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Extension of point-finite partitions of unity

by

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Abstract. A subspace A of a topological space X is said to be P^γ-embedded (P^γ(point-finite)-embedded) in X if every (point-finite) partition of unity α on A with $|\alpha| \le \gamma$ extends to a (point-finite) partition of unity on X. The main results are: (Theorem A) A subspace A of X is P^γ(point-finite)-embedded in X iff it is P^γ-embedded and every countable intersection B of cozero-sets in X with $B \cap A = \emptyset$ can be separated from A by a cozero-set in X. (Theorem B) The product $A \times [0,1]$ is P^γ(point-finite)-embedded in $X \times [0,1]$ iff $A \times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \le \gamma$ iff A is P^γ-embedded in X and every subset B of X obtained from zero-sets by means of the Suslin operation, with $B \cap A = \emptyset$, can be separated from A by a cozero-set in X. These characterizations are used to answer certain questions of Dydak. In particular, it is shown that, assuming CH, the property of $A \times [0,1]$ to be P^γ(point-finite)-embedded in $X \times [0,1]$ is stronger than that of A being P^γ(point-finite)-embedded in X.

1. Introduction. By a space we mean a topological space. A partition of unity α on a space X is called *point-finite* (resp. *locally finite*) if the family $\{\mathrm{coz}(f) : f \in \alpha\}$ is point-finite (resp. locally finite) in X, where $\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}$. Let A be a subspace of a space X and γ an infinite cardinal. When $\alpha = \{f_\lambda\}$ and $\beta = \{g_\lambda\}$ are partitions of unity on A and X, respectively, we say that β is an *extension* of α if $f_\lambda = g_\lambda|_A$ for each λ. Dydak [3] defined A to be P^γ*(point-finite)-embedded* (resp. P^γ*(locally finite)-embedded*) in X if every point-finite (resp. locally finite) partition of unity α on A, with $|\alpha| \le \gamma$, extends to a point-finite (resp. locally finite) partition of unity on X. Extensive studies of P^γ(locally finite)-embedding have been made by Dydak [3], [4] and the second author [21] and [23].

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In this paper, we consider P^γ(point-finite)-embeddings. In particular, we prove Theorems A and B stated in the abstract and apply them to answer Dydak's questions concerning P^γ(point-finite)-embeddings stated below.

Recall from [3] that A is P^γ*-embedded* in X if every partition of unity α on A with $|\alpha| \le \gamma$ extends to a partition of unity on X (see [2], [19] for the original definition of P^γ-embedding). Recall from [18] that A is M^γ-embedded in X if for every AR Y with $w(Y) \le \gamma$, every continuous map from A to Y extends continuously over X. It is known (see [3]) that these extension properties are related as follows, where $A \to B$ means that every A-embedded subspace is B-embedded:

$$M^\gamma \to P^\gamma(\text{point-finite}) \to P^\gamma \overset{(*)}{\longleftarrow} P^\gamma(\text{locally finite}).$$

Przymusiński–Wage [15] showed that the arrow $(*)$ cannot be reversed by giving an example of a collectionwise normal space Z having a closed subspace which is not P^ω(locally finite)-embedded, and Dydak [3] showed that the implication "P^γ(locally finite) $\to$ P^γ(point-finite)" is not true in general (see also [24]). In Section 3, we give an example of a subspace which is P^γ(point-finite)-embedded for every γ but not M^ω-embedded (Example 3.4), and prove that every closed subspace of the space Z of Przymusiński–Wage mentioned above is M^γ-embedded for every γ (Example 3.8). These results answer Dydak's questions [3, Problems 12.10 and 12.11] negatively. Moreover, Dydak [3, Problem 13.6] asked: If A is P^γ(point-finite)-embedded in X, is then $A \times [0,1]$ P^γ(point-finite)-embedded in $X \times [0,1]$? It is known that the answers to the similar questions for P^γ-, M^γ- and P^γ(locally finite)-embeddings are all positive (see Alò–Sennott [1], Sennott [18] and Yamazaki [21], respectively). As an application of Theorem B, we show that the answer is negative for P^γ(point-finite)-embeddings under the assumption of the continuum hypothesis (Examples 3.5 and 3.7).

For a set A, $|A|$ denotes the cardinality of A. As usual, a cardinal is an initial ordinal and an ordinal is identified with the set of smaller ordinals. Let ω denote the first infinite cardinal and ω_1 the first uncountable cardinal. Our terminology and notation follow [5] and [13].

2. P^γ(point-finite)-embeddings and products. A *zero-set* in a space X is a set of the form $f^{-1}(0)$ for some real-valued continuous function f on X and a *cozero-set* is the complement of a zero-set. For a space X, let $\mathcal{Z}(X)$ (resp. $\mathcal{C}oz(X)$) denote the family of all zero-sets (resp. cozero-sets) in X. A set $A \subseteq X$ is called a *Suslin-$\mathcal{Z}$-set* in X if there exists a family $\{Z_\sigma : \sigma \in {}^{<\omega}\omega\} \subseteq \mathcal{Z}(X)$ such that $A = \bigcup_{t \in {}^\omega\omega} \bigcap_{n<\omega} Z_{t|n}$, where ${}^\alpha\omega$ denotes the set of all maps from α to ω and ${}^{<\omega}\omega = \bigcup_{n<\omega} {}^n\omega$ (see [16]). All Baire sets, i.e., members of the smallest σ-algebra including $\mathcal{Z}(X)$, are Suslin-$\mathcal{Z}$-sets. As usual, we call a Suslin-$\mathcal{Z}$-set and a Baire set in a metric space an *analytic* *set* and a *Borel set*, respectively. Now, we consider the following conditions on a subspace A of a space X:

(b_1) For every Suslin-$\mathcal{Z}$-set B in X with $B \cap A = \emptyset$, there exists $U \in \mathcal{C}oz(X)$ such that $B \subseteq U$ and $U \cap A = \emptyset$.
(b_2) For every Baire set B in X with $B \cap A = \emptyset$, there exists $U \in \mathcal{C}oz(X)$ such that $B \subseteq U$ and $U \cap A = \emptyset$.
(b_3) For every countable family $\{G_n : n < \omega\} \subseteq \mathcal{C}oz(X)$ with $\bigcap_{n<\omega} G_n \cap A = \emptyset$, there exists $U \in \mathcal{C}oz(X)$ such that $\bigcap_{n<\omega} G_n \subseteq U$ and $U \cap A = \emptyset$.

Evidently, (b_1) implies (b_2) and (b_2) implies (b_3) (see Remark 2.8 below). Now, we prove the theorems announced in the abstract.

THEOREM 2.1. *Let A be a subspace of a space X and γ an infinite cardinal. Then the following are equivalent*:

(1) *A is P^γ(point-finite)-embedded in X*,
(2) *A is P^γ-embedded in X and P^ω(point-finite)-embedded in X*,
(3) *A is P^γ-embedded in X and satisfies* (b_3) *in X.*

Proof. (1)⇒(2): Obvious. (2)⇒(3): To prove that A satisfies (b_3) in X, take a countable family $\{G_n : n < \omega\} \subseteq \mathcal{C}oz(X)$ with $\bigcap_{n<\omega} G_n \cap A = \emptyset$. We may assume that $G_{n+1} \subseteq G_n$ for each $n < \omega$ and $G_0 = X$. Take continuous functions $f_n : X \to [0, 1/2^n]$, $n < \omega$, with $G_n = \mathrm{coz}(f_n)$, and define $f = \sum_{n<\omega} f_n$. Note that $f(x) > 0$ for all $x \in X$. For each $n < \omega$, define a function $f_n^* : A \to [0,1]$ by $f_n^*(x) = f_n(x)/f(x)$ for $x \in A$. Then $\{f_n^* : n < \omega\}$ is a point-finite partition of unity on A. Since A is P^ω(point-finite)-embedded in X, there exists a point-finite partition of unity $\{g_n : n < \omega\}$ on X such that $g_n|_A = f_n^*$ for each $n < \omega$. Let

$$U = \bigcup_{n<\omega} \{x \in X : g_n(x) \cdot f(x) \neq f_n(x)\}.$$

Then $U \in \mathcal{C}oz(X)$, $\bigcap_{n<\omega} G_n \subseteq U$ since $\{g_n : n < \omega\}$ is point-finite, and $U \cap A = \emptyset$. Hence, A satisfies (b_3) in X.

(3)⇒(1): Let $\alpha = \{f_\lambda : \lambda \in \Lambda\}$ be a point-finite partition of unity on A with $|\Lambda| \leq \gamma$. Since A is P^γ-embedded in X, α extends to a partition of unity $\beta = \{g_\lambda : \lambda \in \Lambda\}$ on X. Let $B = \{x \in X : \beta \text{ is not point-finite at } x\}$. We show that B is the countable intersection of cozero-sets in X. For each $n < \omega$ and each $x \in X$, define

$$k_n(x) = \max\Big\{ \sum_{\lambda\in\delta} g_\lambda(x) : \delta \subseteq \Lambda,\ |\delta| \leq n\Big\}.$$

Since β is a partition of unity, the functions $k_n : X \to [0,1]$, $n < \omega$, are continuous, and for each $x \in X$, $|\{\lambda \in \Lambda : g_\lambda(x) > 0\}| \leq n$ if and only if $k_n(x) = 1$. This implies that $X \setminus B = \bigcup_{n<\omega} k_n^{-1}(1)$, and hence, B is the intersection of countably many cozero-sets in X. Since $B \cap A = \emptyset$ and A satisfies (b_3) in X, we can find a continuous function $h : X \to [0,1]$ such that $B \subseteq \mathrm{coz}(h)$ and $\mathrm{coz}(h) \cap A = \emptyset$. For each $\lambda \in \Lambda$, define a function g_λ^* on X by $g_\lambda^*(x) = \max\{g_\lambda(x) - h(x), 0\}$ for $x \in X$. Then $\{\mathrm{coz}(g_\lambda^*) : \lambda \in \Lambda\}$ is point-finite in X, because if $h(x) = 0$, then $g_\lambda^*(x) = g_\lambda(x)$ and $x \notin B$; and if $h(x) > 0$, then only finitely many g_λ's exceed h at x since $\sum_{\lambda \in \Lambda} g_\lambda(x) = 1$. Since $g_\lambda^*(x) \le g_\lambda(x)$ for each $\lambda \in \Lambda$ and each $x \in X$, it follows from [4, Corollary 2.6] that the function $\sum_{\lambda \in \Lambda} g_\lambda^*$ is continuous. Fix an arbitrary $\mu \in \Lambda$ and define

$$g_\mu^{**}(x) = g_\mu^*(x) + 1 - \sum_{\lambda \in \Lambda} g_\lambda^*(x) \quad \text{for } x \in X.$$

Finally, putting $g_\lambda^{**} = g_\lambda^*$ for each $\lambda \in \Lambda \setminus \{\mu\}$, we obtain a point-finite partition of unity $\{g_\lambda^{**} : \lambda \in \Lambda\}$ on X extending α. Hence, A is P^γ(point-finite)-embedded in X. ∎

We turn to considering the problem when $A \times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for all (or certain) compact Hausdorff spaces Y. We need the following result due to Alò and Sennott [1] as a lemma.

LEMMA 2.2 (Alò–Sennott). *Let A be a P^γ-embedded subspace of a space X, where γ is an infinite cardinal. Then $A \times Y$ is P^γ-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \le \gamma$.*

Since the countable union of cozero-sets is a cozero-set, we have the following corollary from Theorem 2.1 and Lemma 2.2.

COROLLARY 2.3. *Let A be a P^γ(point-finite)-embedded subspace of a space X, where γ is an infinite cardinal. Then $A \times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for every countable, compact metric space Y.*

The next lemma is well known, but we can find no good reference.

LEMMA 2.4 (folklore). *Let X and Y be spaces and $\mathrm{pr}_X : X \times Y \to X$ the projection.*

(1) *If Y is separable, then pr_X carries cozero-sets to cozero-sets.*
(2) *If Y is compact, then pr_X carries cozero-sets to cozero-sets and carries zero-sets to zero-sets.*
(3) *If Y is compact, then pr_X carries Suslin-$\mathcal{Z}$-sets to Suslin $\mathcal{Z}$-sets.*

Proof. (1) Let D be a countable dense set in Y. Then, for every cozero-set G in $X \times Y$, $\mathrm{pr}_X[G] = \bigcup_{y \in D}\{x \in X : \langle x, y\rangle \in G\} \in \mathcal{C}oz(X)$.

(2) This follows from the fact that if Y is compact, then for every real-valued continuous function h on $X \times Y$, the functions f and g on X defined by $f(x) = \sup\{h(x,y) : y \in Y\}$ and $g(x) = \inf\{h(x,y) : y \in Y\}$ for $x \in X$ are continuous (see [6, Lemma 1.1]).

(3) This is a consequence of (2) since we can assume that $Z_{t|n} \subseteq Z_{t|m}$ whenever $m < n$ in the definition of a Suslin-$\mathcal{Z}$-set. ∎

The next lemma is due to the referee's suggestion.

LEMMA 2.5. *For every Suslin-$\mathcal{Z}$-set B in a space X, there exists a continuous map $f : X \to Q$, where Q is the Hilbert cube, such that $B = f^{-1}[S]$ for some analytic set S in Q.*

Proof. This is a consequence of the following observation: For any collection of countably many zero-sets $Z_n = f_n^{-1}(0)$, $n < \omega$, in X, consider the diagonal map $f = \triangle_{n<\omega} f_n : X \to Q$. Then each Z_n is the inverse image of a closed set in Q. ∎

Now, combining Lemma 2.5 with Theorem 2.1, Lemmas 2.2 and 2.4, we have the following theorem.

THEOREM 2.6. *Let A be a subspace of a space X and γ an infinite cardinal. Then the following are equivalent*:

(1) *$A \times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$,*
(2) *$A \times [0,1]$ is P^γ(point-finite)-embedded in $X \times [0,1]$,*
(3) *$A \times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for some uncountable, compact metric space Y,*
(4) *A is P^γ-embedded in X and satisfies (b_1) in X.*

Proof. (1)⇒(2)⇒(3): Obvious. (3)⇒(4): Assume that $A\times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for some uncountable, compact metric space Y. As every P^γ(point-finite)-embedded subspace is P^γ-embedded, it suffices to show that A satisfies (b_1) in X. Let B be a Suslin-$\mathcal{Z}$-set in X with $B \cap A = \emptyset$. Then, by Lemma 2.5, there exists a continuous map $f : X \to Q$ such that $B = f^{-1}[S]$ for some analytic set S in Q. It is known that S is the projection of a G_δ-set G in $Q \times \mathbb{K}$, where $\mathbb{K}$ is the Cantor set (see [11]). Since $\mathbb{K}$ can be embedded in Y, we regard G as a G_δ-set in $Q \times Y$. Put $H = (f \times \mathrm{id}_Y)^{-1}[G]$, where id_Y is the identity of Y. Then H is the intersection of countably many cozero-sets in $X \times Y$ and $B = \mathrm{pr}_X[H]$. Since $A \times Y$ satisfies (b_3) in $X \times Y$ by Theorem 2.1, there exists $U \in \mathcal{C}oz(X \times Y)$ such that $H \subseteq U$ and $U \cap (A \times Y) = \emptyset$. Finally, put $V = \mathrm{pr}_X[U]$. Then $V \in \mathcal{C}oz(X)$ by Lemma 2.4(2), $B \subseteq V$ and $V \cap A = \emptyset$. Hence, A satisfies (b_1) in X.

(4)⇒(1): Let Y be a compact Hausdorff space with $w(Y) \leq \gamma$. By Theorem 2.1 and Lemma 2.2, it suffices to show that $A \times Y$ satisfies (b_3) in $X \times Y$. Let B be the intersection of countably many cozero-sets in $X \times Y$ with $B \cap (A \times Y) = \emptyset$. Then it follows from Lemma 2.4(3) that $\mathrm{pr}_X[B]$ is a Suslin-$\mathcal{Z}$-set in X with $B \cap A = \emptyset$. Since A satisfies (b_1) in X, there exists $U \in \mathcal{C}oz(X)$ such that $\mathrm{pr}_X[B] \subseteq U$ and $U \cap A = \emptyset$. Putting $V = U \times Y$, we obtain $V \in \mathcal{C}oz(X \times Y)$ such that $B \subseteq V$ and $V \cap (A \times Y) = \emptyset$. Hence, $A \times Y$ satisfies (b_3) in $X \times Y$. ∎

The reader might ask if "compact metric" can be replaced by "compact Hausdorff" in condition (3) of Theorem 2.6. In Remark 3.6 below, we show that metrizability of Y is essential in this condition.

The following corollary can be proved similarly to (3)⇒(4) in Theorem 2.6 if we use Lemma 2.4(1) and the fact that every analytic set in Q is the projection of a zero-set in $Q \times \mathbb{P}$, where $\mathbb{P}$ is the space of irrational numbers (see [11]).

Corollary 2.7. *Let A be a P^γ-embedded closed subspace of a space X, where γ is an infinite cardinal, and assume that either $X \times \mathbb{P}$ is normal or $(X \setminus A) \times \mathbb{P}$ is Lindelöf. Then $A \times Y$ is P^γ(point-finite)-embedded in $X \times Y$ for every compact Hausdorff space Y with $w(Y) \leq \gamma$.*

In the remaining part of this section, we consider the relationship between conditions (b_i), $i = 1, 2, 3$, and the following conditions, from the literature, on a subspace A of a space X.

- (a_γ) For every γ-separable continuous pseudometric ϱ on X, there exists $F \in \mathcal{Z}(X)$ such that $A \subseteq F \subseteq \{x \in X : (\exists\, y \in A)(\varrho(x, y) = 0)\}$.
- (c) For every $B \in \mathcal{Z}(X)$ with $B \cap A = \emptyset$, there exists $U \in \mathcal{C}oz(X)$ such that $B \subseteq U$ and $U \cap A = \emptyset$.

Here, a pseudometric d on a space X is called *γ-separable* if the weight of the pseudometric space (X, d) is not greater than γ. Sennott [18] proved that A is M^γ-embedded in X if and only if A is P^γ-embedded in X and satisfies (a_γ) in X. On the other hand, it is known (see [8, Theorem 1.18]) that A satisfies (c) in X if every real-valued continuous function on A extends continuously over X, or equivalently, A is P^ω-embedded in X (see [7] and [9]). Obviously, (b_3) implies (c).

Proposition 2.8. *(a_ω) implies (b_1).*

Proof. Assume that a subspace A of a space X satisfies (a_ω) in X. Let B be a Suslin-$\mathcal{Z}$-set in X with $B \cap A = \emptyset$. Then, by Lemma 2.5, there exists a continuous map f from X to the Hilbert cube Q such that $B = f^{-1}[f[B]]$. Let d be the metric on Q, and define $\varrho(x, y) = d(f(x), f(y))$ for $x, y \in X$. Then ϱ is an ω-separable continuous pseudometric on X such that $\{x \in X : (\exists y \in A)(\varrho(x, y) = 0)\} \cap B = \emptyset$. Hence, by (a_ω), we can find $U \in \mathcal{C}oz(X)$ such that $B \subseteq U$ and $U \cap A = \emptyset$. ∎

Remark 2.9. Summing up the above observations, we have the implications

$$(a_\omega) \Rightarrow (b_1) \Rightarrow (b_2) \Rightarrow (b_3) \Rightarrow (c)$$

In the next section, we give examples showing that $(b_1) \not\Rightarrow (a_\omega)$ and $(c) \not\Rightarrow (b_3)$, and show that $(b_3) \not\Rightarrow (b_2)$ and $(b_2) \not\Rightarrow (b_1)$ assuming the continuum hypothesis.

3. Examples. As stated in the introduction, we now apply Theorems 2.1 and 2.6 to answer Dydak's questions [3, Problems 12.10, 12.11 and 13.6]. Throughout this section, $\mathbb{R}$ denotes the set of real numbers endowed with the Euclidean topology τ. When $A \subseteq X \subseteq \mathbb{R}$, X_A denotes the space with the underlying set X and with the topology $\{U \cup K : U \in \tau_X,\ K \subseteq X \setminus A\}$, where τ_X is the subspace topology on X induced from τ. It is known that X_A is a paracompact Hausdorff space (see [5, Example 5.1.22]). We begin by determining when the subspace A satisfies (b_i), $i = 1, 2, 3$, in X_A in terms of subsets of $\mathbb{R}$.

LEMMA 3.1. *Let $A \subseteq X \subseteq \mathbb{R}$ and $S \subseteq X$. Then*:

(1) *S is a cozero-set in X_A if and only if there exist an open set U in $\mathbb{R}$ and an F_σ-set F in $\mathbb{R}$ such that $S \cap A \subseteq U \cap X \subseteq S$, $S \subseteq F$ and $F \cap A = S \cap A$.*

(2) *If S is a Baire set in X_A, then there exists a Borel set B in $\mathbb{R}$ such that $S \subseteq B$ and $B \cap A = S \cap A$.*

(3) *If S is a Suslin-$\mathcal{Z}$-set in X_A, then there exists an analytic set B in $\mathbb{R}$ such that $S \subseteq B$ and $B \cap A = S \cap A$.*

Proof. (1) First, observe that (i) a set $S \subseteq X$ is open in X_A if and only if there exists an open set U in $\mathbb{R}$ with $S \cap A \subseteq U \cap X \subseteq S$, and (ii) a set $S \subseteq X$ is closed in X_A if and only if there exists a closed set F in $\mathbb{R}$ such that $S \subseteq F$ and $S \cap A = F \cap A$. Now, (1) follows from (i) and (ii) since, by the normality of X_A, $S \in \mathcal{C}oz(X_A)$ if and only if S is an open F_σ-set in X_A.

(2) Let $\mathcal{S}$ be the family of all sets $S \subseteq X$ such that there exist Borel sets B and B' in $\mathbb{R}$ such that $S \cap A \subseteq B \cap X \subseteq S$, $S \subseteq B'$ and $B' \cap A = S \cap A$. Then $\mathcal{S}$ is a σ-algebra of subsets of X, and $\mathcal{C}oz(X_A) \subseteq \mathcal{S}$ by (1). Hence, all Baire sets in X_A belong to $\mathcal{S}$, from which (2) can be deduced.

(3) If S is a Suslin-$\mathcal{Z}$-set in X_A, then there exists $\{Z_\sigma : \sigma \in {}^{<\omega}\omega\} \subseteq \mathcal{Z}(X_A)$ such that $S = \bigcup_{t\in{}^\omega\omega} \bigcap_{n<\omega} Z_{t|n}$. Define $B = \bigcup_{t\in{}^\omega\omega} \bigcap_{n<\omega} \mathrm{cl}_{\mathbb{R}} Z_{t|n}$. Then B is an analytic set in $\mathbb{R}$ with $S \subseteq B$. Since $\mathrm{cl}_{\mathbb{R}} Z_\sigma \cap A = Z_\sigma \cap A$ for each $\sigma \in {}^{<\omega}\omega$, $B \cap A = S \cap A$. ∎

PROPOSITION 3.2. *Let $A \subseteq X \subseteq \mathbb{R}$. Then*:

(1) *A satisfies (b_1) in X_A if and only if for every analytic set B in $\mathbb{R}$ with $B \cap A = \emptyset$, there exists an F_σ-set F in $\mathbb{R}$ such that $B \cap X \subseteq F$ and $F \cap A = \emptyset$.*

(2) *A satisfies (b_2) in X_A if and only if for every Borel set B in $\mathbb{R}$ with $B \cap A = \emptyset$, there exists an F_σ-set F in $\mathbb{R}$ such that $B \cap X \subseteq F$ and $F \cap A = \emptyset$.*

(3) *A satisfies* (b_3) *in* X_A *if and only if for every countable family* $\{F_n : n < \omega\}$ *of* F_σ*-sets in* $\mathbb{R}$ *such that* $\bigcap_{n<\omega} F_n \cap A = \emptyset$ *and* $F_n \cap A$ *is open in* A *for each* $n < \omega$, *there exists an* F_σ*-set* F *in* $\mathbb{R}$ *such that* $\bigcap_{n<\omega} F_n \cap X \subseteq F$ *and* $F \cap A = \emptyset$.

Proof. (1) Assume that A satisfies (b_1) in X_A and let B be an analytic set in $\mathbb{R}$ with $B \cap A = \emptyset$. Since the topology of $\mathbb{R}_A$ is finer than that of $\mathbb{R}$, B is a Suslin-$\mathcal{Z}$-set in $\mathbb{R}_A$, and hence, $B \cap X$ is also a Suslin-$\mathcal{Z}$-set in X_A. Thus, it follows from (b_1) that there exists $U \in \mathcal{C}oz(X_A)$ such that $B \cap X \subseteq U$ and $U \cap A = \emptyset$. By Lemma 3.1(1), there exists an F_σ-set F in $\mathbb{R}$ such that $U \subseteq F$ and $F \cap A = U \cap A$. Then $B \cap X \subseteq F$ and $F \cap A = \emptyset$. Conversely, assume that A satisfies the latter condition in (1) and let C be a Suslin-$\mathcal{Z}$-set in X_A with $C \cap A = \emptyset$. Then, by Lemma 3.1(3), there exists an analytic set H in $\mathbb{R}$ such that $C \subseteq H$ and $H \cap A = \emptyset$. By the assumption, we can find an F_σ-set F in $\mathbb{R}$ such that $H \cap X \subseteq F$ and $F \cap A = \emptyset$. Since $F \cap X \in \mathcal{C}oz(X_A)$ by Lemma 3.1(1), A satisfies (b_1) in X_A. (2) can be proved similarly to (1) using Lemma 3.1(2) instead of Lemma 3.1(3).

(3) Assume that A satisfies (b_3) in X_A and let $\{F_n : n < \omega\}$ be a countable family of F_σ-sets in $\mathbb{R}$ such that $\bigcap_{n<\omega} F_n \cap A = \emptyset$ and $F_n \cap A$ is open in A for each $n < \omega$. For each $n < \omega$, since X_A is normal, we can find $E_n \in \mathcal{C}oz(X_A)$ such that $F_n \cap X \subseteq E_n$ and $E_n \cap A = F_n \cap A$. Since $\bigcap_{n<\omega} E_n \cap A = \emptyset$, it follows from (b_3) that there exists $U \in \mathcal{C}oz(X_A)$ such that $\bigcap_{n<\omega} E_n \subseteq U$ and $U \cap A = \emptyset$. By Lemma 3.1(1), there exists an F_σ-set F in $\mathbb{R}$ such that $U \subseteq F$ and $F \cap A = U \cap A$. Then

$$\bigcap_{n<\omega} F_n \cap X \subseteq \bigcap_{n<\omega} E_n \subseteq U \subseteq F$$

and $F \cap A = \emptyset$. Conversely, assume that A satisfies the latter condition in (3), and take $\{G_n : n < \omega\} \subseteq \mathcal{C}oz(X_A)$ such that $\bigcap_{n<\omega} G_n \cap A = \emptyset$. For each $n < \omega$, by Lemma 3.1(1), there exists an F_σ-set H_n in $\mathbb{R}$ such that $G_n \subseteq H_n$ and $H_n \cap A = G_n \cap A$. Since $\bigcap_{n<\omega} H_n \cap A = \emptyset$ and $H_n \cap A$ is open in A for each $n < \omega$, it follows from our assumption that there exists an F_σ-set H in $\mathbb{R}$ such that $\bigcap_{n<\omega} H_n \cap X \subseteq H$ and $H \cap A = \emptyset$. Then $H \cap X \in \mathcal{C}oz(X_A)$ by Lemma 3.1(1), $\bigcap_{n<\omega} G_n \subseteq H \cap X$ and $(H \cap X) \cap A = \emptyset$. Hence, A satisfies (b_3) in X_A. ∎

Now, we are in a position to construct examples. A subspace A of a space X is said to be P*-embedded* in X if it is P^γ-embedded in X for every γ. M- and P(*point-finite*)*-embeddings* are defined similarly. If $A \subseteq X \subseteq \mathbb{R}$, then the closed subspace A of X_A is always P-embedded in X_A, since X_A is paracompact. The last statement of the following example was proved by the second author in [24]; however, now it is an immediate consequence of Proposition 3.2(3) and Theorem 2.1.

EXAMPLE 3.3. *Let $\mathbb{Q}$ be the set of rational numbers. Then $\mathbb{Q}$ fails to satisfy (b_3) in $\mathbb{R}_{\mathbb{Q}}$. Hence, $\mathbb{Q}$ is not P^ω(point-finite)-embedded in $\mathbb{R}_{\mathbb{Q}}$.*

Example 3.3 shows that $(c) \not\Rightarrow (b_3)$ in general. Recall from [5, 5.5.4] that there exists a set $A \subseteq \mathbb{R}$, called a *Bernstein set*, such that every compact set in $\mathbb{R}$ contained in either A or $\mathbb{R} \setminus A$ is countable.

EXAMPLE 3.4. *Let A be a Bernstein set in $\mathbb{R}$. Then A satisfies (b_1) in $\mathbb{R}_A$ but fails to satisfy $(a)_\omega$ in $\mathbb{R}_A$. Hence, $A \times Y$ is P(point-finite)-embedded in $\mathbb{R}_A \times Y$ for every compact Hausdorff space Y, but A is not M^ω-embedded in $\mathbb{R}_A$.*

Proof. Let B be an analytic set in $\mathbb{R}$ with $B \cap A = \emptyset$. Then B must be countable, since every uncountable analytic set in $\mathbb{R}$ contains a Cantor set (see [10, Theorem 94]). By Proposition 3.2(1), this implies that A satisfies (b_1) in $\mathbb{R}_A$. On the other hand, the Euclidean metric d on $\mathbb{R}$ is an ω-separable continuous pseudometric on $\mathbb{R}_A$ and $\{x \in \mathbb{R}_A : (\exists y \in A)(d(x,y) = 0)\} = A$. Since A is not a zero-set in $\mathbb{R}_A$, A does not satisfy $(a)_\omega$ in $\mathbb{R}_A$ (see also [18, Corollary 5 to Theorem 1]). ∎

EXAMPLE 3.5. *Under CH, there exist sets A and X with $A \subseteq X \subseteq \mathbb{R}$ such that A satisfies (b_2) in X_A but fails to satisfy (b_1) in X_A. Hence, A is P(point-finite)-embedded in X_A, but $A \times [0,1]$ is not P^ω(point-finite)-embedded in $X \times [0,1]$.*

Proof. By [10, Corollary to Lemma 39.4], there exists an analytic set B in $\mathbb{R}$ such that $\mathbb{R} \setminus B$ is not analytic. Put $A = \mathbb{R} \setminus B$ and let $\mathcal{B}$ be the family of all Borel sets in $\mathbb{R}$ containing A. Since $|\mathcal{B}| = 2^\omega$, we can enumerate $\mathcal{B}$ as $\{B_\alpha : \alpha < \omega_1\}$ by CH. Then $\bigcap_{\beta<\alpha} B_\beta \cap B$ is uncountable for each $\alpha < \omega_1$, because A is not a Borel set. Thus, we can choose inductively a point

$$x_\alpha \in \Big(\bigcap_{\beta<\alpha} B_\beta \cap B\Big) \setminus \{x_\beta : \beta < \alpha\}$$

for each $\alpha < \omega_1$. Put $X = A \cup \{x_\alpha : \alpha < \omega_1\}$. Then, since $X \setminus B_\alpha$ is countable for each $\alpha < \omega$, it follows from Proposition 3.2(2) that A satisfies (b_2) in X_A. On the other hand, since B is an analytic set in $\mathbb{R}$ and $B_\alpha \cap B \neq \emptyset$ for each $\alpha < \omega_1$, Proposition 3.2(1) shows that A does not satisfy (b_1) in X_A. ∎

REMARK 3.6. Let X_A be the space defined in Example 3.5, and let $\Omega = \omega_1 + 1$ with the usual order topology. Now, by proving that $A \times \Omega$ is P(point-finite)-embedded in $X_A \times \Omega$, we show that the assumption of metrizability of Y is essential in condition (3) of Theorem 2.6. By Lemma 2.2, $A \times \Omega$ is P-embedded in $X_A \times \Omega$. Thus, by Theorem 2.1, it suffices to show that $A \times \Omega$ satisfies (b_3) in $X_A \times \Omega$. Take a countable family $\{G_n : n < \omega\} \subseteq \mathcal{C}oz(X_A \times \Omega)$ with $\bigcap_{n<\omega} G_n \cap (A \times \Omega) = \emptyset$. Put $A_n = \{x \in A : \langle x, \omega_1\rangle \notin G_n\}$ for each $n < \omega$. Since each A_n is separable and each G_n is an F_σ-set, we can find $\alpha < \omega_1$ such that $G_n \cap (A_n \times (\Omega \setminus \alpha)) = \emptyset$ for each $n < \omega$. Here, we may assume that α is an isolated ordinal. For each $n < \omega$, put

$$H_n = \mathrm{pr}_{X_A}[G_n \cap (X_A \times (\Omega \setminus \alpha))].$$

Then $H_n \in \mathcal{C}oz(X_A)$ by Lemma 2.4(2), and $\bigcap_{n<\omega} H_n \cap A = \emptyset$ as $H_n \cap A_n = \emptyset$ for each $n < \omega$. Since A satisfies (b_3) in X_A, there exists $U \in \mathcal{C}oz(X_A)$ such that $\bigcap_{n<\omega} H_n \subseteq U$ and $U \cap A = \emptyset$. On the other hand, since α is countable compact metrizable, it follows from Corollary 2.3 and Theorem 2.1 that there exists $V \in \mathcal{C}oz(X_A \times \alpha)$ such that $\bigcap_{n<\omega} G_n \cap (X_A \times \alpha) \subseteq V$ and $V \cap (A \times \alpha) = \emptyset$. Finally, putting $W = (U \times (\Omega \setminus \alpha)) \cup V$, we obtain a cozero-set W in $X_A \times \Omega$ such that $\bigcap_{n<\omega} G_n \subseteq W$ and $W \cap (A \times \Omega) = \emptyset$. Hence, $A \times \Omega$ satisfies (b_3) in $X_A \times \Omega$.

EXAMPLE 3.7. *Under CH, there exist sets A and X with $A \subseteq X \subseteq \mathbb{R}$ such that A satisfies (b_3) in X_A but fails to satisfy (b_2) in X_A.*

Proof. Following [10], Σ^0_3 denotes the family of all sets which can be written as the union of countably many G_δ-sets in $\mathbb{R}$, and Π^0_4 denotes the family of all sets which can be written as the intersection of countably many members of Σ^0_3. By [10, Corollary to Lemma 39.1] there exists a Borel set A in $\mathbb{R}$ such that $A \notin \Pi^0_4$. Now, let $\mathcal{B}$ be the family of all members of Π^0_4 containing A. Since $|\mathcal{B}| = 2^\omega$, we can enumerate $\mathcal{B}$ as $\{B_\alpha : \alpha < \omega_1\}$ by CH. Then $\bigcap_{\beta<\alpha} B_\beta \setminus A$ is uncountable for each $\alpha < \omega_1$, because $A \notin \Pi^0_4$. Hence, we can define a set $X = A \cup \{x_\alpha : \alpha < \omega_1\}$ similarly to the proof of Example 3.5. Since $X \setminus B_\alpha$ is countable for each $\alpha < \omega_1$, it follows from Proposition 3.2(3) that A satisfies (b_3) in X_A. On the other hand, $\mathbb{R} \setminus A$ is a Borel set in $\mathbb{R}$, but $(\mathbb{R} \setminus A) \cap B_\alpha \neq \emptyset$ for each $\alpha < \omega_1$. Hence, A does not satisfy (b_2) in X_A by Proposition 3.2(2). ∎

A similar example to Examples 3.5 and 3.7 was constructed by Michael [12] for a countable non-G_δ-set A to show that the product of a Lindelöf space X_A with $\mathbb{P}$ is not necessarily normal under CH.

In [15, Example 3], Przymusiński and Wage constructed an example of a collectionwise normal space Z having a closed subspace K which is not P^ω(locally finite)-embedded in Z. Finally, we show that an M-embedded subspace is not necessarily P^ω(locally finite)-embedded by proving the following:

EXAMPLE 3.8. *Every closed subspace A of the collectionwise normal space Z of Przymusiński–Wage is M-embedded in Z.*

Proof. The space Z is constructed from a subspace W of Rudin's Dowker space of [17]. All we need to know about Z is that every G_δ-set in W is open and that Z is the union of W and another space Y, where W is a G_δ-set in Z and Y is an open (in Z) set which is the topological sum of subspaces of W.

From these facts, if a set G is the union of G_δ-sets in Z, then both $G \cap W$ and $G \cap Y$ are G_δ-sets in Z, and therefore, G is a G_δ-set in Z. Now, let A be a closed subspace of Z. Since Z is collectionwise normal, it follows from [19, Theorem 5.2] that A is P-embedded in Z. To show that A satisfies (a_γ) in Z for every infinite cardinal γ, let ϱ be a γ-separable continuous pseudometric on Z. Then the set $L = \{x \in Z : (\exists y \in A)(\varrho(x,y) = 0)\}$ is a G_δ-set in Z since it is the union of G_δ-sets in Z. Thus, by the normality of Z, there exists a zero-set F in Z such that $A \subseteq F \subseteq L$. Hence, A is M-embedded in Z. ∎

4. Another application and questions. By AR we mean an absolute retract for the class of metrizable spaces. In [14] Morita proved that a subspace A of a space X is P^γ-embedded in X if and only if for every complete AR Y with $w(Y) \leq \gamma$, every continuous map from A to Y extends continuously over X. As another application of Theorem 2.1, we prove the following theorem by a similar argument to the proofs of Morita's theorems in [14] (see also [9, Theorems 2.8 and 2.14]). We now call a metrizable space X *σ-complete* if there exist a metric d on X, which induces the topology of X, and a countable cover $\{X_n : n < \omega\}$ of X such that each X_n is a complete subspace of the metric space (X, d).

THEOREM 4.1. *Let A be a subspace of a space X and γ an infinite cardinal. Then the following are equivalent*:

(1) *A is P^γ(point-finite)-embedded in X*,
(2) *for every σ-complete* AR *Y with $w(Y) \leq \gamma$, every continuous map from A to Y extends continuously over X*,
(3) *for every Banach space B and every convex F_σ-set Y in B with $w(Y) \leq \gamma$, every continuous map from A to Y extends to a continuous map from X to Y.*

Proof. (1)⇒(2): Let $f : A \to Y$ be a continuous map to a σ-complete AR Y with $w(Y) \leq \gamma$. We consider Y a metric space having a countable cover by complete subspaces. Then, by Kuratowski–Wojdysławski's theorem (see [9]), there exist a Banach space B and an isometrical embedding $i : Y \to B$ such that $w(Z) \leq \gamma$, where Z is the convex hull of $i[Y]$. We identify Y and $i[Y]$. Since A is P^γ-embedded in X and $w(\mathrm{cl}_B Z) \leq \gamma$, f extends to a continuous map $g : X \to B$ with $g[X] \subseteq \mathrm{cl}_B Z$ by Morita's theorem mentioned above. Since Y is an F_σ-set in B, $g^{-1}[Y]$ is a countable union of zero-sets in X such that $A \subseteq g^{-1}[Y]$. Since A is P^γ(point-finite)-embedded in X, it follows from Theorem 2.1 that there exists a continuous function $\varphi : X \to [0,1]$ such that the set $F = \varphi^{-1}(0)$ satisfies $A \subseteq F \subseteq g^{-1}[Y]$. Consider the diagonal map

$$h = g \triangle \varphi : X \to B \times [0,1]$$

and let p, q denote the projections of $B \times [0,1]$ onto B and $[0,1]$, respectively. Then $h[F] = h[X] \cap q^{-1}(0)$ is closed in $h[X]$ and $p[h[F]] = g[F] \subseteq Y$. Since Y is an AR, the restriction $p|_{h[F]}$ can be extended to a continuous map $p^* : h[X] \to Y$. Then $p^* \circ h : X \to Y$ is a continuous extension of $(p \circ h)|_A = g|_A = f$.

The implication (2)⇒(3) follows from the fact that every convex F_σ-set in a Banach space is a σ-complete AR. For a set S, let $\ell_1(S)$ be the Banach space of all real-valued functions v on S such that $\|v\| \equiv \sum_{s\in S} |v(s)| < \infty$, and Δ_S the subspace of $\ell_1(S)$ consisting of all $v \in \ell_1(S)$ such that $v(s) = 0$ for all but finitely many $s \in S$, $v \geq 0$, and $\sum_{s\in S} v(s) = 1$. Dydak [3] proved that A is P^γ(point-finite)-embedded in X if (and only if) for every set S with $|S| \leq \gamma$, every continuous map from A to Δ_S extends to a continuous map from X to Δ_S. Since Δ_S is a convex F_σ-set in $\ell_1(S)$, we have the final implication (3)⇒(1). ■

REMARK 4.2. By Hausdorff's extension theorem, a metrizable space is σ-complete if and only if it has a countable cover by closed completely metrizable subspaces. The term "σ-complete" was used by A. H. Stone in [20, Lemma 4] without an explicit definition.

We conclude the paper with some open questions.

QUESTION 4.3. *Does there exist an example in ZFC of a P-embedded subspace which satisfies (b_3) but not (b_2)? Does there exist an example in ZFC of a P-embedded subspace which satisfies (b_2) but not (b_1)?*

The next question was first asked by the second author in [22, Problem 2.3.4], which asks if there is a P^γ(locally finite)-embedding analogue of Theorem 2.1.

QUESTION 4.4. *Let A be a subspace of a space X and γ an uncountable cardinal. Is then A P^γ(locally finite)-embedded in X if A is P^γ- and P^ω(locally finite)-embedded in X?*

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