

Lifting di-analytic involutions of compact Klein surfaces to extended-Schottky uniformizations

by

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Abstract. Let S be a compact Klein surface together with a di-analytic involution $\kappa : S \rightarrow S$. The lowest uniformizations of S are those whose deck group is an extended-Schottky group, that is, an extended Kleinian group whose orientation preserving half is a Schottky group. If S is a bordered compact Klein surface, then it is well known that κ can be lifted with respect to a suitable extended-Schottky uniformization of S . In this paper, we complete the above lifting property by proving that if S is a closed Klein surface, then κ can also be lifted to a suitable extended-Schottky uniformization.

1. Introduction. A *classical Riemann surface* is a topological surface without boundary together with an analytic structure. A *closed Riemann surface* is a compact Riemann surface. A *compact Klein surface* is either a non-orientable compact surface or a bordered orientable surface together with a *di-analytic structure*, that is, a collection of smooth coordinate charts so that each transition function is either conformal or anticonformal. A *closed Klein surface* is an unbordered compact Klein surface.

The basic theory of compact Klein surfaces and the functorial equivalence between them and real algebraic curves can be found in [AG]. The case of bordered compact Klein surfaces has been well studied in the literature (see for instance [AG, BCG, BCNS, Ha, Ma]). On the other hand, the case of closed Klein surfaces has not been studied in detail.

An anticonformal automorphism of a closed Riemann surface will be called a *reflection* if it has fixed points (for instance, $z \mapsto \bar{z}$) and an *imaginary reflection* otherwise (for instance, $z \mapsto -1/\bar{z}$). Each connected component of fixed points of a reflection τ is called an *oval* or a *mirror*. A *di-analytic automorphism* of a compact Klein surface is a self-homeomorphism which in local coordinates is either conformal or anticonformal.

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If S is a compact Klein surface, then there is a closed Riemann surface S^+ , an anticonformal involution $\tau : S^+ \rightarrow S^+$ and a regular (branched) cover map $P : S^+ \rightarrow S$ with deck group $\langle \tau \rangle$. The surface S^+ is known as a *double oriented cover* of S and the genus of S^+ is known as the *algebraic genus* of S . Moreover, τ is a reflection if and only if S has non-empty boundary; in fact, the number of boundary components of S is equal to the number of ovals of τ . A classical result of Harnack [Ha] states that the number of ovals is at most $g + 1$, where g is the genus of S^+ . Any two double oriented covers of S are conformally equivalent closed Riemann surfaces.

We denote by $\widehat{\mathbb{M}}$ the group of conformal (*Möbius transformations*) and anticonformal (*extended Möbius transformations*) automorphisms of the Riemann sphere $\widehat{\mathbb{C}}$ and by \mathbb{M} its index two subgroup of Möbius transformations. If $G < \widehat{\mathbb{M}}$, then we set $G^+ = G \cap \mathbb{M}$. If G contains extended Möbius transformations, then G^+ is of index two in G and is called the *orientation preserving half* of G . A *Kleinian group* is a discrete subgroup of \mathbb{M} and an *extended Kleinian group* is a discrete subgroup G of $\widehat{\mathbb{M}}$ with $G \neq G^+$. If G is either a Kleinian group or an extended Kleinian group, then its *region of discontinuity* is the open subset (maybe empty) $\Omega \subset \widehat{\mathbb{C}}$ of points on which G acts discontinuously. If $K_1 < K_2 < \widehat{\mathbb{M}}$ and K_1 has finite index in K_2 , then one is a (extended) Kleinian group if and only if the other is; in which case both have the same region of discontinuity [M1]. It follows that G is an extended Kleinian group if and only if G^+ is a Kleinian group; both share the same region of discontinuity. Let G be an (extended) Kleinian group and let Δ be a G -invariant connected component of its region of discontinuity Ω . We say that a Kleinian group G^+ *acts freely on* Δ if no element of G^+ has fixed points in it (if $\Delta = \Omega$, then we just say that G^+ acts freely).

A *uniformization of a closed Riemann surface* S^+ is a tuple $(\Delta, G^+, P : \Delta \rightarrow S^+)$, where G^+ is a Kleinian group, Δ is a G^+ -invariant connected component of its region of discontinuity, G^+ acts freely on Δ and $P : \Delta \rightarrow S^+$ is a regular cover map with G^+ as deck group. The collection of uniformizations of S^+ is partially ordered: a uniformization $(\Delta_1, G_1^+, P_1 : \Delta_1 \rightarrow S^+)$ is higher than a uniformization $(\Delta_2, G_2^+, P_2 : \Delta_2 \rightarrow S^+)$ if there is a covering map $Q : \Delta_1 \rightarrow \Delta_2$ so that $P_1 = P_2 \circ Q$. It is a well known fact [M3] that the lowest uniformizations of S^+ are given by the *Schottky uniformizations*, that is, when G^+ is a Schottky group (in this case, Δ is equal to the region of discontinuity Ω of G^+). If $\tau : S^+ \rightarrow S^+$ is a conformal (respectively, anticonformal) involution, then it is possible to find a Schottky uniformization, say $(\Omega, G^+, P : \Omega \rightarrow S^+)$, so that there is a Möbius transformation (respectively, an extended Möbius transformation) $\widehat{\tau}$ so that $\widehat{\tau}(\Omega) = \Omega$ and $\tau \circ P = P \circ \widehat{\tau}$ (we say that τ *lifts* with respect to the above Schottky uniformization). This result is due to Koebe [Ko2] in the case that

τ is a reflection, to [HM1] in the case that τ is an imaginary reflection, and to [K, H4] in the case that τ is a conformal involution. If K is the group generated by G^+ and $\hat{\tau}$, then K is a Kleinian group (respectively, extended Kleinian group) containing G^+ as an index two subgroup.

A *uniformization of a compact Klein surface* S is a tuple $(\Delta, G, P : \Delta \rightarrow S)$, where G is an extended Kleinian group, Δ is a G -invariant connected component of its region of discontinuity, G^+ acts freely on Δ and $P : \Delta \rightarrow S$ is a regular (branched) cover map with G as deck group. Again, the collection of uniformizations of S is partially ordered. In [HM2] it was proved that the lowest uniformizations of closed Klein surfaces are given by the *Klein–Schottky uniformizations* (that is, G is a Klein–Schottky group). In [H5] it is proved that the lowest uniformizations of compact Klein surfaces are given by the *extended-Schottky uniformizations* (that is, G is an extended-Schottky group).

Let S be a compact Klein surface and let $\kappa : S \rightarrow S$ be a di-analytic involution. We say that κ *lifts* with respect to an extended-Schottky uniformization of S , say $(\Omega, G, P : \Delta \rightarrow S)$, if there is either a Möbius transformation or an extended Möbius transformation, say $\hat{\kappa}$, so that $\hat{\kappa}(\Omega) = \Omega$ and $\kappa \circ P = P \circ \hat{\kappa}$.

If S has non-empty boundary, then it was proved in [H3] that κ can be lifted to a suitable extended-Schottky uniformization. In this paper, we complete this lifting property to the closed Klein surface case.

THEOREM 1.1. *Let S be a compact Klein surface and let $\kappa : S \rightarrow S$ be a di-analytic involution. Then there is an extended-Schottky uniformization $(\Omega, G, P : \Omega \rightarrow S)$ for which κ lifts.*

Notice that, by the definition of the lifting property, Theorem 1.1 is equivalent to the following.

THEOREM 1.2. *Let S be a compact Klein surface and let $\kappa : S \rightarrow S$ be a di-analytic involution. Then there exists an extended Kleinian group K , with region of discontinuity Ω , containing as an index two subgroup an extended-Schottky group G , such that $S = \Omega/G$ and $K/G = \langle \kappa \rangle$.*

By the Poincaré extension [M1], every (extended) Möbius transformation extends naturally as an isometry of the hyperbolic three-space \mathbb{H}^3 . The interior M^0 of a handlebody M , say of genus g , admits many different hyperbolic structures. Each of these structures on M^0 is of the form \mathbb{H}^3/G^+ , where G^+ is a discrete group of Möbius transformations isomorphic to the free group of rank g . The conformal boundary of such a hyperbolic structure is given by the Riemann surface (it may be empty) Ω/G^+ , where Ω is the region of discontinuity of G^+ . The hyperbolic structures for which the conformal boundary of M is a closed Riemann surface are exactly the ones

when G^+ is a Schottky group of rank g . In this three-dimensional setting, Theorem 1.1 is equivalent to the following.

THEOREM 1.3. *Let S^+ be a closed Riemann surface, let $\tau : S^+ \rightarrow S^+$ be an anticonformal involution and let $\kappa^+ : S^+ \rightarrow S^+$ be a conformal involution such that $\tau \circ \kappa^+ = \kappa^+ \circ \tau$. Then there is a handlebody M , whose interior M^0 has a hyperbolic structure with conformal boundary S^+ , such that both κ^+ and τ extend continuously as hyperbolic isometries of M^0 .*

This paper is organized as follows. In Section 2 we recall most of the extra definitions we need in the rest of the paper. In particular, in Section 2.7.2 we recall an invariant of di-analytic involutions, called the species, which will be useful in the proof of Theorem 1.1, which is provided in Section 3. As mentioned above, we only need to consider the case of closed Klein surfaces. The proof is done by (i) explicit constructions of extended Kleinian groups containing a Klein–Schottky group as an index two subgroup, (ii) the use of notion of species and (iii) the use of quasiconformal deformation theory. It would be interesting to provide the geometric structure of all those extended Kleinian groups containing an index two extended-Schottky subgroup. This will be pursued elsewhere. Finally, in Section 4 we provide a sufficient condition for the oriented double cover of a closed Klein surface to be hyperelliptic. This is related to a result due to Maskit [M2] which states necessary and sufficient conditions for the oriented double cover of a bordered compact Klein surface of genus zero to be hyperelliptic. We also describe some other properties which may be of interest.

2. Preliminaries. In this section we provide some basic definitions, not already provided in the introduction, and previous results needed in this paper.

2.1. Di-analytic maps. A *di-analytic map* between two compact Klein surfaces is a continuous map which is, in each local chart, either conformal or anticonformal. Two compact Klein surfaces are called *di-analytically equivalent* if there is a di-analytic homeomorphism between them. A *di-analytic automorphism* of a compact Klein surface is a di-analytic self-homeomorphism of it. We denote by $\text{Aut}(S)$ the group of di-analytic automorphisms of S . If S^+ is a closed Riemann surface, then $\text{Aut}(S^+)$ is the full group of conformal and anticonformal automorphisms of S ; in this case, we denote by $\text{Aut}^+(S^+)$ its subgroup of conformal automorphisms.

Let S be a compact Klein surface of algebraic genus $g \geq 2$ and let $\pi : S^+ \rightarrow S$ be a double oriented cover of S . Any di-analytic automorphism $\kappa : S \rightarrow S$ can be lifted to a conformal automorphism $\kappa^+ : S^+ \rightarrow S^+$, that is, $\pi \circ \kappa^+ = \kappa \circ \pi$. Another lifting of κ is provided by the anticonformal automorphism $\tau \circ \kappa^+$. If κ^+ is a hyperelliptic involution, then we say that κ

is a hyperelliptic involution of S . Since the hyperelliptic involution is unique on closed Riemann surfaces of genus $g \geq 2$, the hyperelliptic di-analytic involution is unique on compact Klein surfaces of algebraic genus $g \geq 2$.

There is a natural equivalence between pairs (S, κ) , where S is a compact Klein surface and $\kappa : S \rightarrow S$ is a di-analytic automorphism of S , with triples (S^+, τ, κ^+) , where S^+ is a closed Riemann surface, $\kappa^+ : S^+ \rightarrow S^+$ is a conformal automorphism and $\tau : S^+ \rightarrow S^+$ is an anticonformal automorphism so that $\tau \circ \kappa^+ = \kappa^+ \circ \tau$.

2.2. Schottky and extended-Schottky groups. A *Schottky group* of rank g is a Kleinian group G^+ with non-empty region of discontinuity Ω , isomorphic to a free group of rank g and containing no parabolic transformations. In this case, Ω/G^+ turns out to be a closed Riemann surface of genus g . Conversely, the retrosection theorem [B, Ko1] asserts that for every closed Riemann surface S there is a Schottky group G^+ so that Ω/G^+ is conformally equivalent to S .

An *extended-Schottky group* of rank g is an extended Kleinian group G whose orientation preserving half G^+ is a Schottky group of rank g ; if G contains no reflections, then it is called a *Klein-Schottky group* of rank g .

If Ω is the region of discontinuity of an extended-Schottky group G , say of rank g , then Ω/G is a compact Klein surface of algebraic genus g (a closed Klein surface if and only if G is a Klein-Schottky group) and $S^+ = \Omega/G^+$ is its double oriented cover, which admits an anticonformal involution τ (induced by any element of $G - G^+$) so that $S = S^+/\langle \tau \rangle$. Conversely, if S is a compact Klein surface of algebraic genus g , then the results in [HM1, Ko1] and quasiconformal deformation theory assert that there is an extended-Schottky group G of rank g so that Ω/G is di-analytically equivalent to S .

2.3. Geometric structure of Klein-Schottky groups. In [HM1], it was observed that a Klein-Schottky group of rank g is a free product, in the sense of the Klein-Maskit combination theorems, of some m cyclic groups, where the generators are imaginary reflections, and some n cyclic groups, where the generators are glide-reflections (extended Möbius transformations whose squares are hyperbolic), so that $g = 2n + m - 1$. We say that such a Klein-Schottky group is an (m, n) -*Klein-Schottky group*. See Figure 1 for an example of the structure of a $(3, 2)$ -Klein-Schottky group.

An (m, n) -Klein-Schottky group is isomorphic to the group $\mathbb{Z}_2 * \overset{m}{\cdots} * \mathbb{Z}_2 * \mathbb{Z} * \overset{n}{\cdots} * \mathbb{Z}$, so that the pair (m, n) is an algebraic (and also a geometric) invariant of a Klein-Schottky group.

In the case of more general extended-Schottky groups, a similar decomposition is known [HG], but we do not need it in this paper.

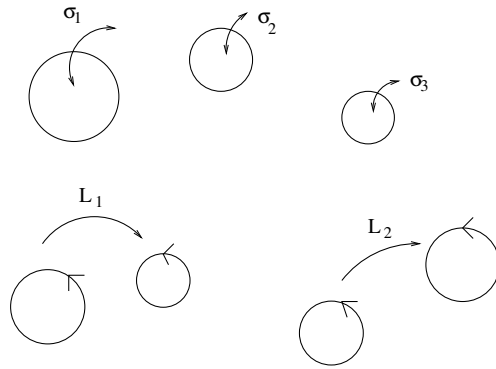


Fig. 1. A (3, 2)-Klein-Schottky group

2.4. Lifting automorphisms. Let $(\Delta, G, P : \Delta \rightarrow S)$ be a uniformization of S (either a closed Riemann surface or a compact Klein surface). A group $H < \text{Aut}(S)$ is said to *lift with respect to* $(\Delta, G, P : \Delta \rightarrow S)$ if for every $h \in H$ there is either a Möbius or an extended Möbius transformation \hat{h} such that $\hat{h}(\Delta) = \Delta$ and $h \circ P = P \circ \hat{h}$.

2.5. The case of closed Riemann surfaces. Let S^+ be a closed Riemann surface of genus $g \geq 2$ and $H < \text{Aut}^+(S^+)$. It is well known that if H lifts with respect to some Schottky uniformization of S^+ , then $|H| \leq 12(g - 1)$ [H6, Z]. Moreover, each lifting \hat{h} is a Möbius transformation. The group K generated by all these liftings is a Kleinian group containing a Schottky group G^+ of rank g as a normal subgroup so that $H \cong K/G^+$. Necessary and sufficient conditions for H to be lifted with respect to a suitable Schottky uniformization of S^+ were provided in [H1].

2.6. The case of compact bordered Klein surfaces. Let S be a compact bordered Klein surface and let $H < \text{Aut}(S)$ be a group of di-analytic automorphisms of S . Let us consider a double oriented cover S^+ , an anticonformal involution $\tau : S^+ \rightarrow S^+$ and a branched cover $\pi : S^+ \rightarrow S$ with $\text{Deck}(\pi) = \langle \tau \rangle$. We may lift H to obtain a group $\hat{H} < \text{Aut}^+(S^+)$. In [H3] it was proved that there is a Schottky uniformization $(\Omega, G^+, P : \Omega \rightarrow S^+)$ such that \hat{H} lifts. Let K be the extended Kleinian group generated by G^+ and all the liftings of the elements of \hat{H} ; so $G^+ \triangleleft K$. Let G be the subgroup of K generated by G^+ and the liftings of τ . Then G turns out to be an extended-Schottky group with G^+ as its orientation preserving half. In this case, $(\Omega, G, \pi \circ P : \Omega \rightarrow S)$ provides an extended-Schottky uniformization of S for which H lifts. This, in particular, provides (i) Theorem 1.1 in the case that S has non-empty boundary, and (ii) the well known fact that the order of the full group of di-analytic automorphisms of a bordered compact Klein surface of algebraic genus $g \geq 2$ is at most $12(g - 1)$ [Ma].

2.7. The case of closed Klein surfaces. Let S be a closed Klein surface and let us consider a double oriented cover S^+ , an anticonformal involution $\tau : S^+ \rightarrow S^+$ and a branched cover $\pi : S^+ \rightarrow S$ with $\text{Deck}(\pi) = \langle \tau \rangle$.

There are examples of groups $H < \text{Aut}(S)$ which cannot be realized by Klein–Schottky uniformizations of S ; for instance, if S has algebraic genus $g \geq 2$ and $|H| > 12(g - 1)$.

Let $\kappa : S \rightarrow S$ be a given di-analytic involution. As already mentioned, κ lifts to a conformal involution $\kappa^+ : S^+ \rightarrow S^+$ and to an anticonformal involution $\tau \circ \kappa^+ : S^+ \rightarrow S^+$, both commuting with τ . In particular, $\pi \circ \kappa^+ = \kappa \circ \pi$ and $\langle \kappa^+, \tau \rangle \cong \mathbb{Z}_2^2$.

It is always possible to find a Kleinian group K_0 , with region of discontinuity Ω , containing a Schottky group G^+ of index two such that $S^+ = \Omega/G^+$ and with the property that κ^+ is induced by $K_0 - G^+$. But it is not clear at this point that we may find an extension K that contains K_0 of index two and G^+ as a normal subgroup such that $\langle \kappa^+, \tau \rangle$ is induced by $K - G^+$. Theorem 1.1 states that this is possible.

If we set $\widehat{S} = S^+/\langle \kappa^+ \rangle$, then τ descends to an anticonformal involution $\widehat{\tau} : \widehat{S} \rightarrow \widehat{S}$. Of course, we have the natural identifications of $\widehat{S}/\langle \widehat{\tau} \rangle$ with $S/\langle \kappa \rangle$ and with $S^+/\langle \tau, \kappa^+ \rangle$ and also the commutative diagram

$$(2.1) \quad \begin{array}{ccc} S^+ & \longrightarrow & \widehat{S} = S^+/\langle \kappa^+ \rangle \\ \downarrow & & \downarrow \\ S = S^+/\langle \tau \rangle & \longrightarrow & S/\langle \kappa \rangle = \widehat{S}/\langle \widehat{\tau} \rangle = S^+/\langle \kappa^+, \tau \circ \kappa^+ \rangle \end{array}$$

2.7.1. Fixed points. The di-analytic involution κ may have isolated and non-isolated fixed points. The components of non-isolated fixed points are simple closed curves called *ovals* of κ . Let h be the number of isolated fixed points and let l be the number of ovals. We say in this case that κ is an (h, l) -involution. An $(h, 0)$ -involution will also be called an h -involution.

Notice (see diagram (2.1)) that κ is an (h, l) -involution if and only if $S/\langle \kappa \rangle$ is a compact Klein orbifold with exactly l border components and h branch values of order 2; moreover, $l > 0$ if and only if $\tau \circ \kappa^+$ is a reflection.

Let us assume that κ is an (h, l) -involution. In this case, $\kappa^+ : S^+ \rightarrow S^+$ has exactly $2h$ fixed points which are permuted by the imaginary reflection τ . It is well known [FK] that $\kappa^+ : S^+ \rightarrow S^+$ has at most $2g + 2$ fixed points (so $h \leq g + 1$), and if it has exactly $2g + 2$ fixed points, then S^+ is hyperelliptic and κ^+ is the hyperelliptic involution, that is, κ is the hyperelliptic involution of S if $h = g + 1$.

PROPOSITION 2.1. *If the closed Klein surface S of algebraic genus g admits an h -involution, then g is odd; in particular, h is even.*

Proof. Assume that g is even. Let γ be the genus of $\widehat{S} = S^+ / \langle \kappa^+ \rangle$; that is, $4\gamma = 2g + 2 - 2h$ (by the Riemann–Hurwitz formula); hence h is odd. As $l = 0$, $S / \langle \kappa \rangle$ is closed, so $\kappa^+ \circ \tau$ and $\widehat{\tau}$ are both imaginary reflections (see diagram (2.1)).

If γ is even, then we can find a single dividing loop \widehat{W} on \widehat{S} , where \widehat{W} is $\widehat{\tau}$ -invariant, and \widehat{W} does not pass through the projection of any of the fixed points of κ^+ . Then there must be exactly h projections of these fixed points on each side of \widehat{W} . However, since \widehat{W} is dividing, and h is odd, this means that \widehat{W} lifts to a single loop W on S^+ , where W double covers \widehat{W} . Since $\widehat{\tau}$ preserves \widehat{W} , and acts as an involution on it, its lift τ has order 4 in its action on W , which contradicts our assumption that τ is an involution.

If γ is odd, then there are two homologically dependent simple disjoint loops, \widehat{W}_1 and \widehat{W}_2 , on \widehat{S} , where $\widehat{\tau}$ keeps each of these loops invariant, and acts as an involution on each of them. As above, we can assume that no projection of a fixed point of κ^+ lies on either of these loops; so there are exactly h of these points on either side of this pair of loops. Since h is odd, a simple loop, homologous to the sum of \widehat{W}_1 and \widehat{W}_2 , lifts to a single loop that double covers it; from this it is not hard to conclude that each of these two loops lifts to a single loop that double covers it. As above, this implies that τ has order 4, contrary to our assumption that τ has order 2. ■

2.7.2. Species of di-analytic involutions. In [BCNS, S] a topological invariant for the (h, l) -involution $\kappa : S \rightarrow S$, called the *species* of κ , is defined as follows.

Let s^- be the number of ovals of fixed points of κ which lift to exactly one loop on S^+ (*twisted ovals*) and let s^+ be the number of ovals of fixed points of κ which lift to exactly two loops on S^+ (*untwisted ovals*); so $l = s^- + s^+$.

If $S / \langle \kappa \rangle$ is either (i) orientable or (ii) non-orientable and $h + s^- > 0$, then the species of κ is defined by the tuple $(\pm; h; \{s^-, s^+\})$ with “+” in case (i) and “−” in case (ii). In [BCNS] it is proved that

$$s^- + h \equiv 0 \pmod{2}.$$

If $S / \langle \kappa \rangle$ is non-orientable and $h + s^- = 0$, then the definition of the species of κ is more technical. Define values $r_1, r_2 \in \{0, 1\}$ so that $r_1 = 0$ if and only if $S^+ / \langle \tau \circ \kappa^+ \rangle$ is orientable (i.e. $\tau \circ \kappa^+$ is a reflection and its ovals disconnect S^+). Let Γ be an extended Kleinian group keeping invariant the hyperbolic plane \mathbb{H}^2 and such that $S = \mathbb{H}^2 / \Gamma$; many authors then say that Γ is a non-euclidean crystallographic (NEC) group. There is an anticonformal isometry $\tilde{\kappa}$ of \mathbb{H}^2 which is a lift of κ . Set $\Delta = \langle \Gamma, \tilde{\kappa} \rangle$; then $S / \langle \kappa \rangle = \mathbb{H}^2 / \Delta$. There is a surjective homomorphism $\phi : \Delta \rightarrow \langle \kappa \rangle$ with kernel Γ . We set $r_2 = 0$ if and only if the number of canonical glide-reflection generators of Δ that are mapped by ϕ to κ is odd. Notice that if $r_1 = 0$, then Δ has no

glide-reflections, so $r_2 = 0$. The species of κ , in this case, is defined by the tuple $(-; r_1, r_2; (s^+))$.

THEOREM 2.2 ([BCNS]). *The species classify κ up to topological equivalence.*

3. Proof of Theorem 1.1. As previously observed, we only need to consider the case of closed Klein surfaces. Let S be a closed Klein surface and $\kappa : S \rightarrow S$ be some (h, l) -involution.

By Theorem 2.2, we only need to construct an extended Kleinian group K , with region of discontinuity Ω , containing as an index two subgroup a Klein–Schottky group G such that Ω/G is a closed Klein surface of the same algebraic genus as S and with the property that K/G induces an (h, l) -involution with the same species as κ . Once this is done, the same theorem, together with quasiconformal deformation theory, provides the proof of our theorem.

We notice that our construction, in each case, provides a family depending on the correct number of parameters which is invariant under quasiconformal deformation (with a suitable normalization). In particular, this also provides explicit descriptions of the moduli of closed Klein surfaces in terms of Klein–Schottky groups.

We divide this section into three subsections, one for the case that $S/\langle\kappa\rangle$ is closed, that is, κ is an h -involution, and the other two for the case that $S/\langle\kappa\rangle$ is bordered, that is, κ is an (h, l) -involution with $l > 0$.

3.1. Closed situation. Let us assume that κ is an h -involution and let us recall the diagram (2.1). As κ has only isolated fixed points, the quotient $S/\langle\kappa\rangle$ is compact without boundary. Since $S/\langle\kappa\rangle = S^+/\langle\kappa^+, \tau \circ \kappa^+\rangle$, it follows that the anticonformal involution $\tau \circ \kappa^+$ must be an imaginary reflection. As a consequence of Proposition 2.1, h is even. In this case, $S/\langle\kappa\rangle = S^+/\langle\tau, \kappa^+\rangle$ is topologically equivalent to the connected sum of $p \geq 1$ real projective planes, having $h \geq 0$ branch values of order 2. The only species that may appear in this case for κ are:

$$\begin{aligned} &(-; h = 2m > 0; \{s^- = 0, s^+ = 0\}), \\ &(-; r_1 = 1; r_2 = 0; (s^+ = 0)), \\ &(-; r_1 = 1; r_2 = 1; (s^+ = 0)). \end{aligned}$$

In this case, S has algebraic genus $g = 2p + h - 3$.

3.1.1. Construction of the extended Kleinian group K . Choose $p + m$ pairwise disjoint circles, say $L_1, \dots, L_m, C_1, \dots, C_p$, bounding a common domain D of connectivity $p + m$. If $h = 0$, then there are no L_j 's.

For $i = 1, \dots, m$ and $j = 1, \dots, p$, let θ_i (respectively, σ_j) be an elliptic transformation of order 2 (respectively, an imaginary reflection) with $\theta_i(L_i) = L_i$ (respectively, $\sigma_j(C_j) = C_j$) and $\theta_i(D) \cap D = \emptyset$ (respectively, $\sigma_j(D) \cap D = \emptyset$).

Let K be the group generated by $\theta_1, \dots, \theta_m, \sigma_1, \dots, \sigma_p$ (see Figure 2 in the case $m = 3$ and $p = 2$).

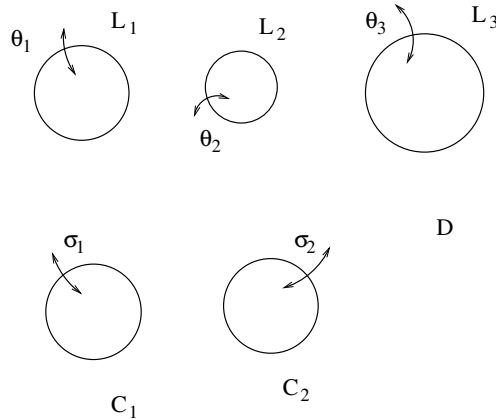


Fig. 2. The group K for $m = 3$ and $p = 2$

Then $K \cong \mathbb{Z}_2 * \dots * \mathbb{Z}_2$ is an extended Kleinian group with connected region of discontinuity Ω and Ω/K is a non-orientable orbifold which is topologically the connected sum of p real projective planes with h orbifold points of order 2.

3.1.2. Construction of Klein–Schottky groups. If $m > 0$, then let $G_0 \subset K$ be the group generated by the transformations (see Figure 3)

$$\sigma_1, \dots, \sigma_p, \quad \theta_1 \circ \sigma_1 \circ \theta_1, \dots, \theta_1 \circ \sigma_p \circ \theta_1, \quad \theta_1 \circ \theta_2, \dots, \theta_1 \circ \theta_m.$$

If $m = 0$, then let $G_1 \subset K$ be the subgroup generated by the transformations (see Figure 4)

$$\sigma_2, \dots, \sigma_p, \quad \sigma_1 \circ \sigma_2 \circ \sigma_1, \dots, \sigma_1 \circ \sigma_p \circ \sigma_1.$$

If $m = 0$ and $p \geq 2$, then let $G_2 \subset K$ be the subgroup generated by the transformations (see Figure 5)

$$\sigma_3, \dots, \sigma_p, \quad \sigma_1 \circ \sigma_3 \circ \sigma_1, \dots, \sigma_1 \circ \sigma_p \circ \sigma_1, \quad \sigma_2 \circ \sigma_1.$$

The group G_0 is a Klein–Schottky group of type $(2p, m - 1)$, G_1 is a Klein–Schottky group of type $(2p - 2, 0)$ and G_2 is a Klein–Schottky group of type $(2p - 4, 1)$. In this way, the closed Klein surface $S_j^* = \Omega/G_j$ is a closed Klein surface of algebraic genus g , for each $j \in \{0, 1, 2\}$.

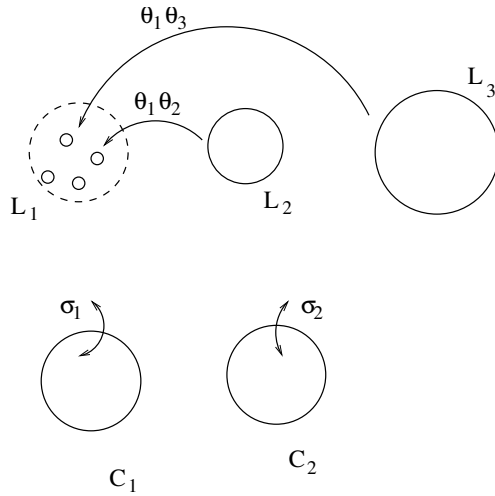


Fig. 3. The group G_0 for $m = 3$ and $p = 2$. Two small circles inside the disc bounded by L_1 correspond to the reflections $\theta_1 \circ \sigma_1 \circ \theta_1$ and $\theta_1 \circ \sigma_2 \circ \theta_1$.

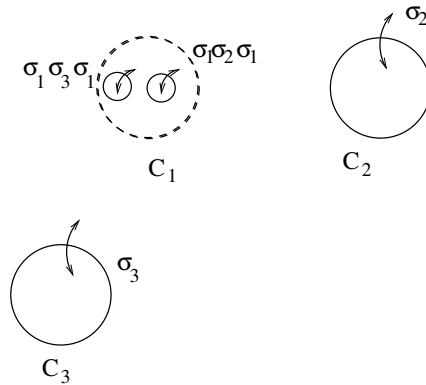


Fig. 4. The group G_1 for $m = 0$ and $p = 3$

We observe that, for $j \in \{0, 1, 2\}$, the group K/G_j is of order two generated by the class of θ_1 (for $j = 0$) and the class of σ_1 (for $j = 2, 3$), say $\kappa_j^* : S_j^* \rightarrow S_j^*$. By the construction, the involution $\kappa_0^* : S_0 \rightarrow S_0$ has the species $(-; h > 0; \{0, 0\})$, and $\kappa_1^* : S_1^* \rightarrow S_1^*$ and $\kappa_2^* : S_2^* \rightarrow S_2^*$ have the species $(-; r_1 = 1; r_2 = 0; (s^+ = 0))$ and $(-; r_1 = 1; r_2 = 1; (s^+ = 0))$, respectively, as desired.

3.2. Bordered orientable situation. Let us assume now that κ is an (h, l) -involution with $l > 0$ and $S/\langle \kappa \rangle$ is orientable. In this case we have

$$\begin{cases} l = s^+ + s^- > 0, \\ s^- + h \equiv 0 \pmod{2}. \end{cases}$$

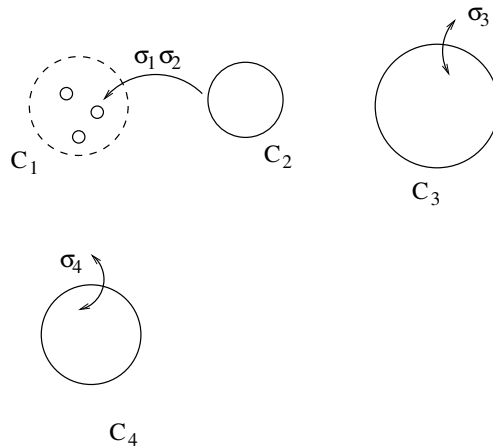


Fig. 5. The group G_2 for $m = 0$ and $p = 4$. Two small circles inside the disc bounded by C_1 correspond to the reflections $\sigma_1 \circ \sigma_3 \circ \sigma_1$ and $\sigma_1 \circ \sigma_4 \circ \sigma_1$.

The species of κ in this case has the form $(+; h; \{s^-, s^+\})$ and the quotient $S/\langle \kappa \rangle$ is a compact orientable surface of topological genus $\gamma \geq 0$ with l boundary components and h branch values of order 2. It follows that S has algebraic genus $g = 4\gamma + 2l + h - 3$. Set $m = [h/2]$, where $[x]$ denotes the integral part of x .

3.2.1. The construction of the group K . Let us consider a collection of $2\gamma + m + s^+ + 2s^-$ circles, say $A_1, A'_1, \dots, A_\gamma, A'_\gamma, B_1, \dots, B_m, C_1, \dots, C_{s^+}, D, E_1, \dots, E_{2s^- - 2}, B$ such that

- (i) $A_1, A'_1, \dots, A_\gamma, A'_\gamma, B_1, \dots, B_m, C_1, \dots, C_{s^+}, D$ are pairwise disjoint, and (with the exception of D) all contained in the same component of the complement of D ;
- (ii) $A_1, A'_1, \dots, A_\gamma, A'_\gamma, B_1, \dots, B_m, C_1, \dots, C_{s^+}, E_1, \dots, E_{2s^- - 2}, B$ are pairwise disjoint;
- (iii) E_j and B both intersect D orthogonally;
- (iv) the circles $E_1, \dots, E_{2s^- - 2}, B$ are ordered consecutively in counter-clockwise order with respect to the circle D .

The A_j 's and A'_j 's are only considered if $\gamma > 0$. The B_j 's are only considered if $m > 0$. The C_j 's are only considered if $s^+ > 0$. The E_j 's are only considered if $s^- > 1$. The circle B is only considered if $h = 2m + 1$ and $s^- > 0$, and D is only considered if $s^- > 0$.

Let \mathcal{D} be the common domain bounded by the circles $A_1, A'_1, \dots, A_\gamma, A'_\gamma, B_1, \dots, B_m, C_1, \dots, C_{s^+}, D, E_1, \dots, E_{2s^- - 2}$ and B . This is a domain of finite connectivity $2\gamma + m + s^+$ if $s^- = 0$ and of connectivity $2\gamma + m + s^+ + 1$ if $s^- > 0$.

If $s^- > 0$, then let \mathbb{H} be either of the two disc components of $\widehat{\mathbb{C}} - D$.

Choose γ loxodromic transformations, say $\alpha_1, \dots, \alpha_\gamma$, m elliptic transformations of order 2, say $\theta_1, \dots, \theta_m$, s^+ reflections, say $\tau_1, \dots, \tau_{s^+}$, $s^- - 1$ hyperbolic transformations, say $\beta_1, \dots, \beta_{s^- - 1}$, an elliptic transformation of order 2, say θ , and a reflection, say τ_0 , so that:

- (i) $\alpha_j(\mathcal{D}) \cap \mathcal{D} = \emptyset$, $\theta_k(\mathcal{D}) \cap \mathcal{D} = \emptyset$, $\beta_i(\mathcal{D}) \cap \mathcal{D} = \emptyset$, $\theta(\mathcal{D}) \cap \mathcal{D} = \emptyset$;
- (ii) $\alpha_j(A_j) = A'_j$, $\theta_k(B_k) = B_k$, $\beta_i(E_{2i}) = E_{2i-1}$,
- (iii) τ_j has C_j as the set of fixed points, $\theta(B) = B$ and τ_0 has D as the circle of fixed points;
- (iv) if $s^- > 0$, then both β_j and θ keep \mathbb{H} invariant.

Observe that, if $s^- > 0$, then the group F generated by $\theta, \beta_1, \dots, \beta_{2s^- - 2}$ is a Fuchsian group so that \mathbb{H}/F is a sphere with s^- holes and one special point of order 2.

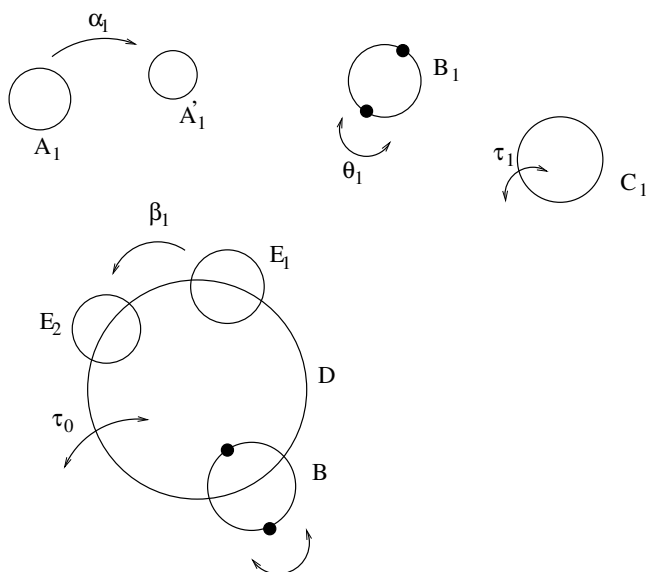


Fig. 6. The group K for $\gamma = m = s^+ = 1$, $h = 3$ and $s^- = 2$

Let K be the group generated by the transformations (see Figure 6)

$$\alpha_1, \dots, \alpha_\gamma, \theta_1, \dots, \theta_m, \tau_1, \dots, \tau_{s^+}, \beta_1, \dots, \beta_{s^- - 1}, \theta, \tau_0.$$

It follows that K is an extended Kleinian group with connected region of discontinuity Ω so that Ω/K is topologically equivalent to $S/\langle \kappa \rangle$. The genus γ is provided by the loxodromic transformations $\alpha_1, \dots, \alpha_\gamma$; the points of order 2 are provided by the elliptic transformations $\theta_1, \dots, \theta_m$ and B ; and the boundary loops are provided by the reflections $\tau_1, \dots, \tau_{s^+}$ and the subgroup generated by the reflection τ_0 and the hyperbolic transformations $\beta_1, \dots, \beta_{s^- - 1}$.

Moreover, K is the free product, in the sense of Klein–Maskit’s combination theorems [M1], of the γ cyclic groups generated by the loxodromic transformations $\alpha_1, \dots, \alpha_\gamma$, the m cyclic groups of order two generated by the elliptic transformations $\theta_1, \dots, \theta_m$, the s^+ cyclic groups of order two generated by the reflections $\tau_1, \dots, \tau_{s^+}$ and the group generated by τ_0 and F (which is an HNN-extension of F by τ_0).

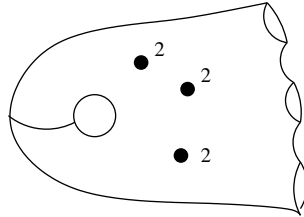


Fig. 7. The quotient S/K for $\gamma = m = s^+ = 1$, $h = 3$ and $s^- = 2$

Let us consider the surjective homomorphism $\Phi : K \rightarrow \langle a : a^2 = 1 \rangle \cong \mathbb{Z}_2$ defined by $\Phi(\alpha_j) = 1$ for $j \in \{1, \dots, \gamma\}$, $\Phi(\theta_j) = a$ for $j \in \{1, \dots, m\}$, $\Phi(\tau_j) = a$ for $j \in \{0, 1, \dots, s^+\}$, and $\Phi(\theta) = a$. Let us consider the index two subgroup $G = \ker(\Phi)$.

3.2.2. *Case $h = 2m \geq 0$, $s^- = 0$.* The index two subgroup $G = \ker(\Phi)$ is generated by the transformations

$\alpha_1, \dots, \alpha_\gamma, \tau_1 \circ \alpha_1 \circ \tau_1, \dots, \tau_1 \circ \alpha_\gamma \circ \tau_1, \theta_1 \circ \tau_1, \dots, \theta_m \circ \tau_1, \tau_2 \circ \tau_1, \dots, \tau_{s^+} \circ \tau_1$ (deleting the elements $\theta_k \circ \tau_1$ in case $h = 0$) and it is a $(0, 2\gamma + m + s^+ - 1)$ -Klein–Schottky group of genus g such that the surface $S^* = \Omega/G$ admits an involution $\kappa^* : S^* \rightarrow S^*$ (induced by τ_1) such that $S^*/\langle \kappa^* \rangle = \Omega/K$ and κ^* has species $(+; h; \{0, s^+\})$ as desired.

3.2.3. *Case $h = 2m \geq 0$, $s^- = 2t > 0$.* The index two subgroup $G = \ker(\Phi)$ is generated by the transformations

$$\alpha_1, \dots, \alpha_\gamma, \tau_0 \circ \alpha_1 \circ \tau_0, \dots, \tau_0 \circ \alpha_\gamma \circ \tau_0, \tau_0 \circ \theta_1, \dots, \tau_0 \circ \theta_m, \tau_0 \circ \tau_1, \dots, \tau_0 \circ \tau_{s^+}, \tau_0 \circ \beta_1, \dots, \tau_0 \circ \beta_{s^- - 1}$$

and it is a $(0, 2\gamma + m + l - 1)$ -Klein–Schottky group such that $S^* = \Omega/G$ is a non-orientable surface of algebraic genus g admitting an involution $\kappa^* : S^* \rightarrow S^*$ (induced by τ_0) so that $S^*/\langle \kappa^* \rangle = \Omega/K$ and with species as desired.

3.2.4. *Case $h = 2m+1$, $s^- = 2t+1$.* The index two subgroup $G = \ker(\Phi)$ is generated by the transformations

$$\alpha_1, \dots, \alpha_\gamma, \tau_0 \circ \alpha_1 \circ \tau_0, \dots, \tau_0 \circ \alpha_\gamma \circ \tau_0, \tau_0 \circ \theta_1, \dots, \tau_0 \circ \theta_m, \tau_0 \circ \tau_1, \dots, \tau_0 \circ \tau_{s^+}, \tau_0 \circ \beta_1, \dots, \tau_0 \circ \beta_{s^- - 1}, \tau_0 \circ \theta,$$

and it is a $(1, 2\gamma + m + l - 1)$ -Klein-Schottky group such that $S^* = \Omega/G$ is a non-orientable surface of algebraic genus g admitting an involution $\kappa^* : S^* \rightarrow S^*$ (induced by τ_0) so that $S^*/\langle \kappa^* \rangle = \Omega/K$ and with species as desired.

3.3. Bordered non-orientable situation. Let us assume now that κ is an (h, l) -involution with $l > 0$ and $S/\langle \kappa \rangle$ is non-orientable. In this case, again we have

$$\begin{cases} l = s^+ + s^- > 0, \\ s^- + h \equiv 0 \pmod{2}. \end{cases}$$

The quotient $S/\langle \kappa \rangle$ is a compact non-orientable surface of topological genus $p \geq 1$ with l boundary components and h branch values of order 2. It follows that the algebraic genus of S is $g = 2p + 2l + h - 3$.

3.3.1. Case $h = s^- = 0$. The species in this case has the form $(-; r_1, r_2; (s^+))$, where $s^+ > 0$. The construction is the same as for 3.2.2 with $h = 0$, $\gamma = p$, eliminating the circles A'_1, \dots, A'_γ and replacing each loxodromic transformation α_j by an imaginary reflection η_j that interchanges the inside and outside of the circle A_j .

3.3.2. Case $h = 2l > 0, s^- = 2t > 0$. The species in this case has the form $(-; h; \{s^-, s^+\})$. The construction is the same as for 3.2.3 with $\gamma = p$, eliminating the circles A'_1, \dots, A'_γ and replacing each α_j by an imaginary reflection η_j that interchanges the inside and outside of A_j .

3.3.3. Case $h = 2l + 1, s^- = 2t + 1$. The species in this case has the form $(-; h; \{s^-, s^+\})$. The construction is the same as for 3.2.4 with $\gamma = p$, eliminating the circles A'_1, \dots, A'_γ and replacing each α_j by an imaginary reflection η_j that interchanges the inside and outside of A_j .

4. Hyperelliptic surfaces. In [M2] B. Maskit obtained the following geometrical description of a hyperelliptic Riemann surface.

THEOREM 4.1 (Maskit [M2]). *Let L_1, \dots, L_{g+1} be pairwise disjoint circles on the Riemann sphere bounding a common region D . Let G be the extended-Schottky group generated by the reflections in these circles and let $S^+ = \Omega(G^+)/G^+$. Then S^+ is hyperelliptic if and only if there is a circle C orthogonal to every $L_i, i = 1, \dots, g + 1$.*

One can view the above as a statement concerning Riemann surfaces admitting maximal reflections, that is, reflections with the maximal number of fixed curves.

One direction of Theorem 4.1 generalizes to Riemann surfaces admitting imaginary reflections; it is not known if the other direction also holds. First, let us notice that an imaginary reflection has no fixed points on

the Riemann sphere $\widehat{\mathbb{C}}$, but it has exactly one fixed point on the hyperbolic space \mathbb{H}^3 as it is the product of a reflection in a geodesic plane in \mathbb{H}^3 with a rotation of angle π along a geodesic line orthogonal to the plane.

THEOREM 4.2. *Let L_1, \dots, L_{g+1} be pairwise disjoint topological circles on the Riemann sphere, bounding a common region D , where, for $j = 1, \dots, g+1$, there is an imaginary reflection $\sigma_j : L_j \rightarrow L_j$ with $\sigma_j(D) \cap D = \emptyset$. Let $G = \langle \sigma_1, \dots, \sigma_{g+1} \rangle$ be the Klein–Schottky group generated by these imaginary reflections and set $S^+ = \Omega(G^+)/G^+$. If there is a hyperbolic plane $\Sigma \subset \mathbb{H}^3$ containing the fixed point of every σ_j , $j = 1, \dots, g + 1$, then S^+ is hyperelliptic.*

Proof. The proof is essentially obvious. Let τ_0 denote the reflection in the hyperbolic plane Σ . Since the fixed point of σ_j lies on Σ , we have $\tau_0 \circ \sigma_i = \sigma_i \circ \tau_0$, $i = 1, \dots, g + 1$. It follows that, for each $i = 1, \dots, g + 1$, $\tau_0 \circ \sigma_i$ normalizes G^+ , and so it projects to a conformal homeomorphism. Of course, $\tau_0 \circ \sigma_i$ and $\tau_0 \circ \sigma_k$ project to the same conformal homeomorphism $j : S^+ \rightarrow S^+$. Since τ_0 and σ_i are commuting involutions, j is an involution.

In the special case that the L_i are all circles orthogonal to C (the boundary circle of Σ), it is clear that the fixed points of $\tau_0 \circ \sigma_i$ lie on L_i ; hence these $2g + 2$ fixed points project to $2g + 2$ distinct points on S , from which it follows that j has at least $2g + 2$ fixed points. This can occur only if j has exactly $2g + 2$ fixed points, and is the hyperelliptic involution.

Since the general case is a quasiconformal deformation of this special case, we again find that j has at least $2g + 2$ fixed points, and the same result holds. ■

Let us consider a pair (S, j) , where S is a hyperelliptic Klein surface of algebraic genus $g \geq 2$ and j is the hyperelliptic involution. Let S^+ be a double oriented Riemann surface of S and $\tau : S^+ \rightarrow S^+$ be an imaginary reflection so that there is a regular di-analytic covering $\pi : S^+ \rightarrow S$ whose covering group is generated by τ . As previously noted, there is a lifting $j^+ : S^+ \rightarrow S^+$ which turns out to be the hyperelliptic involution. The quotient $\widehat{S} = S^+ / \langle j^+ \rangle$ is the Riemann sphere with exactly $2h$ branch values of order 2 so that $h = g + 1$. In this way, we may now consider the hyperelliptic Riemann surface S^+ admitting an anticonformal automorphism.

THEOREM 4.3. *Let S^+ be a hyperelliptic Riemann surface of genus $g \geq 2$, with a hyperelliptic involution j , and assume that S^+ admits an imaginary reflection $\tau : S^+ \rightarrow S^+$.*

- (1) If g is even, then $j \circ \tau$ is a reflection.
- (2) If $j \circ \tau$ is a reflection, then
 - (2.1) for g even, it has exactly one dividing oval of fixed points;
 - (2.2) for g odd, it has exactly two non-dividing ovals of fixed points, both of them dividing S^+ .

Moreover, $S^+ / \langle j, \tau \rangle$ turns out to be

- (3) a closed disc with exactly $g+1$ points of order 2 in its interior if $j \circ \tau$ is a reflection;
- (4) a projective plane with exactly $g+1$ points of order 2 in its interior if $j \circ \tau$ is an imaginary reflection.

REMARK 4.4. If the hyperelliptic Riemann surface S^+ has odd genus, it may happen that there is an imaginary reflection $\tau : S^+ \rightarrow S^+$ for which $j \circ \tau : S^+ \rightarrow S^+$ still is an imaginary reflection. For instance, the hyperelliptic Riemann surface of genus 3 defined by the algebraic curve

$$w^2 = (z^4 - a^4)(z^4 - 1/a^4) \quad (a^8 \neq 1)$$

admits three distinct imaginary reflections. Two of these have the property that their composition is the hyperelliptic involution; the third, when composed with the hyperelliptic involution, has two ovals of fixed points. The first one, $z \mapsto -1/\bar{z}$, operates separately on each of the two sheets and has no fixed points. The second is obtained from the first by composition with the sheet interchange (the hyperelliptic involution), and the third is the composition of the sheet interchange with the reflection $z \mapsto 1/\bar{z}$. Again, the map $z \mapsto 1/\bar{z}$ operates separately on each of the sheets, and has an oval of fixed points on each of them.

COROLLARY 4.5. Let S_t^+ be a hyperelliptic Riemann surface, of fixed genus $g \geq 2$, with a hyperelliptic involution $j_t : S_t^+ \rightarrow S_t^+$, admitting an imaginary reflection $\tau_t : S_t^+ \rightarrow S_t^+$, for $t = 1, 2$. If either

- (1) both $j_1 \circ \tau_1$ and $j_2 \circ \tau_2$ are reflections, or
- (2) both are imaginary reflections,

then there is an orientation-preserving homeomorphism $\omega : S_1^+ \rightarrow S_2^+$, where ω conjugates j_1 to j_2 and τ_1 to τ_2 .

4.1. Proof of Theorem 4.3 and Corollary 4.5. Let us denote by $Q : S^+ \rightarrow \widehat{\mathbb{C}}$ a branched holomorphic covering of degree two. Since the hyperelliptic involution $j : S^+ \rightarrow S^+$ is unique in the group of conformal automorphisms of S^+ , $j \circ \tau$ is an anticonformal involution and, in particular, τ descends to an anticonformal involution $\tilde{\tau} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ that permutes the branch values of Q (the projection of the fixed points of j). Clearly, $\tilde{\tau}$ is an imaginary reflection if and only if $j \circ \tau$ is also an imaginary reflection.

Let us assume $\tilde{\tau}$ is an imaginary reflection. In this case, we may assume $\tilde{\tau}(z) = -1/\bar{z}$. An algebraic curve representing S^+ is then given as

$$y^2 = x \prod_{j=1}^g (x - a_j)(x + 1/\bar{a}_j)$$

where a_j are different and contained in

$$\Delta = \{z \in \mathbb{C} : 0 < |z| \leq 1\}.$$

In this case, τ or $j \circ \tau$ has the form

$$\mathcal{N} := \begin{cases} x \mapsto \frac{-1}{\bar{x}}, \\ y \mapsto (-1)^{(g+1)/2} \frac{\bar{y}}{\bar{x}^{g+1}} \left(\prod_{j=1}^g \frac{a_j}{\bar{a}_j} \right)^{1/2}. \end{cases}$$

If g is even, then \mathcal{N} is clearly not of order 2 (its square is the hyperelliptic involution). It follows in this case that both $\tilde{\tau}$ and $j \circ \tau$ are reflections. We have proved part (1) and (4) of Theorem 4.3. Observe also that we may write down a quasiconformal diffeomorphism $\omega : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ so that ω normalizes $x \mapsto -1/\bar{x}$ and sends the points $a_1, \dots, a_g, -1/\bar{a}_1, \dots, -1/\bar{a}_g$ to $(g + 1)$ -roots of unity. This observation also gives the proof of half of Corollary 4.5.

To prove part (2) and (3) of Theorem 4.3, we must assume that $j \circ \tau$ is a reflection. We may also assume $\tilde{\tau}(z) = \bar{z}$. If some of the branch values of Q are on the real line, then S^+ will be represented by an algebraic curve of the form

$$y^2 = (x - \lambda_1) \dots (x - \lambda_{2r})(x - a_1)(x - \bar{a}_1) \dots (x - a_s)(x - \bar{a}_s)$$

where $\lambda_j \in \mathbb{R}$ and $\text{Im}(a_j) > 0$. But in this case the representations of τ and $j \circ \tau$ are given by

$$\mathcal{L} := \begin{cases} x \mapsto \bar{x}, \\ y \mapsto \bar{y}, \end{cases} \quad \mathcal{M} := \begin{cases} x \mapsto \bar{x}, \\ y \mapsto -\bar{y}. \end{cases}$$

Both have fixed points, for instance, $x = \lambda_1, y = 0$, a contradiction to the fact that τ is an imaginary reflection.

It follows that S^+ is represented by an algebraic curve of the form

$$y^2 = (x - a_1)(x - \bar{a}_1) \dots (x - a_{g+1})(x - \bar{a}_{g+1})$$

where $\text{Im}(a_j) > 0$. In this case, \mathcal{L} represents $j \circ \tau$ and \mathcal{M} represents τ , and the result follows easily. This also yields the other half of Corollary 4.5.

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References

- [AG] N. L. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces*, Lecture Notes in Math. 219, Springer, 1971.
- [B] L. Bers, *Automorphic forms for Schottky groups*, Adv. Math. 16 (1975), 332–361.
- [BCG] E. Bujalance, F. J. Cirre and J. M. Gamboa, *Double coverings of hyperelliptic real algebraic curves*, J. Pure Appl. Algebra 212 (2008), 2011–2026.
- [BCNS] E. Bujalance, A. F. Costa, S. M. Natanzon and D. Singerman, *Involutions of compact Klein surfaces*, Math. Z. 211 (1992), 461–478.
- [FK] H. M. Farkas and I. Kra, *Riemann Surfaces*, 2nd ed., Grad. Texts in Math. 71, Springer, 1991.
- [Ha] A. Harnack, *Über die Vieltheiligkeit der ebenen algebraischen Kurven*, Math. Ann. 10 (1876), 189–199.
- [H1] R. A. Hidalgo, *Automorphisms groups of Schottky type*, Ann. Acad. Sci. Fenn. Math. 30 (2005), 183–204.
- [H2] —, *Γ -Hyperelliptic Riemann surfaces*, Rev. Proyecciones 17 (1998), 77–117.
- [H3] —, *Schottky uniformizations and Riemann matrices of maximally symmetric Riemann surfaces of genus 5*, *ibid.* 20 (2001), 93–126.
- [H4] —, *On τ -hyperelliptic Schottky groups*, Notas Soc. Mat. Chile 8 (1989), 27–36.
- [H5] —, *Lowest uniformizations of compact Klein surfaces*, preprint.
- [H6] —, *On the $12(g - 1)$ bound*, C. R. Math. Rep. Acad. Sci. Canada 18 (1996), 39–42.
- [HG] G. Gromadzki and R. A. Hidalgo, *Components of fixed points of symmetries on handlebodies*, preprint.
- [HM1] R. A. Hidalgo and B. Maskit, *On Klein–Schottky groups*, Pacific J. Math. (2) 220 (2005), 313–328.
- [HM2] —, —, *Lowest uniformizations of compact real surfaces*, in: Contemp. Math. 397, Amer. Math. Soc., 2006, 145–152.
- [K] L. Keen, *On hyperelliptic Schottky groups*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 165–174.
- [Ko1] P. Koebe, *Über die Uniformisierung der algebraischen Kurven II*, Math. Ann. 69 (1910), 1–81.
- [Ko2] —, *Über die Uniformisierung reeller algebraischer Kurven*, Nachr. Akad. Wiss. Göttingen 1907, 177–190.
- [M1] B. Maskit, *Kleinian Groups*, Grundlehren Math. Wiss. 287, Springer, Berlin, 1988.
- [M2] —, *Remarks on m -symmetric Riemann surfaces*, in: Contemp. Math. 211, Amer. Math. Soc., 1997, 433–445.
- [M3] —, *A theorem on planar covering surfaces with applications to 3-manifolds*, Ann. of Math. (2) 81 (1965), 341–355.
- [Ma] C. L. May, *Automorphisms of compact Klein surfaces with boundary*, Pacific J. Math. 59 (1975), 199–210.

- [S] W. Scherrer, *Zur Theorie der endlichen Gruppen topologischer Abbildungen von geschlossenen Flächen in sich*, Comment. Math. Helv. 1 (1929), 69–119.
- [Z] B. Zimmermann, *Über Homöomorphismen n -dimensionaler Henkelkörper und endliche Erweiterungen von Schottky-Gruppen*, *ibid.* 56 (1981), 474–486.

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