FUNDAMENTA
MATHEMATICAE
176 (2003)

More reflections on compactness

by

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Abstract. We consider the question of when $X_M = X$, where X_M is the elementary submodel topology on $X \cap M$, especially in the case when X_M is compact.

1. Introduction. We are interested in the extent to which the part of a topological space reflected in an elementary submodel captures the whole space. Given a topological space $\langle X, \mathcal{T} \rangle \in M$, an elementary submodel of some $H(\theta)$, we define X_M to be $X \cap M$ with the topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. For a careful treatment of elementary submodels, see [8]. For an investigation of how X and M constrain X_M, see [7]. Here we are interested in what conditions on X_M ensure that $X = X_M$. This line of investigation was started in [13] and continued in [12] and [14]. Sample results include:

THEOREM 1.1. $X = X_M$ *provided any of the following conditions hold*:

(a) [13] X_M *is locally compact* T_2, *hereditarily Lindelöf and uncountable.*

(b) [12] X_M *is locally compact* T_2, *locally hereditarily Lindelöf and connected.*

(c) [14] X_M *is homeomorphic to* D^κ, *where* D *is the* 2-*point discrete space*, *and* κ *is less than the first inaccessible cardinal.*

Large cardinals in fact appear frequently in these three papers and will appear here as well. The basic reference is [9].

We will add several more sufficient conditions to this list, e.g.

2000 *Mathematics Subject Classification*: Primary 54A10, 03C62, 54D30; Secondary 54A25, 54B99, 54D65, 54G12.

Key words and phrases: elementary submodel, compact, countable tightness, reflection, separability, first countable.

The first author was supported by the University of São Paulo and NSERC grant A-7354.

The second author's work was supported by NSERC grant A-7354 and Fapesp grant 2000/09908-5.

(d) *X_M is uncountable, compact, T_2, and first countable.*

(e) *X_M is compact, T_2, with countable tightness, countable cellularity, and no isolated points.*

(f) *X_M is compact, T_2, separable, uncountable, and with no isolated points.*

(g) *X_M is compact, T_2, extremally disconnected, and $w(X)$ is less than the first inaccessible.*

When thinking about X_M characterizing X, it is perhaps more natural to think of homeomorphism rather than equality, yet all the previous results have concluded that in fact $X = X_M$. This is no accident:

THEOREM 1.2. *Suppose $0^\#$ does not exist. If X_M is homeomorphic to X, then in fact $X = X_M$.*

$0^\#$ is a set of natural numbers, the existence of which has large cardinal strength. $V = L$ implies $0^\#$ does not exist. The only consequence of its non-existence that we will use is given by the following lemma:

LEMMA 1.3 ([11]). *If $0^\#$ does not exist and $|M| \geq \lambda$, then $\lambda \subseteq M$.*

Let us state an easy, useful lemma:

LEMMA 1.4 ([13]). *If $A, B \in M$, $|A| \leq |B|$, and $B \subseteq M$, then $A \subseteq M$.*

To prove Theorem 1.2, note that since X_M is homeomorphic to X, $|X_M| = |X|$, so $|M| \geq |X \cap M| = |X|$. Therefore $|X| \subseteq M$. Since $X \in M$, we have $|X| \in M$, but then it follows by Lemma 1.4 that $X \subseteq M$, so $X \cap M = X$. We also have $w(X) \leq |M|$, where w is the least cardinal of a base of X, since $w(X_M) \leq |M|$. Again it follows that there is a base $\mathcal{B}$ for X included in M. This implies that X_M is a subspace of X and so equals X.

A "no large cardinal" hypothesis is in fact needed for Theorem 1.2. Recall

DEFINITION 1.5. A cardinal κ is *Jónsson* if any model of size κ has a proper elementary submodel of size κ.

EXAMPLE 1.6. Suppose κ is a Jónsson cardinal (see [9, Section 8]). Then it is standard that there is an elementary submodel N of $H(\kappa^+)$ such that $\kappa \in N$, $|\kappa \cap N| = \kappa$ but $\kappa \not\subseteq N$. Take $X = \kappa$ with the discrete topology. Then X_N is homeomorphic to X but $X_N \neq X$.

Also we can take Y to be the one-point compactification of X. Then we will have Y_N compact, Y_N homeomorphic to Y but $Y_N \neq Y$.

2. Exponential results. For the rest of the paper, we are mainly interested in the case of X_M being compact T_2. In a number of situations, that plus some other simple conditions will ensure that $X_M = X$.

When X_M is compact T_2 we have a useful relationship between X and X_M:

LEMMA 2.1 ([6], [7]). *If X_M is compact T_2, so is X. Also, there is a perfect map π from X onto X_M defined by $\pi^{-1}(\{x\}) = \bigcap\{U \in \mathcal{T} \cap M : x \in U\}$.*

For our first result we need the following lemma which improves a result in [14]:

LEMMA 2.2. *Suppose $X_M = X$. If $(X^\kappa)_M$ is compact T_2, then $(X^\kappa)_M$ is homeomorphic to $X^{\kappa\cap M}$. If in addition $\kappa \subseteq M$, then $(X^\kappa)_M = X^\kappa$.*

Proof. First note that $(X^\kappa)_M$ compact T_2 implies the same for X^κ and thus X is compact T_2. The proof of the lemma is the same as the one in [14] for the case of X being the two-point discrete space D.

Define $h : (X^\kappa)_M \to X^{\kappa\cap M}$ by $h(f) = f\restriction(\kappa \cap M)$. Since $(X^\kappa)_M$ is compact T_2, to show that h is a homeomorphism it is enough to show h is continuous, one-one and has a dense image.

If $f, g \in X^\kappa \cap M$ are such that $f \neq g$, by elementarity, there is $\alpha \in \kappa\cap M$ such that $f(\alpha) \neq g(\alpha)$, so h is one-one.

Now let V_p be a usual (non-empty) basic open set of $X^{\kappa\cap M}$. Note that since $X_M = X$, V_p is such that $\mathrm{proj}_\alpha(V_p) \in M$. Let d_p be the set of all α's such that $\mathrm{proj}_\alpha(V_p) \neq X$. We then have $h^{-1}(V_p) = \{f \in X^\kappa \cap M : f(\alpha) \in \mathrm{proj}_\alpha(V_p)$ for each $\alpha \in d_p\}$, which is open in $(X^\kappa)_M$. Also, $d_p \subseteq \kappa \cap M$, so the function f defined by $f(\alpha) = x$ for some $x \in \mathrm{proj}_\alpha(V_p) \cap M$ if $\alpha \in d_p$, and $f(\alpha) = y$ if $\alpha \notin d_p$ (y any fixed element of $X \cap M$), is in M. Note that $\mathrm{proj}_\alpha(V_p) \cap M \neq \emptyset$ because $\mathrm{proj}_\alpha(V_p) \in M$ by our assumption. But then $h(f) \in V_p$ and we are done. ∎

We can now show:

THEOREM 2.3. *Suppose $0^\#$ does not exist, $\kappa \in M$, and $X_M = X$. If $(X^\kappa)_M$ is compact, then $(X^\kappa)_M = X^\kappa = (X_M)^\kappa$.*

This was proved in [14] for $X = D$ with the additional assumption that $|M| \geq \kappa$. The new formulation shows that the existence of a compact $(D^\kappa)_M \neq D^\kappa$ necessarily has large cardinal strength. It was shown in [14] that such a D^κ exists if there is a 2-huge cardinal.

Proof of Theorem 2.3. From the previous lemma it follows that if $(X^\kappa)_M$ is compact, then $|M| \geq |X^{\kappa\cap M}| \geq 2^{\kappa\cap M}$. We claim $\kappa \subseteq M$.

If not, there is a minimal ordinal $\alpha < \kappa$ such that $\alpha \subseteq M$ but $\alpha \notin M$. Then $|M| \geq 2^{\kappa\cap M} \geq 2^{|\alpha|} \geq |\alpha|^+$. By the non-existence of $0^\#$, $|\alpha|^+ \subseteq M$. But $\alpha \in |\alpha|^+ \subseteq M$, so $\alpha \in M$, a contradiction. ∎

We can drop the assumption "$0^\#$ does not exist" if we assume κ to be less than the first inaccessible cardinal, improving another result in [14]:

THEOREM 2.4. *Suppose that κ is less than the first inaccessible cardinal, $\kappa \in M$, and $X_M = X$. If $(X^\kappa)_M$ is compact, then $(X^\kappa)_M = X^\kappa = (X_M)^\kappa$.*

Proof. Suppose not; then without loss of generality we can suppose κ to be the minimum cardinal such that $\kappa \in M$, $(X^\kappa)_M$ compact, but $(X^\kappa)_M \neq X^\kappa$. By Lemma 2.2 it is enough to show that $\kappa \subseteq M$ to get a contradiction.

Since κ is less than the first inaccessible cardinal, either there is $\lambda < \kappa$ such that $2^\lambda \geq \kappa$, or κ is a singular cardinal.

Suppose first that there is such a λ. Then by elementarity, we can pick $\lambda \in M$. Now, $\lambda < \kappa$ implies that X^κ can be mapped onto X^λ; thus, by elementarity, $(X^\kappa)_M$ can be mapped onto $(X^\lambda)_M$. We then see that $(X^\lambda)_M$ is compact. By the minimality hypothesis on κ we have $(X^\lambda)_M = X^\lambda$. It follows that $X^\lambda \subseteq M$. Since $2^\lambda \geq \kappa$, we must have $\kappa \subseteq M$ and we are done.

As in [14], if κ is singular, note that since $\kappa \in M$, we have $\mathrm{cf}(\kappa) \in M$, which implies $X^{\mathrm{cf}(\kappa)} \in M$. As before, we see that $(X^{\mathrm{cf}(\kappa)})_M$ is compact and thus, by the minimality of κ, we have $(X^{\mathrm{cf}(\kappa)})_M = X^{\mathrm{cf}(\kappa)}$. Therefore $X^{\mathrm{cf}(\kappa)} \subseteq M$, so $\mathrm{cf}(\kappa) \subseteq M$. But then there is a set S of cardinals cofinal in κ included in M. Note that, as before, for each $\lambda \in S$, we have $(X^\lambda)_M$ compact and $(X^\lambda)_M = X^\lambda$, which implies $\lambda \subseteq M$. Hence $\kappa \subseteq M$. ∎

LEMMA 2.5. *If κ is infinite and $(X^\kappa)_M$ is compact, then $2^\lambda \subseteq M$, where X is a topological space with at least two points, $\lambda < \kappa$, and λ is less than the first inaccessible cardinal.*

Proof. Since X has at least two points, D^κ can be embedded in X^κ. Therefore, there is a closed subset F of X^κ that can be mapped onto D^κ. By elementarity we can pick $F \in M$. Then, again by elementarity, F_M is a closed subset of $(X^\kappa)_M$ so it is compact. Now, since F can be mapped onto D^κ, by elementarity, F_M can be mapped onto $(D^\lambda)_M$. We conclude then that $(D^\lambda)_M$ is compact. Since λ is less than the first inaccessible cardinal, by 2.4 we have $(D^\lambda)_M = D^\lambda$, which implies $2^\lambda \subseteq M$. ∎

COROLLARY 2.6. *Suppose X is separable with at least two points and $0^\#$ does not exist. If $(X^\kappa)_M$ is compact, $\kappa \in M$, then $(X^\kappa)_M = X^\kappa = (X_M)^\kappa$.*

Proof. First recall that X will also be compact. Since X is separable and regular, $w(X) \leq 2^{\aleph_0}$. By the previous lemma we have $2^{\aleph_0} \subseteq M$, so X_M is a subspace of X. It will be a dense subspace because X is separable and therefore by compactness $X = X_M$. ∎

The same proof shows:

COROLLARY 2.7. *Suppose X is separable with at least two points and κ is less than the first inaccessible cardinal, $\kappa \in M$. If $(X^\kappa)_M$ is compact, then $(X^\kappa)_M = X^\kappa = (X_M)^\kappa$.*

3. Some examples. A distributed preprint of [14] claimed that if X_M is compact, then it is a retract of X, and that if X_M is compact and separable, then $X_M = X$. Both assertions are refuted by the following example:

EXAMPLE 3.1. Let $\mathcal{A}$ be a maximal almost disjoint family of subsets of ω. Form $\Psi(\mathcal{A}) = \omega \cup \mathcal{A}$, where points in ω are isolated, and a neighborhood of $A \in \mathcal{A}$ is $\{\mathcal{A}\}$ together with a cofinite piece of A. Let X be the one-point compactification of $\Psi(\mathcal{A})$. Let M be an elementary submodel of some $H(\theta)$, with X, $\mathcal{A} \in M$ and $|M| < |\mathcal{A}|$.

It is easy to see that X_M is compact and separable and not equal to X. In [7] we gave an incorrect proof that X_M is not a subspace of X, so we will put a correct proof here. It follows that X_M is certainly not a retract of X.

To see that X_M is not a subspace of X, pick $A \in \mathcal{A} \setminus M$ and consider $V = \{\infty\} \cup \Psi(\mathcal{A}) \setminus (\{A\} \cup A) = \{\infty\} \cup (\mathcal{A} \setminus \{A\}) \cup \omega \setminus A$. We will show that $V \cap M$ is not open in X_M. Let $W \in \mathcal{T} \cap M$ be such that $\infty \in W$. Then $W = \{\infty\} \cup \Psi(\mathcal{A}) \setminus K$, where K is a compact set in $\Psi(\mathcal{A}) \cap M$. We will show that $W \cap M \not\subseteq V \cap M$ by looking at $W \cap \omega$ and $V \cap \omega$.

Note that, since K is compact and is in M, $K \cap \omega$ is finite or covered by a set of the form $(B_1 \setminus F_1) \cup \ldots \cup (B_n \setminus F_n)$, with $B_i \in M$ and F_i finite for each $i \leq n$. Since A is infinite, $K \cap \omega$ finite implies that $W \cap M \not\subseteq V \cap M$. If the second case happens we also have $W \cap M \not\subseteq V \cap M$, but now it is because $\mathcal{A}$ is an almost disjoint family (so there is $i \in A$ such that $i \notin B_j$ for each $j \leq n$).

EXAMPLE 3.2. It is possible to have X_M be a compact subspace of X, without having $X_M = X$. Let X be the one-point compactification of the disjoint sum of $(2^{\aleph_0})^+$ copies of $[0, 1]$. Let M be a countably closed elementary submodel of size $2^{\aleph_0}$ containing X. Then X_M is as desired.

EXAMPLE 3.3. X_M can be compact and connected yet not equal to X. Let X be the long closed interval of length $(2^{\aleph_0})^+ + 1$. Take a countably closed M of size $2^{\aleph_0}$ and such that $M \cap (2^{\aleph_0})^+$ is an ordinal α. Then X_M is homeomorphic to the long closed interval of length $\alpha + 1$, so it is compact and not equal to X.

4. Separability and maps onto I^κ. The following two results will be used several times:

THEOREM 4.1. *Let $X \in M$, with X_M compact and T_2. If $\chi(X) \leq \kappa$ and $\kappa \subseteq M$, then $X_M = X$.*

Proof. Fix for each $x \in X$ a neighborhood base $\mathcal{B}_x$ of x of size $\leq \kappa$. Then for each $x \in M$, we can take $\mathcal{B}_x \in M$ and, since $\kappa \subseteq M$, $\mathcal{B}_x \subseteq M$. Therefore $\{x\} = \bigcap \mathcal{B}_x \supseteq \bigcap\{V \in \mathcal{T} \cap M : x \in V\}$. But this says that $\pi^{-1}(\{x\}) = \{x\}$, with π defined as in 2.1. Thus π is the identity homeomorphism. ∎

COROLLARY 4.2. *Assume $0^\#$ does not exist. If $|M| \geq \kappa$, $\chi(X) \leq \kappa$, and X_M is compact and T_2, then $X_M = X$.*

Proof. $|M| \geq \kappa$, so $\kappa \subseteq M$. ∎

Corollary 4.3. *If X_M is compact T_2 and X is first countable, then $X_M = X$.*

Corollary 4.4. *Suppose X is first countable and $0^\#$ does not exist. If $(X^\kappa)_M$ is compact, $\kappa \in M$, then $(X^\kappa)_M = X^\kappa$.*

Proof. Since $(X^\kappa)_M$ is compact, X^κ and therefore X will also be compact. By 4.3 we then have $X = X_M$. The result follows from 2.3. ∎

Using 2.4 we can similarly show:

Corollary 4.5. *Suppose X is first countable and κ is less than the first inaccessible cardinal. If κ is infinite and $(X^\kappa)_M$ is compact, then $(X^\kappa)_M = X^\kappa$.*

The second result that we will often use is:

Lemma 4.6. *If X_M is compact T_2 and with no isolated points, then $2^{\aleph_0} \subseteq M$.*

Proof. If X_M is compact T_2 and has no isolated points, then X_M can be mapped onto $D^{\aleph_0}$ (see [14]). But then X can be mapped onto $D^{\aleph_0}$ (by Lemma 2.1). By elementarity X_M can be mapped onto $(D^{\aleph_0})_M$ and the result follows as in the proof of Lemma 2.5. ∎

Corollary 4.7. *If X_M is uncountable, first countable, compact and T_2, then $2^{\aleph_0} \subseteq M$.*

Proof. This is because X_M includes a compact perfect set. ∎

Theorem 4.8. *Suppose $0^\#$ does not exist or κ is less than the first inaccessible. Also suppose there is an $f : X \to I^\kappa$ onto and either $\chi(X) \leq 2^\kappa$ or $|f^{-1}(x)| \leq 2^\kappa$ for every $x \in I^\kappa$. If X_M is compact T_2, then $X = X_M$.*

Proof. Since X maps onto I^κ, taking $f \in M$, by elementarity, X_M maps onto $(I_\kappa)_M$, which is therefore compact T_2. By Corollary 2.6 we then have $(I^\kappa)_M = I^\kappa$ and so $I^\kappa \subseteq M$; in particular $2^\kappa \subseteq M$.

If $\chi(X) \leq 2^\kappa$, by Theorem 4.1, we conclude that $X = X_M$. Suppose $|f^{-1}(x)| \leq 2^\kappa$ for every $x \in I^\kappa$. Now, $x \in I^\kappa$ implies $x \in M$ and therefore $f^{-1}(x) \in M$. Next, $|f^{-1}(x)| \leq 2^\kappa$ and $2^\kappa \subseteq M$ give us $f^{-1}(x) \subseteq M$ for each $x \in I^\kappa$, so $X \subseteq M$. As above, we then have $X = X_M$. ∎

Example 6.4 in Section 6 shows that the hypothesis on the cardinality of $f^{-1}(x)$ cannot be removed in the previous result.

Theorem 4.9. *Suppose $2^\omega \subseteq M$. If X_M is compact, T_2, and separable, then $X = X_M$.*

Proof. First assume X maps onto I^{2^ω}, whence by elementarity X_M maps onto $(I^{2^\omega})_M$, which will then be compact. By Corollary 2.7, we will have $(I^{2^\omega})_M = I^{2^\omega}$ so $2^{2^\omega} \subseteq M$.

Now, X_M is compact separable, so $w(X_M) \leq 2^\omega$, and so X_M has no left- or right-separated subspace of size $(2^\omega)^+$. By [13] it follows that X has no left- or right-separated subspace of size $(2^\omega)^+$, thus $|X| \leq 2^{2^\omega}$. But, since $2^{2^\omega} \subseteq M$, and X is compact, we have $X = X_M$.

Now consider the other case. Note that, since $2^\omega \subseteq M$ and $c(X_M) = \omega$, we must have $c(X) = \omega$. For if $c(X) > \omega$ then X has a cellular family of size ω_1, and by elementarity we can take this family in M. Next, $\omega_1 \subseteq M$ implies that this family is included in M, so it will be a family of ω_1 pairwise disjoint open subsets of X_M, contradicting $c(X_M) = \omega$.

If X does not map onto I^{2^ω}, then in particular X does not map onto $I^{(2^\omega)^+}$, so by a result in [5] (corollary after 3.20, page 71), we have $|RO(X)| \leq (2^\omega)^{c(X)} = (2^\omega)^\omega = 2^\omega$. But then $w(X) \leq 2^\omega$, and by 4.1, $X = X_M$. ∎

COROLLARY 4.10. *If X_M is compact, T_2, separable, uncountable, and with no isolated points, then $X = X_M$.*

COROLLARY 4.11. *If X_M is compact, T_2, separable, uncountable, and X_M maps onto $D^{\aleph_0}$, then $X = X_M$.*

Because of Example 3.1, to get X_M compact separable implies $X = X_M$, we must assume $2^{\aleph_0} \subseteq M$ (or something that implies this). Thus assuming not CH, for instance, it is possible to get X_M compact separable but $X \neq X_M$.

As mentioned above, it is not true that X_M separable implies X separable. But in view of 4.9 and 4.10 it is natural to ask if X_M compact separable implies X separable, without the two necessarily being equal. We have:

THEOREM 4.12. *Let M be uncountable. Then $\omega_1 \subseteq M$ if and only if whenever X_M is compact, T_2, separable, and uncountable, then X is separable.*

We first prove the forward direction. We need the following result:

LEMMA 4.13. *Suppose $X \neq X_M$, X_M compact, separable, and uncountable. Then X is a scattered space.*

Proof. If X is not scattered, it has a closed subspace F with no isolated points. We may take $F \in M$. Then F maps onto $I^{\aleph_0}$ so, as before, we get $2^{\aleph_0} \subseteq M$ and hence, by 4.9, $X = X_M$. ∎

We can now prove the forward implication of Theorem 4.12:

Proof. If $X = X_M$, then we are done, so assume $X \neq X_M$. By the previous result, X must then be a scattered space.

Note that X must have at most countably many isolated points. Since $\omega_1 \subseteq M$, if X had uncountably many isolated points, by elementarity X_M would also have uncountably many isolated points, but we are assuming X_M is separable.

Now X is a scattered space, so the first level of X is this countable set of isolated points. On the other hand, for scattered spaces, the set of isolated points has to be dense in the space (otherwise there would be an open set V disjoint from it; but X scattered implies this open set has an isolated point; contradiction). We conclude then that X is separable. ∎

To prove the other implication, we use a variation of 3.1:

EXAMPLE 4.14. Suppose there is an uncountable model M such that $\omega_1 \not\subseteq M$. Then $M \cap \omega_1$ is countable. Take a family $\mathcal{A}$ of countable almost disjoint subsets of ω_1 such that $\mathcal{A} \cap M$ is uncountable. Define $\Psi(\mathcal{A})$ as in Example 3.1, but with ω_1 instead of ω, and take for X the one-point compactification of this space. As before, we see that X_M is compact. But here X_M is separable and X is not.

The following isolated result may be of interest:

THEOREM 4.15. *Suppose X_M is homeomorphic to $\beta N \setminus N$. Then $X = X_M$.*

Proof. Let Y be a countable dense subspace of $I^{2^{\aleph_0}}$, where $I = [0,1]$. Let f map the countable discrete space N onto Y. Then f extends to $\overline{f}$ mapping βN onto $\beta Y = I^{2^{\aleph_0}}$. The restriction $\overline{f}\restriction(\beta N \setminus N)$ must also map onto $I^{2^{\aleph_0}}$, since $I^{2^{\aleph_0}}$ has no countable open sets. Thus X_M and hence X must map onto $I^{2^{\aleph_0}}$. The result now follows from 4.8. (Note that since $|X| \geq |X_M| = 2^{2^{\aleph_0}}$, and $\theta \geq (2^{2^{\aleph_0}})^+$, it follows that $2^{2^{\aleph_0}}$, I and $I^{2^{\aleph_0}}$ are all in $H(\theta)$ and hence, by definability, in M.) ∎

5. First countability and countable tightness.

It is not true in general that X_M first countable implies X first countable [6]. However, in the case of X_M compact it does, i.e., the next result shows that it is enough to assume X_M first countable in 4.3.

THEOREM 5.1. *Suppose X_M is uncountable, compact, T_2, and first countable. Then $X = X_M$.*

COROLLARY 5.2. *If X_M is uncountable, compact, and T_2, then X is first countable if and only if X_M is.*

In order to prove this we first have to establish several useful results. In [15], S. Todorčević defined:

DEFINITION 5.3. Suppose X is a topological space and $F, G \subseteq X$. We say that $\langle F, G\rangle$ is *regular* if F is closed, G is open and $F \subseteq G$.

DEFINITION 5.4. A sequence $\langle F_\alpha, G_\alpha\rangle$, $\alpha < \theta$, of regular pairs of X is called *free* if for any finite subsets K, L of θ with $K < L$, we have $\bigcap_{\alpha\in K} F_\alpha \cap \bigcap_{\beta\in L}(X \setminus G_\beta) \neq \emptyset$.

Note that for a compact T_2 space X, there is a free θ-sequence of regular pairs if and only if there is a free sequence $\{x_\alpha : \alpha < \theta\}$ (in the usual sense). Thus for X compact T_2 we deduce that X has countable tightness if and only if there are no free ω_1-sequences of regular pairs of X ([15]).

THEOREM 5.5. *If X_M is compact with countable tightness and $\omega_1 \subseteq M$, then X also has countable tightness.*

Proof. Suppose X has uncountable tightness. Since X_M is compact, X is compact, so there is a free sequence $\langle F_\alpha, G_\alpha \rangle$, $\alpha < \omega_1$, of regular pairs of X. By elementarity we can take this sequence in M, and because $\omega_1 \subseteq M$, we get $\langle F_\alpha, G_\alpha \rangle \in M$ for every $\alpha < \omega_1$.

By elementarity (and because $\langle F_\alpha, G_\alpha \rangle \in M$ for every $\alpha < \omega_1$), $\langle F_\alpha, G_\alpha \rangle$, $\alpha < \omega_1$, is a sequence of regular pairs of X_M and M thinks that it is free. Since the definition of a free sequence of regular pairs just talks about finite subsets of ω_1, it follows that the sequence is really free (i.e., it is free in V). But this contradicts the assumption that X_M has countable tightness. ∎

We note that the assumption $\omega_1 \subseteq M$ is essential:

EXAMPLE 5.6. Assume that there is an uncountable model M such that $\omega_1 \in M$ but $M \cap \omega_1 = \alpha < \omega_1$. Take $X = \omega_1 + 1$ with the usual order topology. Then X has uncountable tightness, but X_M (which, by [7], is homeomorphic to $\alpha + 1$) is compact and has countable tightness.

We can now prove Theorem 5.1:

Proof of Theorem 5.1. By 4.1 it suffices to show X is first countable. Also, by 4.7 we have $2^{\aleph_0} \subseteq M$.

Suppose that X is not first countable. Then by for example [1], X has a subspace Y of size $\aleph_1$ which is not first countable. Taking $Y \in M$, we see that it is enough to show $\overline{Y} = \overline{Y}_M$. Indeed, $\overline{Y}_M$ is first countable (because it is a subspace of X_M which is first countable) and $\overline{Y}$ is not, so if we show $\overline{Y} = \overline{Y}_M$, we have a contradiction. Without loss of generality, we can take $X = \overline{Y}$, i.e., we can assume $d(X) \leq \aleph_1$.

Let D be dense in X, $D \in M$, $|D| \leq \aleph_1$. Note that because $2^{\aleph_0} \subseteq M$, we have $D \subseteq M$.

Now, by 5.5, we have $t(X) = \aleph_0$. Thus we can write $X = \overline{D} = \bigcup\{\overline{E} : E \in [D]^\omega\}$.

Then $X_M = \bigcup\{\overline{E}_M : E \in [D]^\omega \cap M\}$ (since $D \cap M = D$). Each such $\overline{E}_M = \overline{E}$ (by 4.9). Thus $X_M = \bigcup\{\overline{E} : E \in [D]^\omega \cap M\}$. But $D \in M$ so $[D]^\omega \in M$. Also, $|[D]^\omega| \leq 2^{\aleph_0}$, so $[D]^\omega \subseteq M$ (because $2^{\aleph_0} \subseteq M$), which implies $[D]^\omega \cap M = [D]^\omega$. We then conclude that $X = X_M$. ∎

Now we want to weaken character to tightness. Let X be a topological space and $x \in X$. Recall that a *local π-base for x* is a collection $\mathcal{V}$ of non-empty open sets in X such that if U is any open neighborhood of x, then there is $V \in \mathcal{V}$ such that $V \subseteq U$. We can then define $\pi\chi(x, X) = \min\{|\mathcal{V}| : \mathcal{V}$ is a local π-base for $x\}$, and $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$. We will denote by $h\pi\chi(X)$ the hereditary π-character, i.e. $\sup\{\pi\chi(Y) : Y \subseteq X\}$.

THEOREM 5.7. *Suppose* $2^{\aleph_0} \subseteq M$. *If* X_M *is compact* T_2 *with countable tightness and countable cellularity*, *then* $X_M = X$.

Proof. First note that, since X_M has countable tightness, by Theorem 5.5, X must also have countable tightness. For compact T_2 spaces tightness is equal to $h\pi\chi$ (see for example [4] or [5]). Since X_M is compact, X is also compact [6] and therefore $h\pi\chi(X) = \omega$. In particular, $\pi\chi(X) = \omega$.

Note that, as in the proof of 4.9, $c(X_M) = \omega$ implies that $c(X) = \omega$.

A result of Shapirovskiĭ (see for example [4] or [5]) entails that for a regular space Y, $w(Y) \leq \pi\chi(Y)^{c(Y)}$. Thus, $w(X) \leq 2^{\aleph_0}$, which by Theorem 4.1 implies $X = X_M$ (since $2^{\aleph_0} \subseteq M$). ∎

COROLLARY 5.8. *If* X_M *is compact* T_2 *with countable tightness*, *countable cellularity and no isolated points*, *then* $X_M = X$.

COROLLARY 5.9. *Assume* $0^{\#}$ *does not exist. Suppose* X *has countable tightness and countable cellularity. If* $(X^\kappa)_M$ *is compact* T_2, *then* $(X^\kappa)_M = X^\kappa$.

COROLLARY 5.10. *Assume* κ *is less than the first inaccessible. Suppose* X *has countable tightness and countable cellularity. If* $(X^\kappa)_M$ *is compact* T_2, *then* $(X^\kappa)_M = X^\kappa$.

REMARK 5.11. Example 3.1 shows that the hypothesis of having no isolated points cannot be removed from Corollary 5.8.

6. Scattered spaces. Corollary 4.3 says that if X is compact and first countable, then there is no elementary submodel M such that X_M is compact and different from X. It is natural to ask if the opposite can happen, i.e., X_M be compact for every M. After Piotr Koszmider informed us of a Boolean version (for generic extensions) of the following—which he attributed to folklore—we proved the following result, which was also proved by him independently:

THEOREM 6.1. *If* X *is a compact* T_2 *scattered space*, *then* X_M *is compact for every elementary submodel* M *such that* $X \in M$.

Proof. We first recall that if X is a compact scattered T_2 space, then X is zero-dimensional. This is folklore but we sketch the proof here for completeness. For $x \in X$, let A be the intersection of all clopen subsets of X that contain x. Since X is compact, it is enough to show that $A = \{x\}$. If not, there is $y \in A \setminus \{x\}$ such that $\{y\}$ is an isolated point in A. Let

V_1 and V_2 be disjoint open sets separating the closed disjoint sets $A \setminus \{y\}$ and $\{y\}$, respectively. If W is a clopen set containing x included in $V_1 \cup V_2$ (there is one because A is compact), we can show that $W \cap V_1$ is a clopen set containing x, but not containing y, contradicting $y \in A$.

Without loss of generality we can suppose $X = \{x_\alpha : \alpha \leq \kappa\}$ (here κ could be an ordinal). Also, since X is a compact T_2 scattered space, for every $\alpha \leq \kappa$ there is a clopen set U_α such that $U_\alpha \subseteq \{x_\beta : \beta \leq \alpha\}$, and such that $\mathcal{V}_\alpha = \{U_\alpha \setminus \bigcup_{\gamma \in F} U_\gamma : F \subseteq \alpha, F \text{ finite}\}$ forms a base at x_α. To see this, use the fact that X is zero-dimensional. For each x_α pick a clopen neighborhood U_α of x_α witnessing that x_α is an isolated point in the subspace $\{x_\beta : \beta \geq \alpha\}$ and then use compactness of X to show that $\mathcal{V}_\alpha$ forms a base at x_α.

Let M be an elementary submodel such that $X \in M$, and thus $\kappa \in M$. We want to show that X_M is compact. The perfect map π defined as in 2.1 exists for any compact T_2 $X \in M$ [7], but need not have all of X as domain. We will show it does in our situation, and hence X_M is compact. It suffices to show that for every $y \in X$ there is an $x \in X \cap M$ such that $y \in K_x = \bigcap\{U \in \mathcal{T} \cap M : x \in U\}$.

If $y = x_\alpha$ for $\alpha \in M$, we do not have anything to prove. Suppose then that $y = x_\alpha$ for $\alpha \notin M$. Let $\beta = \min((\kappa + 1 \setminus \alpha) \cap M)$. We will show that in this case $x_\alpha \in K_{x_\beta}$.

For that we first note that for each $V \in \mathcal{V}_\beta$, that is, $V = U_\beta \setminus \bigcup_{\gamma \in F} U_\gamma$ with F a finite subset of β, the set V is in M if and only if $F \subseteq M$. That is because, if $V \in M$, we can reflect to M the following sentence: "there is an $F \subseteq \kappa$ such that F is finite and $V = U_\beta \setminus \bigcup_{\gamma \in F} U_\gamma$".

Now to show that $x_\alpha \in K_{x_\beta}$, fix $V \in \mathcal{V}_\beta \cap M$. Thus $V = U_\beta \setminus \bigcup_{\gamma \in F} U_\gamma$ for some finite $F \subseteq \beta \cap M$. But by the definition of β, $\beta \cap M < \alpha$. Thus by the choices of the U_γ's we must then have $x_\alpha \in V$. ∎

EXAMPLE 6.2. Let X be the long line of length $(2^{\aleph_0})^+ + 1$. Take M and N to be two elementary submodels both including $[0,1]$ but such that $\mathrm{cf}(M \cap (2^{\aleph_0})^+) = \omega$ and $\mathrm{cf}(N \cap (2^{\aleph_0})^+) = \omega_1$. Then X_M and X_N are both compact and non-homeomorphic. If we take a model P such that $[0,1] \not\subseteq P$ we will find that X_P is not compact.

COROLLARY 6.3. *If X is compact scattered and Y is compact, then $(X \times Y)_M$ is compact for every elementary submodel M such that $X, Y \in M$ and $Y_M = Y$.*

Proof. This is because, by elementarity, $(X \times Y)_M = X_M \times Y_M$. ∎

EXAMPLE 6.4. Let X be the space obtained by replacing each point of I^κ by a compact scattered space of size $> 2^\kappa$ and let M be an elementary submodel of cardinality 2^κ. Then X_M is compact but $X_M \neq X$, yet X can be mapped onto I^κ.

We now show that actually the other direction of 6.1 also holds:

THEOREM 6.5. *If there is a countable elementary submodel M such that $X \in M$ and X_M is compact T_2, then X is scattered.*

Proof. Suppose not. By elementarity we can fix $F \in M$ such that F is closed and F has no isolated points. Then F_M is closed in X_M and therefore it is compact T_2. But F_M is countable and every compact T_2 countable space must have an isolated point (by the Baire Category Theorem). So F_M has an isolated point, which implies, by elementarity, that F has an isolated point, a contradiction. ∎

COROLLARY 6.6. *The following are equivalent*:

(a) *X is compact*, *scattered and T_2*;

(b) *there is a countable elementary submodel M such that X_M is compact T_2*;

(c) *X_M is compact T_2 for every elementary submodel M such that $X \in M$.*

7. Extremally disconnected spaces and Boolean algebras. For our first result we need the following lemma:

LEMMA 7.1. *If X_M is extremally disconnected*, *then so is X.*

Proof. For a point $x \in X \cap M$, let $\mathcal{B}_x$ be a base at x in X. By elementarity we can take $\mathcal{B}_x \in M$ and also $\mathcal{B}_x \cap M$ is a base at x in X_M.

Suppose X is not extremally disconnected. Then there is an open set V in X such that $\overline{V}$ is not open in X. This means that

$$(\exists V \in \mathcal{T})(\exists x \in X)[(\forall V_x \in \mathcal{B}_x,\ V_x \cap V \neq \emptyset) \text{ and } (\forall V_x \in \mathcal{B}_x,\ V_x \not\subseteq \overline{V})].$$

By elementarity we then have

$$\begin{aligned}(\exists V \in \mathcal{T} \cap M)(\exists x \in X_M)[(\forall V_x \in \mathcal{B}_x \cap M,\ V_x \cap V \cap M \neq \emptyset)&\\ \text{and } (\forall V_x \in \mathcal{B}_x \cap M,\ V_x \cap M \not\subseteq \operatorname{cl}_{X_M} V)].&\end{aligned}$$

Thus $V \cap M$ is an open set in X_M whose closure is not open, contradicting the fact that X_M is extremally disconnected. ∎

THEOREM 7.2. *Assume $0^\#$ does not exist. If X_M is compact and extremally disconnected*, *then $X_M = X$.*

Proof. This is obvious if X is finite. If X is infinite, X is extremally disconnected by the previous lemma, so by the Balcar–Franěk Theorem (see for example [3, 6.2.G, p. 372]), $|X| = 2^{w(X)}$ and X maps onto $D^{w(X)}$. Then since X_M is compact, $(D^{w(X)})_M$ is compact, so by Theorem 2.6, $2^{w(X)} \subseteq M$, so X and $\mathcal{T}$ are included in M, so $X = X_M$. ∎

Corollary 7.3. *If X_M is compact and extremally disconnected and $w(X)$ is less than the first inaccessible, then $X_M = X$.*

Proof. This follows from 2.4. ∎

In [14], it is shown that if there is a 2-*huge cardinal*, then there is a κ such that $X = (D^\kappa)_M$ is compact but $\neq D^\kappa$. The Stone space $E(X)$ of the regular open algebra of X will be an example of a space X such that X_M is compact and extremally disconnected but $\neq X$.

Definition 7.4. λ is a 2-*huge cardinal* if there is an elementary embedding $j : V \to N$, an inner model, with critical point λ, such that ${}^{j(j(\lambda))}N \subseteq N$.

Theorem 7.5. *If λ is 2-huge, then there is an elementary submodel M such that $(E(D^\lambda))_M$ is compact and extremally disconnected but $\neq E(D^\lambda)$.*

Proof. Arguing in analogy to [14], observe that

$$\begin{aligned} E(D^{j(\lambda)}) \cap j''V_{j(\lambda)} &= \{j(S) : j(S) \in E(D^{j(\lambda)}) \text{ and } S \in V_{j(\lambda)}\} \\ &= \{j(S) : S \in E(D^\lambda) \text{ and } S \in V_{j(\lambda)}\} \\ &= j''(E(D^\lambda)). \end{aligned}$$

The second equality follows by elementarity; the third since $j(\lambda)$ is much bigger than λ. The space $j''(E(D^\lambda))$ is compact T_2 and extremally disconnected; it is the same as $(E(D^{j(\lambda)}))_{j''V_{j(\lambda)}}$, since that is a weaker T_2 topology on the set $j''(E(D^\lambda))$.

As in [14], we note that $j''V_{j(\lambda)}$ and $V_{j(j(\lambda))}$ are in N and that the proof that the former is an elementary submodel of the latter can be carried out in N. Thus N thinks there is an elementary submodel M of $H(j(j(\lambda)))$ such that $(E(D^{j(\lambda)}))_M$ is compact T_2 and extremally disconnected but $\neq E(D^{j(\lambda)})$ (since $j(\lambda)$ is much bigger than λ). By elementarity, there is then in V an elementary submodel M' of $H(j(\lambda))$ such that $(E(D^\lambda))_{M'}$ is compact but $\neq E(D^\lambda)$. ∎

We now look at Boolean algebras. We would like to thank Piotr Koszmider for helping get these results. If A is a Boolean algebra, we will denote by $S(A)$ the Stone space of A. Here we will always assume that M is an elementary submodel such that $A \in M$. Note that in general $S(A)_M$ is homeomorphic to $S(A \cap M) \cap M$.

Lemma 7.6. *If $S(A)_M$ is compact then $S(A)_M$ is homeomorphic to $S(A \cap M)$.*

Proof. Define $f : S(A)_M \to S(A \cap M)$ by $f(u) = u \cap M$.

First note that f is well defined. If $u \in S(A)_M$, then u is an ultrafilter in A and $u \in M$. Then by elementarity we deduce that $u \cap M$ is an ultrafilter in $A \cap M$.

Also, f is 1-1: if $u, v \in S(A) \cap M$ and $u \neq v$, by elementarity we have $u \cap M \neq v \cap M$.

It is easy to see that f is continuous.

Since $S(A)_M$ is compact, it is enough to show that $f(S(A)_M)$ is dense in $S(A \cap M)$. In fact, let $a \in A \cap M$ and $V(a) = \{u \in S(A \cap M) : a \in u\}$ be a basic open set. Since $a \in M$, by elementarity there is an ultrafilter $u \in S(A) \cap M$ such that $a \in u$. But then $u \in S(A)_M$ and $f(u) \in V(a)$, and we are done. ∎

Theorem 7.7. *Assume $0^{\#}$ does not exist. Let A be a Boolean algebra. If $A \cap M$ is complete and $S(A)_M$ is compact, then $A = A \cap M$.*

Proof. Since $A \cap M$ is complete, $S(A \cap M)$ is extremally disconnected. But then, by Lemma 7.6, $S(A)_M$ is also extremally disconnected. Thus $S(A)_M$ is compact extremally disconnected, so by Theorem 7.2, $S(A)_M = S(A)$. We then find, by the lemma again, that $S(A)$ is homeomorphic to $S(A \cap M)$, which implies that $A = A \cap M$. This follows from a result in Boolean algebra, but we will prove it here for completeness. Suppose not, and let $a \in A \setminus M$. Let $[a]_A$ denote the corresponding clopen set in $S(A)$ and f be the homeomorphism between $S(A)$ and $S(A \cap M)$ (which takes u to $u \cap M$). Then $f([a]_A)$ is clopen so there is $b \in A \cap M$ such that $f([a]_A) = [b]_{A \cap M}$. We will show that $a = b$, which is a contradiction. If there is $u \in [b]_A \setminus [a]_A$, then $u \cap M \in [b]_{A \cap M}$ and $u \notin [a]_A$; but this implies $f(u) \in [b]_{A \cap M}$ and $f(u) \notin f([a]_A)$, contradiction. If there is $u \in [a]_A \setminus [b]_A$, then $f(u) \in f([a]_A)$, but $f(u) \notin [b]_{A \cap M}$, again a contradiction. ∎

Example 7.8. Let λ be 2-huge and A be the regular open algebra of D^{λ}. By Theorem 7.5 there is an elementary submodel M such that $S(A)_M$ is compact extremally disconnected but $\neq S(A)$ (and they are not even homeomorphic). Since $S(A)_M$ is compact, it is homeomorphic to $S(A \cap M)$ by Lemma 7.6, so $S(A \cap M)$ is extremally disconnected. Thus A is such that $S(A)_M$ is compact, $A \cap M$ is complete, but $A \neq A \cap M$.

The next example (due to Piotr Koszmider) shows that the hypothesis of $S(A)_M$ being compact is essential:

Example 7.9. Let $\mathcal{B}'$ be a measure algebra such that $\mathcal{B} = \mathcal{B}'/\text{null}$ is c.c.c., has size $> 2^{\aleph_0}$ and is complete. Let M be a countably closed elementary submodel of size $2^{\aleph_0}$. Then $\mathcal{B} \neq \mathcal{B} \cap M$, but $\mathcal{B} \cap M$ is complete. This is because $\mathcal{B}$ has the countable chain condition, M is countably closed and a Boolean algebra is complete if and only if every antichain has a supremum.

We would like to thank Ofélia Alas for her useful comments on an earlier version of this paper.

Remark. After this paper was completed, K. Kunen [10] achieved much sharper bounds for the consistency strength of the existence of a compact $(D^\kappa)_M \neq D^\kappa$. Thus the results here depending on that consistency strength can also be sharpened. Another recent result is due to E. T. Eisworth [2], who proved the converse of Example 1.6, i.e. that if there is an M and a space $X \in M$ such that X_M is homeomorphic to X but $\neq X$, then there is a Jónsson cardinal.

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Received 15 September 2001;
in revised form 4 October 2002