Extension of point-finite partitions of unity

by

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Abstract. A subspace $A$ of a topological space $X$ is said to be $P^\gamma$-embedded ($P^\gamma$(point-finite)-embedded) in $X$ if every (point-finite) partition of unity $\alpha$ on $A$ with $|\alpha| \leq \gamma$ extends to a (point-finite) partition of unity on $X$. The main results are: (Theorem A) A subspace $A$ of $X$ is $P^\gamma$(point-finite)-embedded in $X$ iff it is $P^\gamma$-embedded and every countable intersection $B$ of cozero-sets in $X$ with $B \cap A = \emptyset$ can be separated from $A$ by a cozero-set in $X$. (Theorem B) The product $A \times [0, 1]$ is $P^\gamma$(point-finite)-embedded in $X \times [0, 1]$ if a disjoint $A \times Y$ is $P^\gamma$(point-finite)-embedded in $X \times Y$ for every compact Hausdorff space $Y$ with $w(Y) \leq \gamma$ iff $A$ is $P^\gamma$-embedded in $X$ and every subset $B$ of $X$ obtained from zero-sets by means of the Suslin operation, with $B \cap A = \emptyset$, can be separated from $A$ by a cozero-set in $X$. These characterizations are used to answer certain questions of Dydak. In particular, it is shown that, assuming CH, the property of $A \times [0, 1]$ to be $P^\gamma$(point-finite)-embedded in $X \times [0, 1]$ is stronger than that of $A$ being $P^\gamma$(point-finite)-embedded in $X$.

1. Introduction. By a space we mean a topological space. A partition of unity $\alpha$ on a space $X$ is called point-finite (resp. locally finite) if the family $\{\text{coz}(f) : f \in \alpha\}$ is point-finite (resp. locally finite) in $X$, where $\text{coz}(f) = \{x \in X : f(x) \neq 0\}$. Let $A$ be a subspace of a space $X$ and $\gamma$ an infinite cardinal. When $\alpha = \{f_\lambda\}$ and $\beta = \{g_\lambda\}$ are partitions of unity on $A$ and $X$, respectively, we say that $\beta$ is an extension of $\alpha$ if $f_\lambda = g_\lambda|_A$ for each $\lambda$. Dydak [3] defined $A$ to be $P^\gamma$(point-finite)-embedded (resp. $P^\gamma$(locally finite)-embedded) in $X$ if every point-finite (resp. locally finite) partition of unity $\alpha$ on $A$, with $|\alpha| \leq \gamma$, extends to a point-finite (resp. locally finite) partition of unity on $X$. Extensive studies of $P^\gamma$(locally finite)-embedding have been made by Dydak [3], [4] and the second author [21] and [23].

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In this paper, we consider $P^\gamma$(point-finite)-embeddings. In particular, we prove Theorems A and B stated in the abstract and apply them to answer Dydak’s questions concerning $P^\gamma$(point-finite)-embeddings stated below.

Recall from [3] that $A$ is $P^\gamma$-embedded in $X$ if every partition of unity $\alpha$ on $A$ with $|\alpha| \leq \gamma$ extends to a partition of unity on $X$ (see [2], [19] for the original definition of $P^\gamma$-embedding). Recall from [18] that $A$ is $M^\gamma$-embedded in $X$ if for every AR $Y$ with $w(Y) \leq \gamma$, every continuous map from $A$ to $Y$ extends continuously over $X$. It is known (see [3]) that these extension properties are related as follows, where $A \rightarrow B$ means that every $A$-embedded subspace is $B$-embedded:

$$M^\gamma \rightarrow P^\gamma(\text{point-finite}) \rightarrow P^\gamma \leftarrow (\ast) P^\gamma(\text{locally finite}).$$

Przymusiński–Wage [15] showed that the arrow $(\ast)$ cannot be reversed by giving an example of a collectionwise normal space $Z$ having a closed subspace which is not $P^\omega$(locally finite)-embedded, and Dydak [3] showed that the implication “$P^\gamma$(locally finite) $\rightarrow P^\gamma$(point-finite)” is not true in general (see also [24]). In Section 3, we give an example of a subspace which is $P^\gamma$(point-finite)-embedded for every $\gamma$ but not $M^\omega$-embedded (Example 3.4), and prove that every closed subspace of the space $Z$ of Przymusiński–Wage mentioned above is $M^\gamma$-embedded for every $\gamma$ (Example 3.8). These results answer Dydak’s questions [3, Problems 12.10 and 12.11] negatively. Moreover, Dydak [3, Problem 13.6] asked: If $A$ is $P^\gamma$(point-finite)-embedded in $X$, is then $A \times [0,1]$ $P^\gamma$(point-finite)-embedded in $X \times [0,1]$? It is known that the answers to the similar questions for $P^\gamma$, $M^\gamma$- and $P^\gamma$(locally finite)-embeddings are all positive (see Alò–Sennott [1], Sennott [18] and Yamazaki [21], respectively). As an application of Theorem B, we show that the answer is negative for $P^\gamma$(point-finite)-embeddings under the assumption of the continuum hypothesis (Examples 3.5 and 3.7).

For a set $A$, $|A|$ denotes the cardinality of $A$. As usual, a cardinal is an initial ordinal and an ordinal is identified with the set of smaller ordinals. Let $\omega$ denote the first infinite cardinal and $\omega_1$ the first uncountable cardinal. Our terminology and notation follow [5] and [13].

2. $P^\gamma$(point-finite)-embeddings and products. A zero-set in a space $X$ is a set of the form $f^{-1}(0)$ for some real-valued continuous function $f$ on $X$ and a cozero-set is the complement of a zero-set. For a space $X$, let $\mathcal{Z}(X)$ (resp. $\mathcal{C}(X)$) denote the family of all zero-sets (resp. cozero-sets) in $X$. A set $A \subseteq X$ is called a Suslin-$\mathcal{Z}$-set in $X$ if there exists a family $\{Z_\sigma : \sigma \in \omega \}$ such that $A = \bigcup_{\sigma \in \omega} \bigcap_{\tau \leq \sigma} Z_\tau$, where $\omega \omega$ is the set of all maps from $\alpha$ to $\omega$ and $\omega \omega = \bigcup_{\sigma \in \omega} \omega \omega$ (see [16]). All Baire sets, i.e., members of the smallest $\sigma$-algebra including $\mathcal{Z}(X)$, are Suslin-$\mathcal{Z}$-sets. As usual, we call a Suslin-$\mathcal{Z}$-set and a Baire set in a metric space an analytic
set and a Borel set, respectively. Now, we consider the following conditions on a subspace $A$ of a space $X$:

(b$_1$) For every Suslin-$Z$-set $B$ in $X$ with $B \cap A = \emptyset$, there exists $U \in \text{Coz}(X)$ such that $B \subseteq U$ and $U \cap A = \emptyset$.

(b$_2$) For every Baire set $B$ in $X$ with $B \cap A = \emptyset$, there exists $U \in \text{Coz}(X)$ such that $B \subseteq U$ and $U \cap A = \emptyset$.

(b$_3$) For every countable family $\{G_n : n < \omega\} \subseteq \text{Coz}(X)$ with $\bigcap_{n<\omega} G_n \cap A = \emptyset$, there exists $U \in \text{Coz}(X)$ such that $\bigcap_{n<\omega} G_n \subseteq U$ and $U \cap A = \emptyset$.

Evidently, (b$_1$) implies (b$_2$) and (b$_2$) implies (b$_3$) (see Remark 2.8 below). Now, we prove the theorems announced in the abstract.

**Theorem 2.1.** Let $A$ be a subspace of a space $X$ and $\gamma$ an infinite cardinal. Then the following are equivalent:

1. $A$ is $P^\gamma$(point-finite)-embedded in $X$,
2. $A$ is $P^\gamma$-embedded in $X$ and $P^\omega$(point-finite)-embedded in $X$,
3. $A$ is $P^\gamma$-embedded in $X$ and satisfies (b$_3$) in $X$.

**Proof.** (1)$\Rightarrow$(2): Obvious. (2)$\Rightarrow$(3): To prove that $A$ satisfies (b$_3$) in $X$, take a countable family $\{G_n : n < \omega\} \subseteq \text{Coz}(X)$ with $\bigcap_{n<\omega} G_n \cap A = \emptyset$. We may assume that $G_{n+1} \subseteq G_n$ for each $n < \omega$ and $G_0 = X$. Take continuous functions $f_n : X \to [0, 1/2^n]$, $n < \omega$, with $G_n = \text{coz}(f_n)$, and define $f = \sum_{n<\omega} f_n$. Note that $f(x) > 0$ for all $x \in X$. For each $n < \omega$, define a function $f_n^* : A \to [0, 1]$ by $f_n^*(x) = f_n(x)/f(x)$ for $x \in A$. Then $\{f_n^* : n < \omega\}$ is a point-finite partition of unity on $A$. Since $A$ is $P^\omega$(point-finite)-embedded in $X$, there exists a point-finite partition of unity $\{g_n : n < \omega\}$ on $X$ such that $g_n|_A = f_n^*$ for each $n < \omega$. Let

$$U = \bigcup_{n<\omega} \{x \in X : g_n(x) \cdot f(x) \neq f_n(x)\}.$$ 

Then $U \in \text{Coz}(X)$, $\bigcap_{n<\omega} G_n \subseteq U$ since $\{g_n : n < \omega\}$ is point-finite, and $U \cap A = \emptyset$. Hence, $A$ satisfies (b$_3$) in $X$.

(3)$\Rightarrow$(1): Let $\alpha = \{f_\lambda : \lambda \in A\}$ be a point-finite partition of unity on $A$ with $|A| \leq \gamma$. Since $A$ is $P^\gamma$-embedded in $X$, $\alpha$ extends to a partition of unity $\beta = \{g_\lambda : \lambda \in A\}$ on $X$. Let $B = \{x \in X : \beta \text{ is not point-finite at } x\}$. We show that $B$ is the countable intersection of cozero-sets in $X$. For each $n < \omega$ and each $x \in X$, define

$$k_n(x) = \max \left\{ \sum_{\lambda \in \delta} g_\lambda(x) : \delta \subseteq A, |\delta| \leq n \right\}.$$ 

Since $\beta$ is a partition of unity, the functions $k_n : X \to [0, 1]$, $n < \omega$, are continuous, and for each $x \in X$, $|\{\lambda \in A : g_\lambda(x) > 0\}| \leq n$ if and only if $k_n(x) = 1$. This implies that $X \setminus B = \bigcup_{n<\omega} k_n^{-1}(1)$, and hence, $B$ is the
intersection of countably many cozero-sets in $X$. Since $B \cap A = \emptyset$ and $A$ satisfies $(b_3)$ in $X$, we can find a continuous function $h : X \to [0, 1]$ such that $B \subseteq \text{coz}(h)$ and $\text{coz}(h) \cap A = \emptyset$. For each $\lambda \in \Lambda$, define a function $g_\lambda^*$ on $X$ by $g_\lambda^*(x) = \max\{g_\lambda(x) - h(x), 0\}$ for $x \in X$. Then $\{\text{coz}(g_\lambda^*) : \lambda \in \Lambda\}$ is point-finite in $X$, because if $h(x) = 0$, then $g_\lambda^*(x) = g_\lambda(x)$ and $x \notin B$; and if $h(x) > 0$, then only finitely many $g_\lambda$’s exceed $h$ at $x$ since $\sum_{\lambda \in \Lambda} g_\lambda(x) = 1$. Since $g_\lambda^*(x) \leq g_\lambda(x)$ for each $\lambda \in \Lambda$ and each $x \in X$, it follows from [4, Corollary 2.6] that the function $\sum_{\lambda \in \Lambda} g_\lambda^*$ is continuous. Fix an arbitrary $\mu \in \Lambda$ and define

$$g_{\mu}^{**}(x) = g_{\mu}^*(x) + 1 - \sum_{\lambda \in \Lambda} g_\lambda^*(x) \quad \text{for } x \in X.$$ 

Finally, putting $g_\lambda^{**} = g_\lambda^*$ for each $\lambda \in \Lambda \setminus \{\mu\}$, we obtain a point-finite partition of unity $\{g_\lambda^{**} : \lambda \in \Lambda\}$ on $X$ extending $\alpha$. Hence, $A$ is $P^\gamma$(point-finite)-embedded in $X$.

We turn to considering the problem when $A \times Y$ is $P^\gamma$(point-finite)-embedded in $X \times Y$ for all (or certain) compact Hausdorff spaces $Y$. We need the following result due to Alô and Sennott [1] as a lemma.

**Lemma 2.2** (Alô–Sennott). Let $A$ be a $P^\gamma$-embedded subspace of a space $X$, where $\gamma$ is an infinite cardinal. Then $A \times Y$ is $P^\gamma$-embedded in $X \times Y$ for every compact Hausdorff space $Y$ with $w(Y) \leq \gamma$.

Since the countable union of cozero-sets is a cozero-set, we have the following corollary from Theorem 2.1 and Lemma 2.2.

**Corollary 2.3.** Let $A$ be a $P^\gamma$(point-finite)-embedded subspace of a space $X$, where $\gamma$ is an infinite cardinal. Then $A \times Y$ is $P^\gamma$(point-finite)-embedded in $X \times Y$ for every countable, compact metric space $Y$.

The next lemma is well known, but we can find no good reference.

**Lemma 2.4** (folklore). Let $X$ and $Y$ be spaces and $\text{pr}_X : X \times Y \to X$ the projection.

1. If $Y$ is separable, then $\text{pr}_X$ carries cozero-sets to cozero-sets.
2. If $Y$ is compact, then $\text{pr}_X$ carries cozero-sets to cozero-sets and carries zero-sets to zero-sets.
3. If $Y$ is compact, then $\text{pr}_X$ carries Suslin-Z-sets to Suslin Z-sets.

**Proof.** (1) Let $D$ be a countable dense set in $Y$. Then, for every cozero-set $G$ in $X \times Y$, $\text{pr}_X[G] = \bigcup_{y \in D} \{x \in X : (x, y) \in G\} \in \text{Coz}(X)$.

(2) This follows from the fact that if $Y$ is compact, then for every real-valued continuous function $h$ on $X \times Y$, the functions $f$ and $g$ on $X$ defined by $f(x) = \sup\{h(x, y) : y \in Y\}$ and $g(x) = \inf\{h(x, y) : y \in Y\}$ for $x \in X$ are continuous (see [6, Lemma 1.1]).
(3) This is a consequence of (2) since we can assume that \( Z_{t|n} \subseteq Z_{t|m} \) whenever \( m < n \) in the definition of a Suslin-\( Z \)-set. 

The next lemma is due to the referee’s suggestion.

**Lemma 2.5.** For every Suslin-\( Z \)-set \( B \) in a space \( X \), there exists a continuous map \( f : X \to Q \), where \( Q \) is the Hilbert cube, such that \( B = f^{-1}[S] \) for some analytic set \( S \) in \( Q \).

**Proof.** This is a consequence of the following observation: For any collection of countably many zero-sets \( Z_n = f_{n}^{-1}(0) \), \( n \in \omega \), in \( X \), consider the diagonal map \( f = \bigtriangleup_{n<\omega}f_n : X \to Q \). Then each \( Z_n \) is the inverse image of a closed set in \( Q \).

Now, combining Lemma 2.5 with Theorem 2.1, Lemmas 2.2 and 2.4, we have the following theorem.

**Theorem 2.6.** Let \( A \) be a subspace of a space \( X \) and \( \gamma \) an infinite cardinal. Then the following are equivalent:

1. \( A \times Y \) is \( P^\gamma \) (point-finite)-embedded in \( X \times Y \) for every compact Hausdorff space \( Y \) with \( w(Y) \leq \gamma \),
2. \( A \times [0, 1] \) is \( P^\gamma \) (point-finite)-embedded in \( X \times [0, 1] \),
3. \( A \times Y \) is \( P^\gamma \) (point-finite)-embedded in \( X \times Y \) for some uncountable, compact metric space \( Y \),
4. \( A \) is \( P^\gamma \)-embedded in \( X \) and satisfies \( (b_1) \) in \( X \).

**Proof.** (1)\( \Rightarrow \) (2)\( \Rightarrow \) (3): Obvious. (3)\( \Rightarrow \) (4): Assume that \( A \times Y \) is \( P^\gamma \) (point-finite)-embedded in \( X \times Y \) for some uncountable, compact metric space \( Y \). As every \( P^\gamma \) (point-finite)-embedded subspace is \( P^\gamma \)-embedded, it suffices to show that \( A \) satisfies \( (b_1) \) in \( X \). Let \( B \) be a Suslin-\( Z \)-set in \( X \) with \( B \cap A = \emptyset \). Then, by Lemma 2.5, there exists a continuous map \( f : X \to Q \) such that \( B = f^{-1}[S] \) for some analytic set \( S \) in \( Q \). It is known that \( S \) is the projection of a \( G_\delta \)-set \( G \) in \( Q \times \mathbb{K} \), where \( \mathbb{K} \) is the Cantor set (see [11]). Since \( \mathbb{K} \) can be embedded in \( Y \), we regard \( G \) as a \( G_\delta \)-set in \( Q \times Y \). Put \( H = (f \times \text{id}_Y)^{-1}[G] \), where \( \text{id}_Y \) is the identity of \( Y \). Then \( H \) is the intersection of countably many cozero-sets in \( X \times Y \) and \( B = \text{pr}_X[H] \). Since \( A \times Y \) satisfies \( (b_3) \) in \( X \times Y \) by Theorem 2.1, there exists \( U \in \text{Coz}(X \times Y) \) such that \( H \subseteq U \) and \( U \cap (A \times Y) = \emptyset \). Finally, put \( V = \text{pr}_X[U] \). Then \( V \in \text{Coz}(X) \) by Lemma 2.4(2), \( B \subseteq V \) and \( V \cap A = \emptyset \). Hence, \( A \) satisfies \( (b_1) \) in \( X \).

(4)\( \Rightarrow \) (1): Let \( Y \) be a compact Hausdorff space with \( w(Y) \leq \gamma \). By Theorem 2.1 and Lemma 2.2, it suffices to show that \( A \times Y \) satisfies \( (b_3) \) in \( X \times Y \). Let \( B \) be the intersection of countably many cozero-sets in \( X \times Y \) with \( B \cap (A \times Y) = \emptyset \). Then it follows from Lemma 2.4(3) that \( \text{pr}_X[B] \) is a Suslin-\( Z \)-set in \( X \) with \( B \cap A = \emptyset \). Since \( A \) satisfies \( (b_1) \) in \( X \), there exists \( U \in \text{Coz}(X) \) such that \( \text{pr}_X[B] \subseteq U \) and \( U \cap A = \emptyset \). Putting \( V = U \times Y \),
we obtain \( V \subseteq Coz(X \times Y) \) such that \( B \subseteq V \) and \( V \cap (A \times Y) = \emptyset \). Hence, \( A \times Y \) satisfies \((b_3)\) in \( X \times Y \). ■

The reader might ask if “compact metric” can be replaced by “compact Hausdorff” in condition (3) of Theorem 2.6. In Remark 3.6 below, we show that metrizability of \( Y \) is essential in this condition.

The following corollary can be proved similarly to (3)⇒(4) in Theorem 2.6 if we use Lemma 2.4(1) and the fact that every analytic set in \( Q \) is the projection of a zero-set in \( Q \times \mathbb{P} \), where \( \mathbb{P} \) is the space of irrational numbers (see [11]).

**Corollary 2.7.** Let \( A \) be a \( P^\gamma \)-embedded closed subspace of a space \( X \), where \( \gamma \) is an infinite cardinal, and assume that either \( X \times \mathbb{P} \) is normal or \((X \setminus A) \times \mathbb{P} \) is Lindelöf. Then \( A \times Y \) is \( P^\gamma \) (point-finite) -embedded in \( X \times Y \) for every compact Hausdorff space \( Y \) with \( w(Y) \leq \gamma \).

In the remaining part of this section, we consider the relationship between conditions \((b_i)\), \( i = 1, 2, 3 \), and the following conditions, from the literature, on a subspace \( A \) of a space \( X \).

\( (a_\gamma) \) For every \( \gamma \)-separable continuous pseudometric \( \varrho \) on \( X \), there exists \( F \in Z(X) \) such that \( A \subseteq F \subseteq \{ x \in X : (\exists y \in A)(\varrho(x, y) = 0) \} \).

\( (c) \) For every \( B \in Z(X) \) with \( B \cap A = \emptyset \), there exists \( U \in Coz(X) \) such that \( B \subseteq U \) and \( U \cap A = \emptyset \).

Here, a pseudometric \( d \) on a space \( X \) is called \( \gamma \)-separable if the weight of the pseudometric space \((X, d)\) is not greater than \( \gamma \). Sennott [18] proved that \( A \) is \( M^\gamma \)-embedded in \( X \) if and only if \( A \) is \( P^\gamma \)-embedded in \( X \) and satisfies \((a_\gamma)\) in \( X \). On the other hand, it is known (see [8, Theorem 1.18]) that \( A \) satisfies \((c)\) in \( X \) if every real-valued continuous function on \( A \) extends continuously over \( X \), or equivalently, \( A \) is \( P^\omega \)-embedded in \( X \) (see [7] and [9]). Obviously, \((b_3)\) implies \((c)\).

**Proposition 2.8.** \((a_\omega)\) implies \((b_1)\).

**Proof.** Assume that a subspace \( A \) of a space \( X \) satisfies \((a_\omega)\) in \( X \). Let \( B \) be a Suslin-Z-set in \( X \) with \( B \cap A = \emptyset \). Then, by Lemma 2.5, there exists a continuous map \( f \) from \( X \) to the Hilbert cube \( Q \) such that \( B = f^{-1}[f[B]] \). Let \( d \) be the metric on \( Q \), and define \( \varrho(x, y) = d(f(x), f(y)) \) for \( x, y \in X \). Then \( \varrho \) is an \( \omega \)-separable continuous pseudometric on \( X \) such that \( \{ x \in X : (\exists y \in A)(\varrho(x, y) = 0) \} \cap B = \emptyset \). Hence, by \((a_\omega)\), we can find \( U \in Coz(X) \) such that \( B \subseteq U \) and \( U \cap A = \emptyset \). ■

**Remark 2.9.** Summing up the above observations, we have the implications

\( (a_\omega) \Rightarrow (b_1) \Rightarrow (b_2) \Rightarrow (b_3) \Rightarrow (c) \)
In the next section, we give examples showing that \((b_1) \not\Rightarrow (a_\omega)\) and \((c) \not\Rightarrow (b_3)\), and show that \((b_3) \not\Rightarrow (b_2)\) and \((b_2) \not\Rightarrow (b_1)\) assuming the continuum hypothesis.

3. Examples. As stated in the introduction, we now apply Theorems 2.1 and 2.6 to answer Dydak’s questions [3, Problems 12.10, 12.11 and 13.6]. Throughout this section, \(\mathbb{R}\) denotes the set of real numbers endowed with the Euclidean topology \(\tau\). When \(A \subseteq X \subseteq \mathbb{R}\), \(X_A\) denotes the space with the underlying set \(X\) and with the topology \(\{U \cup K : U \in \tau_X, K \subseteq X \setminus A\}\), where \(\tau_X\) is the subspace topology on \(X\) induced from \(\tau\). It is known that \(X_A\) is a paracompact Hausdorff space (see [5, Example 5.1.22]). We begin by determining when the subspace \(A\) satisfies \((b_i)\), \(i = 1, 2, 3\), in \(X_A\) in terms of subsets of \(\mathbb{R}\).

**Lemma 3.1.** Let \(A \subseteq X \subseteq \mathbb{R}\) and \(S \subseteq X\). Then:

1. \(S\) is a cozero-set in \(X_A\) if and only if there exist an open set \(U\) in \(\mathbb{R}\) and an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(S \cap A \subseteq U \cap X \subseteq S\), \(S \subseteq F\) and \(F \cap A = S \cap A\).
2. If \(S\) is a Baire set in \(X_A\), then there exists a Borel set \(B\) in \(\mathbb{R}\) such that \(S \subseteq B\) and \(B \cap A = S \cap A\).
3. If \(S\) is a Suslin-Z-set in \(X_A\), then there exists an analytic set \(B\) in \(\mathbb{R}\) such that \(S \subseteq B\) and \(B \cap A = S \cap A\).

**Proof.**  (1) First, observe that (i) a set \(S \subseteq X\) is open in \(X_A\) if and only if there exists an open set \(U\) in \(\mathbb{R}\) with \(S \cap A \subseteq U \cap X \subseteq S\), and (ii) a set \(S \subseteq X\) is closed in \(X_A\) if and only if there exists a closed set \(F\) in \(\mathbb{R}\) such that \(S \subseteq F\) and \(S \cap A = F \cap A\). Now, (1) follows from (i) and (ii) since, by the normality of \(X_A\), \(S \in Coz(X_A)\) if and only if \(S\) is an open \(F_\sigma\)-set in \(X_A\).

(2) Let \(S\) be the family of all sets \(S \subseteq X\) such that there exist Borel sets \(B\) and \(B'\) in \(\mathbb{R}\) such that \(S \cap A \subseteq B \cap X \subseteq S\), \(S \subseteq B'\) and \(B' \cap A = S \cap A\). Then \(S\) is a \(\sigma\)-algebra of subsets of \(X\), and \(Coz(X_A) \subseteq S\) by (1). Hence, all Baire sets in \(X_A\) belong to \(S\), from which (2) can be deduced.

(3) If \(S\) is a Suslin-Z-set in \(X_A\), then there exists \(\{Z_\sigma : \sigma \in <\omega_\omega\} \subseteq Z(X_A)\) such that \(S = \bigcup_{\tau \in <\omega} \bigcap_{n \in \omega} Z_{\tau[n]}\). Define \(B = \bigcup_{\tau \in <\omega} \bigcap_{n \in \omega} \text{cl}_R Z_{\tau[n]}\). Then \(B\) is an analytic set in \(\mathbb{R}\) with \(S \subseteq B\). Since \(\text{cl}_R Z_\sigma \cap A = Z_\sigma \cap A\) for each \(\sigma \in <\omega_\omega\), \(B \cap A = S \cap A\).

**Proposition 3.2.** Let \(A \subseteq X \subseteq \mathbb{R}\). Then:

1. \(A\) satisfies \((b_1)\) in \(X_A\) if and only if for every analytic set \(B\) in \(\mathbb{R}\) with \(B \cap A = \emptyset\), there exists an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(B \cap X \subseteq F\) and \(F \cap A = \emptyset\).
2. \(A\) satisfies \((b_2)\) in \(X_A\) if and only if for every Borel set \(B\) in \(\mathbb{R}\) with \(B \cap A = \emptyset\), there exists an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(B \cap X \subseteq F\) and \(F \cap A = \emptyset\).
(3) A satisfies \((b_3)\) in \(X_A\) if and only if for every countable family \(\{F_n : n < \omega\}\) of \(F_\sigma\)-sets in \(\mathbb{R}\) such that \(\bigcap_{n<\omega} F_n \cap A = \emptyset\) and \(F_n \cap A\) is open in \(A\) for each \(n < \omega\), there exists an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(\bigcap_{n<\omega} F_n \cap X \subseteq F\) and \(F \cap A = \emptyset\).

**Proof.** (1) Assume that \(A\) satisfies \((b_1)\) in \(X_A\) and let \(B\) be an analytic set in \(\mathbb{R}\) with \(B \cap A = \emptyset\). Since the topology of \(\mathbb{R}_A\) is finer than that of \(\mathbb{R}\), \(B\) is a Suslin-\(Z\)-set in \(\mathbb{R}_A\), and hence, \(B \cap X\) is also a Suslin-\(Z\)-set in \(X_A\). Thus, it follows from \((b_1)\) that there exists \(U \in Coz(X_A)\) such that \(B \cap X \subseteq U\) and \(U \cap A = \emptyset\). By Lemma 3.1(1), there exists an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(U \subseteq F\) and \(F \cap A = U \cap A\). Then \(B \cap X \subseteq F\) and \(F \cap A = \emptyset\). Conversely, assume that \(A\) satisfies the latter condition in (1) and let \(C\) be a Suslin-\(Z\)-set in \(X_A\) with \(C \cap A = \emptyset\). Then, by Lemma 3.1(3), there exists an analytic set \(H\) in \(\mathbb{R}\) such that \(C \subseteq H\) and \(H \cap A = \emptyset\). By the assumption, we can find an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(H \cap X \subseteq F\) and \(F \cap A = \emptyset\). Since \(F \cap X \in Coz(X_A)\) by Lemma 3.1(1), \(A\) satisfies \((b_1)\) in \(X_A\). (2) can be proved similarly to (1) using Lemma 3.1(2) instead of Lemma 3.1(3).

(3) Assume that \(A\) satisfies \((b_3)\) in \(X_A\) and let \(\{F_n : n < \omega\}\) be a countable family of \(F_\sigma\)-sets in \(\mathbb{R}\) such that \(\bigcap_{n<\omega} F_n \cap A = \emptyset\) and \(F_n \cap A\) is open in \(A\) for each \(n < \omega\). For each \(n < \omega\), since \(X_A\) is normal, we can find \(E_n \in Coz(X_A)\) such that \(F_n \cap X \subseteq E_n\) and \(E_n \cap A = F_n \cap A\). Since \(\bigcap_{n<\omega} E_n \cap A = \emptyset\), it follows from \((b_3)\) that there exists \(U \in Coz(X_A)\) such that \(\bigcap_{n<\omega} E_n \subseteq U\) and \(U \cap A = \emptyset\). By Lemma 3.1(1), there exists an \(F_\sigma\)-set \(F\) in \(\mathbb{R}\) such that \(U \subseteq F\) and \(F \cap A = U \cap A\). Then

\[
\bigcap_{n<\omega} F_n \cap X \subseteq \bigcap_{n<\omega} E_n \subseteq U \subseteq F
\]

and \(F \cap A = \emptyset\). Conversely, assume that \(A\) satisfies the latter condition in (3), and take \(\{G_n : n < \omega\} \subseteq Coz(X_A)\) such that \(\bigcap_{n<\omega} G_n \cap A = \emptyset\). For each \(n < \omega\), by Lemma 3.1(1), there exists an \(F_\sigma\)-set \(H_n\) in \(\mathbb{R}\) such that \(G_n \subseteq H_n\) and \(H_n \cap A = G_n \cap A\). Since \(\bigcap_{n<\omega} H_n \cap A = \emptyset\) and \(H_n \cap A\) is open in \(A\) for each \(n < \omega\), it follows from our assumption that there exists an \(F_\sigma\)-set \(H\) in \(\mathbb{R}\) such that \(\bigcap_{n<\omega} H_n \cap X \subseteq H\) and \(H \cap A = \emptyset\). Then \(H \cap X \in Coz(X_A)\) by Lemma 3.1(1), \(\bigcap_{n<\omega} G_n \subseteq H \cap X\) and \((H \cap X) \cap A = \emptyset\). Hence, \(A\) satisfies \((b_3)\) in \(X_A\). ■

Now, we are in a position to construct examples. A subspace \(A\) of a space \(X\) is said to be \(P\)-embedded in \(X\) if it is \(P\)-\(\gamma\)-embedded in \(X\) for every \(\gamma\). \(M\)- and \(P\)(point-finite)-embeddings are defined similarly. If \(A \subseteq X \subseteq \mathbb{R}\), then the closed subspace \(A\) of \(X_A\) is always \(P\)-embedded in \(X_A\), since \(X_A\) is paracompact. The last statement of the following example was proved by the second author in [24]; however, now it is an immediate consequence of Proposition 3.2(3) and Theorem 2.1.
Example 3.3. Let $\mathbb{Q}$ be the set of rational numbers. Then $\mathbb{Q}$ fails to satisfy $(b_3)$ in $\mathbb{R}_\mathbb{Q}$. Hence, $\mathbb{Q}$ is not $P^\omega$(point-finite)-embedded in $\mathbb{R}_\mathbb{Q}$.

Example 3.3 shows that $(c) \not\equiv (b_3)$ in general. Recall from [5, 5.5.4] that there exists a set $A \subseteq \mathbb{R}$, called a Bernstein set, such that every compact set in $\mathbb{R}$ contained in either $A$ or $\mathbb{R} \setminus A$ is countable.

Example 3.4. Let $A$ be a Bernstein set in $\mathbb{R}$. Then $A$ satisfies $(b_1)$ in $\mathbb{R}_A$ but fails to satisfy $(a)_\omega$ in $\mathbb{R}_A$. Hence, $A \times Y$ is $P$(point-finite)-embedded in $\mathbb{R}_A \times Y$ for every compact Hausdorff space $Y$, but $A$ is not $M^\omega$-embedded in $\mathbb{R}_A$.

Proof. Let $B$ be an analytic set in $\mathbb{R}$ with $B \cap A = \emptyset$. Then $B$ must be countable, since every uncountable analytic set in $\mathbb{R}$ contains a Cantor set (see [10, Theorem 94]). By Proposition 3.2(1), this implies that $A$ satisfies $(b_1)$ in $\mathbb{R}_A$. On the other hand, the Euclidean metric $d$ on $\mathbb{R}$ is an $\omega$-separable continuous pseudometric on $\mathbb{R}_A$ and \{ $x \in \mathbb{R}_A$ : ($\exists y \in A$)(d(x, y) = 0) \} = A. Since $A$ is not a zero-set in $\mathbb{R}_A$, $A$ does not satisfy $(a)_\omega$ in $\mathbb{R}_A$ (see also [18, Corollary 5 to Theorem 1]).

Example 3.5. Under CH, there exist sets $A$ and $X$ with $A \subseteq X \subseteq \mathbb{R}$ such that $A$ satisfies $(b_2)$ in $X_A$ but fails to satisfy $(b_1)$ in $X_A$. Hence, $A$ is $P$(point-finite)-embedded in $X_A$, but $A \times [0, 1]$ is not $P^\omega$(point-finite)-embedded in $X \times [0, 1]$.

Proof. By [10, Corollary to Lemma 39.4], there exists an analytic set $B$ in $\mathbb{R}$ such that $\mathbb{R} \setminus B$ is not analytic. Put $A = \mathbb{R} \setminus B$ and let $B$ be the family of all Borel sets in $\mathbb{R}$ containing $A$. Since $|B| = 2^\omega$, we can enumerate $B$ as \{ $B_\alpha : \alpha < \omega_1$ \} by CH. Then $\bigcap_{\beta < \alpha} B_\beta \cap B$ is uncountable for each $\alpha < \omega_1$, because $A$ is not a Borel set. Thus, we can choose inductively a point

$$x_\alpha \in \left( \bigcap_{\beta < \alpha} B_\beta \cap B \right) \setminus \{ x_\beta : \beta < \alpha \}$$

for each $\alpha < \omega_1$. Put $X = A \cup \{ x_\alpha : \alpha < \omega_1 \}$. Then, since $X \setminus B_\alpha$ is countable for each $\alpha < \omega$, it follows from Proposition 3.2(2) that $A$ satisfies $(b_2)$ in $X_A$. On the other hand, since $B$ is an analytic set in $\mathbb{R}$ and $B_\alpha \cap B = \emptyset$ for each $\alpha < \omega_1$, Proposition 3.2(1) shows that $A$ does not satisfy $(b_1)$ in $X_A$.

Remark 3.6. Let $X_A$ be the space defined in Example 3.5, and let $\Omega = \omega_1 + 1$ with the usual order topology. Now, by proving that $A \times \Omega$ is $P$(point-finite)-embedded in $X_A \times \Omega$, we show that the assumption of metrizability of $Y$ is essential in condition (3) of Theorem 2.6. By Lemma 2.2, $A \times \Omega$ is $P$-embedded in $X_A \times \Omega$. Thus, by Theorem 2.1, it suffices to show that $A \times \Omega$ satisfies $(b_3)$ in $X_A \times \Omega$. Take a countable family \{ $G_n : n < \omega$ \} $\subseteq \mathcal{C}oz(X_A \times \Omega)$ with $\bigcap_{n < \omega} G_n \cap (A \times \Omega) = \emptyset$. Put $A_n = \{ x \in A : \langle x, \omega_1 \rangle \not\in G_n \}$ for each $n < \omega$. Since each $A_n$ is separable and each $G_n$ is an $F_\sigma$-set, we can
find \( \alpha < \omega_1 \) such that \( G_n \cap (A_n \times (\Omega \setminus \alpha)) = \emptyset \) for each \( n < \omega \). Here, we may assume that \( \alpha \) is an isolated ordinal. For each \( n < \omega \), put
\[ H_n = \text{pr}_{X_A}[G_n \cap (X_A \times (\Omega \setminus \alpha))]. \]
Then \( H_n \in \text{Coz}(X_A) \) by Lemma 2.4(2), and \( \bigcap_{n<\omega} H_n \cap A = \emptyset \) as \( H_n \cap A_n = \emptyset \) for each \( n < \omega \). Since \( A \) satisfies \((b_3)\) in \( X_A \), there exists \( U \in \text{Coz}(X_A) \) such that \( \bigcap_{n<\omega} H_n \subseteq U \) and \( U \cap A = \emptyset \). On the other hand, since \( \alpha \) is countable compact metrizable, it follows from Corollary 2.3 and Theorem 2.1 that there exists \( V \in \text{Coz}(X_A \times \alpha) \) such that \( \bigcap_{n<\omega} G_n \cap (X_A \times \alpha) \subseteq V \) and \( V \cap (A \times \alpha) = \emptyset \). Finally, putting \( W = (U \times (\Omega \setminus \alpha)) \cup V \), we obtain a cozero-set \( W \) in \( X_A \times \Omega \) such that \( \bigcap_{n<\omega} G_n \subseteq W \) and \( W \cap (A \times \Omega) = \emptyset \). Hence, \( A \times \Omega \) satisfies \((b_3)\) in \( X_A \).

**Example 3.7.** Under CH, there exist sets \( A \) and \( X \) with \( A \subseteq X \subseteq \mathbb{R} \) such that \( A \) satisfies \((b_3)\) in \( X_A \) but fails to satisfy \((b_2)\) in \( X_A \).

**Proof.** Following [10], \( \Sigma^0_3 \) denotes the family of all sets which can be written as the union of countably many \( G_\beta \)-sets in \( \mathbb{R} \), and \( \Pi^0_4 \) denotes the family of all sets which can be written as the intersection of countably many members of \( \Sigma^0_3 \). By [10, Corollary to Lemma 39.1] there exists a Borel set \( A \) in \( \mathbb{R} \) such that \( A \notin \Pi^0_4 \). Now, let \( \mathcal{B} \) be the family of all members of \( \Pi^0_4 \) containing \( A \). Since \( |\mathcal{B}| = 2^\omega \), we can enumerate \( \mathcal{B} \) as \( \{B_\alpha : \alpha < \omega_1\} \) by CH. Then \( \bigcap_{\beta<\alpha} B_\beta \setminus A \) is uncountable for each \( \alpha < \omega_1 \), because \( A \notin \Pi^0_4 \). Hence, we can define a set \( X = A \cup \{x_\alpha : \alpha < \omega_1\} \) similarly to the proof of Example 3.5. Since \( X \setminus B_\alpha \) is countable for each \( \alpha < \omega_1 \), it follows from Proposition 3.2(3) that \( A \) satisfies \((b_3)\) in \( X_A \). On the other hand, \( \mathbb{R} \setminus A \) is a Borel set in \( \mathbb{R} \), but \( (\mathbb{R} \setminus A) \cap B_\alpha \neq \emptyset \) for each \( \alpha < \omega_1 \). Hence, \( A \) does not satisfy \((b_2)\) in \( X_A \) by Proposition 3.2(2).  

A similar example to Examples 3.5 and 3.7 was constructed by Michael [12] for a countable non-\( G_\delta \)-set \( A \) to show that the product of a Lindelöf space \( X_A \) with \( \mathbb{P} \) is not necessarily normal under CH.  

In [15, Example 3], Przymusiński and Wage constructed an example of a collectionwise normal space \( Z \) having a closed subspace \( K \) which is not \( P^\omega(\text{locally finite}) \)-embedded in \( Z \). Finally, we show that an \( M \)-embedded subspace is not necessarily \( P^\omega(\text{locally finite}) \)-embedded by proving the following:

**Example 3.8.** Every closed subspace \( A \) of the collectionwise normal space \( Z \) of Przymusiński–Wage is \( M \)-embedded in \( Z \).

**Proof.** The space \( Z \) is constructed from a subspace \( W \) of Rudin’s Dowker space of [17]. All we need to know about \( Z \) is that every \( G_\delta \)-set in \( W \) is open and that \( Z \) is the union of \( W \) and another space \( Y \), where \( W \) is a \( G_\delta \)-set in \( Z \) and \( Y \) is an open (in \( Z \)) set which is the topological sum of subspaces of \( W \).
From these facts, if a set $G$ is the union of $G_\delta$-sets in $Z$, then both $G \cap W$ and $G \cap Y$ are $G_\delta$-sets in $Z$, and therefore, $G$ is a $G_\delta$-set in $Z$. Now, let $A$ be a closed subspace of $Z$. Since $Z$ is collectionwise normal, it follows from [19, Theorem 5.2] that $A$ is $P$-embedded in $Z$. To show that $A$ satisfies $(a_\gamma)$ in $Z$ for every infinite cardinal $\gamma$, let $\rho$ be a $\gamma$-separable continuous pseudometric on $Z$. Then the set $L = \{ x \in Z : (\exists y \in A)(\rho(x, y) = 0) \}$ is a $G_\delta$-set in $Z$ since it is the union of $G_\delta$-sets in $Z$. Thus, by the normality of $Z$, there exists a zero-set $F$ in $Z$ such that $A \subseteq F \subseteq L$. Hence, $A$ is $M$-embedded in $Z$. 

4. Another application and questions. By AR we mean an absolute retract for the class of metrizable spaces. In [14] Morita proved that a subspace $A$ of a space $X$ is $P^{\gamma}$-embedded in $X$ if and only if for every complete AR $Y$ with $w(Y) \leq \gamma$, every continuous map from $A$ to $Y$ extends continuously over $X$. As another application of Theorem 2.1, we prove the following theorem by a similar argument to the proofs of Morita’s theorems in [14] (see also [9, Theorems 2.8 and 2.14]). We now call a metrizable space $X$ $\sigma$-complete if there exist a metric $d$ on $X$, which induces the topology of $X$, and a countable cover $\{ X_n : n < \omega \}$ of $X$ such that each $X_n$ is a complete subspace of the metric space $(X,d)$.

**Theorem 4.1.** Let $A$ be a subspace of a space $X$ and $\gamma$ an infinite cardinal. Then the following are equivalent:

1. $A$ is $P^{\gamma}$ (point-finite)-embedded in $X$,
2. for every $\sigma$-complete AR $Y$ with $w(Y) \leq \gamma$, every continuous map from $A$ to $Y$ extends continuously over $X$,
3. for every Banach space $B$ and every convex $F_\sigma$-set $Y$ in $B$ with $w(Y) \leq \gamma$, every continuous map from $A$ to $Y$ extends to a continuous map from $X$ to $Y$.

**Proof.** $(1) \Rightarrow (2)$: Let $f : A \to Y$ be a continuous map to a $\sigma$-complete AR $Y$ with $w(Y) \leq \gamma$. We consider $Y$ a metric space having a countable cover by complete subspaces. Then, by Kuratowski–Wojdysławski’s theorem (see [9]), there exist a Banach space $B$ and an isometrical embedding $i : Y \to B$ such that $w(Z) \leq \gamma$, where $Z$ is the convex hull of $i[Y]$. We identify $Y$ and $i[Y]$. Since $A$ is $P^{\gamma}$-embedded in $X$ and $w (\text{cl}_B Z) \leq \gamma$, $f$ extends to a continuous map $g : X \to B$ with $g[X] \subseteq \text{cl}_B Z$ by Morita’s theorem mentioned above. Since $Y$ is an $F_\sigma$-set in $B$, $g^{-1}[Y]$ is a countable union of zero-sets in $X$ such that $A \subseteq g^{-1}[Y]$. Since $A$ is $P^{\gamma}$ (point-finite)-embedded in $X$, it follows from Theorem 2.1 that there exists a continuous function $\varphi : X \to [0, 1]$ such that the set $F = \varphi^{-1}(0)$ satisfies $A \subseteq F \subseteq g^{-1}[Y]$. Consider the diagonal map.
\[ h = g \triangledown \varphi : X \to B \times [0, 1] \]

and let \( p, q \) denote the projections of \( B \times [0, 1] \) onto \( B \) and \( [0, 1] \), respectively. Then \( h[F] = h[X] \cap q^{-1}(0) \) is closed in \( h[X] \) and \( p[h[F]] = g[F] \subseteq Y \). Since \( Y \) is an AR, the restriction \( p[h[F]] \) can be extended to a continuous map \( p^* : h[X] \to Y \). Then \( p^* \circ h : X \to Y \) is a continuous extension of \( (p \circ h)|_A = g|_A = f \).

The implication \((2) \Rightarrow (3)\) follows from the fact that every convex \( F_\sigma \)-set in a Banach space is a \( \sigma \)-complete AR. For a set \( S \), let \( \ell_1(S) \) be the Banach space of all real-valued functions \( v \) on \( S \) such that \( \|v\| = \sum_{s \in S} |v(s)| < \infty \), and \( \Delta_S \) the subspace of \( \ell_1(S) \) consisting of all \( v \in \ell_1(S) \) such that \( v(s) = 0 \) for all but finitely many \( s \in S \), \( v \geq 0 \), and \( \sum_{s \in S} v(s) = 1 \). Dydak [3] proved that \( A \) is \( P^\gamma \)(point-finite)-embedded in \( X \) if (and only if) for every set \( S \) with \( |S| \leq \gamma \), every continuous map from \( A \) to \( \Delta_S \) extends to a continuous map from \( X \) to \( \Delta_S \). Since \( \Delta_S \) is a convex \( F_\sigma \)-set in \( \ell_1(S) \), we have the final implication \((3) \Rightarrow (1)\).

** Remark 4.2.** By Hausdorff’s extension theorem, a metrizable space is \( \sigma \)-complete if and only if it has a countable cover by closed completely metrizable subspaces. The term “\( \sigma \)-complete” was used by A. H. Stone in [20, Lemma 4] without an explicit definition.

We conclude the paper with some open questions.

** Question 4.3.** Does there exist an example in ZFC of a \( P \)-embedded subspace which satisfies \((b_3)\) but not \((b_2)\)? Does there exist an example in ZFC of a \( P \)-embedded subspace which satisfies \((b_2)\) but not \((b_1)\)?

The next question was first asked by the second author in [22, Problem 2.3.4], which asks if there is a \( P^\gamma \)(locally finite)-embedding analogue of Theorem 2.1.

** Question 4.4.** Let \( A \) be a subspace of a space \( X \) and \( \gamma \) an uncountable cardinal. Is then \( A \) \( P^\gamma \)(locally finite)-embedded in \( X \) if \( A \) is \( P^\gamma \)- and \( P^\omega \)(locally finite)-embedded in \( X \)?

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