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# On second order Thom–Boardman singularities

by

**László M. Fehér** and **Balázs Kőműves** (Budapest)

**Abstract.** We derive closed formulas for the Thom polynomials of two families of second order Thom–Boardman singularities $\Sigma^{i,j}$. The formulas are given as linear combinations of Schur polynomials, and all coefficients are nonnegative.

**1. Introduction.** In the fifties, R. Thom [Tho56] defined, for generic smooth (resp. analytic) maps $f : N^n \to P^p$ between real (resp. complex) manifolds, singularity subsets $\Sigma^I(f) \subset N$ for each partition $I = (i_1, \ldots, i_k)$, by setting inductively

$$\Sigma^i(f) = \{x \in N : \operatorname{corank}(d_x f) = i\},$$
$$\Sigma^{I,j}(f) = \Sigma^j(f|_{\Sigma^I(f)}).$$

Later J. M. Boardman [Boa67] gave a more precise and also more general definition in terms of jet bundles. Thom also observed that for generic maps, the cohomology classes $[\Sigma^I(f)] \in H^*(N)$ represented by (the closures of) these subsets depend only on the Stiefel–Whitney (resp. Chern) classes of the bundles $TN$ and $f^*TP$, and that they are given by a universal polynomial $[\Sigma^I(n,p)](c(TN), c(f^*TP))$ in these classes, now called the Thom polynomial of the singularity. In fact, the same holds for many other singularity classes, that is, for (the closure of) a submanifold of some jet space invariant under the left-right action of jets of (germs of) biholomorphisms; and for *stable* classes of singularities, these polynomials depend only the Chern classes of the virtual normal bundle $f^*TP - TN \in K(N)$ and on the codimension $r = p - n$ (see Proposition 3.5). Since the Thom–Boardman singularities are stable, their Thom polynomials can be expressed as polynomials $[\Sigma^I(r)]$ in the formal variables $c_1, c_2, c_3, \ldots$.

So far, very few of these polynomials are known explicitly. The Thom polynomials for $\Sigma^i$ were calculated by Porteous [Por71]; Ronga [Ron72] gave

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an algorithm to calculate the classes for $\Sigma^{i,j}$, but this algorithm is inefficient even using today's top personal computers, and the only known formula derived from it gives the Thom polynomials of $\Sigma^{1,1}$ (for all $r$). M. Kazarian in [Kaz] simplified the argument of Ronga and also gave slightly different versions of the original algorithm. Recently the Thom polynomials for $\Sigma^{1,1,1}$ were calculated in [BFR02] using the methods of restriction equations; and some other sporadic results are known (for example $\Sigma^{1,1,1,1}$ in the case $r=0$, see [Gaf83]; $\Sigma^{1^k}$ for $k\le 8$, $r=0$, see [Rim01]). The present work can be considered as the first step of the more ambitious quest for a *formula* for the Thom polynomials of *all* second order Thom–Boardman singularities $\Sigma^{i,j}$. We would like to emphasize that it was not a priori clear that such formulas exist: in fact, the method of restriction equations [Rim01] suggested just the contrary, since the complexity of singularities with codimension smaller than that of $\Sigma^{i,j}(r)$ is rapidly increasing with $i$.

The success of the method of restriction equations (for a more detailed account than in [Rim01] see [FR04]) encouraged us to try it in this situation as well. To our surprise we realized that the restriction equation we started to study is a consequence of the fact mentioned above, that these singularities are stable. It turned out that in essence, everything needed to produce our formulas has been known for at least 20 years.

From the results of Porteous [Por71] and Ronga [Ron72] one can deduce that in some sense it is natural to write the polynomials $[\Sigma^{i,j}(r)]$ in terms of Schur polynomials. After finishing this work we were informed by Piotr Pragacz that he also used Schur polynomial methods to calculate other Thom polynomials [Pra]. Theorems 4.8 and 4.10 give the formulas as (nonnegative) linear combinations of Schur polynomials. To state them, we introduce the notations

$$E_{\lambda/\mu}(n) := \det\left[\binom{\lambda_k+n-k}{\mu_l+n-l}\right]_{k,l\le n}, \qquad F_{\lambda/\mu}(n) := \det\left[\left\{\begin{matrix}\lambda_k+n-k\\ \mu_l+n-l\end{matrix}\right\}\right]_{k,l\le n}$$

where $\lambda$ and $\mu$ are partitions, and $\left\{\begin{smallmatrix}n\\ k\end{smallmatrix}\right\} = \sum_{j=0}^{k}\binom{n}{j}$.

THEOREM 4.8. *The Thom polynomial of the second order Thom–Boardman singularity $\Sigma^{i,j}(-i+1)$ is*

$$[\Sigma^{i,j}(-i+1)] = \sum_{\mu\subset\delta} 2^{|\mu|-j(j-1)/2}\cdot E_{\delta/\mu}(i)\cdot s_{(d-|\mu|,\widetilde{\mu})}$$

*where $\delta$ is the "staircase" partition $\delta=(j,j-1,\dots,2,1)$, and $d$ is the codimension of $\Sigma^{i,j}(-i+1)$, that is, $d=i+j(j+1)/2$.*

Note that these are the simplest singularities which can occur for negative codimension maps. In the setting of $f: N^n\to P^{n-i+1}$ described above, $s_\lambda$ becomes $s_\lambda(f^*TP-TN)$.

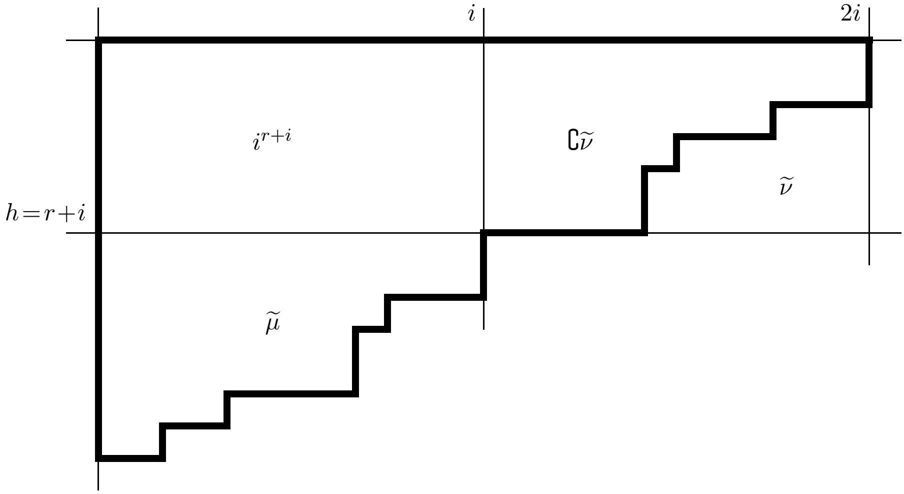


Fig. 1. The partition $(i^{r+i}+\mathsf{C}\widetilde{\nu},\widetilde{\mu})$

THEOREM 4.10. *The Thom polynomial of* $\Sigma^{i,1}(r)$ *is*

$$[\Sigma^{i,1}(r)] = \sum_{(\nu,\mu)\in I} F_{\nu/\mu}(i)\cdot s_{(i^{r+i}+\mathsf{C}\widetilde{\nu},\widetilde{\mu})}$$

*where* $I=\{(\nu,\mu)\ :\ \nu\subset(r+i)^i,\ l(\mu)\le i,\ |\nu|-|\mu|=i-1\}$ (*see Figure* 1).

The second order Thom–Boardman singularities are indexed by the three parameters $i$, $j$ and $r$ (although to us, it seems to be more natural to use $h=r+i$ instead of $r$). Theorems 4.8 and 4.10 each provide a closed formula for a two-parameter family. This furnishes two "transversal" planes in the 3-dimensional space of parameters. In the latter case we get a family where $r$ can run independently, giving infinitely many examples of *Thom series* (see [FR]). Namely, if we fix $i$ and $j$, the Thom polynomials for different $r$'s should fit together into one series

$$\mathrm{Ts}(\Sigma^{i,j}) = \sum_{\gamma} c_\gamma\cdot d_\gamma$$

where $\gamma$ runs over nonincreasing $\mathbb{Z}$-valued sequences of length $i+k=i+ij-\binom{j}{2}$ with fixed sum $|\gamma|=j(j-i)\le 0$. Here, $c_\gamma$ are (integer) coefficients—*independent of* $r$—and $d_\gamma$ are "renormalized" Schur polynomials $d_\gamma = s_{(h^{i+k}+\gamma)^\sim}$. Furthermore, Theorem 4.2 says that $c_\gamma=0$ unless $\gamma_l\ge 0$ for $l\le i$ and $\gamma_l\le 0$ for $l>i$. In this notation, Theorem 4.10 becomes

$$\mathrm{Ts}(\Sigma^{i,1}) = \sum_{(\nu,\mu)\in I'} F_{\nu/\mu}(i)\cdot d_{(\mu,-\nu_i,-\nu_{i-1},\dots,-\nu_1)}$$

with $I'=\{(\nu,\mu) : l(\nu)\le i,\ l(\mu)\le i,\ |\nu|-|\mu|=i-1\}$. Theorem 4.8 provides the "lowest terms" for the Thom series $\mathrm{Ts}(\Sigma^{i,j})$.

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*Notations.* A *partition* is a nonincreasing sequence of positive integers $\mu = (\mu_1 \geq \cdots \geq \mu_n > 0)$; the *length* of a partition is denoted by $l(\mu) = n$, its *weight* by $|\mu| = \sum \mu_i$. We adopt the convention that $\mu_i = 0$ if $i > l(\mu)$. The dual (or conjugate) partition is denoted by $\widetilde{\mu}$, i.e. $\widetilde{\mu}_j = \max\{k : \mu_k \geq j\}$; note that $l(\widetilde{\mu}) = \mu_1$. $\lambda \pm \mu$ denotes the sequence given by pointwise addition (resp. subtraction); while $\lambda + \mu$ is again a partition, $\lambda - \mu$ is often not. $(\lambda, \mu)$ denotes the concatenation, i.e. $(\lambda_1, \ldots, \lambda_{l(\lambda)}, \mu_1, \ldots, \mu_{l(\mu)})$. Finally, $n^k$ is the "block" partition $(n, n, \ldots, n)$ ($k$ times); and for $\lambda \subset n^k$ we denote its "complement" by $\complement\lambda$, i.e. $(\complement\lambda)_j = n - \lambda_{k+1-j}$ (we omit the block itself from the notation, as it will always be clear from the context).

Let $c_1, c_2, \ldots$ and $s_1, s_2, \ldots$ be two sequences of (formal) variables related by the equation

$$(1 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots) \cdot (1 - s_1 t + s_2 t^2 - s_3 t^3 + - \cdots) = 1.$$

The *Schur polynomial* $s_\lambda$ is then the determinant $s_\lambda = \det[s_{\lambda_i - i + j}] = \det[c_{\widetilde{\lambda}_i - i + j}]$ (we adopt the convention that $c_0 = 1$ and $c_k = 0$ for $k < 0$; similarly for $s_0$ and $s_{<0}$). If we set the degree of $c_k$ (resp. $s_k$) to $k$, then $s_\lambda$ will be a homogeneous polynomial of degree $|\lambda|$.

If $E \to X$ is a complex vector bundle, or more generally, $E \in K(X)$, we can interpret these sequences as its Chern and Segre classes; the resulting expression is denoted by $s_\lambda(E)$, and is an element of $H^*(X)$, the singular cohomology group with integer coefficients. Note that $s_\lambda(-E) = s_{\widetilde{\lambda}}(E^*)$ where $E^* = \mathrm{Hom}(E, \mathbb{C})$ is the dual bundle of $E$.

The Littlewood–Richardson coefficients will be denoted by $c^\lambda_{\mu,\nu}$, i.e. we have the expansion $s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$.

**2. Thom polynomials.** We use the general framework of "Thom polynomials for group actions" introduced by M. Kazarian [Kaz01]; see also [FR04].

Let $\varrho : G \to \mathrm{GL}(V)$ be a representation of the Lie group $G$ on the vector space $V$. Then any closed invariant subvariety $\Sigma$ of $V$ represents an equivariant cohomology class $[\Sigma] \in H^*_G(V) \cong H^*(BG)$. We sometimes call this class the Thom polynomial because $H^*(BG)$ is (at least rationally) a polynomial ring, and the Thom polynomials of singularities are special cases where $V$ is the (infinite-dimensional) vector space of holomorphic germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, and $G$ is the "left-right" group $\mathcal{A}(n,p) = j^\infty \mathrm{Diff}(n) \times j^\infty \mathrm{Diff}(p)$ where $j^\infty \mathrm{Diff}(n)$ denotes the group of germs of biholomorphisms of $(\mathbb{C}^n, 0)$. Some caution is required in this case since there is no natural topology defined on the group $\mathcal{A}(n,p)$, so the classifying space $B\mathcal{A}(n,p)$ is not defined. As explained in [Rim02], one can work with the classifying space of the subgroup of linear germs $\mathrm{GL}_n \times \mathrm{GL}_p \subset \mathcal{A}(n,p)$ instead. We have $H^*(B(\mathrm{GL}_n \times \mathrm{GL}_p)) \cong \mathbb{Z}[a_1,\dots,a_n,b_1,\dots,b_p]$ and one can interpret the variables as Chern classes: $a_i = c_i(A)$ and $b_i = c_i(B)$ for the tautological vector bundles $A^n \to B\mathrm{GL}_n$ and $B^p \to B\mathrm{GL}_p$.

The Thom polynomial has a geometric meaning. Suppose that $E \to X$ is a $\varrho$-bundle, i.e. $E$ is of the form $P \times_\varrho V$ for some principal $G$-bundle $P \to X$. Then we can define a subset $\Sigma(E)$ of the total space of $E$, the union of $\Sigma$-points in each fiber.

Proposition 2.1. *If $\sigma : X \to E$ is a generic section (transversal to $\Sigma(E)$) then for the cohomology class $[\Sigma(\sigma)] \in H^*(X)$ defined by*

$$\Sigma(\sigma) := \{x \in X : \sigma(x) \in \Sigma(E)\}$$

*we have*

$$[\Sigma(\sigma)] = k^*[\Sigma]$$

*where $k : X \to BG$ is the classifying map of the principal $G$-bundle $P \to X$.*

**3. Thom polynomials of $\Sigma^{i,j}$.** This approach to $\Sigma^{i,j}$ singularities is fairly standard (see e.g. [Ron72, AVGL91, Kaz]).

Our goal is to reduce the situation to the study of a finite-dimensional representation. This is possible since the $\Sigma^{i,j}$ class of a map germ depends only on its first and second partial derivatives. The difficulty is that the second partial derivatives depend on the choice of the local coordinate system in a complicated (nonlinear) way. Let $f : N^n \to P^p$ be a holomorphic map of complex manifolds. Let $J^2(f) \to N$ denote the bundle of second jets along $f$, i.e. the fiber $J^2_x(f)$ over $x \in N$ is the set of 2-jets of map germs from $(N,x)$ to $(P,f(x))$. In other words, we identify two map germs if their first and second partial derivatives at $x \in N$ agree for some local coordinate system. The map $f$ defines a section $j^2(f)$ of $J^2(f)$ by taking the 2-jet of $f$ at every point $x \in N$. It is easy to see that

$$J^2_x(f) \cong \mathrm{Hom}(T_xN, T_{f(x)}P) \oplus \mathrm{Hom}(\mathrm{Sym}^2(T_xN), T_{f(x)}P).$$

This diffeomorphism depends on the local coordinate system chosen, however it can be shown that

$$J^2(f) \cong \mathrm{Hom}(TN, f^*(TP)) \oplus \mathrm{Hom}(\mathrm{Sym}^2(TN), f^*(TP))$$

as a fiber bundle. Let us fix such an isomorphism $\varphi$ for now. Then with every point $x \in N$ we associate two linear maps: $\mathrm{d}_x f : T_xN \to T_{f(x)}P$ and $\mathrm{d}^2_x f : \mathrm{Sym}^2(T_xN) \to T_{f(x)}P$. The map $\mathrm{d}^2_x f$ depends on the choice of $\varphi$ but (as can be easily checked) the induced map

$$\widehat{\mathrm{d}^2_x f} : \mathrm{Sym}^2(\mathrm{Ker}(\mathrm{d}_x f)) \to \mathrm{Coker}(\mathrm{d}_x f)$$

does not. The map $\widehat{\mathrm{d}_x^2 f}$ was introduced by I. R. Porteous in [Por71, Proposition 2.1], and called the *second intrinsic derivative*, to give a simpler definition for the $\Sigma^{i,j}$ classes. To formulate the result we need a definition. Let $\alpha \in \mathrm{Hom}(\mathrm{Sym}^2 A, B)$ be a linear map where $A, B$ are vector spaces. Since $\mathrm{Sym}^2 A \cong (A \otimes A)/\Lambda^2 A$, there is a canonical inclusion $\mathrm{Hom}(\mathrm{Sym}^2 A, B) \subset \mathrm{Hom}(A \otimes A, B)$, so $\alpha$ defines an element $\widetilde{\alpha} \in \mathrm{Hom}(A, A^* \otimes B)$.

Definition 3.1. $\mathrm{corank}_2(\alpha) := \mathrm{corank}(\widetilde{\alpha})$.

Proposition 3.2 (J. M. Boardman [Boa67, Theorem 7.15]).

$$\Sigma^{i,j}(f) = \{x \in N : \mathrm{corank}(\mathrm{d}_x f) = i,\ \mathrm{corank}_2(\widehat{\mathrm{d}_x^2 f}) = j\}.$$

We would like to use this form to show that the Thom polynomial of $\Sigma^{i,j}$ agrees with the Thom polynomial corresponding to the following finite-dimensional representation. Let $A$ and $B$ be complex vector spaces and let $V = \mathrm{Hom}(A, B) \oplus \mathrm{Hom}(\mathrm{Sym}^2 A, B)$. The group $G = \mathrm{GL}(A) \times \mathrm{GL}(B)$ acts on $V$.

Definition 3.3. Let $A$, $B$ be vector spaces and define the subvarieties

$$\begin{aligned} \Sigma^{\bullet,j}(A,B) &:= \{\alpha \in \mathrm{Hom}(\mathrm{Sym}^2 A, B) : \mathrm{corank}_2(\alpha) = j\}, \\ \Sigma^{i,j}(A,B) &:= \{(\alpha_1, \alpha_2) \in \mathrm{Hom}(A,B) \oplus \mathrm{Hom}(\mathrm{Sym}^2 A, B) : \mathrm{corank}(\alpha_1) = i, \\ &\qquad\qquad \widehat{\alpha}_2 \in \Sigma^{\bullet,j}(\mathrm{Ker}(\alpha_1), \mathrm{Coker}(\alpha_1))\}, \end{aligned}$$

where $\widehat{\alpha}_2 : \mathrm{Sym}^2(\mathrm{Ker}(\alpha_1)) \to \mathrm{Coker}(\alpha_1)$ is the obvious map induced by $\alpha_2$.

Notice that $\overline{\Sigma}^{i,j}(A,B)$ is a $G = \mathrm{GL}(A) \times \mathrm{GL}(B)$-invariant subvariety of the vector space $\mathrm{Hom}(A,B) \oplus \mathrm{Hom}(\mathrm{Sym}^2 A, B)$ so we can define the Thom polynomial $[\Sigma^{i,j}(A,B)] \in H^*(BG)$ (we use the convention that the Thom polynomial of an invariant subset is the equivariant cohomology class defined by its closure).

As explained above, the jet bundle $J^2(f)$ is isomorphic to a vector bundle associated to this representation and the principal $\mathrm{GL}_n \times \mathrm{GL}_p$-bundle corresponding to the vector bundles $TN$ and $f^*TP$. Proposition 3.2 shows that $x \in \Sigma^{i,j}(f) \Leftrightarrow x \in \Sigma^{i,j}(j^2(f))$. By the Thom transversality theorem for generic $f$ the section $j^2(f)$ is also generic, so Proposition 2.1 gives $[\Sigma^{i,j}(f)] = [\Sigma^{i,j}(A^n, B^p)](c(TN), c(f^*TP))$. (Notice that pulling back by the classifying map means substitution of the corresponding Chern classes.) Or, using the notation $\Sigma^{i,j}(n,p)$ of the introduction:

Proposition 3.4.

$$\begin{aligned} [\Sigma^{i,j}(n,p)] = [\Sigma^{i,j}(A^n, B^p)] &\in H^*(B(\mathrm{GL}(A) \times \mathrm{GL}(B))) \\ &\cong \mathbb{Z}[a_1, \dots, a_n, b_1, \dots, b_p]. \end{aligned}$$

We will continue with the formalism that $A^n$ resp. $B^p$ will denote complex vector spaces equipped with the standard representation of $\mathrm{GL}(A) \cong$

$\mathrm{GL}_n$ (resp. $\mathrm{GL}_p$). We will think of them either as vector spaces, representations or equivariant vector bundles over the one-point space; from this last viewpoint they have (equivariant) Chern classes $a_1, \dots, a_n$ (resp. $b_1, \dots, b_p$), which we will often treat as formal variables.

As mentioned before, the polynomials $[\Sigma^{i,j}(n,p)]$ are stable in the following sense:

PROPOSITION 3.5. *There exist polynomials $[\Sigma^{i,j}(r)] \in \mathbb{Z}[c_1, c_2, \dots]$ such that for all pairs $(n,p)$ of natural numbers with $p-n=r$ we have*

$$[\Sigma^{i,j}(n,p)] = [\Sigma^{i,j}(r)](B^p - A^n),$$

*where the right hand side means that we substitute $c_k(B^p - A^n)$ for $c_k$ in the polynomial $[\Sigma^{i,j}(r)]$.*

A more general theorem is due to J. Damon. A proof can be found e.g. in [FR04]. Here $B-A$ denotes the difference in the Grothendieck group; $c_i(B-A)$ can be interpreted as the $i$th Taylor coefficient of the formal power series $(\sum_{k\ge 0} b_k t^k)/(\sum_{l\ge 0} a_l t^l)$. For example,

$$\begin{aligned} c_1(B-A) &= b_1 - a_1, \\ c_2(B-A) &= b_2 - a_2 + a_1^2 - a_1 b_1, \end{aligned}$$

and so on.

**4. Calculation of the Thom polynomials.** It follows from Definition 3.3 that $\Sigma^{i,j}(A^n, B^p)$ is empty if $n < i$ and for $n = i$ it has a particularly simple structure:

$$\Sigma^{i,j}(A^i, B^{r+i}) = \{(0, \alpha_2) : \alpha_2 \in \Sigma^{\bullet,j}(A^i, B^{r+i})\},$$

so its Thom polynomial is a product:

$$[\Sigma^{i,j}(A^i, B^{r+i})] = e\big(\mathrm{Hom}(A^i, B^{r+i})\big) \cdot [\Sigma^{\bullet,j}(A^i, B^{r+i})],$$

where $e(\mathrm{Hom}(A^i, B^{r+i}))$ is the Thom polynomial of $\{0\} \subset \mathrm{Hom}(\mathbb{C}^i, \mathbb{C}^{r+i})$—the (equivariant) *Euler class* of this representation—and $[\Sigma^{\bullet,j}(A^i, B^{r+i})]$ is the Thom polynomial of the subvariety $\overline{\Sigma}^{\bullet,j}(A^i, B^{r+i}) \subset \mathrm{Hom}(\mathrm{Sym}^2 A^i, B^{r+i})$. Since the Euler class of a representation is the product of its weights we see that $e = \prod(\beta_j - \alpha_i)$ where $\alpha_i$ and $\beta_j$ are the *Chern roots* of $A$ and $B$, i.e. $a_k$ is the $k$th elementary symmetric polynomial of the variables $\alpha_i$, and similarly $b_k$ is the $k$th elementary symmetric polynomial of the variables $\beta_j$. Comparing this with Proposition 3.5 we get the equations

$$[\Sigma^{i,j}(r)](B^{r+i-1} - A^{i-1}) = 0, \tag{1}$$

$$[\Sigma^{i,j}(r)](B^{r+i} - A^{i}) = e(\mathrm{Hom}(A,B)) \cdot [\Sigma^{\bullet,j}(A,B)]. \tag{2}$$

Equations (1) and (2) can also be interpreted as *restriction equations* in the sense of [FR04], the right hand side of equation (2) being an "incidence class" in the sense of [Rim01]. From this point of view equation (1) follows from the obvious fact that $\Sigma^{i,j}(f) \subset \Sigma^i(f)$ for any holomorphic map $f$, and equation (2) is a consequence of the local behavior of the set $\Sigma^{i,j}(f)$ at a point of $\Sigma^i(f)$. This was indeed our first approach and we only later realized that (1) and (2) also follow from stability.

The idea of our calculation is to solve the system of equations (1) and (2). To do that, we have to study the homomorphism

$$\varrho_{n,p} : \mathbb{Z}[c_1, c_2, \dots] \to \mathbb{Z}[a_1, \dots, a_n, b_1, \dots, b_p]$$

sending $c_i$ to $c_i(B^p - A^n)$. Elements in the image of $\varrho_{n,p}$ are called *super-symmetric polynomials* (or *Schur functions in difference of alphabets*). The following proposition states some of the fundamental properties of super-symmetric polynomials (in other words, of the map $\varrho_{n,p}$).

PROPOSITION 4.1.

(i) $\mathrm{Ker}(\varrho_{n-1,p-1}) = \langle s_\lambda : n^p \subset \lambda \rangle$, *where* $\langle\ \rangle$ *means the generated* $\mathbb{Z}$-*module.*

(ii) *Suppose that* $\lambda$ *is a partition containing the "block" partition* $n^p$. *Also assume that* $(n+1)^{p+1} \not\subset \lambda$, *i.e.* $\lambda$ *is of the form* $(n^p + \beta, \alpha)$, *where* $l(\beta) \le p$ *and* $\alpha_1 \le n$; *i.e.* $\lambda_i = n + \beta_i$ *for* $i \le p$ *and* $\lambda_i = \alpha_{i-p}$ *for* $i > p$. *Then*

$$s_\lambda(B^p - A^n) = \begin{cases} (-1)^{|\alpha|} e(\mathrm{Hom}(A,B)) s_{\widetilde{\alpha}}(A) s_\beta(B), & (n+1)^{p+1} \not\subset \lambda, \\ 0, & (n+1)^{p+1} \subset \lambda. \end{cases}$$

The proof can be found e.g. in [FP98, §3.2]. Part (i) is a corollary of a result of Pragacz [Pra88] on universally supported classes (*avoiding ideal* in the terminology of [FR04]) for $\Sigma^i$; part (ii) is sometimes called the *factorization formula*.

From this formula it is clear that the system of equations above does not have a unique solution: If we write the solution as a linear combination of Schur polynomials, we will have an ambiguity in the terms $s_\lambda$ where $(i+1)^{r+i+1} \subset \lambda$. But it is also clear that all the other terms are uniquely determined by our equations. To our surprise, these ambiguous terms turn out to be zero:

THEOREM 4.2. *Write the universal polynomial* $[\Sigma^{i,j}(r)]$ *as a linear combination of Schur polynomials*: $[\Sigma^{i,j}(r)] = \sum e^\lambda s_\lambda$, *where* $e^\lambda$ *are* (*integer*) *coefficients. Then* $e^\lambda = 0$ *if* $(i+1)^{r+i+1} \subset \lambda$.

The proof, which is based on Ronga's [Ron72] pushforward formula for $[\Sigma^{i,j}(r)]$, is given in Section 5 (Theorem 5.1). The same proof yields some more vanishing results, but we do not need them.

Theorem 4.2, Proposition 4.1 and the two equations above together imply the following:

COROLLARY 4.3. *Write the polynomial* $\left[\Sigma^{\bullet,j}(A^i,B^{r+i})\right]$ *as a linear combination of products of Schur polynomials in the variables* $a_i$ *and* $b_i$:

$$[\Sigma^{\bullet,j}(A^i,B^{r+i})]=\sum e_{\alpha,\beta}s_\alpha(A)s_\beta(B).$$

*Then*

$$[\Sigma^{i,j}(r)]=\sum(-1)^{|\alpha|}e_{\alpha,\beta}s_{(i^{r+i}+\beta,\widetilde{\alpha})}.$$

Calculating $[\Sigma^{\bullet,j}(A^i,B^{r+i})]$ *in this form* seems to be a very difficult problem in general, although the difficulties are purely combinatorial, as we have the following pushforward formula (see also Lemma 5.2 about calculation of pushforward).

PROPOSITION 4.4. *Let* $\pi:\mathrm{Gr}_j(A^i)\to\mathrm{pt}$ *denote the projection map from the Grassmannian of* $j$*-planes in* $A$ *to the one-point space*, *and* $0\to R^j\to\pi^*A\to Q^{i-j}\to 0$ *the tautological exact sequence* (*of equivariant vector bundles*) *over* $\mathrm{Gr}_j(A)$. *Then*

$$[\Sigma^{\bullet,j}(A,B)]=\pi_*c_{\mathrm{top}}(\pi^*B\otimes(R\otimes Q+\mathrm{Sym}^2R)^*),$$

*where the equation lives in* $H^*_{\mathrm{GL}_i\times\mathrm{GL}_h}(\mathrm{pt})$ (*in particular*, $c_{\mathrm{top}}$ *is the equivariant top Chern class*).

Let us emphasize that in the light of Corollary 4.3 above, this formula, while much simpler than Ronga's or Kazarian's pushforward formulas for $[\Sigma^{i,j}(r)]$, contains the same amount of information. The proof of the proposition is analogous to [LP00, §3]; we do not repeat it here, as we will not use this formula in the rest of the paper.

There are two particular cases when we know the solution in the required form, namely, the cases $r+i=1$ and $j=1$. It is not hard to see why these are simpler from the pushforward formula above.

THEOREM 4.5.

$$[\Sigma^{\bullet,j}(A^i,L^1)]=2^j\cdot s_\delta(A^*\otimes\sqrt{L})\quad\textit{where}\quad\delta=(j,j-1,\dots,2,1).$$

Note that the line bundle $L$ has no square root, so the formula above should be understood formally: the only Chern root of $\sqrt{L}$ is $\beta/2$ where $\beta=\beta_1$ is the Chern root of $L$, and then the Chern roots of $A^*\otimes\sqrt{L}$ are $-\alpha_1+\beta/2,\dots,-\alpha_n+\beta/2$.

*Proof.* Notice that the elements of $\mathrm{Hom}(\mathrm{Sym}^2\mathbb{C}^i,\mathbb{C})$ can be identified with symmetric $i\times i$ matrices and then $\mathrm{corank}_2=\mathrm{corank}$, so the Thom polynomial in question is given by the *twisted symmetric degeneracy locus* formula ([HT84], [JLP82], [Pra90], [Ful96]). A general explanation of twisting can be found in [FNR05]. ∎

THEOREM 4.6. $[\Sigma^{\bullet,1}(A^i,B^{r+i})]=c_{i(r+i-1)+1}(A^*\otimes B-A)$.

*Proof.* The codimension of $\Sigma^{\bullet,1}(V^n,W^p)\subset \operatorname{Hom}(\operatorname{Sym}^2 V,W)$ is $pn-n+1$, which equals the codimension of $\Sigma^1(V,V^*\otimes W)\subset \operatorname{Hom}(V,V^*\otimes W) = \operatorname{Hom}(V\otimes V,W)$; so exactly as noted in [LP00, §1.11.1], where a similar degeneracy locus problem is considered, we are in the situation of the Giambelli–Thom–Porteous formula:

$$[\Sigma^{\bullet,1}(V^n,W^p)] = [\Sigma^1(V,V^*\otimes W)] = c_{pn-n+1}(V^*\otimes W - V). \ \blacksquare$$

According to Corollary 4.3, the only thing we need is the separation of variables in the formulas of Theorems 4.5 and 4.6. We will use the following lemma, due to Lascoux.

LEMMA 4.7 ([Las78]). *Set*

$$E_{\lambda/\mu}(n) = \det\left[\binom{\lambda_i+n-i}{\mu_j+n-j}\right]_{i,j\le n}.$$

(1) *Let $A^n$ and $B^p$ be an $n$-dimensional and a $p$-dimensional vector bundle, respectively. Then*

$$\sum_k c_k(A\otimes B) = \sum_{\mu\subset\lambda\subset p^n} E_{\lambda/\mu}(n)s_\mu(A)s_{\complement\widetilde{\lambda}}(B).$$

(2) *Furthermore, if $L$ is a line bundle and $\lambda$ is partition with $l(\lambda)\le n$, then*

$$s_\lambda(A\otimes L) = \sum_{\mu\subset\lambda} E_{\lambda/\mu}(n)\cdot c_1(L)^{|\lambda|-|\mu|}\cdot s_\mu(A).$$

REMARK. The coefficients $E_{\lambda/\mu}(n)$ are known to be nonnegative. This is for example a very special case of [Pra96, Corollary 7.2] which says that if we set

$$s_\lambda(S^{\mu_1}E_1\otimes\cdots\otimes S^{\mu_k}E_k) = \sum e^{\boldsymbol{\nu}}_{\lambda,\boldsymbol{\mu}} s_{\nu_1}(E_1)\cdots s_{\nu_k}(E_k),$$

where $S^\mu$ is the Schur functor associated to the partition $\mu$, then all the coefficients $e^{\boldsymbol{\nu}}_{\lambda,\boldsymbol{\mu}}$ will be nonnegative.

A more concrete way to see this nonnegativity is via the following formula (which also motivates our notation): Suppose that $n\ge l(\lambda),l(\mu)$ (if this is not the case, one should take $(\lambda_1,\dots,\lambda_n)$ and $(\mu_1,\dots,\mu_n)$ instead of $\lambda$ and $\mu$ on the r.h.s.); then

$$E_{\lambda/\mu}(n) = s_{\lambda/\mu}\left(1,\frac{1}{2!},\frac{1}{3!},\frac{1}{4!},\dots\right)\cdot\prod_{(i,j)\in\lambda/\mu}(n-i+j),$$

where we substitute $1/k!$ for the $k$th *elementary symmetric polynomial* in the skew Schur polynomial $s_{\lambda/\mu}$. The proof of the formula is a straightforward computation (one observes that in the expansion of the determinant $E_{\lambda/\mu}(n)$ each term is the polynomial $\prod(n-i+j)$ up to a scalar factor). Another corollary is that $E_{\lambda/\mu}(n)=0$ if $\mu\not\subset\lambda$.

The lemma, together with Theorem 4.5 and the framework built above, immediately implies one of our main theorems:

THEOREM 4.8. *The Thom polynomial of the second order Thom–Boardman singularity $\Sigma^{i,j}(-i+1)$ is*

$$[\Sigma^{i,j}(-i+1)] = \sum_{\mu\subset\delta} 2^{|\mu|-j(j-1)/2}\cdot E_{\delta/\mu}(i)\cdot s_{(d-|\mu|,\widetilde{\mu})}$$

*where $\delta$ is the "staircase" partition $\delta=(j,j-1,\ldots,2,1)$ and*

$$d = \operatorname{codim}\Sigma^{i,j}(-i+1) = i+|\delta| = i+\binom{j+1}{2}.$$

Similarly, Theorem 4.6 leads to

THEOREM 4.9. *Using the shorthand notation $h=r+i$, we have*

$$[\Sigma^{i,1}(r)] = \sum_{(\lambda,\mu)\in J} s_{(i^h+\lambda,\mu)}\cdot \sum_{x\in\{0,1\}^{l(\mu)}} E_{\widetilde{\complement\lambda}/(\mu-x)^{\sim}}(i)$$

*where $J=\{(\lambda,\mu):\lambda\subset i^h,\ \mu_1\le i,\ |\lambda|+|\mu|=ih-i+1,$ and $\mu-x$ is a valid partition$\}$.*

*Proof.* According to Corollary 4.3, to solve the equation for $[\Sigma^{i,1}]$ all we have to do is to expand $c_{ih-i+1}(A^*\otimes B-A)$ into a linear combination of products of Schur polynomials. For convenience, we calculate the total Chern class

$$\sum_{m\ge0} c_m(A^*\otimes B-A) = \Big(\sum_{k\ge0} c_k(A^*\otimes B)\Big)\cdot\Big(\sum_{l\ge0} c_l(-A)\Big).$$

Using Lemma 4.7, the Pieri formula, and

$$c(-A)=\sum_{l\ge0} c_l(-A)=\sum_{k\ge0}(-1)^k s_k(A),$$

we get

$$c(A^*\otimes B-A) = \sum_{\mu\subset\lambda\subset i^h}\ \sum_{x\in\{0,1\}^{l(\mu)}} (-1)^{|\mu+x|} E_{\widetilde{\lambda}/\widetilde{\mu}}(i)\cdot s_{(\mu+x)^{\sim}}(A)s_{\complement\lambda}(B),$$

where the second sum runs over 0-1 sequences such that $\mu+x$ is a valid partition. From this the theorem follows directly, by using the fact that $E_{\lambda/\mu}(k)=0$ if $\mu\not\subset\lambda$ and $k\ge l(\lambda),l(\mu)$. ∎

Note that in both cases, the Thom polynomial is a nonnegative linear combination of Schur polynomials. Based on computational evidence, we can formulate the following

CONJECTURE. *The Thom polynomials of all Thom–Boardman classes can be written as nonnegative linear combinations of Schur polynomials.*

The nonnegativity of coefficients of Thom polynomials has also been observed by Pragacz in [Pra].

With some work, we can get a more compact formula. Recall the shorthand notations

$$\left\{ {n \atop k} \right\} := \sum_{j=0}^{k} \binom{n}{j} \quad \text{and} \quad F_{\lambda/\mu}(n) := \det \left[ \left\{ {\lambda_k + n - k \atop \mu_l + n - l} \right\} \right]_{k,l \le n}.$$

Note that the numbers $\left\{ {n \atop k} \right\}$ also form a Pascal-like triangle:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 2 & & & & \\
 & & & 1 & & 3 & & 4 & & & \\
 & & 1 & & 4 & & 7 & & 8 & & \\
 & 1 & & 5 & & 11 & & 15 & & 16 & \\
1 & & 6 & & 16 & & 26 & & 31 & & 32
\end{array}$$

Theorem 4.10.

$$[\Sigma^{i,1}(r)] = \sum_{(\nu,\mu)\in I} F_{\nu/\mu}(i) \cdot s_{(i^h + \mathsf{C}\widetilde{\nu}, \widetilde{\mu})}$$

*where* $I = \{(\nu,\mu) : \nu \subset h^i,\ l(\mu) \le i,\ \text{and } |\nu| - |\mu| = i - 1\}$.

Remark. Note that the coefficients do not depend on the relative codimension $r = h - i$. This is not a coincidence, and a similar phenomenon occurs for a large class of singularities; see [FR] and the discussion in the introduction.

*Proof.* According to Theorem 4.9, the coefficient $a_{\nu,\mu}$ of $s_{(i^h + \mathsf{C}\widetilde{\nu}, \widetilde{\mu})}$ is a sum which we can rewrite as follows:

$$a_{\nu,\mu} = \sum_{x \in \{0,1\}^{\mu_1}} E_{\nu/(\widetilde{\mu} - x)^{\sim}}(i) = \sum_{\alpha_1 = \mu_2}^{\mu_1} \sum_{\alpha_2 = \mu_3}^{\mu_2} \cdots \sum_{\alpha_i = 0}^{\mu_i} E_{\nu/\alpha}(i).$$

Expanding the determinant $E_{\nu/\alpha}(i)$ and rearranging the sums yields

$$a_{\nu,\mu} = \det \left[ \left\{ {\nu_k + i - k \atop \mu_l + i - l} \right\} - \left\{ {\nu_k + i - k \atop \mu_{(l+1)} + i - (l+1)} \right\} \right]_{k,l \le i}.$$

Observe that $a_{\nu,\mu}$ is of the form $\det(A - B)$ where

$$B_{k,l} = \begin{cases} A_{k,l+1} & \text{if } l < n, \\ 0 & \text{if } l = n. \end{cases}$$

It is then an easy exercise to prove that $\det(A + \beta B) = \det(A)$ for any $\beta \in \mathbb{C}$. ∎

**5. Review of Ronga's formula.** Ronga's result [Ron72], expressed in our language, is the following. Let $D^n \to M$ be a rank $n$ complex vector bundle, $p : \mathrm{Gr}_i(D) \to M$ the bundle of Grassmannians of $i$-planes in $D$, and let $E \to \mathrm{Gr}_i(D)$ be the tautological subbundle over $\mathrm{Gr}_i(D)$; finally, $\pi : \mathrm{Gr}_j(E) \to \mathrm{Gr}_i(D)$ is the bundle of Grassmannians of $j$-planes in $E$ and $0 \to R^j \to \pi^* E \to Q^{i-j} \to 0$ the tautological exact sequence of vector bundles over $\mathrm{Gr}_j(E)$. Furthermore, introduce the shorthand notations $h = r + i$ and $k = ij - \binom{j}{2}$. Now

$$\begin{aligned}&[\Sigma^{i,j}(r)](-D)\\ &\quad= (-1)^{hk} p_* \pi_* [s_{(n+r)^i}(\pi^* E^*) \cdot s_{h^k}(R \otimes Q + \mathrm{Sym}^2 R + \pi^* p^* D - \pi^* E)]\end{aligned}$$

where the left hand side means that we insert the Chern classes of $-D$ (which are the same as the Segre classes of $D^*$) into the universal polynomial $[\Sigma^{i,j}(r)]$. This formula is not very well suited for direct computations; nevertheless, we can use it to get some qualitative information about these Thom polynomials.

THEOREM 5.1. *Write the universal polynomial* $[\Sigma^{i,j}(r)]$ *as a linear combination of Schur polynomials*: $[\Sigma^{i,j}(r)] = \sum e^\lambda s_\lambda$, *where* $e^\lambda$ *are* (*integer*) *coefficients. Then* $e^\lambda = 0$ *if* $\lambda$ *satisfies any of the following three conditions*:

(a) $i^h \not\subset \lambda$,
(b) $(i+1)^{h+1} \subset \lambda$,
(c) $\lambda_1 > i + k$.

We will use the following well known lemma (see [JLP82]) about push-forwards (or Gysin maps):

LEMMA 5.2. *Let* $E^n \to M$ *be a complex vector bundle*, $\pi : \mathrm{Gr}_r(E) \to M$ *the bundle of Grassmannians of* $r$*-planes in* $E$, *and* $0 \to R^r \to \pi^* E \to Q^q \to 0$ *the tautological exact sequence of bundles over* $\mathrm{Gr}_r(E)$. *Then*

$$\pi_*[s_\mu(R) s_\nu(Q)] = s_{(\nu - r^q, \mu)}(E).$$

REMARK. This formula should be understood as follows: $(\nu - r^q, \mu)$ is very often not a valid partition; but we can extend the definition of the Schur polynomials to arbitrary sequences. Every such "generalized Schur polynomial" is either zero or a "usual Schur polynomial" up to sign. For example for the particular case $\nu = 0$ the formula gives

$$\pi_* s_\mu(R) = s_{(-r^q, \mu)}(E) = (-1)^{qr} s_{\mu - q^r}(E),$$

which is zero if $q^r \not\subset \mu$ (this special case was also proved in [Ron72]). Note also that $\pi_*$ is an $H^*(M)$-module homomorphism.

*Proof of Theorem 5.1.* With some abuse of notation, we will omit pull-backs from the formulas; that is, we will simply write $E$ instead of $\pi^* E$ and so on. All three claims will be consequences of the following computation. First, using the expansion

$$s_\lambda(A+B)=\sum_{\mu,\nu} c^\lambda_{\mu,\nu}\cdot s_\mu(A)s_\nu(B),$$

which, for the special case $\lambda=h^k$ gives $s_{h^k}(A+B)=\sum_{\mu\subset h^k} s_\mu(A)s_{\complement\mu}(B)$, we get

$$[\Sigma^{i,j}(r)](-D)=(-1)^{hk}\sum_{\lambda\subset h^k} p_*[s_{(n+r)^i}(E^*)s_\lambda(D-E)\cdot\pi_* s_{\complement\lambda}(R\otimes Q+\operatorname{Sym}^2 R)].$$

We are not interested in the exact result of the inner pushforward; instead we just set

$$(-1)^{hk}\cdot\pi_* s_{\complement\lambda}(R\otimes Q+\operatorname{Sym}^2 R)=\sum_{l(\mu)\le i} f^\mu_\lambda\cdot s_\mu(E),$$

where $f^\mu_\lambda$ are some coefficients. Using the above expansion again, now for $s_\lambda(D-E)$, we get

$$[\Sigma^{i,j}(r)](-D)=\sum_{\lambda\subset h^k}\sum_{\alpha,\beta\subset\lambda}\sum_{l(\mu)\le i} c^\lambda_{\alpha,\beta} f^\mu_\lambda\cdot s_\alpha(D)\cdot p_*[s_{(n+r)^i}(E^*)s_{\widetilde\beta}(E^*)s_\mu(E)].$$

Using the Littlewood–Richardson rule, Lemma 5.2 and the fact that the rank of $E$ is $i$, we find immediately that

$$p_*[s_{(n+r)^i}(E^*)s_{\widetilde\beta}(E^*)s_\mu(E)]=\sum_{l(\gamma)\le i} g_\gamma\cdot s_{h^i+\gamma}(D),$$

where the $g_\gamma$'s are integer coefficients. Now, we see that $[\Sigma^{i,j}](-D)$ is a linear combination of terms of the form $s_\alpha(D)s_{(h^i+\gamma)}(D)$, where $\alpha\subset h^k$ and $l(\gamma)\le i$. From the Littlewood–Richardson rule it follows directly that the expansion of such a term satisfies the *duals* of all three claims of the theorem, that is, the duals of the partitions appearing in the expansions satisfy the three conditions; thus, by the identity $s_\lambda(-D)=s_{\widetilde\lambda}(D^*)=(-1)^{|\lambda|}s_{\widetilde\lambda}(D)$ the theorem follows. ∎

**6. Examples.** The Thom polynomials of the singularities $\Sigma^{i,j}(-i+1)$ for $j\le 2$ are

$$[\Sigma^{i,0}]=s_i,$$
$$[\Sigma^{i,1}]=is_{i+1}+2s_{i,1},$$
$$[\Sigma^{i,2}]=\binom{i+1}{3}s_{i+3}+(i^2-1)s_{i+2,1}+2(i+1)s_{i+1,2}+2(i-1)s_{i+1,1,1}+4s_{i,2,1}.$$

*Morin singularities.* Recall that the Morin singularity $A_2(r)$, where $r$ is the relative codimension $r=p-n$, is $A_2(r)=\Sigma^{1,1}(r)$ for $r$ nonnegative and

$A_2(r) = \Sigma^{1-r,1}(r)$ for $r$ negative. We have

$$[A_2(r)] = \begin{cases} \displaystyle\sum_{k=0}^{r+1} 2^k s_{2^{r+1-k},1^{2k}} & \text{if } r \geq 0, \\ 2s_{1-r,1} + (1-r)s_{2-r} & \text{if } r \leq 0. \end{cases}$$

The $r \geq 0$ case is already known (see [Ron72]).

*Thom polynomials of $\Sigma^{2,1}$.* Let $h = r+2$. Then

$$[\Sigma^{2,1}(r)] = \sum_K \left( \begin{Bmatrix} a+1 \\ d+1 \end{Bmatrix} \begin{Bmatrix} b \\ c \end{Bmatrix} - \begin{Bmatrix} a+1 \\ c \end{Bmatrix} \begin{Bmatrix} b \\ d+1 \end{Bmatrix} \right) \cdot s_{(h+d,h+c,h-b,h-a)^\sim}$$

where $K = \{(a,b,c,d) \in \mathbb{N}^4 : b \leq a \leq h,\ c \leq d,\ c+d = a+b-1\}$.

Remark. We can state the analogous theorems for real singularities using cohomology with $\mathbb{Z}_2$-coefficients, by replacing Chern classes with the corresponding Stiefel–Whitney classes. The Thom polynomials for the *real* $\Sigma^{i,1}(-i+1)$ were already calculated by Thom in [Tho56].

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Department of Analysis
Eötvös University
Budapest, Hungary
E-mail: lfeher@cs.elte.hu

Department of Mathematics
Central European University
Budapest, Hungary
E-mail: komuves@renyi.hu

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