rightarrow G$ is (f_i, δ) -crooked for each $i = 1, 2, \dots$, then ω is (f, δ) -crooked.

The proof of the following proposition is similar to that of Proposition 4 of [10]. For the sake of completeness we provide the proof.

PROPOSITION 2.6. Let $f : G \rightarrow G$ be a map such that for each $\delta > 0$, there is an integer $n > 0$ such that each map $\omega : [0, 1] \rightarrow G$ is (f^n, δ) -crooked. Then the inverse limit (G, f) is hereditarily indecomposable.

Proof. Let X denote the inverse limit (G, f) . Suppose that A and B are subcontinua of X such that $A \cap B \neq \emptyset$, $B \setminus A \neq \emptyset$ and $A \setminus B \neq \emptyset$. Let k be an integer so large that, if $\pi_k : X \rightarrow G$ is the k th-coordinate projection, then $\pi_k(A) \setminus \pi_k(B) \neq \emptyset$ and $\pi_k(B) \setminus \pi_k(A) \neq \emptyset$. Let $\delta > 0$ be such that there exists $x \in \pi_k(A) \setminus N_\delta(\pi_k(B))$ and $y \in \pi_k(B) \setminus N_\delta(\pi_k(A))$. Let n be an integer such that each map $\omega : [0, 1] \rightarrow G$ is (f^n, δ) -crooked. Let $\omega : [0, 1] \rightarrow \pi_{n+k}(A \cup B)$

be an embedding so that $f^n(\omega(0)) = x$, $f^n(\omega(1)) = y$ and $\pi_{n+k}(A) \cap \omega([0, 1])$ is a connected set. Then ω is not (f^n, δ) -crooked. This is a contradiction.

The following proposition is an analogue of Proposition 6 of [10].

PROPOSITION 2.7. *Let $f : G \rightarrow G$ be a map which satisfies condition (*). Then for each $\varepsilon > 0$, there exists an integer $n > 0$ such that for each subcontinuum A with $\text{diam}(A) \geq \varepsilon$, $f^n(A) = G$.*

Proof. Choose pairwise disjoint arcs K_1, \dots, K_n in G with $\text{diam}(K_i) \leq \varepsilon/4$ for each $i = 1, \dots, n$ such that each continuum in $G \setminus (K_1 \cup \dots \cup K_n)$ has diameter $< \varepsilon/2$. Let m_i be a positive integer such that $f^{m_i}(K_i) = G$. Let $m = \max\{m_1, \dots, m_n\}$. If A is a continuum in G with $\text{diam}(A) \geq \varepsilon$, then A contains K_i for some $i = 1, \dots, n$. Hence $f^m(A) \supset f^m(K_i) = f^{m-m_i}(f^{m_i}(K_i)) = f^{m-m_i}(G) = G$.

3. Proof of Main Theorem. Our proof of the Main Theorem follows that in [10]. We prove that each piecewise linear map of a graph which satisfies (*) can be perturbed slightly so as to have a local expansion property (Lemma 3.2) and further to have a certain crookedness (Lemma 3.3). The second perturbation corresponds to Lemma in [10] (pp. 1167–1168) and the first perturbation provides a situation to which the second perturbation can be applied. Once we obtain these results, the remaining part of the proof proceeds as in [10]. First we prove the following lemma which is an analogue of Proposition 5 of [10].

By a *fan*, we mean a wedge of finitely many arcs. A map $f : X \rightarrow Y$ of a compact, metric space X to a continuum Y is said to be *weakly confluent* if for each subcontinuum $K \subset Y$, there exists a subcontinuum $L \subset f^{-1}(K)$ such that $f(L) = K$. It is well known that every map of a continuum onto an arc is weakly confluent.

CONVENTION. Throughout this section, we fix a convex metric ϱ and a triangulation \mathcal{T} on a compact connected graph G such that each edge e of \mathcal{T} is isometric to the unit interval. Let Id_H denote the identity map on a subgraph H .

In what follows, we often consider a map $g_R : R \rightarrow R^*$ defined on an edge R of a triangulation of G to a subgraph $R^* \subset G$. We apply Definition 2.2 to g_R with the domain R of g_R being regarded as the unit interval to say simply that “ g_R is $(\text{Id}_{R^*}, \gamma/5)$ -crooked” etc.

Note that the numbers $\gamma/5$ and $\varepsilon/4k$ in the statement of Lemma 3.1 are chosen for a technical reason for later use.

LEMMA 3.1. *Let G be the above compact connected graph with a triangulation \mathcal{T} . For each well-chosen k , ε and γ with $k \geq 1$ an integer,*

$0 < \gamma < \varepsilon/4k$ and $0 < \varepsilon < 1/4$, there exist a piecewise linear map $g : G \rightarrow G$ and a subdivision \mathcal{T}' of \mathcal{T} which satisfy the following conditions:

- (1) Each edge of \mathcal{T}' has length $\gamma/5$.
- (2) For each edge R of \mathcal{T}' , we have $g(R) = R^*$, where R^* is the largest subpolyhedron of \mathcal{T}' contained in the closure of the $\varepsilon/2$ -neighborhood of R . In particular, $\text{diam}(g(R)) \geq \varepsilon/2$.
- (3) For each edge e of \mathcal{T} , there exists an edge R of \mathcal{T}' such that each edge of \mathcal{T}' which meets R^* is contained in e .
- (4) For each subcontinuum A of G with $\text{diam}(A) \geq \gamma/5$ and for each positive number q , we have

$$A \subset g(A) \subset g(N_q(A)) \subset N_{q+\gamma/5}(g(A)).$$

- (5) For each subcontinuum B of G with $\text{diam}(B) \leq \gamma/5$,

$$\text{diam}(g(B)) \geq 2k \text{diam}(B) \geq 2 \text{diam}(B).$$
- (6) Each map $\omega : [0, 1] \rightarrow G$ with $\text{diam}(\omega([0, 1])) < \gamma/5$ is $(g, 2\gamma/5)$ -crooked.
- (7) $\varrho(g, \text{Id}_G) < \varepsilon/2 + \gamma/5$.

REMARK. The following fact, an easy consequence of (1) and the definition of R^* , is important in the proof below: If R_1 and R_2 are adjacent edges of \mathcal{T}' , then the Hausdorff distance of R_1^* and R_2^* is at most $\gamma/5$.

Proof of Lemma 3.1. Let $G, \mathcal{T}, k, \varepsilon$ and γ be as in the hypothesis. By replacing γ by a smaller number if necessary, we may suppose that $1/\gamma$ is an integer. Let \mathcal{T}' be a subdivision of G satisfying (1) and (3). Notice that for each edge R of \mathcal{T}' , R^* is a fan containing at most one vertex of \mathcal{T} . Denote the set of all edges of \mathcal{T} by $\{e_1, \dots, e_m\}$. For each $j = 1, \dots, m$, take an edge R_j of \mathcal{T}' satisfying (3). Then

$$\text{cl}\left(G - \bigcup_{j=1}^m R_j\right) = \bigcup_{i=1}^s T_i,$$

where the T_i 's are pairwise disjoint fans such that T_i contains exactly one vertex v_i of \mathcal{T} . Label the edges of \mathcal{T} which meet v_i as $e_{i,1}, \dots, e_{i,k_i}$.

Let R be an edge of \mathcal{T}' other than R_1, \dots, R_m . Then $R \subset T_i$ for some i and hence $R^* \subset \bigcup_{j=1}^{k_i} e_{i,j}$ by the definition of R^* . Let $v_{R,0}$ and $v_{R,1}$ be the endpoints of R , where $v_{R,0}$ is the endpoint of R closer to v_i .

Next we define a piecewise linear map $\phi_R : [0, 1] \rightarrow R^*$ as follows. If $R^* \subset e_{i,j}$ for some j , then let a_t be the endpoint of R^* closer to $v_{R,t}$ for $t = 0, 1$. The map $\phi_R : [0, 1] \rightarrow R^*$ is the simplest piecewise linear map which carries the 5-tuple $(0, 1/4, 1/2, 3/4, 1)$ onto $(v_{R,0}, a_1, a_0, a_1, v_{R,1})$ (see Figure 1).

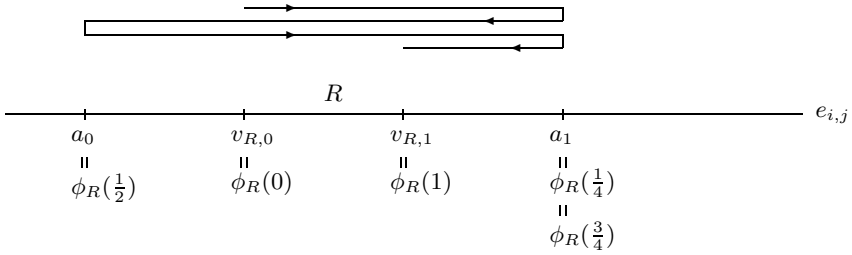


Fig. 1

Now suppose R^* is not a subset of $e_{i,j}$ for any j . The subset R^* is a fan with the unique branch point v_i . Let a_j be the endpoint of R^* in $e_{i,j}$ for $j = 1, \dots, k_i$ (note that a_j is distinct from v_i). Let $\xi = 2k_i$ and let $\phi_R : [0, 1] \rightarrow R^*$ be the simplest piecewise linear map which carries the $(\xi + 1)$ -tuple $(0, 1/\xi, 2/\xi, \dots, (\xi - 1)/\xi, 1)$ onto $(v_{R,0}, a_1, a_2, \dots, a_{k_i}, a_{k_i-1}, \dots, a_1, v_{R,1})$ (see Figure 2).

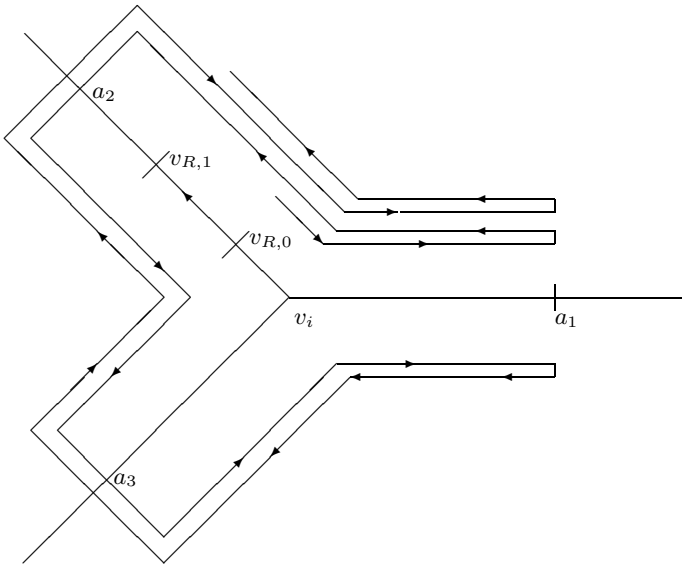


Fig. 2

Next we define an $(\text{Id}_{[0,1]}, p)$ -crooked piecewise linear map $b_R : R \rightarrow [0, 1]$ with $b_R(v_{R,0}) = 0$ and $b_R(v_{R,1}) = 1$, where the positive number p is chosen so that

$$(8) \quad \varrho(z_1, z_2) < p \Rightarrow \varrho(\phi_R(z_1), \phi_R(z_2)) < \gamma/5,$$

where $z_1, z_2 \in R$. Hence by Proposition 2.4(2), the map $g_R = \phi_R \circ b_R : R \rightarrow R^*$ is $(\text{Id}_{R^*}, \gamma/5)$ -crooked.

We define $b_R : R \rightarrow [0, 1]$ to be a nowhere locally constant simplicial map with respect to some subdivisions \mathcal{P} and \mathcal{Q} of R and $[0, 1]$ respectively. Let

$v_{R,0} = p_0, p_1, \dots, p_{M-1}, p_M = v_{R,1}$ be the vertices of \mathcal{P} so that p_i and p_{i+1} are adjacent. We also assume that the midpoint of R is a vertex p_μ of \mathcal{P} . Similarly let $0 = q_0 < q_1 < \dots < q_N = 1$ be the vertices of \mathcal{Q} . The map b_R satisfies the following conditions:

- (9) (crookedness in the sense of Bing [3]) If $b_R(p_s) = q_u$ and $b_R(p_t) = q_v$ with $v - u \geq 5$, then there exist x and y with $s < x < y < t$ such that $b_R(p_x) = q_{v-1}$ and $b_R(p_y) = q_{u+1}$, and
- (10) $b_R([v_{R,0}, p_\mu]) \supset [0, 1/2]$ and $b_R([p_\mu, v_{R,1}]) \supset [1/2, 1]$, where $[v_{R,0}, p_\mu]$ is the segment in R connecting $v_{R,0}$ with p_μ , etc.

We further choose \mathcal{P} and \mathcal{Q} so that $\varrho(p_i, p_{i+1}) = \gamma/5M$ for $i = 1, \dots, M - 1$ and $q_{j+1} - q_j = 1/N$ for $j = 1, \dots, N - 1$. By the crookedness condition (9), M/N becomes arbitrarily large as M increases to infinity. So the derivative of b_R becomes arbitrarily large in absolute value for large N .

As we stated above, we define $g_R = \phi_R \circ b_R : R \rightarrow R^*$ for each edge R other than R_1, \dots, R_m . Recall from the construction that ϕ_R ‘‘contracts’’ subintervals of $[0, 1]$ by a factor of order at most $1/\varepsilon$. Therefore, by taking N large enough, we obtain the following condition:

- (11) $\text{diam}(g_R(B)) \geq 4k \text{diam}(B)$ for each subinterval B in R .

Also we may require

- (12) g_R is $(\text{Id}_{R^*}, \gamma/5)$ -crooked on each interval J of R .

Indeed, if $b_R(J)$ contains five consecutive edges of \mathcal{Q} , then (9) guarantees the desired crookedness. Otherwise, $\text{diam}(b_R(J)) < 6/N$ and hence $\text{diam}(g(J)) < \gamma/5$, by taking sufficiently large N . It follows that $g_R|J$ is trivially $(\text{Id}_{R^*}, \gamma/5)$ -crooked.

By the construction of ϕ_R , we have

- (13) $g_R(B) = R^*$ for each subinterval B containing an endpoint of R and having length at least $\gamma/10$.

We next define $g_{R_i} : R_i \rightarrow R_i^*$ for $i \in \{1, \dots, m\}$. Let P and Q be the edges of \mathcal{T}' adjacent to R_i . Note that $P^* \cup R_i^* \cup Q^* \subset e_i$ by (3). We suppose $P \subset T_k$ and $Q \subset T_j$ for some $k \neq j$.

The vertex $v_{P,0}$ is closer to v_k than $v_{P,1}$ is, and $v_{Q,0}$ is closer to v_j than $v_{Q,1}$ is. So $v_{P,1}$ is the common endpoint of P and R_i , and $v_{Q,1}$ is the common endpoint of Q and R_i . Let a_k be the endpoint of R_i^* closer to v_k , and a_j the endpoint of R_i^* closer to v_j . Let $\phi_{R_i} : [0, 1] \rightarrow R_i^*$ be the simplest piecewise linear map which carries $(0, 1/5, 2/5, 3/5, 4/5, 1)$ onto $(v_{P,1}, a_j, a_k, a_j, a_k, v_{Q,1})$. Let $b_{R_i} : R_i \rightarrow [0, 1]$ be an $(\text{Id}_{[0,1]}, p)$ -crooked piecewise linear map with $b_{R_i}(v_{P,1}) = 0$ and $b_{R_i}(v_{Q,1}) = 1$ for p as in (8). With the same conditions as used in defining b_R above, let $g_{R_i} = \phi_{R_i} \circ b_{R_i}$. Then

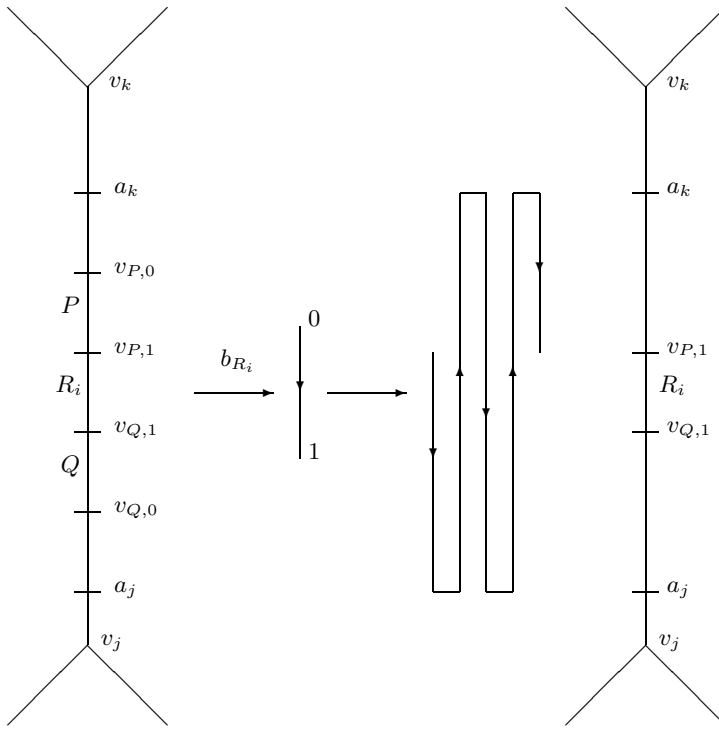


Fig. 3

- (14) g_{R_i} is $(\text{Id}_{R_i^*}, \gamma/5)$ -crooked on each subinterval of R_i , and
- (15) $\text{diam}(g_{R_i}(B)) \geq 4k \text{diam}(B)$ for each subinterval B of R_i (see Figure 3).

Let $g : G \rightarrow G$ be a map defined by $g = \bigcup \{g_R : R \in \mathcal{R}_{\mathcal{T}'}\}$. Then g is well defined and continuous since the endpoints of R are fixed by g_R for each edge R of \mathcal{T}' .

Condition (5) follows from (11) and (15) above. Also (7) follows from the fact that $g(R) = R^*$ and by definition of R^* for an edge R of \mathcal{T}' .

Now we are going to prove (4). Let A be a subcontinuum of G with $\text{diam}(A) \geq \gamma/5$ and let $q > 0$. As each edge of \mathcal{T} has length $\gamma/5$, there exist R in \mathcal{T}' and v an endpoint of R such that the component B of v in $R \cap A$ has diameter at least $\gamma/10$. Hence by (10), $R^* \subset g(A)$. Let $q = \alpha\gamma/5 + \beta$, where α is a nonnegative integer and $0 \leq \beta \leq \gamma/5$, and take a point $x \in A$ with $\varrho(x, A \cap R) = q$. Since g maps adjacent edges of \mathcal{T}' to within Hausdorff distance $\gamma/5$ of each other (see the Remark before the proof), we have $\varrho(g(x), R^*) \leq \alpha\gamma/5 + \beta + \gamma/5 \leq q + \gamma/5$. Thus $g(N_q(A)) \subset N_{q+\gamma/5}(g(A))$. The other inclusions follow easily from the construction of g . This proves (4).

It remains to show that (6) is satisfied. Let $\omega : [0, 1] \rightarrow G$ be a map with $\text{diam}(\omega([0, 1])) < \gamma/5$. If $\text{diam} g(\omega([0, 1])) \leq 2\gamma/5$, then ω is trivially $(g, 2\gamma/5)$ -crooked. If $\text{diam} g(\omega([0, 1])) > 2\gamma/5$, we have two cases to consider.

CASE A: $\omega([0, 1]) \subset R$ for some edge R of \mathcal{T}' . As $g|_{\omega([0,1])} = gR|_{\omega([0,1])}$ is $(\text{Id}_{R^*}, \gamma/5)$ -crooked by (12) and (14), there exist intervals K_1 and K_2 in $\omega([0, 1])$ disjoint except possibly for a common endpoint such that

$$gR(\omega([0, 1])) \subset \text{cl}(N_{\gamma/5}(gR(K_j))), \quad j = 1, 2.$$

Using weak confluence of ω , we find intervals I_1 and I_2 in $[0, 1]$ disjoint except possibly for a common endpoint such that $\omega(I_j) = K_j$. This proves that ω is $(g, \gamma/5)$ -crooked in this case.

CASE B: $\omega([0, 1])$ is not contained in any edge of \mathcal{T}' . As $\text{diam}(\omega([0, 1])) < \gamma/5$, there is a vertex v of \mathcal{T}' such that $v \in \omega([0, 1]) \subset \text{int}(\text{st}(v, \mathcal{T}'))$, where $\text{st}(v, \mathcal{T}')$ is the union of the edges of \mathcal{T}' having v as an endpoint. By construction of g , it follows that there is R with v being an endpoint such that $\text{cl}(N_{\gamma/5}(g(R \cap \omega([0, 1]))) \supset g(\omega([0, 1]))$. Since $R \cap \omega([0, 1])$ is a subinterval of R and g is $(\text{Id}_{R^*}, \gamma/5)$ -crooked on all subintervals of R , by (11) and (12), there exist intervals K and L , disjoint except possibly for a common endpoint, such that

$$g(R \cap \omega([0, 1])) \subset \text{cl}(N_{\gamma/5}(K)) \cap \text{cl}(N_{\gamma/5}(L)).$$

Hence $g(\omega([0, 1])) \subset \text{cl}(N_{2\gamma/5}(K)) \cap \text{cl}(N_{2\gamma/5}(L))$ and in this case also, ω is $(g, 2\gamma/5)$ -crooked and (6) is proved.

This completes the proof of Lemma 3.1.

REMARK. It follows from the construction of g that, for each edge R of \mathcal{T}' , there exist arcs K_0 and K_1 , disjoint except possibly for a common endpoint, in R such that $g(R) = g(K_j)$ for $j = 0, 1$.

The next lemma describes a perturbation of f of the main theorem to a map f_0 with local expansion property, while retaining condition (*).

LEMMA 3.2. *Let $f : G \rightarrow G$ be a piecewise linear map which satisfies (*). Then, for each δ with $0 < \delta < 1$, there exist a piecewise linear map $f_0 : G \rightarrow G$ and a positive number $\gamma < \delta$ such that*

- (1) $\varrho(f_0, f) < \delta$.
- (2) For each subcontinuum A in G with $\text{diam}(A) < \gamma/5$,

$$\text{diam}(f_0(A)) \geq 2 \text{diam}(A).$$

- (3) The map f_0 satisfies condition (*).

REMARK. 1) As is seen below, $f_0 = f \circ g$, where g is the map obtained in Lemma 3.1. The crookedness condition (9) of Lemma 3.1 plays no role in the proof of this lemma. We will make use of (9) later in Lemma 3.3.

2) The proof below shows that the map f_0 satisfies the following condition: there exists an integer $n > 0$ such that, for every subcontinuum A of G such that $\text{diam}(A) \geq \gamma/5$, we have $f_0^n(A) = G$.

Proof of Lemma 3.2. Observe first that $\text{diam}(G) \geq 1$. As f is piecewise linear and nowhere locally constant, there exist $0 < S_1 < 1 < S_2$ such that S_1^{-1} and S_2 are integers and

$$(4) \quad S_1 \text{diam}(C) \leq \text{diam}(f(C)) \leq S_2 \text{diam}(C)$$

for each subcontinuum C of G .

Let $k = 2/S_1$ and let $\varepsilon > 0$ be a number such that

$$(5) \quad \varepsilon < \delta/2S_2.$$

By Proposition 2.7, there exists a positive integer n such that

$$f^n(A) = G \quad \text{for each subcontinuum } A \text{ of } G \text{ with } \text{diam}(A) > \varepsilon/4.$$

Take a positive number γ so that $1/\gamma$ is an integer and

$$(7) \quad \gamma < \min\{\varepsilon/2, 1/S_2^n\}.$$

Apply Lemma 3.1 to the above k, ε and γ , to obtain a map $g : G \rightarrow G$ which is $(\varepsilon/2 + \gamma/5)$ -close to Id_G and a triangulation \mathcal{T}' of G which satisfy conditions (1)–(7) of Lemma 3.1. Define a map $f_0 : G \rightarrow G$ by $f_0 = f \circ g$. We prove that f_0 satisfies the desired conditions.

Condition (1) follows immediately from (4)–(6) above and the condition (7) of Lemma 3.1.

Let A be a subcontinuum of G with $\text{diam}(A) \geq \gamma/5$. As each edge of \mathcal{T}' has length equal to $\gamma/5$, there exists an edge R of \mathcal{T}' such that $\text{diam}(A \cap R) \geq \gamma/10$. By (10) of Lemma 3.1, we have $R^* \subset g(A)$. Also by (2) of Lemma 3.1, we see that $\text{diam}(g(A)) \geq \varepsilon/2$. By the choice of n , we have $f^n(g(A)) = G$. From $\gamma < 1/S_2^n$ by (7), it follows from (5) and $f^n(g(A)) = G$ that $\text{diam}(f^j(g(A))) \geq \gamma$ for each $j \geq 1$ (recall $\text{diam}(G) \geq 1$). By (4) of Lemma 3.1, $g(f^j(g(A))) \supset f^j(g(A))$ for each $j \geq 1$. Therefore $f_0(f(g(A))) = fgf(g(A)) \supset f^2(g(A))$, and an inductive argument shows that $f_0^{j-1}(f(g(A))) \supset f^j(g(A))$ for each $j \geq 1$. Thus, $f_0^n(A) = f_0^{n-1}(f(g(A))) \supset f^n(g(A)) = G$, as required.

To prove (2), let A be a nondegenerate subcontinuum of G . If $\text{diam}(A) \geq \gamma/5$, then $f_0^n(A) = G$ by the above. Suppose, therefore, that $\text{diam}(A) < \gamma/5$. We prove first that $\text{diam}(f_0(A)) \geq 2 \text{diam}(A)$. By (5) of Lemma 3.1, $\text{diam}(g(A)) \geq 2k \text{diam}(A) > (2/S_1) \text{diam}(A) \geq 4 \text{diam}(A)$. So

$$\begin{aligned} \text{diam}(f_0(A)) &= \text{diam}(f(g(A))) \geq S_1 \text{diam}(g(A)) \\ &\geq S_1 \frac{2}{S_1} \text{diam}(A) = 2 \text{diam}(A), \end{aligned}$$

which proves (2). If $\text{diam}(f_0(A)) \geq \gamma/5$, then $f_0^{n+1}(A) = G$, and if $\text{diam}(f_0(A)) < \gamma/5$, then $\text{diam}(f_0^2(A)) \geq 2^2 \text{diam}(A)$. By repeating this argument finitely many times we see that $f_0^m(A) = G$ for some integer m . So (3) is satisfied.

This completes the proof.

The map f_0 of Lemma 3.2 is further perturbed to obtain, by induction, a sequence of maps so that the limit map has the crookedness property required in the hypothesis of Proposition 2.6. The following lemma describes the general inductive step.

LEMMA 3.3. *Let $h : G \rightarrow G$ be a piecewise linear map which satisfies (*) and the condition: there exists $0 < \beta < 1$ such that*

$$(+)_{\beta} \quad \text{for each subcontinuum } A \text{ with } \text{diam}(A) \leq \beta, \\ \text{diam}(h(A)) \geq 2 \text{diam}(A).$$

Then, for each δ with $0 < \delta < 1$, there exist a map $F : G \rightarrow G$ satisfying (), a positive number $\gamma < \delta$ and an integer $n > 0$ such that*

- (1) $\rho(F, h) < \delta$.
- (2) $F^n(A) = G$ for every subcontinuum A of G with $\text{diam}(A) \geq \gamma/5$.
- (3) The map F satisfies condition $(+)_{\gamma/5}$.
- (4) Each map $\omega : [0, 1] \rightarrow G$ is (F^n, δ) -crooked.

Proof. As in Lemma 3.2, choose S_1 and S_2 so that S_1^{-1} and S_2 are integers, $0 < S_1 < 1 < S_2$ and

$$(5) \quad S_1 \text{diam}(C) \leq \text{diam}(h(C)) \leq S_2 \text{diam}(C) \\ \text{for each subcontinuum } C \text{ of } G.$$

Let $k = 2/S_1$. Next choose ε so that each subcontinuum of G with diameter less than ε contains at most one branch point of G and

$$(6) \quad \varepsilon < \frac{\delta}{2S_2}$$

and an integer n (using Proposition 2.7) in such a way that

$$(7) \quad h^n(A) = G \text{ for each subcontinuum } A \text{ of } G \text{ with } \text{diam}(A) > \varepsilon/4.$$

The choice of a positive number γ is more delicate than in Lemma 3.2. The γ is chosen as follows:

$$(8) \quad \gamma < \min \left\{ \frac{(S_2 - 1)\varepsilon}{8S_2^{n+1}}, \frac{\varepsilon}{4k}, \frac{S_1}{4S_2^n}, \frac{\beta}{4S_2^n} \right\}.$$

We again apply Lemma 3.1 to obtain a map $g : G \rightarrow G$ and a triangulation \mathcal{T}' of G which satisfy conditions (1)–(7) of Lemma 3.1. Define a map $F : G \rightarrow G$ by $F = h \circ g$. We prove that F satisfies the desired conditions.

As h satisfies the hypothesis of Lemma 3.2, conditions (1)–(3) follow from Lemma 3.2. We need only show that (4) holds for F .

We will need the following estimation. Since h satisfies $(+)_\beta$ and $\gamma < \beta/2$ (see (8)), h satisfies

$$(9) \quad \text{diam}(h(A)) \geq 2 \min\{\text{diam}(A), \gamma/5\} \quad \text{for each subcontinuum } A.$$

We also need the following inclusion:

$$(10) \quad \text{For each subcontinuum } A \text{ of } G \text{ with } \text{diam}(A) \geq \gamma/5, \text{ for each positive number } q, \text{ and for each positive integer } j, \text{ we have}$$

$$F^j(N_q(A)) \subset N_{S_2^j(q+2\gamma)}(F^j(A)).$$

Indeed, for each subcontinuum A of G with $\text{diam}(A) \geq \gamma/5$ and each $q > 0$,

$$\begin{aligned} F(N_q(A)) &= h(g(N_q(A))) \subset h(N_{q+\gamma/5}(g(A))) \\ &\subset N_{S_2(q+\gamma/5)}(h(g(A))) = N_{S_2(q+\gamma/5)}(F(A)) \end{aligned}$$

by (5) and Lemma 3.1(4). Inductively, using (5) and Lemma 3.1(4),

$$F^j(N_q(A)) \subset N_\alpha(F^j(A))$$

for each positive integer j , where $\alpha = S_2^j q + \frac{\gamma}{5} \frac{S_2^{j+1} - 1}{S_2 - 1}$. Since

$$\alpha < S_2^j \left(q + \gamma \frac{S_2}{S_2 - 1} \right) \leq S_2^j (q + 2\gamma)$$

(note $S_2 \geq 2$), we have the inclusion (10).

Now, consider a continuous map $\omega : [0, 1] \rightarrow G$. We must prove that ω is (F^n, δ) -crooked. By the remark following Definition 2.2, it suffices to show that there exist two (disjoint except possibly for a common endpoint) intervals I_0 and I_1 in $[0, 1]$ such that $F^n(\omega([0, 1])) \subset N_\delta(F^n(\omega(I_j)))$ for $j = 0, 1$. We may assume that $\text{diam}(F^n(\omega([0, 1]))) \geq 2\delta > \gamma$, or there is nothing to prove. In what follows $\omega([0, 1])$ is denoted by $\text{Im } \omega$ for simplicity. There are two cases to consider.

CASE I: *There exists an edge R of \mathcal{T}' contained in $\text{Im } \omega = \omega([0, 1])$.* If $\text{Im } \omega$ contains no simple closed curve then $\omega|_{\omega^{-1}(R)}$ is weakly confluent. If $\text{Im } \omega$ contains a simple closed curve, let $\tilde{G} \rightarrow G$ be the universal cover of G , $\tilde{\mathcal{T}}'$ the triangulation of \tilde{G} which is the lifting of \mathcal{T}' , and $\tilde{\omega} : [0, 1] \rightarrow \tilde{G}$ a lifting of ω to \tilde{G} . If \tilde{R} is a 1-simplex of $\tilde{\mathcal{T}}'$ contained in $\tilde{\omega}([0, 1])$, then $\tilde{\omega}|_{\tilde{\omega}^{-1}(\tilde{R})}$ is weakly confluent onto \tilde{R} . Without loss of generality, we may suppose that R is the homeomorphic projection of \tilde{R} in G . So $\omega|_{\omega^{-1}(R)}$ is weakly confluent onto R .

As in the remark following Lemma 3.1, there exist (disjoint except possibly for a common endpoint) arcs K_0 and K_1 in R such that $g(R) = g(K_j)$

for $j = 0, 1$. By the weak confluence of ω restricted to the preimage of R , there exist (disjoint except possibly for a common endpoint) arcs I_0 and I_1 in $[0, 1]$ such that $\omega(I_j) = K_j$. By the proof of (2) and (3) of Lemma 3.2 with A replaced by $\omega(I_j)$ and f_0 replaced by $F = f_0 \circ g$ respectively, we have $F^n(\omega(I_j)) = G$ for $j = 0, 1$.

CASE II: $\text{Im } \omega = \omega([0, 1])$ contains no edge of T' . Here our argument is completely parallel to the one of [10, p. 1169]. We divide our consideration into two cases, Case II.a and Case II.b. A detailed argument is provided for Case II.a, while the argument for Case II.b is sketched briefly.

Let m be the largest integer such that $F^m(\text{Im } \omega)$ contains no edge of T' . Then there is a vertex v in T' such that $F^m(\text{Im } \omega) \subset \text{st}(v, T') =$ the union of all edges of T' containing v . Note that in this case $\text{diam}(F^m(\text{Im } \omega)) < 2\gamma/5$.

CASE II.a: $\text{diam}(g(F^m(\text{Im } \omega))) \geq 2\gamma$. Note that $g(F^m(\text{Im } \omega))$ is a tree. As in Case B of Lemma 3.1, there is an edge P of T' which has v as an endpoint such that

$$g(F^m(\text{Im } \omega)) \subset \text{cl}(N_{\gamma/5}(g(P \cap F^m(\text{Im } \omega)))).$$

Since g is $(\text{Id}_{P^*}, 2\gamma/5)$ -crooked for each subinterval of P , there exist (disjoint except possibly for a common endpoint) intervals I_0 and I_1 in $[0, 1]$ such that

$$g(F^m(\text{Im } \omega) \cap P) \subset \text{cl}(N_{2\gamma/5}(g(F^m(\omega(I_j))))) \quad \text{for } j = 0, 1.$$

Hence,

$$(11) \quad g(F^m(\text{Im } \omega)) \subset \text{cl}(N_{\gamma/5}(g(K))) \subset \text{cl}(N_{3\gamma/5}(g(F^m(\omega(I_j)))))$$

for $j = 0, 1$.

So g is $(F^m, 3\gamma/5)$ -crooked. Now

$$\begin{aligned} F^{m+1}(\text{Im } \omega) &= h(g(F^m(\text{Im } \omega))) \\ &\subset h(\text{cl}(N_{3\gamma/5}(g(F^m(\omega(I_j)))))), \quad j = 0, 1, \text{ by (11)} \\ &\subset \text{cl}(N_{S_2 3\gamma/5}(h(g(F^m(\omega(I_j))))) \quad \text{by (5)} \\ &\subset N_{S_2 \gamma}(F^{m+1}(\omega(I_j))), \quad j = 0, 1. \end{aligned}$$

As $\text{diam}(g(F^m(\text{Im } \omega))) \geq 2\gamma$, by (11) we have $\text{diam}(g(F^m(\omega(I_j)))) \geq 4\gamma/5$. By (9), $\text{diam}(F^{m+1}(\omega(I_j))) \geq 2 \min\{\text{diam}(g(F^m(\omega(I_j)))) , \gamma/5\} = 2\gamma/5 > \gamma/5$ for $j = 0, 1$. Now using (11), (5), (9), (10), (8) and (6), one can show that

$$\begin{aligned} F^n(\text{Im } \omega) &\subset F^{n-m-1}(N_{S_2 \gamma}(F^{m+1}(\omega(I_j)))) \\ &\subset N_{S_2^{n-m-1}(S_2 \gamma + 2\gamma)}(F^n(\omega(I_j))) \quad \text{by (9) and (10)} \\ &\subset N_\delta(F^n(\omega(I_j))) \quad \text{by (8) and (6), for } j = 0, 1. \end{aligned}$$

Hence ω is (F^n, δ) -crooked.

CASE II.b: $\text{diam}(g(F^m(\text{Im } \omega))) < 2\gamma$. By the choice of m , $F^{m+1}(\text{Im } \omega)$ contains an edge R of \mathcal{T}' . This together with (5) yields

$$\gamma/5 \leq \text{diam}(F^{m+1}(\text{Im } \omega)) \leq 2S_2\gamma < \varepsilon \quad \text{by (8).}$$

The above inequality means that $F^{m+1}(\text{Im } \omega)$ contains at least a half of an edge of \mathcal{T}' , but still sits in a tree with at most one branch point. Then one can proceed as in Case II.a to conclude that ω is (F^n, δ) -crooked as well.

This proves (5), and hence completes the proof.

Proof of Main Theorem. Suppose that a map $f : G \rightarrow G$ satisfies condition (*) in the Main Theorem and an $\varepsilon > 0$ is given. First, using Lemma 3.2, we construct a map f_0 satisfying (*) and $(+)_\beta$, where f_0 is $\varepsilon/2$ -close to f . Then as in [10], Lemma 3.3 enables us to construct a sequence $\{f_i : G \rightarrow G \mid i \geq 0\}$ of surjections and an increasing sequence $\{n(i) \mid i \geq 0\}$ of integers such that

- (a) $n(0) = 0$.
- (b) $d(f_i, f_{i+1}) < \varepsilon/2^{i+1}$.
- (c) Each map $\omega : [0, 1] \rightarrow G$ is $(f_i^{n(k)}, (\varepsilon/2^k - \varepsilon/2^{k+i}))$ -crooked for each $k = 1, \dots, i + 1$.
- (d) For each subcontinuum A of G with $\text{diam}(A) > \varepsilon/2^k$, $f_i^{n(k)}(A) = G$ for each $k = 0, 1, \dots, i$.

By (b), the limit map $f_\varepsilon = \lim_{i \rightarrow \infty} f_i$ exists and f_ε is $\varepsilon/2$ -close to f_0 . Since f_0 is $\varepsilon/2$ -close to f , f_ε is ε -close to f . Condition (c) and Propositions 2.5 and 2.6 guarantee that (G, f_ε) is hereditarily indecomposable. This completes the proof of the Main Theorem.

4. Applications. For every integer $p \geq 1$, there exists a topologically unique hereditarily indecomposable circle-like continuum $\Sigma(p) = \lim_{\leftarrow} (S_i, f_i : S_{i+1} \rightarrow S_i)$, where each S_i is a simple closed curve and $\text{deg } f_i = p$ ([5] and [11]), and it is called the *pseudo-solenoid* of type p if $p > 1$ and the *pseudo-circle* if $p = 1$. The p -adic solenoid is denoted by $S(p)$ ($S(1)$ is the simple closed curve).

We assume that each simple closed curve under consideration is the unit circle in the complex plane, i.e. $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The map $f_p : S^1 \rightarrow S^1$ defined by $f_p(z) = z^p$ gives the p -adic solenoid as its inverse limit (S^1, f_p) . If $p > 1$, it is easy to see that f_p satisfies the conditions of the Main Theorem. If $p = 1$, it is easy to construct a map $\phi_1 : S^1 \rightarrow S^1$ of degree 1 satisfying the condition of the Main Theorem. Applying the Main Theorem to f_p and ϕ_1 respectively (with the help of the uniqueness of $\Sigma(p)$), we have the following inverse sequence representation of $\Sigma(p)$ by a single

bonding map. The authors are grateful to Professor P. Minc for pointing out the $p = 1$ case.

COROLLARY 4.1. *The pseudo-solenoid or pseudo-circle $\Sigma(p)$ ($p \geq 1$) can be represented as an inverse limit (S^1, g_p) of a map $g_p : S^1 \rightarrow S^1$ of the simple closed curve with degree p .*

Suppose that $g_p : S^1 \rightarrow S^1$ is a map as in the above corollary. By the homotopy lifting property of f_p , it is easy to construct a sequence $\{h_i : S^1 \rightarrow S^1\}$ of surjections such that $h_i \circ g_p = f_p \circ h_{i+1}$. The sequence induces a surjective map $h_\infty = \lim_{\leftarrow} (h_i) : \Sigma(p) \rightarrow S(p)$. Applying Proposition B of [7], we can choose (h_i) so that $h_i = h_{i+1}$ for each i . The next corollary gives a little more information on these maps.

COROLLARY 4.2. *Suppose that $p > 1$. For each $\varepsilon > 0$, there exist surjections $g, h : S^1 \rightarrow S^1$ such that*

- (1) *the inverse limit (S^1, g) is homeomorphic to $\Sigma(p)$,*
- (2) *$h \circ g = f_p \circ h$ and hence h induces a surjection $h_\infty : \Sigma(p) \rightarrow S(p)$,*
- (3) *$\varrho(h, \text{id}) < \varepsilon$.*

Proof. Since f_p is a positively expansive open map, it has the following property called the *pseudo-orbit tracing property*: for each $\varepsilon > 0$, there is a $\delta > 0$ such that for each sequence $\{x_i \mid i \geq 0\}$ of points of S^1 with $\varrho(f_p(x_i), x_{i+1}) < \delta$ for each $i \geq 0$, there exists a point x of S^1 such that $\varrho(f_p^i(x), x_i) < \varepsilon$ for each i .

Apply the Main Theorem to obtain a map $g : S^1 \rightarrow S^1$ such that $\varrho(f_p, g) < \delta$ and (S^1, g) is hereditarily indecomposable, and hence, is the pseudo-solenoid of type p . For each x in S^1 , consider the sequence $\{g^i(x) \mid i \geq 0\}$. Then $\varrho(f(g^i(x)), g^{i+1}(x)) < \delta$ for each $i \geq 0$, and the pseudo-orbit tracing property guarantees the existence of a point $h(x)$ of S^1 such that $\varrho(f_p^i(h(x)), g^i(x)) < \varepsilon$ for each $i \geq 0$. The remaining part of the proof proceeds as those of the topological stability of expansive homeomorphisms with the pseudo-tracing property (see for example, [1, Theorem 5.8, p. 684]).

REMARK. For a connected graph G , Id_G does not satisfy condition (*). However, as in the above argument for the case $p = 1$, the construction of Lemma 3.1 yields a piecewise linear map approximating Id_G and satisfying (*). Thus Id_G can be approximated by a map $h : G \rightarrow G$ such that (G, h) is hereditarily indecomposable.

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