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Anosov theorem for coincidences on nilmanifolds

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Abstract. Suppose that L, L' are simply connected nilpotent Lie groups such that the groups $\gamma_i(L)$ and $\gamma_i(L')$ in their lower central series have the same dimension. We show that the Nielsen and Lefschetz coincidence numbers of maps $f, g : \Gamma\backslash L \to \Gamma'\backslash L'$ between nilmanifolds $\Gamma\backslash L$ and $\Gamma'\backslash L'$ can be computed algebraically as follows:

$$L(f,g) = \det(G_* - F_*), \quad N(f,g) = |L(f,g)|,$$

where F_*, G_* are the matrices, with respect to any preferred bases on the uniform lattices Γ and Γ', of the homomorphisms between the Lie algebras $\mathfrak{L}, \mathfrak{L}'$ of L, L' induced by f, g.

1. Introduction. Let M and N be closed manifolds, and $f, g : M \to N$ continuous maps. Then we define

$$\mathrm{Coin}(f,g) = \{x \in M \mid f(x) = g(x)\},$$

the *coincidence set* of f and g. Coincidence theory for pairs f, g is a natural extension of fixed point theory for a self-map $f : M \to M$. There are well known invariants in coincidence theory which are the Lefschetz coincidence number $L(f,g)$ and Nielsen coincidence number $N(f,g)$.

Suppose that M, N are closed orientable manifolds of the same dimension n. Then $L(f,g)$ is defined and $L(f,g) \neq 0$ implies the existence of a coincidence for any maps f', g' which are homotopic to f, g, respectively. The definition of $L(f,g)$ is in [13, Chap. 7]. The Nielsen coincidence number $N(f,g)$ is a non-negative integer with the property that any two maps f', g' which are homotopic to f, g, respectively, have at least $N(f,g)$ coincidences. In [12], Schirmer shows that if $n \geq 3$, then there are two maps f', g', homotopic to f, g respectively, such that they have exactly $N(f,g)$ coincidences. Therefore, if $n \geq 3$ and $N(f,g) = 0$, then there are coincidence free maps in the homotopy classes of f, g. Thus the Nielsen coincidence number is much

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more powerful than the Lefschetz coincidence number but computing it is very hard.

In [2], Brooks, Brown, Pak and Taylor show that for a self-map $f: M \to M$ on a torus, the Nielsen number $N(f)$ and Lefschetz number $L(f)$ are equal up to sign, i.e.,

$$N(f) = |L(f)| = |\det(I - f_*)|,$$

where $f_* : \pi_1(M) \to \pi_1(M)$ is the homomorphism on $\pi_1(M)$ induced by f. In [1] and [4], this result is extended to compact nilmanifolds. Let L be a connected, simply connected nilpotent Lie group, Γ a uniform lattice of it, and $M = \Gamma\backslash L$ a nilmanifold. Any $f : M \to M$ is homotopic to a map obtained from an endomorphism $F : L \to L$ for which $F(\Gamma) \subset \Gamma$. Let F_* be the corresponding endomorphism of the Lie algebra of L. Then $N(f) = |L(f)| = |\det(I - F_*)|$. In [10], McCord generalized this result to coincidences on nilmanifolds (see also [3], [5], [8] and [14]). If M_1, M_2 are nilmanifolds of the same dimension, then $N(f, g) = |L(f, g)|$ for any $f, g : M_1 \to M_2$.

The purpose of this work is to offer an algebraic computation formula for the Nielsen and Lefschetz coincidence numbers of any pair of continuous maps between nilmanifolds $\Gamma\backslash L$ and $\Gamma'\backslash L'$. Suppose that L, L' are simply connected nilpotent Lie groups such that the groups $\gamma_i(L), \gamma_i(L')$ in the lower central series have the same dimension. Any continuous maps $f, g : \Gamma\backslash L \to \Gamma'\backslash L'$ induce homomorphisms Φ_*, Ψ_* between the Lie algebras $\mathfrak{L}, \mathfrak{L}'$. The uniform lattices Γ and Γ' give rise to preferred bases for $\mathfrak{L}$ and $\mathfrak{L}'$. Let F_*, G_* be the matrices of the homomorphisms Φ_*, Ψ_* with respect to any preferred bases of Γ, Γ'. Then we show that

$$L(f, g) = \det(G_* - F_*), \quad N(f, g) = |L(f, g)|.$$

Since every infra-nilmanifold admits a finite covering by a closed nilmanifold, the averaging formula for Nielsen coincidence numbers on infra-nilmanifolds in [9] will become a practical computation formula.

2. Anosov theorem for coincidences on nilmanifolds. Let $f, g : \Gamma\backslash L \to \Gamma'\backslash L'$ be continuous maps between nilmanifolds $\Gamma\backslash L$ and $\Gamma'\backslash L'$ of the same dimension. In what follows, *we shall fix liftings* $\widetilde{f}, \widetilde{g} : L \to L'$ *of* f, g. Then these liftings define homomorphisms $\varphi, \psi : \Gamma \to \Gamma'$ as follows:

$$\widetilde{f}\gamma = \varphi(\gamma)\widetilde{f}, \quad \widetilde{g}\gamma = \psi(\gamma)\widetilde{g}.$$

By [6], the homomorphisms $\varphi, \psi : \Gamma \to \Gamma'$ extend uniquely to Lie group homomorphisms $\Phi, \Psi : L \to L'$. Then they induce Lie algebra homomorphisms $\Phi_*, \Psi_* : \mathfrak{L} \to \mathfrak{L}'$. Since $\Phi(\Gamma) \subset \Gamma'$ and $\Psi(\Gamma) \subset \Gamma'$, the endomorphisms Φ, Ψ induce maps $\varphi_\#, \psi_\# : \Gamma\backslash L \to \Gamma'\backslash L'$. Furthermore, $\varphi_\#$ and f induce exactly the same homomorphism $\varphi : \Gamma \to \Gamma'$, and $\psi_\#$ and g induce exactly the same homomorphism $\psi : \Gamma \to \Gamma'$. Since $\Gamma\backslash L$ and $\Gamma'\backslash L'$ are $K(\pi, 1)$-manifolds, $\varphi_\#$ and f are homotopic and $\psi_\#$ and g are homotopic. Since the Nielsen and Lefschetz coincidence numbers are homotopy invariants, we may assume in what follows that f, g are induced by homomorphisms Φ, Ψ between the universal covering nilpotent Lie groups L and L'.

The homomorphisms $\varphi, \psi : \Gamma \to \Gamma'$ define the *Reidemeister action* of Γ on Γ' as follows:

$$\Gamma \times \Gamma' \to \Gamma', \qquad (\gamma, \gamma') \mapsto \psi(\gamma)\gamma'\varphi(\gamma)^{-1}.$$

Denote the set of Reidemeister classes of Γ' determined by f, g by $\mathcal{R}[f, g]$. Then the coincidence set $\mathrm{Coin}(f, g)$ splits into a disjoint union of coincidence classes

$$\mathrm{Coin}(f, g) = \coprod_{[\gamma'] \in \mathcal{R}[f,g]} p(\mathrm{Coin}(\gamma'\Phi, \Psi)).$$

Let Γ be a uniform lattice of a connected, simply connected nilpotent Lie group L. Then Γ is a finitely generated torsion-free nilpotent group. Recall that the lower central series of Γ is defined inductively via $\gamma_1(\Gamma) = \Gamma$ and $\gamma_{i+1}(\Gamma) = [\gamma_i(\Gamma), \Gamma]$. Suppose that Γ is *c-step nilpotent*, i.e., $\gamma_c(\Gamma) \neq 1$, but $\gamma_{c+1}(\Gamma) = 1$. The *isolator* of a subgroup H of Γ, denoted by $\sqrt[\Gamma]{H}$, is the set $\{x \in \Gamma \mid x^k \in H \text{ for some } k\}$. It is well known ([11, p. 473]) that the sequence

$$\Gamma = \Gamma_1 = \sqrt[\Gamma]{\gamma_1(\Gamma)} \supset \Gamma_2 = \sqrt[\Gamma]{\gamma_2(\Gamma)} \supset \cdots \supset \Gamma_c = \sqrt[\Gamma]{\gamma_c(\Gamma)} \supset \Gamma_{c+1} = 1$$

forms a central series with $\Gamma_i/\Gamma_{i+1} \cong \mathbb{Z}^{k_i}$. Since $\sqrt[\Gamma]{\gamma_i(\Gamma)} = \Gamma \cap \gamma_i(L)$, $\sqrt[\Gamma]{\gamma_i(\Gamma)}$ is a uniform lattice of $\gamma_i(L)$ and hence the nilmanifolds $\sqrt[\Gamma]{\gamma_i(\Gamma)}\backslash\gamma_i(L)$ are naturally sitting inside the nilmanifold $\Gamma\backslash L$. Now we fix the orientations of all manifolds arising in the natural embeddings $\sqrt[\Gamma]{\gamma_i(\Gamma)}\backslash\gamma_i(L) \hookrightarrow \Gamma\backslash L$. This means that the fixed orientation of $\Gamma\backslash L$ induces the fixed orientations of all the submanifolds $\sqrt[\Gamma]{\gamma_i(\Gamma)}\backslash\gamma_i(L)$. We can choose a generating set

$$\mathbf{a} = \{\mathbf{a}_1, \ldots, \mathbf{a}_c\}$$

in such a way that Γ_i is the group generated by $\mathbf{a}_i = \{a_{i1}, \ldots, a_{in_i}\}$ and Γ_{i+1}, and $\{\mathbf{a}_i, \ldots, \mathbf{a}_c\}$ determines the fixed orientation of $\sqrt[\Gamma]{\gamma_i(\Gamma)}\backslash\gamma_i(L)$ for each $i = 1, \ldots, c$. We refer to $\mathbf{a} = \{\mathbf{a}_1, \ldots, \mathbf{a}_c\}$ as a *preferred basis* of Γ.

We use $\mathfrak{L}$ to indicate the Lie algebra of L. This Lie algebra $\mathfrak{L}$ has the same dimension and nilpotency class as L. Moreover, in the case of connected, simply connected nilpotent Lie groups it is known that the exponential map $\exp : \mathfrak{L} \to L$ is a diffeomorphism, We denote its inverse by log. If L' is another connected, simply connected nilpotent Lie group, with Lie algebra $\mathfrak{L}'$, then we have the following properties:

- For any homomorphism $\phi : L \to L'$ of Lie groups, there exists a unique homomorphism $d\phi : \mathfrak{L} \to \mathfrak{L}'$ (differential of ϕ) of Lie algebras, making the following diagram commuting:

$$\begin{array}{ccc} L & \xrightarrow{\phi} & L' \\ {\scriptstyle \log}\downarrow\ \uparrow{\scriptstyle \exp} & & {\scriptstyle \log}\downarrow\ \uparrow{\scriptstyle \exp} \\ \mathfrak{L} & \xrightarrow{d\phi} & \mathfrak{L}' \end{array}$$

- Conversely, for any homomorphism $d\phi : \mathfrak{L} \to \mathfrak{L}'$ of Lie algebras, there exists a unique homomorphism $\phi : L \to L'$ of Lie groups, making the above diagram commuting.

If $\mathbf{a}$ is a preferred basis of Γ, then $\log \mathbf{a} = \{\log \mathbf{a}_1, \ldots, \log \mathbf{a}_c\} \subset \mathfrak{L}$ can be regarded as a basis for the vector space $\mathfrak{L}$. We also call it *preferred*. In particular, if Γ is a uniform lattice of $\mathbb{R}^d$ then every preferred basis $\mathbf{a}$ of Γ becomes a preferred basis $\log \mathbf{a} = \mathbf{a}$ for the vector space $\mathbb{R}^d$.

LEMMA 2.1. *Let $M = \Gamma \backslash L$ be a nilmanifold of dimension d and $T = \Gamma' \backslash \mathbb{R}^d$ be a torus. Then for any continuous maps $f, g : M \to T$, we have*

$$L(f,g) = \det(G_* - F_*), \quad N(f,g) = |L(f,g)|,$$

where F_*, G_* are the $d \times d$ matrices, with respect to any preferred bases $\log \mathbf{a}$ *and* $\log \mathbf{a}'$ *of Γ and Γ', of the homomorphisms from $\mathfrak{L}$ to $\mathbb{R}^d$ induced by $f, g : M \to T$.*

Proof. If we assume that M is also a torus then the result is known. Otherwise, the homomorphism $\Psi - \Phi : L \to \mathbb{R}^d$ from the non-abelian Lie group L into the abelian Lie group $\mathbb{R}^d$ must be singular. In this case, $L(f,g) = N(f,g) = \det(G_* - F_*) = 0$. ■

REMARK 2.2. Our original proof was longer, and this one was suggested by the referee.

NOTATION. For the commuting diagram

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & L' \\ \downarrow & & \downarrow \\ \Gamma\backslash L & \xrightarrow{f} & \Gamma'\backslash L' \end{array}$$

we shall use the following notations.

- F_* is the matrix of the homomorphism $\Phi_* : \mathfrak{L} \to \mathfrak{L}'$ with respect to *any preferred bases* $\log \mathbf{a}, \log \mathbf{a}'$ of the uniform lattices Γ, Γ' respectively. That is,

$$F_* = [\Phi_*]_{\log \mathbf{a}}^{\log \mathbf{a}'}.$$

- If $L = L'$, then f_* is the matrix of the homomorphism $\Phi_* : \mathfrak{L} \to \mathfrak{L}$ with respect to an *arbitrarily chosen basis* $\mathfrak{b}$ for $\mathfrak{L}$. That is,

$$f_* = [\Phi_*]_{\mathfrak{b}}^{\mathfrak{b}}.$$

EXAMPLE 2.3. For any $c \in \mathbb{R} - \{0\}$, consider $f, g : \mathbb{Z}\backslash\mathbb{R} \to c\mathbb{Z}\backslash\mathbb{R}$. Then we have a commuting diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\Phi", "\Psi"'] & \mathbb{R} \\
\mathbb{Z} \arrow[u, "\mathrm{inc}"'] \arrow[r, "\varphi", "\psi"'] & c\mathbb{Z} \arrow[u, "\mathrm{inc}"']
\end{tikzcd}$$

Here for some $a, b \in \mathbb{Z}$, $\Phi(x) = a(cx)$ and $\Psi(x) = b(cx)$ ($x \in \mathbb{R}$). A preferred basis of $\mathbb{Z}$ is $\mathbf{a} = \{1\}$ and a preferred basis of $c\mathbb{Z}$ is $\mathbf{a}' = \{c\}$. Thus $F_* = [\Phi_*]_{\mathbf{a}}^{\mathbf{a}'} = [a]$, $G_* = [\Psi_*]_{\mathbf{a}}^{\mathbf{a}'} = [b]$, $f_* = [\Phi_*]_{\mathbf{a}}^{\mathbf{a}} = [\Phi_*]_{\mathbf{a}'}^{\mathbf{a}'} = [ac]$, $g_* = [bc]$ and

$$L(f,g) = \det(G_* - F_*) = b - a, \quad N(f,g) = |b - a|.$$

The following is our main result.

THEOREM 2.4. *Let $M = \Gamma\backslash L$ and $M' = \Gamma'\backslash L'$ be nilmanifolds. Suppose that the groups $\gamma_i(L)$ and $\gamma_i(L')$ in the lower central series of L and L' have the same dimension. Then for any continuous maps $f, g : M \to M'$, we have*

$$L(f,g) = \det(G_* - F_*), \quad N(f,g) = |L(f,g)|.$$

Proof. Suppose that L is a simply connected c-step nilpotent Lie group. Then $\gamma_c(L) \neq 1$ but $\gamma_{c+1}(L) = 1$. Let $L_c = \gamma_c(L)$, $\Gamma_c = \Gamma \cap L_c$, $L'_c = \gamma_c(L')$ and $\Gamma'_c = \Gamma' \cap L'_c$. Note that $\Gamma_c = \Gamma \cap \gamma_c(L) = \sqrt[\Gamma]{\gamma_c(\Gamma)}$ and $\Gamma'_c = \sqrt[\Gamma']{\gamma_c(\Gamma')}$. Now we obtain principal fiber bundles $T \to M \to B$ and $T' \to M' \to B'$, where $T = \Gamma_c\backslash L_c$ and $T' = \Gamma'_c\backslash L'_c$ are tori of the same dimension, and $B = (\Gamma/\Gamma_c)\backslash(L/L_c)$ and $B' = (\Gamma'/\Gamma'_c)\backslash(L'/L'_c)$ are nilmanifolds of the same dimension.

We may assume that the diagram

$$\begin{tikzcd}
L \arrow[r, "\Phi", "\Psi"'] \arrow[d] & L' \arrow[d] \\
\Gamma\backslash L \arrow[r, "f", "g"'] & \Gamma'\backslash L'
\end{tikzcd}$$

is commuting. The restrictions of $\Phi, \Psi : L \to L'$ induce endomorphisms $\widehat{\Phi}, \widehat{\Psi} : L_c \to L'_c$ and hence, in turn, endomorphisms $\overline{\Phi}, \overline{\Psi} : L/L_c \to L'/L'_c$ so that the following diagrams are commuting:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & L_c & \longrightarrow & L & \longrightarrow & L/L_c & \longrightarrow & 1 \\
 & & {\scriptstyle\widehat{\Phi}}\downarrow{\scriptstyle\widehat{\Psi}} & & {\scriptstyle\Phi}\downarrow{\scriptstyle\Psi} & & {\scriptstyle\bar{\Phi}}\downarrow{\scriptstyle\bar{\Psi}} & & \\
1 & \longrightarrow & L'_c & \longrightarrow & L' & \longrightarrow & L'/L'_c & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathfrak{L}_c & \longrightarrow & \mathfrak{L} & \longrightarrow & \mathfrak{L}/\mathfrak{L}_c & \longrightarrow & 1 \\
 & & {\scriptstyle\widehat{\Phi}_*}\downarrow{\scriptstyle\widehat{\Psi}_*} & & {\scriptstyle\Phi_*}\downarrow{\scriptstyle\Psi_*} & & {\scriptstyle\bar{\Phi}_*}\downarrow{\scriptstyle\bar{\Psi}_*} & & \\
1 & \longrightarrow & \mathfrak{L}'_c & \longrightarrow & \mathfrak{L}' & \longrightarrow & \mathfrak{L}'/\mathfrak{L}'_c & \longrightarrow & 1
\end{array}$$

where $\mathfrak{L}_c$ is the Lie algebra of L_c and so on. We choose any preferred basis $\log \mathbf{a} = \{\log \widehat{\mathbf{a}}, \log \overline{\mathbf{a}}\}$ of $\mathfrak{L}$ so that $\log \widehat{\mathbf{a}}$ is a preferred basis for $\mathfrak{L}_c$ and the image of $\log \overline{\mathbf{a}}$ in $\mathfrak{L}/\mathfrak{L}_c$ is a preferred basis for $\mathfrak{L}/\mathfrak{L}_c$. Similarly we choose any preferred basis $\log \mathbf{a}' = \{\log \widehat{\mathbf{a}}', \log \overline{\mathbf{a}}'\}$ of $\mathfrak{L}'$.

Then Φ_* and Ψ_* have matrices of the form

$$F_* = \begin{bmatrix} \widehat{F}_* & * \\ 0 & \overline{F}_* \end{bmatrix}, \quad G_* = \begin{bmatrix} \widehat{G}_* & * \\ 0 & \overline{G}_* \end{bmatrix},$$

where $\widehat{F}_*$, $\widehat{G}_*$, $\overline{F}_*$ and $\overline{G}_*$ are the matrices with respect to the preferred bases $\log \widehat{\mathbf{a}}$, $\log \widehat{\mathbf{a}}'$, the image of $\log \overline{\mathbf{a}}$ and the image of $\log \overline{\mathbf{a}}'$.

Thus $\det(G_* - F_*) = \det(\widehat{G}_* - \widehat{F}_*) \cdot \det(\overline{G}_* - \overline{F}_*)$. Furthermore, $\widehat{\Phi}, \widehat{\Psi}$ map Γ_c into Γ'_c, and $\overline{\Phi}, \overline{\Psi}$ map Γ/Γ_c into Γ'/Γ'_c. Thus they induce maps $\widehat{f}, \widehat{g} : T \to T'$ and $\bar{f}, \bar{g} : B \to B'$ so that the following diagram is commutative:

$$\begin{array}{ccccc}
T & \longrightarrow & M & \longrightarrow & B \\
{\scriptstyle\widehat{f}}\downarrow{\scriptstyle\widehat{g}} & & {\scriptstyle f}\downarrow{\scriptstyle g} & & {\scriptstyle\bar{f}}\downarrow{\scriptstyle\bar{g}} \\
T' & \longrightarrow & M' & \longrightarrow & B'
\end{array}$$

Now we prove the theorem using induction on the nilpotency of L. On the tori, by Lemma 2.1 we have

$$L(\widehat{f}, \widehat{g}) = \det(\widehat{G}_* - \widehat{F}_*), \quad N(\widehat{f}, \widehat{g}) = |L(\widehat{f}, \widehat{g})|,$$

where $\widehat{F}_*$ and $\widehat{G}_*$ are the matrices with respect to the preferred bases $\log \widehat{\mathbf{a}}$ and $\log \widehat{\mathbf{a}}'$ of the vector spaces $\mathfrak{L}_c$ and $\mathfrak{L}'_c$ corresponding to any preferred bases $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{a}}'$ of the uniform lattices Γ_c and Γ'_c, respectively. By the induction hypothesis, we have

$$L(\bar{f}, \bar{g}) = \det(\overline{G}_* - \overline{F}_*), \quad N(\bar{f}, \bar{g}) = |L(\bar{f}, \bar{g})|,$$

where $\overline{F}_*$ and $\overline{G}_*$ are the matrices with respect to the preferred bases $\log \overline{\mathbf{a}}$ and $\log \overline{\mathbf{a}}'$ of the vector spaces $\mathfrak{L}/\mathfrak{L}_c$ and $\mathfrak{L}'/\mathfrak{L}'_c$ corresponding to any preferred bases $\overline{\mathbf{a}}$ and $\overline{\mathbf{a}}'$ of the uniform lattices Γ/Γ_c and Γ'/Γ'_c, respectively. (Here we abuse notation: $\overline{\mathbf{a}} \subset \Gamma$ and the image of $\overline{\mathbf{a}}$ in Γ/Γ_c is the preferred basis $\overline{\mathbf{a}}$.)

If $L(\bar{f},\bar{g})=0$, then $N(\bar{f},\bar{g})=0$ by the induction hypothesis, and hence $(\bar{f},\bar{g})$ is homotopic to a coincidence free pair. This fact follows from the Wecken type theorem if $\dim B=\dim B'\geq 3$ (see [12]). If $\dim B=\dim B' < 3$, then they are T^1 or T^2 and hence this fact is easily deduced. Next, this homotopy may be lifted to give rise to a deformation of (f,g) to a coincidence free pair. Thus $N(f,g)=L(f,g)=0$.

Now assume that $L(f,g)\neq 0$. Then the assumptions of [8, Theorem 6.5] are satisfied, because the second fundamental group of any nilmanifold vanishes and $\mathrm{Coin}(\overline{\Phi},\overline{\Psi})=\{0\}$ by [10, Lemma 2.5]. Thus the product formula $N(f,g)=N(\widehat{f},\widehat{g})\cdot N(\bar{f},\bar{g})$ holds. Since the fibration $T'\to M'\to B'$ is orientable, the formula $L(f,g)=L(\widehat{f},\widehat{g})\cdot L(\bar{f},\bar{g})$ also holds. Hence

$$|L(f,g)|=|L(\widehat{f},\widehat{g})\cdot L(\bar{f},\bar{g})|=N(\widehat{f},\widehat{g})\cdot N(\bar{f},\bar{g})=N(f,g),$$

and

$$L(f,g)=L(\widehat{f},\widehat{g})\cdot L(\bar{f},\bar{g})=\det(\widehat{G}_*-\widehat{F}_*)\cdot\det(\overline{G}_*-\overline{F}_*)=\det(G_*-F_*).$$

Finally, suppose that $\log\mathbf{b}=\{\log\widehat{\mathbf{b}},\log\overline{\mathbf{b}}\}$ and $\log\mathbf{b}'=\{\log\widehat{\mathbf{b}'},\log\overline{\mathbf{b}'}\}$ are other preferred bases of $\mathfrak{L}$ and $\mathfrak{L}'$, respectively. We notice that the transition matrices from one preferred basis to another one (both corresponding to the same uniform lattice) have determinant $+1$ since both preferred bases determine the same orientation. Namely, the transition matrices $[\mathrm{id}]_{\log\mathbf{a}}^{\log\mathbf{b}}$ and $[\mathrm{id}]_{\log\mathbf{b}'}^{\log\mathbf{a}'}$ have determinant $+1$. Since

$$\begin{aligned}
[\Phi_*]_{\log\mathbf{a}}^{\log\mathbf{a}'}&=[\mathrm{id}]_{\log\mathbf{b}'}^{\log\mathbf{a}'}\cdot[\Phi_*]_{\log\mathbf{b}}^{\log\mathbf{b}'}\cdot[\mathrm{id}]_{\log\mathbf{a}}^{\log\mathbf{b}},\\
[\Psi_*]_{\log\mathbf{a}}^{\log\mathbf{a}'}&=[\mathrm{id}]_{\log\mathbf{b}'}^{\log\mathbf{a}'}\cdot[\Psi_*]_{\log\mathbf{b}}^{\log\mathbf{b}'}\cdot[\mathrm{id}]_{\log\mathbf{a}}^{\log\mathbf{b}},
\end{aligned}$$

it follows that $\det(G_*-F_*)$ does not depend on the choice of the pairs of preferred bases $\mathbf{a},\mathbf{a}'$ and $\mathbf{b},\mathbf{b}'$ of Γ,Γ'. This finishes the proof. ∎

3. Example. For $\mathbf{x}=\{x_1,\dots,x_p\}\subset\Gamma$, $\mathbf{X}=\{X_1,\dots,X_p\}\subset\mathfrak{L}$, and a $p\times p$ integral matrix $N=(n_{ij})$, we use the following notations:

$$\begin{aligned}
\mathbf{x}^N&=\{x_1^{n_{11}}x_2^{n_{12}}\cdots x_p^{n_{1p}},\dots,x_1^{n_{p1}}x_2^{n_{p2}}\cdots x_p^{n_{pp}}\},\\
N\mathbf{X}&=\{n_{11}X_1+\dots+n_{1p}X_p,\dots,n_{p1}X_1+\dots+n_{pp}X_p\}.
\end{aligned}$$

Recall that for a uniform lattice Γ of a simply connected nilpotent Lie group L, we let $\Gamma_i=\sqrt[\Gamma]{\gamma_i(\Gamma)}$; then a generating set

$$\mathbf{a}=\{\mathbf{a}_1,\dots,\mathbf{a}_c\}$$

is a preferred basis of Γ if and only if Γ_i is the group generated by $\mathbf{a}_i$ and Γ_{i+1} for each $i=1,\dots,c$.

LEMMA 3.1. *Let Γ and Λ be uniform lattices of a simply connected nilpotent Lie group L. Let $\mathbf{a}=\{\mathbf{a}_1,\dots,\mathbf{a}_c\}$ be a preferred basis of Γ. If $\Lambda\subset\Gamma$,* *then there exists an upper triangular block integral matrix*

$$N = \begin{bmatrix} N_{11} & N_{12} & \dots & N_{1c} \\ 0 & N_{22} & \dots & N_{2c} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & N_{cc} \end{bmatrix}$$

such that the diagonal blocks have positive determinant, $[\Gamma : \Lambda] = \det(N)$ *and* Λ *has a preferred basis*

$$\mathbf{a}^N = \{\mathbf{a}_1^{N_{11}}\mathbf{a}_2^{N_{12}}\cdots\mathbf{a}_c^{N_{1c}}; \mathbf{a}_2^{N_{22}}\mathbf{a}_3^{N_{23}}\cdots\mathbf{a}_c^{N_{2c}}; \dots; \mathbf{a}_c^{N_{cc}}\}.$$

If $\mathbf{b}$ *is any preferred basis of* Λ, *then there exists an upper triangular block integral matrix* N *whose diagonal blocks have positive determinant*, $[\Gamma : \Lambda] = \det(N)$ *and* $\mathbf{b} = \mathbf{a}^N$.

Proof. Let $\Lambda_i = \Lambda \cap \Gamma_i = \Lambda \cap \sqrt[\Gamma]{\gamma_i(\Gamma)}$ for $i = 1, \dots, c$. Then

$$\Lambda = \Lambda_1 \supset \cdots \supset \Lambda_c \supset \Lambda_{c+1} = 1$$

is a central series of Λ with $\Lambda_i/\Lambda_{i+1} \cong \mathbb{Z}^{k_i}$ for each $i = 1, \dots, c$. In fact, there is a natural injection $\Lambda_i/\Lambda_{i+1} \to \Gamma_i/\Gamma_{i+1}$.

Since $\Lambda_c \subset \Gamma_c = \langle \mathbf{a}_c \rangle \cong \mathbb{Z}^{k_c}$, there is an integral matrix N_{cc} with positive determinant such that $\mathbf{a}_c^{N_{cc}}$ is a generating set of Λ_c. Obviously $[\Gamma_c : \Lambda_c] = \det(N_{cc})$. Next we consider the following commuting diagram of homomorphisms:

$$\begin{tikzcd} 1 \arrow[r] & \Gamma_c \arrow[r] & \Gamma_{c-1} \arrow[r] & \Gamma_{c-1}/\Gamma_c \cong \mathbb{Z}^{k_{c-1}} \arrow[r] & 1 \\ 1 \arrow[r] & \Lambda_c \arrow[r] \arrow[u, "\mathrm{inc}"'] & \Lambda_{c-1} \arrow[r] \arrow[u, "\mathrm{inc}"'] & \Lambda_{c-1}/\Lambda_c \cong \mathbb{Z}^{k_{c-1}} \arrow[r] \arrow[u, "\text{natural injection}"'] & 1 \end{tikzcd}$$

Since $\mathbf{a}_{c-1}\Gamma_c$ is a generating set of Γ_{c-1}/Γ_c, we can take $\mathbf{a}'_{c-1} \subset \Lambda_{c-1}$ so that $\mathbf{a}'_{c-1}\Lambda_c$ is a generating set of Λ_{c-1}/Λ_c. Then $\{\mathbf{a}'_{c-1}, \mathbf{a}_c^{N_{cc}}\}$ is a generating set of Λ_{c-1} and $\mathbf{a}'_{c-1} = \mathbf{a}_{c-1}^{N_{c-1,c-1}}\mathbf{a}_c^{N_{c-1,c}}$, where $N_{c-1,c-1}$ and $N_{c-1,c}$ are integral matrices so that $\det(N_{c-1,c-1}) = [\Gamma_{c-1}/\Gamma_c : \Lambda_{c-1}/\Lambda_c]$. Moreover, $[\Gamma_{c-1} : \Lambda_{c-1}] = [\Gamma_{c-1}/\Gamma_c : \Lambda_{c-1}/\Lambda_c] \cdot [\Gamma_c : \Lambda_c] = \det(N_{c-1,c-1})\det(N_{cc})$. Proceeding inductively, we obtain integral matrices $N_{11}, N_{12}, \dots, N_{1c}; N_{22}, N_{23}, \dots, N_{2c}; \dots; N_{cc}$ such that $\det(N_{ii})$ are positive, $[\Gamma : \Lambda] = \det(N_{11})\det(N_{22})\cdots \cdots\det(N_{cc})$ and Λ has a preferred basis

$$\mathbf{a}^N = \{\mathbf{a}_1^{N_{11}}\mathbf{a}_2^{N_{12}}\cdots\mathbf{a}_c^{N_{1c}}; \mathbf{a}_2^{N_{22}}\mathbf{a}_3^{N_{23}}\cdots\mathbf{a}_c^{N_{2c}}; \dots; \mathbf{a}_c^{N_{cc}}\}.$$

This proves the lemma. ■

Let $\Lambda \subset \Gamma$ be uniform lattices of a simply connected nilpotent Lie group L. Let $q : \Lambda\backslash L \to \Gamma\backslash L$ be the covering projection. Then we have the commuting diagrams

$$\begin{array}{ccc} L & \xrightarrow{\text{id}} & L \\ \uparrow{\scriptstyle\text{inc}} & & \uparrow{\scriptstyle\text{inc}} \\ \Lambda & \xrightarrow{\text{inc}} & \Gamma \end{array} \qquad\qquad \begin{array}{ccc} L & \xrightarrow{\text{id}} & L \\ \downarrow & & \downarrow \\ \Lambda\backslash L & \xrightarrow{q} & \Gamma\backslash L \end{array}$$

Let Γ have a preferred basis $\mathbf{a} = \{\mathbf{a}_1, \ldots, \mathbf{a}_c\}$. By Lemma 3.1, Λ has a preferred basis $\mathbf{b} = \mathbf{a}^N$ for some upper triangular block integral matrix N with $[\Gamma : \Lambda] = \det(N)$.

To compare $\log \mathbf{b}$ with $\log \mathbf{a}$, we recall the famous Baker–Campbell–Hausdorff formula:

$$\log(a \cdot b) = \log a * \log b \quad \text{for all } a, b \in L,$$

where

$$A * B = A + B + \frac{1}{2}[A, B] + \sum_{m=3}^{\infty} C_m(A, B).$$

Here $C_m(A, B)$ stands for a rational combination of m-fold Lie brackets in A and B. Since our Lie algebra is nilpotent, the sum involved in $A * B$ is always finite.

Since $\mathbf{a}_i \subset \gamma_i(L)$, we have $\log \mathbf{a}_i \subset \log \gamma_i(L) = \gamma_i(\mathfrak{L})$. So, $[\log \mathbf{a}_i, \log \mathbf{a}_{i+1}] \subset \gamma_{i+2}(\mathfrak{L})$. This implies that $[\log \mathbf{a}_i, \log \mathbf{a}_{i+1}]$ is a rational matrix linear combination of $\log \mathbf{a}_j$ where $j > i + 1$. Thus

$$\begin{aligned}\log(\mathbf{a}_i^{N_i} \mathbf{a}_{i+1}^{N_{i+1}} \cdots \mathbf{a}_c^{N_c}) &= N_i \log \mathbf{a}_i \\ &\quad + \text{a rational matrix linear combination of } \log \mathbf{a}_{i+1}, \ldots, \log \mathbf{a}_c.\end{aligned}$$

Therefore,

$$\det([q_*]_{\log \mathbf{b}}^{\log \mathbf{a}}) = \det([\text{id}]_{\log \mathbf{b}}^{\log \mathbf{a}}) = \det \begin{bmatrix} N_{11} & * & \ldots & * \\ 0 & N_{22} & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & N_{cc} \end{bmatrix} = [\Gamma : \Lambda].$$

We single this fact out as a lemma.

LEMMA 3.2. *Let $\Lambda \subset \Gamma$ be uniform lattices of a simply connected nilpotent Lie group L. Let $q : \Lambda\backslash L \to \Gamma\backslash L$ be the covering projection. For any preferred bases $\mathbf{a}$ and $\mathbf{b} = \mathbf{a}^N$ of Γ and Λ, respectively, we have*

$$\det([q_*]_{\log \mathbf{b}}^{\log \mathbf{a}}) = \det([\text{id}]_{\log \mathbf{b}}^{\log \mathbf{a}}) = [\Gamma : \Lambda].$$

The following is practically useful in computing the Nielsen and Lefschetz coincidence numbers on some nilmanifolds.

Corollary 3.3. *Let $\Gamma\backslash L$ and $\Gamma'\backslash L$ be nilmanifolds, i.e., Γ and Γ' are uniform lattices of the connected simply connected nilpotent Lie group L. Suppose $\Gamma\cap\Gamma'$ is a uniform lattice of L. Then for any continuous maps $f,g:\Gamma\backslash L\to\Gamma'\backslash L$, we have*

$$L(f,g)=\frac{[\Gamma':\Gamma\cap\Gamma']}{[\Gamma:\Gamma\cap\Gamma']}\det(g_*-f_*),\quad N(f,g)=|L(f,g)|.$$

Proof. Let $\Lambda:=\Gamma\cap\Gamma'$ be the uniform lattice of L. Thus Λ has finite index in both Γ and Γ'. Choose preferred bases $\mathbf{b},\mathbf{a},\mathbf{a}'$ of the uniform lattices Λ,Γ,Γ', respectively. Then by Lemma 3.1 we have $\mathbf{b}=\mathbf{a}^N=\mathbf{a}'^{N'}$ for some upper triangular block integral matrices N,N' with $[\Gamma:\Lambda]=\det(N)$ and $[\Gamma':\Lambda]=\det(N')$. The endomorphisms Φ_*,Ψ_* induced by f,g on the vector space $\mathfrak{L}$ with various preferred bases yield the commuting diagram

$$\begin{array}{ccc}
(\mathfrak{L},\log\mathbf{a}) & \xrightarrow[\Psi_*]{\Phi_*} & (\mathfrak{L},\log\mathbf{a}')\\
\uparrow{\scriptstyle\mathrm{id}} & & \uparrow{\scriptstyle\mathrm{id}}\\
(\mathfrak{L},\log\mathbf{b}) & \xrightarrow[\Psi_*]{\Phi_*} & (\mathfrak{L},\log\mathbf{b})
\end{array}$$

The corresponding matrices thus satisfy

$$[\Phi_*]^{\log\mathbf{a}'}_{\log\mathbf{a}}\cdot[\mathrm{id}]^{\log\mathbf{a}}_{\log\mathbf{b}}=[\mathrm{id}]^{\log\mathbf{a}'}_{\log\mathbf{b}}\cdot[\Phi_*]^{\log\mathbf{b}}_{\log\mathbf{b}},$$
$$[\Psi_*]^{\log\mathbf{a}'}_{\log\mathbf{a}}\cdot[\mathrm{id}]^{\log\mathbf{a}}_{\log\mathbf{b}}=[\mathrm{id}]^{\log\mathbf{a}'}_{\log\mathbf{b}}\cdot[\Psi_*]^{\log\mathbf{b}}_{\log\mathbf{b}},$$

or

$$F_*\cdot[\mathrm{id}]^{\log\mathbf{a}}_{\log\mathbf{b}}=[\mathrm{id}]^{\log\mathbf{a}'}_{\log\mathbf{b}}\cdot f_*,\qquad G_*\cdot[\mathrm{id}]^{\log\mathbf{a}}_{\log\mathbf{b}}=[\mathrm{id}]^{\log\mathbf{a}'}_{\log\mathbf{b}}\cdot g_*$$

By Lemma 3.2,

$$\det([\mathrm{id}]^{\log\mathbf{a}}_{\log\mathbf{b}})=[\Gamma:\Lambda],\qquad \det([\mathrm{id}]^{\log\mathbf{a}'}_{\log\mathbf{b}})=[\Gamma':\Lambda].$$

Theorem 2.4, together with the above observation, yields

$$\begin{aligned}
L(f,g)&=\det(G_*-F_*)\\
&=\frac{\det([\mathrm{id}]^{\log\mathbf{a}'}_{\log\mathbf{b}})}{\det([\mathrm{id}]^{\log\mathbf{a}}_{\log\mathbf{b}})}\det(g_*-f_*)=\frac{[\Gamma':\Gamma\cap\Gamma']}{[\Gamma:\Gamma\cap\Gamma']}\det(g_*-f_*),\\
N(f,g)&=|L(f,g)|.
\end{aligned}$$

This finishes the proof. ∎

Example 3.4. Let L be the 3-dimensional Heisenberg group. That is,

$$L=\left\{\begin{bmatrix}1&x&z\\0&1&y\\0&0&1\end{bmatrix}\;\middle|\;x,y,z\in\mathbb{R}\right\}.$$

We denote this general element by $\{x, y, z\}$. For any integer $k > 0$, we consider the subgroups $\mathit{\Gamma}_k = \{\{m, n, l/k\} \mid m, n, l \in \mathbb{Z}\}$ of L. These are uniform lattices of L, and every uniform lattice of L is isomorphic to some $\mathit{\Gamma}_k$.

Let $\mathit{\Phi} : L \to L$ be an endomorphism. Then we have a commuting diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & [L, L] & \longrightarrow & L & \longrightarrow & L/[L, L] & \longrightarrow & 1 \\
 & & \Big\downarrow \widehat{\mathit{\Phi}} & & \Big\downarrow \mathit{\Phi} & & \Big\downarrow \overline{\mathit{\Phi}} & & \\
1 & \longrightarrow & [L, L] & \longrightarrow & L & \longrightarrow & L/[L, L] & \longrightarrow & 1
\end{array}$$

Since $[L, L] = \{\{0, 0, z\} \mid z \in \mathbb{R}\}$ and $L/[L, L] \cong \{\{x, y, 0\} \mid x, y \in \mathbb{R}\}$, $\mathit{\Phi}$ must send $\{x, y, z\}$ to $\{\alpha x+\gamma y, \beta x+\delta y, \eta z+\varphi(x, y, z)\}$ for some $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{R}$. In particular, $\mathit{\Phi}(\{x, y, 0\} \cdot \{0, 0, z\}) = \mathit{\Phi}(\{x, y, 0\}) \cdot \mathit{\Phi}(\{0, 0, z\})$ implies that $\varphi(x, y, z) = \varphi(x, y, 0)$. Thus $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a function depending only on x and y. Comparing the images of $\{x, y, 0\} = \{0, y, 0\} \cdot \{x, 0, 0\}$ and $\{x, y, xy\} = \{x, 0, 0\} \cdot \{0, y, 0\}$ under $\mathit{\Phi}$ shows that $\alpha\delta - \beta\gamma = \eta$.

Suppose $\mathit{\Phi}$ maps $\mathit{\Gamma}_k$ into $\mathit{\Gamma}_{k'}$. Then $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\mathit{\Phi}(\{0, 0, 1/k\}) = \{0, 0, \eta/k\} = \{0, 0, l/k'\}$ for some $l \in \mathbb{Z}$. Thus, if $(k, k') = m$, i.e., $k = ms$, $k' = mt$ and $(s, t) = 1$, then η is a multiple of s and l is a multiple of t.

Let $\mathit{\Phi}, \mathit{\Psi} : L \to L$ be the endomorphisms of L given by

$$\begin{aligned}
\mathit{\Phi}(\{x, y, z\}) &= \{2x - 2y, 2x + y, 6z + 2x^2 - 4xy - y^2\}, \\
\mathit{\Psi}(\{x, y, z\}) &= \{3y, x + y, -3z + 3xy + \tfrac{3}{2}y^2\}.
\end{aligned}$$

Then $\mathit{\Phi}(\mathit{\Gamma}_6) \subset \mathit{\Gamma}_4$ and $\mathit{\Psi}(\mathit{\Gamma}_6) \subset \mathit{\Gamma}_4$. Thus the endomorphisms $\mathit{\Phi}, \mathit{\Psi} : L \to L$ induce $f, g : \mathit{\Gamma}_6 \backslash L \to \mathit{\Gamma}_4 \backslash L$ so that the following diagram commutes:

$$\begin{array}{ccc}
L & \xrightarrow[\mathit{\Psi}]{\mathit{\Phi}} & L \\
\Big\downarrow & & \Big\downarrow \\
\mathit{\Gamma}_6 \backslash L & \xrightarrow[g]{f} & \mathit{\Gamma}_4 \backslash L
\end{array}$$

Since $\mathit{\Gamma}_2 = \mathit{\Gamma}_4 \cap \mathit{\Gamma}_6$, by Corollary 3.3 the Lefschetz and Nielsen coincidence numbers of f, g are given by

$$L(f, g) = \frac{2}{3} \det(g_* - f_*), \quad N(f, g) = |L(f, g)|,$$

where f_*, g_* are the matrices of the differentials of $\mathit{\Phi}, \mathit{\Psi}$ with respect to any basis of $\mathfrak{L}$.

We take an ordered (linear) basis for the Lie algebra $\mathfrak{L}$ of L as follows:

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that this basis for $\mathfrak{L}$ is obtained from the preferred basis $\{0,0,1\}$, $\{1,0,0\}$, $\{0,1,0\}$ for Γ_1. With respect to this basis, the differentials of Φ and Ψ are

$$f_* = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 2 & 1 \end{bmatrix}, \quad g_* = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Therefore, the Lefschetz and Nielsen coincidence number of the maps $f, g : \Gamma_6\backslash L \to \Gamma_4\backslash L$ are

$$L(f,g) = \frac{2}{3}\det(g_* - f_*) = \frac{2}{3}(-45) = -30, \qquad N(f,g) = 30.$$

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