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On the continuity of the Hausdorff dimension of the Julia–Lavaurs sets

by

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Abstract. Let $f_0(z) = z^2 + 1/4$. We denote by $\mathcal{E}_0$ the set of parameters $\sigma \in \mathbb{C}$ for which the critical point 0 escapes from the filled-in Julia set $K(f_0)$ in one step by the Lavaurs map g_σ. We prove that if $\sigma_0 \in \partial\mathcal{E}_0$, then the Hausdorff dimension of the Julia–Lavaurs set $J_{0,\sigma}$ is continuous at σ_0 as the function of the parameter $\sigma \in \overline{\mathcal{E}_0}$ if and only if $\mathrm{HD}(J_{0,\sigma_0}) \geq 4/3$. Since $\mathrm{HD}(J_{0,\sigma}) > 4/3$ on a dense set of parameters which correspond to preparabolic points, the lower semicontinuity implies the continuity of $\mathrm{HD}(J_{0,\sigma})$ on an open and dense subset of $\partial\mathcal{E}_0$.

1. Introduction. For a polynomial f we define the *filled-in Julia set* $K(f)$ as the set of points that do not escape to infinity under iteration of f. The boundary of $K(f)$ is called the *Julia set* of f and denoted by $J(f)$.

Put $f_\varepsilon(z) = z^2+1/4+\varepsilon$. The function $\varepsilon \mapsto \mathrm{HD}(J(f_\varepsilon))$, where HD denotes the Hausdorff dimension, is real-analytic on the set of hyperbolic parameters (see [4]). We can also consider the function $\varepsilon \mapsto J(f_\varepsilon)$ with values in $\mathcal{K}(\mathbb{C})$, the space of nonempty compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric. We know from [1] that this function is continuous on the set of hyperbolic parameters.

Let us consider the set of real nonnegative parameters ε. The maps f_ε are hyperbolic provided $\varepsilon > 0$, whereas f_0 has a parabolic fixed point at $1/2$, with multiplier 1. It is of interest to study the behaviour of $J(f_\varepsilon)$ when $\varepsilon \searrow 0$. We know that the function $\varepsilon \mapsto J(f_\varepsilon)$ is not continuous at zero from the right. The possible limits of $J(f_\varepsilon)$ in the space $\mathcal{K}(\mathbb{C})$, which occur after passing to a subsequence, are called *Julia–Lavaurs sets*. These sets depend on a parameter $\sigma \in \mathbb{R}$ and will be denoted by $J_{0,\sigma}$. For background information see [1]. We recall some facts in the next section.

The function $\varepsilon \mapsto \mathrm{HD}(J(f_\varepsilon))$ is not continuous at zero either (see [2]). If we choose a sequence of parameters ε_n so that $J(f_{\varepsilon_n})$ converges in the

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Hausdorff metric to $J_{0,\sigma}$ for some σ, then also $\mathrm{HD}(J(f_{\varepsilon_n}))$ converges to $\mathrm{HD}(J_{0,\sigma})$. It is a famous problem whether the Hausdorff dimension of $J_{0,\sigma}$ depends on the parameter σ, and whether the limit $\lim_{\varepsilon\searrow 0} d(\varepsilon)$ exists.

We can also define Julia–Lavaurs sets for parameters $\sigma \in \mathbb{C}$, using the so-called Lavaurs map g_σ (Julia–Lavaurs sets depend in fact only on the class of σ in the space $\mathbb{C}/\mathbb{Z}$). A parameter $\sigma \in \mathbb{C}$ is called *hyperbolic* if the critical point 0 escapes from $K(f_0)$ under iteration of g_σ. The set of hyperbolic parameters is denoted by $\mathcal{E}$. We are interested in its connected component containing $\mathbb{R}$, which is denoted by $\mathcal{E}_0$. Equivalently $\sigma \in \mathcal{E}_0$ if and only if $g_\sigma(0) \notin K(f_0)$. The Hausdorff dimension of $J_{0,\sigma}$ can be treated as a function of the parameter: $\sigma \mapsto \mathrm{HD}(J_{0,\sigma})$. We will consider $\sigma \in \overline{\mathcal{E}_0}$.

M. Urbański and M. Zinsmeister proved in [5] that if $\sigma_n \in \mathcal{E}_0$ converges to $\sigma_0 \in \partial\mathcal{E}_0$ along an external ray, then $\mathrm{HD}(J_{0,\sigma_n})$ converges to $\mathrm{HD}(J_{0,\sigma_0})$. They also showed that $\mathrm{HD}(J_{0,\sigma})$ is real-analytic on the set $\mathcal{E}$ (see [6]). We will prove the following:

THEOREM 1.1. *The function* $\sigma \mapsto \mathrm{HD}(J_{0,\sigma})$ *defined on the set* $\overline{\mathcal{E}_0}$ *is continuous at* $\sigma_0 \in \partial\mathcal{E}_0$ *if and only if* $\mathrm{HD}(J_{0,\sigma_0}) \geq 4/3$. *Moreover the function* $\sigma \mapsto \max\{\mathrm{HD}(J_{0,\sigma}), 4/3\}$ *is continuous on* $\partial\mathcal{E}_0$.

As a corollary we will get continuity on an open and dense subset of $\partial\mathcal{E}_0$.

We will use the following notation: $A \asymp B$ means that $K^{-1} < A/B < K$, where the constant $K > 1$ does not depend on the parameter.

2. The Julia–Lavaurs set. Now we introduce so-called Fatou coordinates. Note that the point $1/2$, which is a parabolic fixed point of f_0, is a common point of the boundaries of the disks $B(3/8, 1/8)$ and $B(5/8, 1/8)$.

There exist holomorphic injective maps

$$\phi^- : B(3/8, 1/8) \to \mathbb{C}, \qquad \phi^+ : B(5/8, 1/8) \to \mathbb{C}$$

(attracting and repelling Fatou coordinate) for which

$$\begin{aligned} &\phi^-(f_0(z)) = \phi^-(z) + 1 \quad \text{provided } z \in B(3/8, 1/8)\\ &\phi^+(f_0(z)) = \phi^+(z) + 1 \quad \text{provided } z, f_0(z) \in B(5/8, 1/8). \end{aligned} \tag{2.1}$$

Moreover there exists a constant $M > 0$ such that

$$\begin{aligned} &\phi^-(B(3/8, 1/8)) \supset \{z : \mathrm{Re}(z) > M\},\\ &\phi^+(B(5/8, 1/8)) \supset \{z : \mathrm{Re}(z) < -M\}. \end{aligned} \tag{2.2}$$

We can also assume that $\phi^-(3/8) = \phi^+(5/8) = 0$.

Using (2.1) the function ϕ^- can be extended to the union of the preimages of $B(3/8, 1/8)$ under iterations of f_0, that is, $\mathrm{Int}\, K(f_0)$. The function obtained is not injective, and has critical points at zero and at each preimage of zero under f_0^n, $n \geq 1$.

Denote by Ψ^+ the inverse function of ϕ^+. So, Ψ^+ is defined only on $\phi^+(B(5/8,1/8))$, but if we take $\phi^+(z)=Z$ then (2.1) leads to $f_0(\Psi^+(Z)) = \Psi^+(Z+1)$. This equation and (2.2) allow us to extend Ψ^+ to the whole $\mathbb{C}$. This new function Ψ^+ has a critical point at Z if and only if $\Psi^+(Z-n)=0$ for some $n\in\mathbb{N}\setminus\{0\}$.

Let T_σ denote the translation $Z\mapsto Z+\sigma$. *The Lavaurs map* g_σ : $\operatorname{Int} K(f_0)\to\mathbb{C}$ is

$$g_\sigma = \Psi^+\circ T_\sigma\circ\phi^-.$$

Note that

$$g_\sigma\circ f_0 = f_0\circ g_\sigma = g_{\sigma+1}. \tag{2.3}$$

The set of critical points of g_σ consists of all points z for which $f_0^k(z)=0$ for all $k\ge 0$ or $g_{\sigma-n}(z)=0$ for all $n\in\mathbb{N}\setminus\{0\}$.

We say that a point $z\in K(f_0)$ *escapes* from $K(f_0)$ by (f_0,g_σ) if there exists $m\ge 1$ such that $g_\sigma^m(z)$ is well defined and $g_\sigma^m(z)\notin K(f_0)$. The remaining points of $K(f_0)$ are called *non-escaping* (in particular $J(f_0)$ and points whose forward trajectory under g_σ hit $J(f_0)$).

We define the *filled-in Julia–Lavaurs set* $K_{0,\sigma}$ as the set of all non-escaping points from $K(f_0)$.

The boundary $\partial K_{0,\sigma}$ is called the *Julia–Lavaurs set* and is denoted by $J_{0,\sigma}$.

Proposition 2.1 ([1]). *We have*

$$J_{0,\sigma} := \overline{\{z : \exists_{m\in\mathbb{N}},\ g_\sigma^m(z)\in J(f_0)\}}.$$

If $\sigma-\sigma'\in\mathbb{Z}$ then (2.3) leads to $K_{0,\sigma}=K_{0,\sigma'}$. Thus, instead of σ one can consider its class in $\mathbb{C}/\mathbb{Z}$.

A parameter σ is called *hyperbolic* if 0 escapes under g_σ, that is, $0\notin K_{0,\sigma}$. The set of all hyperbolic parameters will be denoted by $\mathcal{E}$.

Theorem 2.2 ([1]). *If $\sigma\in\mathcal{E}$ then*

$$J_{0,\sigma}=K_{0,\sigma}.$$

The map g_σ and the set $J_{0,\sigma}$ can also be defined as follows:

Proposition 2.3 ([5], [1]). *If $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is a sequence of complex numbers with positive real parts which converges to zero in such a way that there exists a sequence of positive integers $N_n\to\infty$ such that*

$$\frac{-\pi}{\sqrt{\varepsilon_n}} + N_n \to \sigma, \tag{2.4}$$

then $f_{\varepsilon_n}^{N_n}$ converges to g_σ almost uniformly on $\operatorname{Int} K(f_0)$. *Moreover, if $\sigma\in\mathcal{E}$ then the set $J(f_{\varepsilon_n})$ converges to $J_{0,\sigma}$ in the space $\mathcal{K}(\mathbb{C})$.*

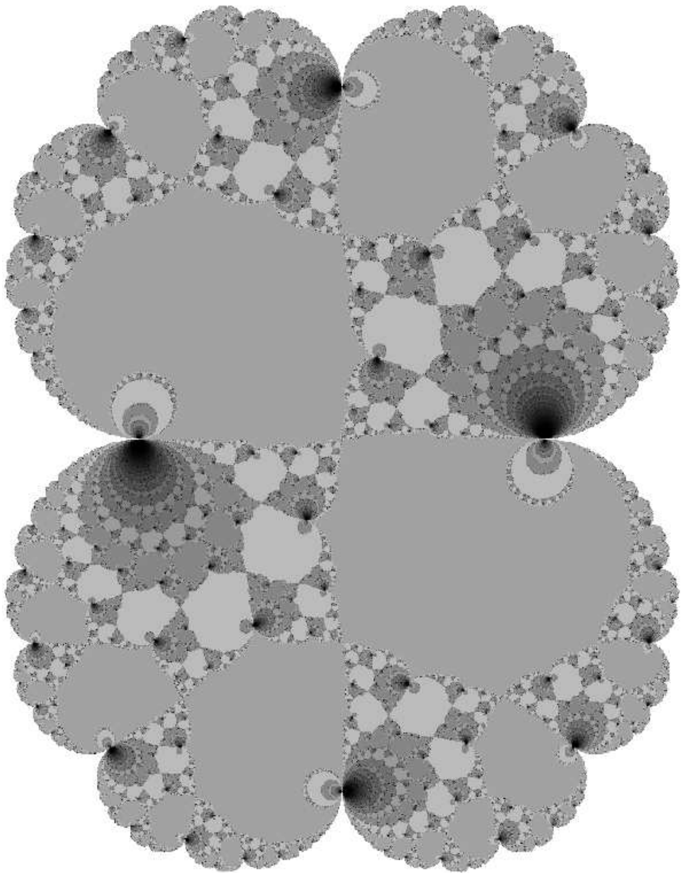

Fig. 1. The Julia–Lavaurs set for $\sigma \in \partial\mathcal{E}_0$

Let us denote by $\mathcal{E}_0$ the connected component of $\mathcal{E}$ containing $\mathbb{R}$. The set $\mathcal{E}_0$ can also be defined as the set of parameters for which 0 escapes in one step, i.e. $g_\sigma(0) \notin K(f_0)$. Further, we will consider the closure of the set $\mathcal{E}_0$.

Proposition 2.4 ([5]). *If $\sigma \in \overline{\mathcal{E}_0}$, then*

$$J_{0,\sigma} = K_{0,\sigma}$$

and the map $\sigma \mapsto J_{0,\sigma}$ is continuous on $\overline{\mathcal{E}}_0$.

If $\sigma \in \mathcal{E}_0$ then the set $g_\sigma^{-1}(\operatorname{Int} K(f_0))$ consists of countably many connected components which are topological disks. For every $x \in J_0$ such that $f_0^n(x) = 1/2$ for all $n \geq 0$ there exist two components of $g_\sigma^{-1}(\operatorname{Int} K(f_0))$ which have x in their boundaries. Denote by $\hat{P}^+_{1/2}$ and $\hat{P}^-_{1/2}$ the closures of the upper and the lower component respectively containing $1/2$. The set

$$\mathcal{P}_{1/2} = (\hat{P}^-_{1/2} \cup \hat{P}^+_{1/2}) \cap K_{0,\sigma}$$

will be called the *butterfly at* $1/2$ and the sets $P^{\pm}_{1/2} = \hat{P}^{\pm}_{1/2} \cap K_{0,\sigma}$ will be called the *wings* of the butterfly at $1/2$.

If x is a preparabolic point, then we define the butterfly $\mathcal{P}_x$, and its wings P^+_x and P^-_x, as the preimages of $\mathcal{P}_{1/2}$, $P^+_{1/2}$ and $P^-_{1/2}$ respectively. The Julia–Lavaurs set is the union of J_0 and the butterflies at $1/2$ and at all the preparabolic points. Clearly all the butterflies are pairwise disjoint if $\sigma \in \mathcal{E}_0$.

If $\sigma \in \partial\mathcal{E}_0$ then the butterflies and their wings can be defined analogously, but they are not pairwise disjoint. For example, if $\operatorname{Im}\sigma > 0$ then we have $P^+_{1/2} \cap P^+_{-1/2} = \{0\}$ (see Figure 1), whereas if $\operatorname{Im}\sigma < 0$ then $P^-_{1/2} \cap P^-_{-1/2} = \{0\}$. Note that $P^+_{-1/2}$ is the lower wing at $-1/2$, since by definition $f_0(P^+_{-1/2}) = P^+_{1/2}$.

3. Partition. First we assume that $\sigma \in \mathcal{E}_0$. Let $A_{0,0}$ denote the closed subset of $J_{0,\sigma}$ which contains $-1/2$ and lies between the external rays of arguments $\frac{1}{3} \cdot 2\pi$, $\frac{2}{3} \cdot 2\pi$. In this section we describe (with more precision) a partition of $A_{0,0}$, which was defined in [5], and denoted by $\{A_n\}$.

We start with a partition of the whole set $J_{0,\sigma}$ (into the sets $A_{0,0}$, $A^s_{p,q}$ where $q \in \mathbb{Z}$ if $|p| \geq 2$, and $q \leq -1$ if $|p| = 1$), called the DSZ partition. It was defined in [2] and also used in [5], but we change the notation.

Let $A^s_{1,-q}$ and $A^s_{-1,-q}$, where $q \geq 1$, be the successive preimages of $A_{0,0}$ under f_0^q, in the upper and lower half-plane respectively. Notice that the union of $A_{0,0}$ and all the sets $A^s_{\pm 1,-q}$ is equal to $J_{0,\sigma} \setminus \mathcal{P}_{1/2}$.

Now we cut $P^+_{1/2}$. For $A^s_{2,0}$ we take the closure of the "first" component of $g_\sigma^{-1}(A_{0,0}) \cap P^+_{1/2}$ (there are infinitely many components), namely the component with the property that any other component is included in $f_0^n(A^s_{2,0})$ for some $n \geq 1$. We define

$$A^s_{2,q} := f_0^q(A^s_{2,0}) \cap P^+_{1/2} \quad \text{for } q \in \mathbb{Z}$$

(note that $g_\sigma(A^s_{2,q}) = A^s_{1,q}$ if $q \leq -1$, and $g_\sigma(A^s_{2,q}) = f_0^q(A_{0,0})$ if $q \geq 0$), next

$$A^s_{p,q} := g_\sigma^{2-p}(A^s_{2,q}) \cap P^+_{1/2} \quad \text{for } p \geq 3.$$

Thus we have defined the partition of $P^+_{1/2}$ into the sets $A^s_{p,q}$ where $p \geq 2$.

The partition of $P^-_{1/2}$ can be described analogously, but we take $p \leq -2$ and replace above p by $|p|$. The sets $A_{0,0}$ and $A^s_{p,q}$ form the DSZ partition of $J_{0,\sigma} \setminus \{1/2\}$. The DSZ partition can also be defined for $\sigma \in \partial\mathcal{E}_0$ by a continuity argument.

Now we describe the partition $\{A_n\}$ of the set $A_{0,0}$.

Step 1. First we use only symmetry and the DSZ partition. Let $s(z) = -z$ and let $A_{p,q} := s(A^s_{p,q})$. Because the set $J_{0,\sigma}$ is symmetric with respect to 0, the family $A_{p,q}$, where $|p| \geq 2$, $q \in \mathbb{Z}$, is a partition of $\mathcal{P}_{-1/2} \subset A_{0,0}$. In order to get a partition of $A_{0,0}$ we define B as the set of all pairs (p,q) such that $q \in \mathbb{Z}$ if $|p| \geq 2$, and $q \leq -2$ if $|p| = 1$. Then

$$A_{0,0} \setminus \{-1/2\} = \bigcup_{(p,q)\in B} A_{p,q}.$$

It follows from the construction that $A_{0,0}$ can be carried onto $A_{p,q}$ by a map $\varphi_{p,q}$ which is a composition of f_0^n, g_σ^{-k} and s (or their suitable inverses). Since $g_\sigma \circ f_0 = f_0 \circ g_\sigma$, the maps $\varphi_{p,q}$ can be written in the following form:

$$\varphi_{p,q} = s \circ f_0^q \circ g_\sigma^{1-|p|} : A_{0,0} \to A_{p,q}.$$

If $\sigma \in \mathbb{R}$, then $\{A_n\}$ can be defined as the family $\{A_{p,q}\}$, $(p,q) \in B$. But later on we will consider parameters in $\partial\mathcal{E}_0$, or close to $\partial\mathcal{E}_0$. In this case the critical point 0 belongs, or is close, to wings of the butterflies, and thus the partition $\{A_{p,q}\}$ must be refined.

Step 3/2. The construction will be carried out under the assumption $\sigma \in \partial\mathcal{E}_0$ and $\operatorname{Im}\sigma > 0$.

If $\sigma \in \mathcal{E}_0$, then σ lies on a curve $\phi^+(\Gamma) \subset \mathcal{E}_0$, which is the image of an external ray $\Gamma \subset \mathbb{C} \setminus K(f_0)$. If Γ lands at $z \in \partial K(f_0)$ and $\sigma_0 = \phi^+(z)$, then the curve $\gamma_{\sigma_0} := \phi^+(\Gamma)$ lands at $\sigma_0 \in \partial\mathcal{E}_0$ (if $\sigma > 0$ then $\sigma_0 > 0$). We define the partition for $\sigma \in \mathcal{E}_0$ analogously as for $\sigma_0 \in \partial\mathcal{E}_0$ (cf. [5]).

Recall that the assumption $\sigma \in \partial\mathcal{E}_0$, $\operatorname{Im}\sigma > 0$ implies $\{0\} = P^+_{1/2} \cap P^+_{-1/2}$. Changing σ by adding an integer (this does not change the set $J_{0,\sigma}$) we can also assume that $0 \in A^s_{2,0}$ (or if zero is a common point of two cylinders then $0 \in A^s_{2,0} \cap A^s_{2,1}$). Thus we also have $0 \in A_{2,0}$ and $g_\sigma(A_{2,0}) = g_\sigma(A^s_{2,0}) = A_{0,0}$, in particular $g_\sigma(0) \in A_{0,0}$.

Step 2. The sets $A_{p,q}$, where $(p,q) \in B$, are called *cylinders of order* 0. We want each piece of the partition $\{A_n\}$ (to be constructed) to be mapped onto a "big" set (not necessarily $A_{0,0}$) with bounded distortion by a composition of f_0^n, g_σ^k and s.

First we assign to $\{A_n\}$ all the cylinders of order 0 except $A_{2,0}$ and its two neighbours, i.e. $A_{2,-1}$ and $A_{2,1}$ which we need to partition additionally.

We assume first that $\sigma \in \partial\mathcal{E}_0$ does not correspond to a preparabolic point, i.e. $g_\sigma(0)$ is not a preparabolic point. We cut the cylinders $A_{2,0}$, $A_{2,\pm1}$ into cylinders of order 1, i.e. we take the images of $A_{p,q}$ under $\varphi_{2,0}$ and $\varphi_{2,\pm1}$. In the next step we keep all cylinders of order 1 except a piece which contains 0 and its two neighbours. We cut these three pieces into cylinders of order 2, and so on.

If a parameter corresponds to a preparabolic point the procedure is the same, until we reach the step for which 0 does not belong to any cylinder of the iterated partition; we then stop the process.

LEMMA 3.1. *Each piece of the partition $\{A_n\}$ can be mapped by a composition of f_0^n, g_σ^k and s with uniformly bounded distortion onto $A_{0,0}$ or $A_{2,-1}$.*

Proof. Let $\sigma \in \partial\mathcal{E}_0$. First we consider the pieces A_n of the form $A_{p,q}$. So we deal with $(p,q) \in B \setminus \{(2,-1),(2,0),(2,1)\}$.

If $p \geq 3$ or $p = 2$ and $q \leq -2$, then the maps $\varphi_{p,q}^{-1}$ do not have bounded distortion, but using $g_\sigma^{p-2} \circ f_0^{1-q}$, the sets $A_{p,q}$ can be mapped onto $A_{2,-1}$.

In all the other cases $\varphi_{p,q}^{-1}$ has bounded distortion. More precisely:

If $p \in \{-1,1\}$, then $A_{p,q}$ can be mapped onto $A_{0,0}$ using $f_0^{-q} \circ s$.

The set $A_{-2,0}^s$ is far from the set of critical points, so each piece $A_{p,q}$, where $p \leq -2$, can be carried onto $A_{-2,0}^s$ with bounded distortion and next, using g_σ, onto $A_{0,0}$.

If $p = 2$ and $q \geq 2$, then $\varphi_{p,q}^{-1}$ has bounded distortion although g_σ and f_0^{-q} do not, because $\varphi_{p,q}^{-1}$ can be written as $f_0 \circ g_{\sigma-3} \circ f_0^{2-q} \circ s$. Indeed if we subtract three from σ then there are no critical points in $A_{2,2}$ (and its neighbours).

Now we consider pieces of $\{A_n\}$ which are cylinders of order greater than 0. Let A_n be a cylinder of order k, and let I be a cylinder of order $k-1$ which contains A_n (I does not belong to $\{A_n\}$). Then I contains zero or is a neighbour of such a cylinder.

It follows from the construction that the distortion of g_σ is uniformly bounded on the pieces of $\{A_n\}$. The image of I under g_σ is a cylinder which has nonempty intersection with the Julia set $J(f_0)$. Such a cylinder can be mapped, using composition of $\varphi_{\pm1,q}^{-1}$, onto $A_{0,0}$ with uniformly bounded distortion, whereas the image of $g_\sigma(A_n)$ is a set $A_{p,q}$ which, as we have shown, can be mapped with bounded distortion onto $A_{0,0}$ or $A_{2,-1}$.

If $\sigma \in \mathcal{E}_0$ then analogous considerations can be carried out. ∎

We define cylinders of order two of the partition $\{A_n\}$ as the preimages of the sets A_n under the maps which carry pieces of the partition onto $A_{0,0}$ or $A_{2,-1}$. The new partition will be denoted by $\{A_n^2\}$. Partitions of higher order can be defined analogously and denoted by $\{A_n^k\}$.

The Schwarz lemma and Koebe distortion theorem imply the following corollary:

COROLLARY 3.2. *Each cylinder of the partition $\{A_n^k\}$ for each $k \geq 1$ can be mapped by compositions of f_0^n, g_σ^m and s with uniformly bounded distortion onto $A_{0,0}$ or $A_{2,-1}$.*

REMARK 3.3. Each piece of $\{A_n^k\}$ can be mapped by a composition of $\varphi_{p,q}^{-1}$ onto $A_{0,0}$, but the distortion is unbounded, so we cannot build an IFS. Nevertheless this system can be improved in order to get a graph directed Markov system as in [3], using the mappings of Corollary 3.2:

Let $\{A_{0,0}, A^s_{2,-1}\}$ be the set of vertices. Now we define the partition $\{\hat{A}_n\}$ of $A_{0,0} \cup A^s_{2,-1}$. On the set $A_{0,0}$ we take $\{A^m_n\}$ for m large enough. Then there exists a family of uniform contractions ϕ^s_n mapping $A_{0,0}$ or $A_{2,-1}$ onto each piece of $\{A^m_n\}$. On the set $A^s_{2,-1}$ we take a symmetric partition with respect to $A_{2,-1}$. Next, composing ϕ^s_n with symmetry if necessary, for each piece of $\{\hat{A}_n\}$ we obtain a contraction $\phi_n : X_n \to \hat{A}_n$, where $X_n \in \{A_{0,0}, A^s_{2,-1}\}$. The family of contractions $S_\sigma = \{\phi_n\}$ forms a simple conformal graph directed Markov system (see [3, pp. 1, 3, 71]).

Remark 3.4. Denote the limit set of S_σ by Λ_σ. The set Λ_σ is equal to $A_{0,0} \cup A^s_{2,-1}$ up to some countable subset which consists of preimages of 0 (in the case $\sigma \in \partial\mathcal{E}_0$) and $-1/2$ under compositions of maps that form our system. In particular, $\mathrm{HD}(\Lambda_\sigma) = \mathrm{HD}(A_{0,0} \cup A^s_{2,-1}) = \mathrm{HD}(J_{0,\sigma})$.

4. Conformal measures and semicontinuity. A Borel probability measure ω on $J(f_\varepsilon)$ is said to be *t-conformal* for f_ε if for every Borel subset $A \subset J(f_\varepsilon)$,

$$\omega(f_\varepsilon(A)) = \int_A |f'_\varepsilon|^t \, d\omega \tag{4.1}$$

provided f_ε is injective on A.

If a measure ω is supported on $J_{0,\sigma}$ then we say that ω is *t-conformal for* (f_0, g_σ) if (4.1) is satisfied for f_0 and, under the assumption $A \subset \operatorname{Int} K(f_0)$, also for g_σ (g_σ is defined only on $\operatorname{Int} K(f_0)$).

Let first $\sigma \in \mathcal{E}_0$. Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ satisfy (2.4). Then $J(f_{\varepsilon_n}) \to J_{0,\sigma}$ in $\mathcal{K}(\mathbb{C})$ and for sufficiently large n the maps f_{ε_n} are hyperbolic. So, there exist d_{ε_n}-conformal measures ω_{ε_n} for f_{ε_n} on the set $J(f_{\varepsilon_n})$, where $d_{\varepsilon_n} = \mathrm{HD}(J(f_{\varepsilon_n}))$. Passing to a subsequence, we can get a weakly convergent sequence of measures which converges to a measure ω_σ supported on the set $J_{0,\sigma}$. Because g_σ is a limit of high iterates of f_{ε_n} (see Proposition 2.3), ω_σ is a d_σ-conformal measure for (f_0, g_σ), where $d_\sigma = \lim_{n\to\infty} d_{\varepsilon_n}$. It follows from [2] that $d_\sigma = \mathrm{HD}(J_{0,\sigma})$ and ω_σ is atomless.

For $\sigma \in \overline{\mathcal{E}_0}$, we conclude from [3] that there exists a d_σ-conformal measure $\tilde{\omega}_\sigma$ for S_σ on its limit set Λ_σ (see Remark 3.4), where $d_\sigma = \mathrm{HD}(J_{0,\sigma})$.

We know that for $\sigma \in \mathcal{E}_0$ the d_σ-conformal measure ω_σ for (f_0, g_σ) is atomless. Thus $\omega_\sigma|_{\Lambda_\sigma}$ is positive and so, after normalization, we get a d_σ-conformal measure of the system S_σ, as its elements are compositions of f_0, g_σ and s. The uniqueness leads to $\tilde{\omega}_\sigma = \omega_\sigma|_{\Lambda_\sigma}$. On the other hand, using f_0^{-1} and s, we can obtain ω_σ as the extension of $\tilde{\omega}_\sigma$ from Λ_σ to $J_{0,\sigma}$. More precisely, we reconstruct ω_σ on the set $A^s_{0,0}$ from $A_{0,0}$ using s (ω_σ coincides with $\tilde{\omega}_\sigma$ on $A^s_{2,-1}$), whereas on the sets $A^s_{\pm1,-1}$ the measure ω_σ is defined as $|(f_0^{-1})' \circ f_0|^{d_\sigma} (f_0^{-1})_* \tilde{\omega}_\sigma$ (up to normalization).

For $\sigma_0 \in \partial\mathcal{E}_0$ we can choose a sequence of parameters $\sigma_n \in \mathcal{E}_0$ which converges to σ_0 as in [5]. Namely, if $\sigma_0 = \phi^+(z)$ and $\Gamma_z \subset \mathbb{C} \setminus K(f_0)$ is an external ray landing at z, then we take a sequence tending to σ_0 along the curve $\gamma_{\sigma_0} = \phi^+(\Gamma_z)$. It was proved in [5] that the sequence of d_{σ_n}-conformal measures $\tilde{\omega}_{\sigma_n}$ weakly converges to an atomless d_{σ_0}-conformal measure on Λ_{σ_0} for S_{σ_0}, which by uniqueness is $\tilde{\omega}_{\sigma_0}$. The fact that $\lim_{n\to\infty} d_{\sigma_n} = d_{\sigma_0}$ leads to $\lim_{n\to\infty} \mathrm{HD}(J_{0,\sigma_n}) = \mathrm{HD}(J_{0,\sigma_0})$. This is the continuity proved in [5].

Next, using f_0^{-1} and s in the same scheme as before (extending $\tilde{\omega}_\sigma$ to ω_σ for $\sigma \in \mathcal{E}_0$), we define ω_{σ_0} as the extension of $\tilde{\omega}_{\sigma_0}$ from Λ_{σ_0} to J_{0,σ_0}. Hence ω_{σ_0} is the weak limit of the extensions ω_{σ_n}, therefore ω_{σ_0} is d_{σ_0}-conformal for (f_0, g_{σ_0}). We see that ω_{σ_0} is atomless, thus we have obtained:

Lemma 4.1. *For every $\sigma \in \overline{\mathcal{E}_0}$ there exists an atomless d_σ-conformal measure ω_σ for (f_0, g_σ), where $d_\sigma = \mathrm{HD}(J_{0,\sigma})$.*

Remark 4.2. This atomless property follows in [5] from uniform (from above) estimates of conformal measures, for σ in the external ray γ_{σ_0}, of appropriate clusters of cylinders. The partitions for $\sigma \in \gamma_{\sigma_0}$ are all combinatorially the same. Compare Remark 5.2.

We shall also use the following lemma (standard in the rational functions setting).

Lemma 4.3. *Let $\nu^\pm$ be $d^\pm$-conformal measures for (f_0, g_σ) on $J_{0,\sigma}$, where $d^+ \geq d^-$. Then, if ν^+ is nonatomic, we have $d^+ = d^-$.*

Proof. For every $\delta > 0$, using cylinders of partitions $\{A_n^k\}$, we can construct a cover $\{B_n\}$ of $A_{0,0}$ (minus some countable subset which consists of preimages of 0 and $-1/2$ under suitable compositions of $\varphi_{p,q}$) such that $\mathrm{diam}(B_n) < \delta$ for every $n \in \mathbb{N}$.

Each point belongs to at most two pieces of $\{B_n\}$, whereas each set B_n can be mapped onto a "big" set (i.e. $A_{0,0}$ or $A_{2,-1}$) using a map φ_n, with bounded distortion. Hence we get

$$\sum_{n\in\mathbb{N}} \mathrm{diam}(B_n)^{d^\pm} \asymp \sum_{n\in\mathbb{N}} \|(\varphi_n^{-1})'\|^{d^\pm}.$$

Next, since $\nu^\pm$ are $d^\pm$-conformal (and $\nu^\pm(A_{0,0})$, $\nu^\pm(A_{2,-1})$ are positive), bounded distortion leads to

$$\sum_{n\in\mathbb{N}} \mathrm{diam}(B_n)^{d^\pm} \asymp \sum_{n\in\mathbb{N}} \|(\varphi_n^{-1})'\|^{d^\pm} \asymp \sum_{n\in\mathbb{N}} \nu^\pm(B_n). \tag{4.2}$$

Since the measure ν^+ is nonatomic, we have $\sum \nu^+(B_n) = \nu^+(A_{0,0}) > 0$. Thus, by (4.2), and next using the assumptions $d^+ \geq d^-$, $\mathrm{diam}(B_n) < \delta$ we get

$$K \leq \sum_{n\in\mathbb{N}} \mathrm{diam}(B_n)^{d^+} \leq \delta^{d^+ - d^-} \sum_{n\in\mathbb{N}} \mathrm{diam}(B_n)^{d^-}$$

for a constant $K > 0$. Letting $\delta \to 0$ we see that if $d^+ > d^-$ then $\delta^{d^+ - d^-} \to 0$, which gives us a contradiction. ∎

Let $\sigma_n \in \overline{\mathcal{E}_0}$ be an arbitrary sequence of parameters tending to $\sigma_0 \in \partial\mathcal{E}_0$ such that the sequence of measures ω_{σ_n} weakly converges to a measure ν_{σ_0}. The limit measure ν_{σ_0} is supported on J_{0,σ_0} and is $d^+_{\sigma_0}$-conformal for (f_0, g_{σ_0}), where $d^+_{\sigma_0} = \lim_{n\to\infty} d_{\sigma_n}$. Denote by $\mathcal{N}_{\sigma_0}$ the set of all possible limit measures for the parameter σ_0.

Recall from [6] that the function $\sigma \mapsto \mathrm{HD}(J_{0,\sigma})$ is real-analytic on $\mathcal{E}_0$.

PROPOSITION 4.4. *The function $\sigma \mapsto \mathrm{HD}(J_{0,\sigma})$ is lower semicontinuous on the set $\overline{\mathcal{E}}_0$.*

Proof. We will consider parameters $\sigma \in \partial\mathcal{E}_0$. In order to obtain lower semicontinuity it is enough to prove that if $\nu_{\sigma_0} \in \mathcal{N}_{\sigma_0}$ is a $d^+_{\sigma_0}$-conformal measure, then

$$\mathrm{HD}(J_{0,\sigma_0}) \le d^+_{\sigma_0}. \tag{4.3}$$

In the same way as in the proof of Lemma 4.3, we can construct a cover $\{B_n\}$ of $A_{0,0}$ such that $\mathrm{diam}(B_n) < \delta$. Next, we derive (4.2) and then conclude that

$$\sum_{n\in\mathbb{N}} \mathrm{diam}(B_n)^{d^+_{\sigma_0}} \le K_1 \sum_{n\in\mathbb{N}} \nu_{\sigma_0}(B_n) \le K_2$$

for certain constants K_1, K_2. We see that (4.3) follows from the above, and the proof finished. ∎

Note that lower semicontinuity implies continuity of $\mathrm{HD}(J_{0,\sigma})$ on a dense G_δ subset of $\partial\mathcal{E}_0$. Theorem 1.1 will yield continuity on an open and dense set.

5. Continuity of the dimension. Now we prove a key fact for obtaining continuity of the Hausdorff dimension whenever $\mathrm{HD}(J_{0,\sigma}) \ge 4/3$. It will allow us to use Lemma 4.3.

LEMMA 5.1. *If $\sigma_0 \in \partial\mathcal{E}_0$ and $\nu_{\sigma_0} \in \mathcal{N}_{\sigma_0}$ is a $d^+_{\sigma_0}$-conformal measure, where $d^+_{\sigma_0} > 4/3$, then ν_{σ_0} is atomless.*

REMARK 5.2. Notice that now the partitions for $\sigma_n \to \sigma_0$ can have different combinatorics, unlike for radial convergence (cf. Remark 4.2). Coping with this is the main novelty of the present paper compared to [5].

Proof of Lemma 5.1. It follows from [2] (and also [5]) that ν_{σ_0} has no atoms at parabolic (preparabolic) points. We must check critical (precritical) points, but it is enough to consider the critical point 0. All the other points belong to the limit set Λ_{σ_0} of S_{σ_0}. In particular the derivatives of compositions of the maps in S_{σ_0} tend to infinity, so there cannot be atoms.

If ν_{σ_0} is a weak limit of a sequence $\{\omega_{\sigma_n}\}_{n\in\mathbb{N}}$, and ν_{σ_0} is $d^+_{\sigma_0}$-conformal, then $d^+_{\sigma_0} = \lim_{n\to\infty} d_{\sigma_n}$. Therefore, we conclude from Proposition 4.4 and the assumption $d^+_{\sigma_0} > 4/3$ that $d_{\sigma_n} > 4/3 + \varepsilon$ for some $\varepsilon > 0$ and all n large enough.

Let $U_\sigma(\sqrt{R})$ denote the connected component of the preimage of the disk $B(g_\sigma(0), R)$ under g_σ, which contains 0. Then, uniformly in σ close to σ_0, for some constant C_1, we have

$$B(0, C_1\sqrt{R}) \subset U_\sigma(\sqrt{R}) \subset g_\sigma^{-1}(B(g_\sigma(0), R))$$

(0 is a critical point of degree two). In order to prove that there is no atom at 0, it is enough to show that $\omega_\sigma(U_\sigma(\sqrt{R})) \to 0$ when $R \to 0$, uniformly in σ, provided $\mathrm{HD}(J_{0,\sigma}) = d_\sigma > 4/3 + \varepsilon$.

Estimations will be carried out under the assumption that $\sigma \in \partial\mathcal{E}_0$ and σ does not correspond to a preparabolic point (i.e. $g_\sigma(0)$ is not a preparabolic point). Later on we will deduce the general case.

For each $k \geq 1$, we will estimate the measure of the union of all cylinders of order k which are pieces of $\{A_n\}$ and have nonempty intersection with $U_\sigma(\sqrt{R}) \cap A_{2,0}$. Let us denote this measure by $\xi_k(\sqrt{R})$. To simplify notation we omit σ (we shall try to find a bound not depending on σ). The remaining cylinders are included in $U_\sigma(\sqrt{R}) \cap A_{2,\pm 1}$, and we can deal with them analogously, so we have $\omega_\sigma(U_\sigma(\sqrt{R})) \asymp \sum_{k=1}^{\infty} \xi_k(\sqrt{R})$.

Since 0 is the critical point of degree two, and for some constant $K > 0$, $\mathrm{diam}(g_\sigma(A_n)) < K \,\mathrm{dist}(g_\sigma(0), g_\sigma(A_n))$, it follows that

$$\mathrm{diam}(A_n) \asymp \mathrm{dist}(g_\sigma(0), g_\sigma(A_n))^{-1/2}\,\mathrm{diam}(g_\sigma(A_n)). \tag{5.1}$$

Step 1. First we estimate $\xi_1(\sqrt{R})$. If $g_\sigma(0) \in A_{1,q^*}$, then we denote by A^* the set $B \setminus \{(1, q^*-1), (1, q^*), (1, q^*+1)\}$. We see that cylinders of order 1 included in $A_{2,0}$ are of the form $g_\sigma^{-1}(A_{p,q})$ where $(p, q) \in A^*$.

Since the Fatou coordinates behave like $-1/z$, analogously to [5, proof of Proposition 5.1] or [2] we have

$$\mathrm{dist}(g_\sigma(0), A_{p,q}) \asymp \left| \frac{1}{q + pi} - \frac{1}{q^* + i} \right| \tag{5.2}$$

provided $g_\sigma(0) \in A_{1,q^*}$, $(p, q) \in A^*$. Next, because the diameters of the cylinders are bounded in the Fatou coordinates, we also have

$$\mathrm{diam}(A_{p,q}) \asymp \frac{1}{p^2 + q^2}. \tag{5.3}$$

If $A_n = g_\sigma^{-1}(A_{p,q})$, $(p, q) \in A^*$, then combining (5.2), (5.3) with (5.1) we get

$$\mathrm{diam}(g_\sigma^{-1}(A_{p,q})) \asymp \left| \frac{1}{q + pi} - \frac{1}{q^* + i} \right|^{-1/2} (p^2 + q^2)^{-1}. \tag{5.4}$$

Because of (5.2), in order to estimate $\xi_1(\sqrt{R})$ we can consider only cylinders with indices from the set $A_R^* := \left\{(p,q) \in A^* : \left|\frac{1}{q+pi} - \frac{1}{q^*+i}\right| < C_2 R\right\}$ for a suitable constant C_2. Then by (5.4), writing for simplicity d instead of d_σ, for some constant K_1 we obtain

$$\xi_1(\sqrt{R}) < K_1 \sum_{(p,q)\in A_R^*} \left|\frac{1}{q+pi} - \frac{1}{q^*+i}\right|^{-d/2} (p^2+q^2)^{-d}.$$

This sum will be estimated from above by an integral. Let $I(z) := -1/z$. Writing Z instead of $q+pi$ and taking

$$B_R^* := \left\{z : \frac{C_3}{(q^*)^2+1} < \left|z - \frac{1}{q^*+i}\right| < C_4 R\right\}$$

(where $C_3/((q^*)^2+1)$ is related to the diameter of A_{1,q^*}, which is omitted), for suitable constants C_3, C_4, K_2 we obtain

$$\xi_1(\sqrt{R}) < K_2 \int\limits_{I(B_R^*)} \left|\frac{1}{Z} - \frac{1}{q^*+i}\right|^{-d/2} |Z|^{-2d}\, dl_2(Z).$$

Taking $Z = I(z)$ and $\frac{1}{q^*+i} = z^*$, we get

$$\xi_1(\sqrt{R}) < K_2 \int\limits_{B_R^*} |z-z^*|^{-d/2}|z|^{2d}\left|\frac{1}{z^2}\right|^2 dl_2(z) = \int\limits_{B_R^*} |z-z^*|^{-d/2}|z|^{2d-4}\, dl_2(z).$$

Next, using the Hölder inequality, we estimate the above by

$$\Big(\int\limits_{B_R^*} |z-z^*|^{-\frac{d}{2}\cdot\frac{8-3d}{d}}\, dl_2(z)\Big)^{\frac{d}{8-3d}} \Big(\int\limits_{B_R^*} |z|^{(2d-4)\cdot\frac{8-3d}{8-4d}}\, dl_2(z)\Big)^{\frac{8-4d}{8-3d}}.$$

Because $B_R^* \subset B(z^*, C_4R)$, the integrals can be taken over $B(z^*, C_4R)$ instead of B_R^*. Next, replacing $|z|$ by $|z-z^*|$ in the second integral, the value can only increase, since the exponent is negative and the integral is taken over a ball centred at z^*. Therefore $\xi_1(\sqrt{R})$ can be estimated from above by

$$\Big(\int\limits_{B(z^*,C_4R)} |z-z^*|^{3d/2-4}\, dl_2(z)\Big)^{\frac{d}{8-3d}} \Big(\int\limits_{B(z^*,C_4R)} |z-z^*|^{3d/2-4} dl_2(z)\Big)^{\frac{8-4d}{8-3d}}.$$

Passing to polar coordinates, and using the assumption that $d = d_\sigma > 4/3$, we get

$$\xi_1(\sqrt{R}) < \int_0^{C_4R} r\cdot r^{3d/2-4} dr < \frac{K_3}{3d/2-2} R^{3d/2-2} < K_4 R^{3d/2-2}. \tag{5.5}$$

Step 2. Now we can estimate the measure of the set $U_\sigma(\sqrt{R}) \cap A_{2,0}$. Let k_0 be the least order of a cylinder which is a piece of $\{A_n\}$ and has nonempty intersection with $U_\sigma(\sqrt{R}) \cap A_{2,0}$. The family of all cylinders of order k_0 which are pieces of $\{A_n\}$ and have nonempty intersection with $U_\sigma(\sqrt{R}) \cap A_{2,0}$ will be denoted by $\{Q_n\}$. We estimate the measure of these cylinders in a similar way to $\xi_1(\sqrt{R})$. Let I_k denote the cylinder of order k containing 0 (I_k does not belong to $\{A_n\}$). Then each Q_n is included in I_{k_0-1} or its two neighbours, which as before can be omitted.

We have $g_\sigma(I_{k_0-1}) \subset A_{0,0}$ and $g_\sigma(Q_n) \cap B(g_\sigma(0), R) \neq \emptyset$. Moreover $g_\sigma(I_{k_0-1})$ can be mapped with bounded distortion onto $A_{0,0}$, while the cylinders $g_\sigma(Q_n)$ are mapped onto the sets $A_{p,q}$, and the union of those images is included in a ball of radius $C_5 R/\operatorname{diam}(g_\sigma(I_{k_0-1}))$ for suitable C_5.

In order to estimate ξ_{k_0}, we use the already obtained estimate for ξ_1, taking into account scaling by $\operatorname{diam}(g_\sigma(I_{k_0-1}))$ (up to a constant). We have

$$\xi_{k_0}(\sqrt{R}) < K_4 \operatorname{diam}(g_\sigma(I_{k_0-1}))^{d/2} \xi_1\bigg(\sqrt{\frac{C_5 R}{\operatorname{diam}(g_\sigma(I_{k_0-1}))}}\bigg),$$

where the factor $\operatorname{diam}(g_\sigma(I_{k_0-1}))^{d/2} = \operatorname{diam}(g_\sigma(I_{k_0-1}))^{d(-1/2+1)}$ comes from (5.1).

In the case of cylinders of order $k > k_0$ we proceed analogously. So, because images of cylinders are included in $A_{0,0}$, we get

$$\xi_k(\sqrt{R}) < K_5 \operatorname{diam}(g_\sigma(I_{k-1}))^{d/2}\, \xi_1\big(\sqrt{\operatorname{diam}(A_{0,0})}\big).$$

The assumption $d = d_\sigma > 4/3+\varepsilon$ implies that the quantity $\xi_1(\sqrt{\operatorname{diam}(A_{0,0})})$ can be estimated by a constant, uniformly in σ. Thus, summing and using (5.5), we obtain

$$\begin{aligned}(5.6)\qquad \omega_\sigma(U_\sigma(\sqrt{R})) < K_6 \operatorname{diam}(g_\sigma(I_{k_0-1}))^{d/2}\bigg(\frac{C_5 R}{\operatorname{diam}(g_\sigma(I_{k_0-1}))}\bigg)^{3d/2-2} \\ + K_7 \sum_{k>k_0} \operatorname{diam}(g_\sigma(I_{k-1}))^{d/2}.\end{aligned}$$

Since the diameter of I_k decreases at least as fast as a geometric sequence, we have

$$\begin{aligned}(5.7)\qquad \omega_\sigma(U_\sigma(\sqrt{R})) < K_8 \operatorname{diam}(g_\sigma(I_{k_0-1}))^{2-d} R^{3d/2-2} \\ + K_9 \operatorname{diam}(g_\sigma(I_{k_0}))^{d/2},\end{aligned}$$

which tends to zero uniformly in σ if $d = d_\sigma > 4/3+\varepsilon$ (we can assume that $\operatorname{diam}(g_\sigma(I_{k_0})) < K_{10} R$). Recall that we assumed that $\sigma \in \partial\mathcal{E}_0$ and σ does not correspond to a preparabolic point.

Step 3. If a parameter is related to a preparabolic point, then the order of cylinders which intersect $U_\sigma(\sqrt{R}) \cap A_{2,0}$ is bounded. First we assume that only cylinders of order 1 have nonempty intersection. In this case we take $q^* = \infty$ (the sets A^* and B are equal), and next $B^*_R = B(z^*, C_4 R) \setminus \{z^*\}$, where $z^* = 0$. Then we can deduce the estimate (5.5).

If all cylinders which intersect $U_\sigma(\sqrt{R}) \cap A_{2,0}$ have the same order then, as before, using bounded distortion we reduce the problem to the case of order 1. If the orders are not the same, then we obtain estimate (5.6), but with a finite series.

If $\sigma \in \mathcal{E}_0$ then the same estimations can be carried out (see [5, Section 6]).

Finally, estimate (5.7) holds for every $\sigma \in \overline{\mathcal{E}_0}$ provided $d = d_\sigma > 4/3$, which ends the proof. ■

Recall that σ corresponds to a preparabolic point if $g_\sigma(0)$ is a preparabolic point. It follows from [5, Proposition 5.1] that

Proposition 5.3. *If σ corresponds to a preparabolic point then* $\mathrm{HD}(J_{0,\sigma}) > 4/3$.

Proof of Theorem 1.1. We conclude from Proposition 4.4 that

$$\liminf_{\sigma_n \to \sigma_0} \mathrm{HD}(J_{0,\sigma_n}) \geq \mathrm{HD}(J_{0,\sigma_0}). \tag{5.8}$$

Next, the fact that parameters which correspond to preparabolic points are dense in $\partial\mathcal{E}_0$ and Proposition 5.3 gives us

$$\limsup_{\sigma_n \to \sigma_0} \mathrm{HD}(J_{0,\sigma_n}) \geq 4/3. \tag{5.9}$$

Now we prove that

$$\limsup_{\sigma_n \to \sigma_0} \mathrm{HD}(J_{0,\sigma_n}) \leq \max\{\mathrm{HD}(J_{0,\sigma_0}), 4/3\}. \tag{5.10}$$

It is enough to show that if $\nu_{\sigma_0} \in \mathcal{N}_{\sigma_0}$ is a $d^+_{\sigma_0}$-conformal measure, then $d^+_{\sigma_0}$ cannot be greater than $\max\{\mathrm{HD}(J_{0,\sigma_0}), 4/3\}$. If it is, then $d^+_{\sigma_0} > 4/3$ and Lemma 5.1 implies that the measure ν_{σ_0} is atomless. So, it follows from Lemmas 4.3 and 4.1 that $d^+_{\sigma_0} = \mathrm{HD}(J_{0,\sigma_0})$, which is a contradiction.

Thus (5.8), (5.10) gives us continuity of the Hausdorff dimension provided $\mathrm{HD}(J_{0,\sigma_0}) \geq 4/3$.

If we assume that $\mathrm{HD}(J_{0,\sigma_0}) < 4/3$, then combining (5.9) with (5.10), we see that

$$\limsup_{\sigma_n \to \sigma_0} \mathrm{HD}(J_{0,\sigma_n}) = 4/3.$$

Therefore the Hausdorff dimension is not continuous, but we obtain continuity of the function $\sigma \mapsto \max\{\mathrm{HD}(J_{0,\sigma}), 4/3\}$. ■

The lower semicontinuity (see Proposition 4.4) implies that the condition $\mathrm{HD}(J_{0,\sigma_0}) > 4/3$ is satisfied on an open subset of $\partial\mathcal{E}_0$. So, because of density of parameters which correspond to preparabolic points and Proposition 5.3, Theorem 1.1 gives us:

Corollary 5.4. *The Hausdorff dimension of $J_{0,\sigma}$ as a function of the parameter $\sigma \in \overline{\mathcal{E}_0}$ is continuous on an open and dense subset of $\partial\mathcal{E}_0$, in particular at all parameters which correspond to preparabolic points.*

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