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The point of continuity property, neighbourhood assignments and filter convergences

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Abstract. We show that for some large classes of topological spaces X and any metric space (Z,d), the point of continuity property of any function $f : X \to (Z,d)$ is equivalent to the following condition:

(∗) For every $\varepsilon > 0$, there is a neighbourhood assignment $(V_x)_{x\in X}$ of X such that $d(f(x), f(y)) < \varepsilon$ whenever $(x,y) \in V_y \times V_x$.

We also give various descriptions of the filters $\mathcal{F}$ on the integers $\mathbb{N}$ for which (∗) is satisfied by the $\mathcal{F}$-limit of any sequence of continuous functions from a topological space into a metric space.

1. Introduction. All the functions considered in this paper are assumed to take their values in metric spaces. Following [22], a topological space X is said to be *weakly separated* if there is a neighbourhood assignment $(V_x)_{x\in X}$ of X such that $x = y$ whenever $(x,y) \in V_y \times V_x$. Here, a *neighbourhood assignment* means that each V_x is an open neighbourhood of x in X. Extending Tkachenko's concept to functions, let us say that a function f from a (topological) space X into a metric space (Z,d) is *weakly separated* if for every $\varepsilon > 0$, there is a neighbourhood assignment $(V_x)_{x\in X}$ of X such that $d(f(x), f(y)) \le \varepsilon$ whenever $(x,y) \in V_y \times V_x$. This class of functions was considered by Lee, Tang and Zhao in [20] for metric spaces X. They proved that a real-valued function on a Polish space is of the first Baire class if and only it is weakly separated. We shall extend this result to a much broader setting.

Let X be a space and (Z,d) be a metric space. Recall that a function $f : X \to Z$ is said to be *cliquish* if for every $\varepsilon > 0$ and every nonempty open set $U \subset X$, there is a nonempty open set $O \subset U$ such that $d(f(x), f(y)) < \varepsilon$ for every $x, y \in O$. Following [17], f is said to be *fragmentable* if f is hereditarily

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cliquish, that is, the restriction $f_{|A}$ of f to any nonempty (closed) subset A of X is cliquish. It is well known [17] that every function f defined on a hereditarily Baire space X is fragmentable if and only if the restriction of f to each nonempty closed subset of X has a point of continuity, i.e., f has the *point of continuity property* (PCP). Moreover, if X is metrizable and Z is the reals $\mathbb{R}$, then f has the PCP if and only if f is the pointwise limit of a sequence of continuous real-valued functions on X, i.e., f is of the first Baire class.

Part of the result of [20] is that every weakly separated function defined on a hereditarily Baire metrizable space X has the PCP, or equivalently, is fragmentable. The assumption of hereditary Baireness of X is essential here; see Propositions 2.1, 2.4 and Examples 2.3 in Section 2 where some auxiliary facts are stated. We shall show in Section 4 (Theorems 4.1 and 4.2) that this part of [20] extends to any hereditarily Baire space X satisfying one of the following conditions:

(i) X is monotonically semistratifiable as defined in [5],
(ii) X is a suborderable monotonic β-space, again as defined in [5],
(iii) X is a monotonic β-space and has a point-countable T_0-separating open collection.

It follows from (i) that every weakly separated function defined on a hereditarily Baire semistratifiable space is σ-discrete in Hansell's sense [12] and F_σ-measurable (Theorem 4.5). Our results also imply that every separately continuous function $f : X \times Y \to Z$ defined on a hereditarily Baire product $X \times Y$ of two semistratifiable spaces X and Y into a metric space (Z, d) is F_σ-measurable and σ-discrete (Corollary 4.6). T. Banakh's question [2] related to this result is answered in the affirmative.

The main results of Section 4 depend on some properties of neighbourhood assignments established in Section 3.

In [20] it is also proved that if X is a Polish space, then every function $f : X \to \mathbb{R}$ of the first Baire class is weakly separated. This leads to the following natural question: for which filters $\mathcal{F}$ on the integers $\mathbb{N}$, is the $\mathcal{F}$-limit of any $\mathcal{F}$-convergent sequence of continuous functions weakly separated? At first glance, one would expect that these filters form a class strictly larger than the class of filters obtained the same way by replacing weakly separated functions by functions of the first Baire class. It turns out that these two classes of filters are identical. More precisely, we shall prove that for every filter $\mathcal{F}$ on $\mathbb{N}$ the following are equivalent:

(i) $\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets (as defined in [19]),
(ii) $\mathcal{F}$ is F_σ-separated from its dual ideal (Lemma 5.4),
(iii) the $\mathcal{F}$-limit of any sequence of continuous functions is F_σ-measurable (Proposition 5.7),

(iv) the $\mathcal{F}$-limit of any sequence of continuous functions is weakly separated (Proposition 5.5).

Similar results on analytic filters in the context of descriptive set theory were recently obtained by M. Laczkovich and I. Recław [18] and by G. Debs and J. Saint Raymond [8].

2. Preliminaries. In what follows, (Z, d) is an arbitrary metric space. Recall that a neighbourhood assignment (neighbornet in [14]) of the space X is a collection $\mathcal{V} = (V_x)_{x\in X}$, where each V_x is an open neighbourhood of x in X; $\mathcal{V}$ is said to be *unsymmetric* [14] if $V_x = V_y$ whenever $(x, y) \in V_y \times V_x$.

PROPOSITION 2.1. *Let $f : X \to Z$ be a fragmentable function. For every $\varepsilon > 0$, there is an unsymmetric neighbourhood assignment $(V_x)_{x\in X}$ of X such that $d(f(x), f(y)) \le \varepsilon$ for each $(x, y) \in V_y \times V_x$. In particular, f is weakly separated.*

Proof. Let $\varepsilon > 0$. Then, following [17], there exist a cardinal number κ and a partition $\mathcal{D} = \{U_\alpha : \alpha < \kappa\}$ of X such that for each $\alpha < \kappa$, the set $\bigcup_{\beta\le\alpha} U_\beta$ is open in X and for every $x, y \in U_\alpha$, $d(f(x), f(y)) \le \varepsilon$. For every $x \in X$, let $\alpha_x < \kappa$ be the unique $\alpha < \kappa$ such that $x \in U_\alpha$. Then the collection $(V_x)_{x\in X}$, where $V_x = \bigcup_{\alpha\le\alpha_x} U_\alpha$, is a neighbourhood assignment of X. Let $x, y \in X$ be such that $x \in V_y$ and $y \in V_x$. Then, clearly, $\alpha_x = \alpha_y$, hence $V_x = V_y$ and $d(f(x), f(y)) \le \varepsilon$. ■

Recall that a space X is said to be a *Baire space* if every countable intersection of dense open subsets of X is a dense set. If every closed subspace of X is Baire, then X is said to be *hereditarily Baire*.

PROPOSITION 2.2. *Let X be a space such that every "unsymmetrically" weakly separated function $f : X \to Z$ is cliquish* (*respectively, fragmentable*)*. Then X is Baire* (*respectively, hereditarily Baire*)*.*

Proof. Let $U \subset \bigcup_{n\ge1} F_n$ be a nonempty open set, where each F_n is closed. Put $F_0 = X$ and for $x \in X$, let $\phi(x) = 0$ if $x \notin U$ and $\phi(x) = n_x$ if $x \in U$, where n_x is the first $n \ge 1$ such that $x \in F_n$. The function $f : X \ni x \mapsto n_x \in \mathbb{N}$ is weakly separated, $\mathbb{N}$ being equipped with the discrete metric. Indeed, let $V_x = X$ if $n_x = 0$ and $V_x = X \setminus \bigcup_{n<n_x} F_n$ if $n_x > 0$. If $x \in V_y$ and $y \in V_x$, then $n_x = n_y$ (and $V_x = V_y$). Hence, there is a nonempty open set $O \subset U$ and $n \ge 1$ such that $n_x = n$ for every $x \in O$, that is, $O \subset F_n$. This shows that X is Baire.

To establish the "parenthetical" implication of the proposition, suppose that X has a closed non-Baire subspace F. Let $f : F \to \mathbb{N}$ be a weakly separated function which is not cliquish. Then the function $g : X \to \mathbb{N}$ given by $g(x) = 0$ for $x \notin F$ and $g(x) = f(x)$ for $x \in F$ is weakly separated and not fragmentable. ■

The converse of Proposition 2.2 is false, that is, it is not true that for every Baire (respectively, hereditarily Baire) space X, any weakly separated function defined on X is cliquish (respectively, fragmentable). The first counterexample that comes to mind is the Sorgenfrey line $\mathbb{S}$ [9]. Indeed, since the space $\mathbb{S}$ is hereditarily Baire and weakly separated (in Tkachenko's sense), any function $f : \mathbb{S} \to Z$ that is not cliquish will do the job. Here are two more counterexamples (see Question 4.8):

Examples 2.3. (1) Let ω_1 denote the first uncountable ordinal. Let X be the set of all $\phi \in \{0,1\}^{\omega_1}$ such that ϕ is constant on some cofinal segment $[\alpha, \omega_1[$, $\alpha < \omega_1$. The set X endowed with the product topology is sequentially compact, $\{0,1\}$ being discrete. For every $\phi \in X$, let α_ϕ be the first $\alpha < \omega_1$ such that ϕ is constant on $[\alpha, \omega_1[$. Define a function $f : X \to \{0,1\}$ by $f(\phi) = \phi(\alpha_\phi)$. Let us show that f is weakly separated. For every $\phi \in X$, let V_ϕ be the set of $\psi \in X$ such that $\phi(\alpha_\phi) = \psi(\alpha_\phi)$. Then $(V_\phi)_{\phi \in X}$ is a neighbourhood assignment of X. Let $\phi \in V_\psi$ and $\psi \in V_\phi$; if e.g. $\alpha_\phi \le \alpha_\psi$, then $f(\phi) = f(\psi)$. However, as is easily seen, f is not cliquish.

(2) Let S and T be two disjoint dense subsets of the reals. Following [3], the space $\mathrm{Bush}(S,T)$ is the set of all functions $f : [0, \omega_1[\to S \cup T$ for which there are $\alpha_f < \omega_1$ and $t_f \in T$ such that $f([0, \alpha_f[) \subset S$ and $f([\alpha_f, \omega_1[) = \{t_f\}$. The space $\mathrm{Bush}(S,T)$ is equipped with the topology induced by the lexicographic order. It is established in [3] that $\mathrm{Bush}(S,T)$ is a Baire space and that for every $f \in \mathrm{Bush}(S,T)$, a basis of neighbourhoods for f is given by sets of the form

$$B(f, \varepsilon) = \{g \in \mathrm{Bush}(S,T) : g_{|[0,\alpha_f[} = f_{|[0,\alpha_f[} \text{ and } |g(\alpha_f) - f(\alpha_f)| < \varepsilon\},$$

where $\varepsilon > 0$. It is easy to see that the function $\phi : \mathrm{Bush}(S,T) \ni f \mapsto f(\alpha_f) \in \mathbb{R}$ is weakly separated and not cliquish. This fact will be used in Remark 3.5 to answer a question in [3].

We end this section with a notational convention and a simple statement that indicates how to track the spaces of interest here, i.e., spaces for which the above mentioned converse of 2.2 holds. Let X be a space. Recall that a collection $\mathcal{C}$ of nonempty open subsets of X is said to be a *π-base at $x \in X$* if every neighbourhood of x contains some member of $\mathcal{C}$. If $\mathcal{C}$ is a π-base at each $x \in X$, then $\mathcal{C}$ is called a *π-base of X*. For every neighbourhood assignment $\mathcal{V} = (V_x)_{x \in X}$ of X, put $\Lambda(\mathcal{V}) = \{(x,y) \in X \times X : (x,y) \in V_y \times V_x\}$. We shall write Λ in place of $\Lambda(\mathcal{V})$ if there is no confusion. Let $\Lambda \circ \Lambda$ denotes the set of all $(x,y) \in X \times X$ for which there is $z \in X$ such that $\{(x,z),(z,y)\} \subset \Lambda$. For every $k \ge 1$, Λ^k stands for the k-fold composition $\Lambda \circ \cdots \circ \Lambda$.

Proposition 2.4. *Let X be a space for which there is $k \ge 1$ such that for every neighbourhood assignment $\mathcal{V}$ of X, the collection of all open sets*

$U \subset X$ such that $U \times U \subset \Lambda^k$ forms a π-base of X. Then every weakly separated function $f : X \to Z$ is cliquish.

Proof. Let $f : X \to Z$ be a weakly separated function, $O \subset X$ a nonempty open set and $\varepsilon > 0$. Let $\mathcal{V} = (V_x)_{x \in X}$ be a neighbourhood assignment of X such that $d(f(x), f(y)) < \varepsilon/k$ for every $(x, y) \in \Lambda$. Choose a nonempty open set $U \subset O$ such that $U \times U \subset \Lambda^k$. Then $d(f(x), f(y)) < \varepsilon$ for every $x, y \in U$. ∎

3. Facts on neighbourhood assignments. This section is devoted entirely to the proofs of some properties of neighbourhood assignments suggested by 2.4. In Section 4, we will give some applications of these studies to the class of generalized ordered spaces and to two classes of spaces introduced by J. Chaber in [5], namely monotonic β-spaces and monotonically semistratifiable spaces. To this end, we shall present the results of this section by using three variants of a game inspired by the monotonic properties introduced in [5]. To each space X we assign three two-person infinite games $\mathcal{J}$, $\mathcal{J}^*$ and $\mathcal{J}^{**}$. For the game $\mathcal{J}$, Player II is assigned an open set $U_0 \subset X$ and a point $x_0 \in U_0$, then Player I chooses a nonempty open set $V_0 \subset U_0$. At the nth stage, $n \in \mathbb{N}$, $n \geq 1$, Player II is assigned an open set $U_n \subset V_{n-1}$ and a point $x_n \in U_n$, then I chooses a nonempty open set $V_n \subset U_n$. Player I wins if either $\bigcap_{n \in \mathbb{N}} U_n = \emptyset$ or the sequence $(x_n)_{n \in \mathbb{N}}$ converges to some point $x \in \bigcap_{n \in \mathbb{N}} U_n$. Otherwise, Player II wins. The games $\mathcal{J}^*$ and $\mathcal{J}^{**}$ differ from $\mathcal{J}$ only in the winning rule: For $\mathcal{J}^{**}$ (respectively, $\mathcal{J}^*$), I wins iff either $\bigcap_{n \in \mathbb{N}} U_n = \emptyset$ or $(x_n)_{n \in \mathbb{N}}$ (respectively, every subsequence of $(x_n)_{n \in \mathbb{N}}$) has at least a cluster point in $\bigcap_{n \in \mathbb{N}} U_n$.

If Player I has a winning strategy in the game $\mathcal{J}^*$ and if X is regular, then Player I has a winning strategy in $\mathcal{J}$ provided that X satisfies one of the following conditions:

- (i) X has an open collection $\mathcal{U}$ which is point-countable and T_0-separating, that is, for each $x \in X$, $\{U \in \mathcal{U} : x \in U\}$ is countable and for any distinct $x, y \in X$ there is $U \in \mathcal{U}$ such that $|U \cap \{x, y\}| = 1$.
- (ii) X has a θ-diagonal [24], that is, there is a function $g : X \times \mathbb{N} \to \mathcal{T}(X)$ (the topology of X) such that $x \in g(n, x)$ and $\bigcap_{n \in \mathbb{N}} g^*(x, n) \subset \{x\}$, where $g^*(x, n) = \bigcup\{g(y, n) : x \in g(y, n) \text{ and } y \in g(x, n)\}$.
- (iii) X has an almost-G_δ-diagonal, that is, there is a sequence $(\mathcal{G}_n)_{n \in \mathbb{N}}$ of open collections such that $X = \overline{\bigcup \mathcal{G}_n}$ for each $n \in \mathbb{N}$, and $\bigcap_{n \in \mathbb{N}} \operatorname{st}(x, \mathcal{G}_n) \subset \{x\}$ for each $x \in X$.

In view of the application in Section 4 we will be mostly interested in spaces satisfying (i). We shall give a proof in this case assuming a condition weaker than (i). Similar proofs also work for spaces satisfying (ii) or (iii).

PROPOSITION 3.1. *Let X be a regular space. Suppose that there is a collection $\{W(n,x) : (n,x) \in \mathbb{N} \times X\}$ of open subsets of X such that for every countable set $D \subset X$ and any distinct $x, y \in \overline{D}$, there are $n \in \mathbb{N}$ and $z \in D$ such that $\{x,y\} \cap \overline{W(n,z)} \neq \emptyset$ and $|\{x,y\} \cap W(n,z)| \leq 1$.*

If Player I has a winning strategy in $\mathcal{J}^*$, then Player I has a winning strategy in $\mathcal{J}$.

Proof. Choose a bijection $\mathbb{N} \ni n \mapsto (\psi(n), \phi(n)) \in \mathbb{N} \times \mathbb{N}$ such that $\phi(n) \leq n$ for each $n \in \mathbb{N}$. Let τ be a winning strategy for Player I in $\mathcal{J}^*$. We define a strategy σ for Player I in $\mathcal{J}$ as follows. Let $n \geq 0$ and denote by (x_n, U_n) the nth move of Player II. Put

$$O_n = \tau((x_0, U_0), \ldots, (x_n, U_n)) \cap W(\psi(n), x_{\phi(n)}).$$

If $O_n = \emptyset$, then using the regularity of X, define $\sigma((x_0, U_0), \ldots, (x_n, U_n))$ to be any nonempty open set whose closure is contained in $\tau((x_0, U_0), \ldots, (x_n, U_n))$. If $O_n \neq \emptyset$, then again using the regularity of X, take $\sigma((x_0, U_0), \ldots, (x_n, U_n))$ to be any nonempty open set whose closure is contained in O_n.

To show that σ is a winning strategy, let $(x_n, U_n)_{n\in\mathbb{N}}$ be a game for Player II against σ such that $\bigcap_{n\in\mathbb{N}} U_n \neq \emptyset$ and let us show that $(x_n)_{n\in\mathbb{N}}$ converges to a point of $\bigcap_{n\in\mathbb{N}} U_n$. Since τ is a winning strategy and the play $(x_n, U_n)_{n\in\mathbb{N}}$ is also compatible with τ, every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a cluster point in $\bigcap_{n\in\mathbb{N}} U_n$. Consequently, to show that $(x_n)_{n\in\mathbb{N}}$ converges in $\bigcap_{n\in\mathbb{N}} U_n$, it suffices to prove that $(x_n)_{n\in\mathbb{N}}$ has at most one cluster point.

Let $x, y \in X$ be two cluster points of $(x_n)_{n\in\mathbb{N}}$. Let $n, m \in \mathbb{N}$ be such that $x \in \overline{W(n, x_m)}$ and let us show that $\{x, y\} \subset W(n, x_m)$; this will imply that $x = y$. Choose $k \in \mathbb{N}$ such that $(n,m) = (\psi(k), \phi(k))$. Since $\overline{U}_{l+1} \subset U_l$ and $x_l \in U_l$ for every $l \in \mathbb{N}$, x belongs to $\bigcap_{l\in\mathbb{N}} U_l$; in particular, $O_k \neq \emptyset$. It follows from the definition of $\sigma((x_0, U_0), \ldots, (x_k, U_k))$ that $\overline{U}_{k+1} \subset W(\psi(k), x_{\phi(k)})$, therefore $\{x, y\} \subset W(n, x_m)$ as claimed. ■

We keep the notation of Proposition 2.4 throughout this section.

PROPOSITION 3.2. *Let X be a Baire space for which Player I has a winning strategy in $\mathcal{J}$. Let $\mathcal{V} = (V_x)_{x\in X}$ be a neighbourhood assignment of X. Then, for every nonempty open set $O \subset X$, there exist $x \in O$ and a nonempty open set $U \subset O$ such that $(x, y) \in \Lambda^2$ for every $y \in U$. In particular, $U \times U \subset \Lambda^4$.*

Proof. Suppose that the result is false for some nonempty open set $O \subset X$. We shall define a strategy σ for Player II in the Banach–Mazur game $\mathrm{BM}(X)$ played on X (see [21]). Let τ be a winning strategy for Player I in $\mathcal{J}$. Choose $x_0 \in O$, put $U_0 = O \cap V_{x_0}$ and define $\sigma(\emptyset) = \tau(x_0, U_0)$. At stage $n \geq 0$, if W_n is the nth move of Player I in the game $\mathrm{BM}(X)$, choose $x_{n+1} \in W_n$ such that $(x_n, x_{n+1}) \notin \Lambda^2$, put $U_{n+1} = W_n \cap V_{x_{n+1}}$ and define

$$\sigma(W_0, \dots, W_n) = \tau((x_0, U_0), \dots, (x_{n+1}, U_{n+1})).$$

Since X is Baire, there is a winning game $(W_n)_{n\in\mathbb{N}}$ for Player I against σ, that is, $\bigcap_{n\in\mathbb{N}} W_n \neq \emptyset$ ([21]). The sequence $(x_n, U_n)_{n\in\mathbb{N}}$ is a play for Player II in the game $\mathcal{J}$ against the strategy τ. It follows that $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in \bigcap_{n\in\mathbb{N}} U_n$. In particular, there is $p \in \mathbb{N}$ such that $x_n \in V_x$ for every $n \geq p$. Since $x \in \bigcap_{n\in\mathbb{N}} V_{x_n}$, it follows that $(x_p, x_{p+1}) \in \Lambda^2$, which is a contradiction. ∎

The following lemma will be used in the proof of Proposition 3.4.

Lemma 3.3. *Let X be a Baire space for which Player I has a winning strategy in the game $\mathcal{J}^{**}$, and let $(V_x)_{x\in X}$ be a neighbourhood assignment of X. Then, for every nonempty open set $O \subset X$, there exists a nonempty open set $U \subset O$ such that for each $x \in U$, the collection of all open sets W satisfying $W \subset \overline{\{y \in X : x \in V_y\}}$ is a π-base at x.*

Proof. Suppose that the claim is false for some nonempty open set $U_0 \subset X$ and let τ denote a winning strategy for Player I in the game $\mathcal{J}^{**}$. Consider the following strategy for Player II in the Banach–Mazur game on X. Choose $x_0 \in U_0$ and define $\sigma(\emptyset) = \tau(x_0, U_0)$. At the nth stage, $n \geq 0$, if W_n is the nth move of Player I, choose a point $x_{n+1} \in W_n$ for which there is an open neighbourhood $U_{n+1} \subset W_n$ in X such that for every nonempty open set $W \subset U_{n+1}$, $W \not\subset \overline{\{y \in X : x_{n+1} \in V_y\}}$. Then

$$\tau((x_0, U_0), \dots, (x_{n+1}, U_{n+1})) \setminus \overline{\{y \in X : x_{n+1} \in V_y\}} \neq \emptyset.$$

Define

$$\sigma(W_0, \dots, W_n) = \tau((x_0, U_0), \dots, (x_{n+1}, U_{n+1})) \setminus \overline{\{y \in X : x_{n+1} \in V_y\}}.$$

Let $(W_n)_{n\in\mathbb{N}}$ be a winning game for Player I against σ. Then, since τ is a winning strategy for Player I in $\mathcal{J}^{**}$, the sequence $(x_n)_{n\in\mathbb{N}}$ has a cluster point $x \in \bigcap_{n\in\mathbb{N}} U_n$. Therefore, there exists $n \in \mathbb{N}$ such that $x_{n+1} \in V_x$. But this is impossible since $x \in U_{n+2} \subset W_{n+1}$ and $W_{n+1} \cap \{y \in X : x_{n+1} \in V_y\} = \emptyset$. This contradiction completes the proof. ∎

Recall that a *generalized ordered space* (GO space) is a topological space X with a linear order $<$ such that the order-convex open subsets of X form a base of X and for every $x \in X$ the intervals $]\leftarrow, x[$ and $]x, \rightarrow[$ are open.

Proposition 3.4. *Let $(V_x)_{x\in X}$ be a neighbourhood assignment of a Baire GO space $(X, <)$ for which Player I has a winning strategy in the game $\mathcal{J}^{**}$. Then for every nonempty open set $O \subset X$ there are a nonempty open set $W \subset O$ and $x \in O$ such that $(x, y) \in \Lambda^3$ for every $y \in W$. In particular, $W \times W \subset \Lambda^6$.*

Proof. Without loss of generality we may assume that for each $x \in X$, V_x is order-convex. Let $U \subset O$ be a nonempty open set satisfying the conclusion of Lemma 3.3. We may also assume that U has no isolated points. Let $x \in U$ and choose a nonempty open set $W \subset U \cap V_x$ such that $W \subset \overline{\{y \in X : x \in V_y\}}$. Let $y \in W$; we will show that $(x,y) \in \Lambda^3$. We only examine the case $x < y$, the case $y < x$ is similar.

If $W \cap V_y \cap \,]y,\to[\, \neq \emptyset$, choose $z \in W \cap V_y \cap \,]y,\to[$ such that $x \in V_z$. Since $W \subset V_x$, we have $z \in V_x$. Also, $y \in V_z$, because V_z is convex and $x \le y \le z$. Thus $(x,z) \in \Lambda$ and $(y,z) \in \Lambda$, and therefore $(x,y) \in \Lambda^2$.

Now, suppose that $W \cap V_y \subset \,]\leftarrow, y]$. Since $y \in U$, there is a nonempty open set $W_y \subset \,]x,\to[\, \cap W \cap V_y$ such that $W_y \subset \overline{\{z \in Y : y \in V_z\}}$. Since there is no isolated points in U, there are $t_1, t_2, t_3 \in W_y$ such that $t_1 < t_2 < t_3 < y$. Hence, in particular $]t_2, y[\, \cap W \neq \emptyset$. This allows us to choose $z_1 \in \,]t_2, y] \cap V_x$ such that $x \in V_{z_1}$, i.e., $(x, z_1) \in \Lambda$. Again, since V_{z_1} is convex and $x \le t_1 \le z_1$, it follows that $t_1 \in V_{z_1}$. So, in particular $]\leftarrow, z_1[\, \cap V_{z_1} \cap W_y \neq \emptyset$. This in turn gives us a point $z_2 \in \,]\leftarrow, z_1] \cap V_{z_1} \cap V_y$ such that $y \in V_{z_2}$, i.e., $(y, z_2) \in \Lambda$. Since V_{z_2} is convex and $z_2 \le z_1 \le y$, we have $z_1 \in V_{z_2}$, and so $(z_1, z_2) \in \Lambda$. Consequently, $(x,y) \in \Lambda^3$. ∎

Remark 3.5. A space X is said to be *ω-Čech-complete* [3] if there is a sequence $(\mathcal{G}_n)_{n\in\mathbb{N}}$ of open covers of X such that for every decreasing sequence $(F_n)_{n\in\mathbb{N}}$ of nonempty closed sets such that each F_n is contained in some $G_n \in \mathcal{G}_n$, we have $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$. Every regular ω-Čech-complete space is hereditarily Baire. It is asked in [3, Question 6.5] for which dense subsets S and T of the reals, the space $\mathrm{Bush}(S,T)$ (defined in Example 2.3(2)) is ω-Čech-complete. The answer is that for any such sets S and T, $\mathrm{Bush}(S,T)$ is not ω-Čech-complete. We propose two different arguments showing that. An easy way to show that $\mathrm{Bush}(S,t)$ is not ω-Čech-complete is to check that the closed subspace $\{f \in \mathrm{Bush}(S,T) : \alpha_f < \omega\}$ of $\mathrm{Bush}(S,T)$ is not Baire. One can also proceed as follows: First notice that for any regular ω-Čech-complete space X, Player I has a winning strategy in the game $\mathcal{J}^{**}$. Therefore, since the function ϕ in Example 2.3(2) is not cliquish, it follows from Propositions 2.4 and 3.4 that Player I does not have a winning strategy in this game played on $\mathrm{Bush}(S,T)$. The second argument additionally shows that $\mathrm{Bush}(S,T)$ is not a monotonic β-space (see below), which improves a result of [3] that $\mathrm{Bush}(S,T)$ is not a β-space in Hodel's sense [13].

4. Weak separability versus fragmentability. In this section, the results of Section 3 are summarized in Theorems 4.1 and 4.2. We shall also give some applications, focusing on monotonically semistratifiable spaces [5]. Recall that a regular space X is said to be *monotonically semistratifiable* (respectively, a *monotonic β-space*) if for every $x \in X$, there is a decreasing sequence $(\mathcal{B}_n(x))_{n\in\mathbb{N}}$ of base of X at the point x such that whenever $y \in B_{n+1} \subset B_n \in \mathcal{B}_n(x_n)$, $n \in \mathbb{N}$, the sequence $(x_n)_{n\in\mathbb{N}}$ converges to y in X (respectively, clusters in X). The concept of monotonic semistratifiable spaces is a common generalization of semistratifiable spaces and spaces having a base of countable order (the so-called BCO spaces). Clearly, for every subspace X of a monotonically semistratifiable space, Player I has a winning strategy in the game $\mathcal{J}$ played on X. Similarly, for every closed subspace X of a monotonic β-space, Player I has a winning strategy in the game $\mathcal{J}^*$ played on X. Consequently, Propositions 2.1, 2.4 and 3.2 yield:

THEOREM 4.1. *Let $f : X \to Z$ be a function. If X is a hereditarily Baire monotonically semistratifiable space, then f is weakly separated if and only if f is fragmentable.*

Similarly, Propositions 2.1, 2.4, and 3.1 or 3.4 yield:

THEOREM 4.2. *Let $f : X \to Z$ be a function, where X is a hereditarily Baire monotonic β-space. If X is either a GO space or has a point-countable T_0-separating open collection, then f is weakly separated if and only if f is fragmentable.*

Before we proceed to derive some further results from Theorem 4.1, let us mention the following question raised in [2] (we have learned from T. Banakh that this is in fact an old problem due to V. K. Maslyuchenko): Does every separately continuous function $f : X \times Y \to Z$ defined on the product of metrizable compacta and acting into a linear metric space belong to Baire class one? It is easy to see that such a function has a separable range. Thus, to give a positive answer to this question, it is enough to show that f is F_σ-measurable, and next to apply Fosgerau's theorem [10]. Recall that a function $f : X \to Z$ is *F_σ-measurable* if for every open set $G \subset Z$, $f^{-1}(G)$ is an F_σ-set in X. One way to prove that the function f in Banakh's question is F_σ-measurable is to show that f has the PCP by use of separate-versus-joint continuity techniques. We propose a different method, within the present framework, which also relies on Fosgerau's theorem but is applicable to a larger class of spaces. Let us first make the following observation:

PROPOSITION 4.3. *Let X and Y be spaces and let Z be metric. Then every separately continuous function $f : X \times Y \to Z$ is weakly separated.*

Proof. Let $f : X \times Y \to (Z, d)$ be a separately continuous function and $\varepsilon > 0$. For each $(x, y) \in X \times Y$, let $V(x, y)$ be an open neighbourhood of (x, y) in $X \times Y$ such that

$$d(f(x, b), f(x, y)) \le \varepsilon/2 \quad\text{and}\quad d(f(x, y), f(a, y)) \le \varepsilon/2$$

for every $(a,b) \in V(x,y)$. Then, for any $(a,b) \in V(x,y)$ and $(x,y) \in V(a,b)$, we have

$$d(f(a,b), f(x,y)) \leq d(f(a,b), f(a,y)) + d(f(a,y), f(x,y)) \leq \varepsilon. \ \blacksquare$$

Proposition 4.3 becomes false under the assumption that f_x is continuous for each $x \in X$ and f^y is fragmentable (hence weakly separated) for every $y \in Y$, even if $X = Y = [0,1]$ and Z is the reals $\mathbb{R}$ (for an example, see [7]). Theorem 4.1 and Proposition 4.3 yield

Corollary 4.4. *Let $f : X \times Y \to Z$ be a separately continuous function, where X and Y are monotonically semistratifiable. If $X \times Y$ is Baire* (*respectively, hereditarily Baire*), *then f is cliquish* (*respectively, fragmentable*).

A collection $(O_i)_{i \in I}$ of subsets of the space X is said to be *discrete* [9] if each point of the space X has a neighbourhood that meets at most one of the sets of $(O_i)_{i \in I}$. A *base* for a function $f : X \to Z$ is a collection $\mathcal{M}$ of subsets of X such that for every open set $G \subset Z$, $f^{-1}(G)$ is a union of sets from $\mathcal{M}$. The function f is said to be *σ-discrete* [12] if f has a base $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ such that for each $n \in \mathbb{N}$ the collection $\mathcal{U}_n$ is discrete. It is shown in [12] that if f has the PCP and X is hereditary subparacompact, then f is F_σ-measurable and σ-discrete. Since semistratifiable spaces are (hereditarily) subparacompact [6] (see also [11]), taking into account Hansell's theorem and the above results (4.1 and 4.3), one can state:

Theorem 4.5. *Every weakly separated function from a semistratifiable hereditarily Baire space into a metric space is F_σ-measurable and σ-discrete.*

Corollary 4.6. *Every separately continuous function $f : X \times Y \to Z$, where X and Y are semistratifiable and $X \times Y$ is hereditarily Baire, is F_σ-measurable and σ-discrete.*

Fosgerau's result (Theorem 1 in [10]) (see also [23]) and Corollary 4.6 allow us to answer Banakh–Maslyuchenko's question as follows:

Corollary 4.7. *Every separately continuous function $f : X \times Y \to Z$, where $X \times Y$ is metrizable and hereditarily Baire and Z is an arcwise connected and locally arcwise connected metric space, is of the first Baire class.*

The above corollary remains true for $X \times Y$ paracompact and semistratifiable (in place of metrizable) by a result of L. Veselý [23].

We conclude this section with the following question:

Question 4.8. Let $f : X \to Z$ be a weakly separated function defined on a compact space X into a metric space. Is it true that f has a point of continuity?

By Propositions 2.4, 3.1 and 3.2, the answer is positive if X is *Corson compact*, i.e., if there is a set I such that X is homeomorphic to a compact subspace of $\{x \in \mathbb{R}^I : \{i \in I : x(i) \neq 0\}$ is countable$\}$, $\mathbb{R}^I$ being equipped with the product topology.

5. Filter convergences. Let X be a space and Z be metric. It is proved in [20] that if $f : X \to Z$ is the pointwise limit of a sequence of continuous functions $f_n : X \to Z$, $n \in \mathbb{N}$, then f is weakly separated. A natural question arises: For what kind of filters $\mathcal{F}$ on the integers $\mathbb{N}$ does this result remain true if convergence with respect to the Fréchet filter is replaced by convergence with respect to $\mathcal{F}$? We shall give a complete answer to this question. In fact it turns out that this class of filters is one of the classes introduced and studied by C. Laflamme in [19]. In addition, we show that these filters are the ones that can be separated by an F_σ-set from their dual ideals. The precise definitions will be given later.

The closely related question for functions of the first Baire class (in place of weakly separated functions) was initially considered by Katětov [15]. This question has been studied from the descriptive set theoretical point of view in [8] and [18] for analytic filters. We shall prove that the filters $\mathcal{F}$ for which the $\mathcal{F}$-limit for any $\mathcal{F}$-convergent sequence of continuous functions is F_σ-measurable are the same as above.

As usual, $2^{\mathbb{N}} = \mathcal{P}(\mathbb{N})$ is the Cantor space and $[\mathbb{N}]^{<\omega}$ denotes the set of finite subsets of $\mathbb{N}$. Let $\mathcal{M} \subset 2^{\mathbb{N}}$. Following [4], $\mathcal{M}$ is said to be *monotone* if $A \in \mathcal{M}$ and $A \subset B \subset \mathbb{N}$ imply $B \in \mathcal{M}$. An open set $\mathcal{G} \subset 2^{\mathbb{N}}$ is said to be *positive* if for each $A \in \mathcal{G}$, there is a finite set $F \subset A$ such that $F \subset B \subset \mathbb{N}$ implies $B \in \mathcal{G}$. In what follows, $\mathcal{M}^\uparrow$ stands for the smallest monotone subset of $2^{\mathbb{N}}$ containing $\mathcal{M}$; that is, $A \in \mathcal{M}^\uparrow$ means $B \subset A \subset \mathbb{N}$ for some $B \in \mathcal{M}$.

The main results in this section (Propositions 5.5 and 5.7) are based on Lemmas 5.3 and 5.4. To establish Lemma 5.4, we need the following result due to D. Cenzer [4, Theorem 3].

LEMMA 5.1. *If $\mathcal{U}$ is a monotone G_δ-set, then $\mathcal{U}$ is a countable intersection of positive open subsets of $2^{\mathbb{N}}$.*

DEFINITION 5.2. Following [19], a filter $\mathcal{F}$ on $\mathbb{N}$ is called *ω-diagonalizable by $\mathcal{F}$-universal sets* if there is a sequence $\mathcal{Z}_n \subset [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$, $n \in \mathbb{N}$, such that:

(i) for every $A \in \mathcal{F}$ and $n \in \mathbb{N}$, there is $F \in \mathcal{Z}_n$ such that $F \subset A$;
(ii) for each $A \in \mathcal{F}$, there is $n \in \mathbb{N}$ such that $F \cap A \neq \emptyset$ for all but finitely many $F \in \mathcal{Z}_n$.

LEMMA 5.3. *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then the following are equivalent:*

(a) *$\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets.*

(b) *There is a function* $\phi : \mathcal{F} \to [\mathbb{N}]^{<\omega}$ *such that* $[\phi(A)\cup\phi(B)]\cap A\cap B \neq \emptyset$ *for every* $A, B \in \mathcal{F}$.

(c) *There is a sequence* $\mathcal{Z}_n \subset [\mathbb{N}]^{<\omega} \setminus \{\emptyset\}$, $n \in \mathbb{N}$, *satisfying condition* (i) *of* 5.2 *and such that for each* $A \in \mathcal{F}$, *there is* $n \in \mathbb{N}$ *so that* $F \cap A \neq \emptyset$ *for every* $F \in \mathcal{Z}_n$.

Proof. To show that (a) implies (b), suppose that $\mathcal{F}$ is ω-diagonalizable by $\mathcal{Z}_n = \{F_n^m : m \in \mathbb{N}\}$, $n \in \mathbb{N}$. We assume that $\mathcal{F}$ includes all cofinite subsets of $\mathbb{N}$. Otherwise, there is $F \in [\mathbb{N}]^{<\omega}$ such that the constant function $\mathcal{F} \ni A \mapsto F \in \mathbb{N}^{<\omega}$ can be taken as ϕ. Let $A \in \mathcal{F}$. Using condition (ii) of 5.2, choose two integers n_A, m_A, with $m_A \geq n_A$, such that $F_{n_A}^m \cap A \neq \emptyset$ for every $m \geq m_A$. Using condition (i) of 5.2 and the fact that $\mathcal{F}$ includes all cofinite subsets of $\mathbb{N}$, for every $k \leq m_A$, let $m(k, A) \geq m_A$ be such that $F_k^{m(k,A)} \subset A$. Define

$$\phi(A) = \bigcup_{k\leq m_A} F_k^{m(k,A)}.$$

To show that ϕ works, let $A, B \in \mathcal{F}$. We may suppose that $m_B \leq m_A$, hence $n_B \leq m_A$. Since $F_{n_B}^{m(n_B,A)} \subset A$ and $F_{n_B}^{m(n_B,A)} \cap B \neq \emptyset$, and since $m(n_B, A) \geq m_A \geq m_B$, it follows that $\phi(A) \cap A \cap B \neq \emptyset$. Consequently, $[\phi(A) \cup \phi(B)] \cap A \cap B \neq \emptyset$.

To show that (b) implies (c), suppose that $\phi : \mathcal{F} \to [\mathbb{N}]^{<\omega}$ is a function satisfying $[\phi(A)\cup\phi(B)]\cap A\cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$. Write $\phi(\mathcal{F}) = \{F_n : n \in \mathbb{N}\}$ and, for each $n \in \mathbb{N}$, define

$$\mathcal{Z}_n = \{[\phi(A) \cup F_n] \cap A : A \in \mathcal{F}\}.$$

Clearly, (i) of 5.2 is satisfied by the sequence $\mathcal{Z}_n$, $n \in \mathbb{N}$. Let $A \in \mathcal{F}$, and let $k \in \mathbb{N}$ be such that $\phi(A) = F_k$. We have $[\phi(B) \cup F_k] \cap B \cap A \neq \emptyset$ for every $B \in \mathcal{F}$, that is, $A \cap F \neq \emptyset$ for every $F \in \mathcal{Z}_k$. The proof is complete because (c) obviously implies (a). ∎

For a set $\mathcal{U} \subset \{0,1\}^{\mathbb{N}}$, let $\mathcal{U}^*$ denote the set $\{\mathbb{N} \setminus A : A \in \mathcal{U}\}$. Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then $\mathcal{F}$ is said to be *separated from its dual* $\mathcal{F}^*$ *by a set* $\mathcal{U} \subset 2^{\mathbb{N}}$ if $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{F}^* \cap \mathcal{U} = \emptyset$. This is the same as saying that $\mathcal{F}$ is separated from $\mathcal{F}^*$ by the set $\mathcal{G} = 2^{\mathbb{N}} \setminus \mathcal{U}^*$. Thus, since the mapping $A \mapsto \mathbb{N} \setminus A$ is a homeomorphism of the Cantor space, $\mathcal{F}$ is separated from $\mathcal{F}^*$ by an F_σ-set if and only if $\mathcal{F}$ is separated from $\mathcal{F}^*$ by a G_δ-set.

The following statement is established in [18] for Borel filters. The proof given in [18] uses Borel determinacy.

LEMMA 5.4. *Let* $\mathcal{F}$ *be a filter on* $\mathbb{N}$. *Then* $\mathcal{F}$ *is* ω*-diagonalizable by* $\mathcal{F}$*-universal sets if and only if* $\mathcal{F}$ *is separated from its dual* $\mathcal{F}^*$ *by a* G_δ*-set* (*equivalently, an* F_σ*-set*) $\mathcal{G} \subset 2^{\mathbb{N}}$.

Proof. Suppose that $\mathcal{F}$ is ω-diagonalizable by $\mathcal{Z}_n$, $n \in \mathbb{N}$. We may assume that the sequence $\mathcal{Z}_n$, $n \in \mathbb{N}$, satisfies condition (c) of Lemma 5.3. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the open (positive) set given by all $A \subset \mathbb{N}$ for which there is $F \in \mathcal{Z}_n$ such that $F \subset A$. Then $\mathcal{F}$ is separated from $\mathcal{F}^*$ by the G_δ-set $\bigcap_{n\in\mathbb{N}} \mathcal{U}_n$ of $2^{\mathbb{N}}$.

Conversely, suppose that $\mathcal{F}$ is separated from $\mathcal{F}^*$ by an F_σ-set $\mathcal{U} \subset \{0,1\}^{\mathbb{N}}$. Notice that $\mathcal{U}^{\uparrow}$ is an F_σ-set too and it separates $\mathcal{F}$ from $\mathcal{F}^*$. So, taking $\mathcal{U}^{\uparrow}$ in place of $\mathcal{U}$, we can assume that $\mathcal{U}$ is monotone. Let $\mathcal{G} = 2^{\mathbb{N}} \setminus \mathcal{U}^*$. Then $\mathcal{G}$ is a monotone G_δ-set of $2^{\mathbb{N}}$ that separates $\mathcal{F}$ from $\mathcal{F}^*$. By Lemma 5.1, we can write $\mathcal{G} = \bigcap_{n\in\mathbb{N}} \mathcal{U}_n$, where each $\mathcal{U}_n$ is an open positive set. For each $A \in \mathcal{F}$ and $n \in \mathbb{N}$, let $F_n^A \subset A$ be a finite set so that $B \in \mathcal{U}_n$ whenever $F_n^A \subset B \subset \mathbb{N}$. Clearly, condition (i) of 5.2 is satisfied by the sequence $\mathcal{Z}_n = \{F_n^A : A \in \mathcal{F}\}$, $n \in \mathbb{N}$. To conclude, suppose to the contrary that (ii) of 5.2 is not satisfied by $(\mathcal{Z}_n)_{n\in\mathbb{N}}$. Let $A \in \mathcal{F}$ be such that for every $n \in \mathbb{N}$ there exists $F_n \in \mathcal{Z}_n$ so that $F_n \cap A = \emptyset$. Since the set $B = \bigcup_{n\in\mathbb{N}} F_n$ belongs to $\mathcal{F}^*$, there is $n \in \mathbb{N}$ such that $B \notin \mathcal{U}_n$. It follows that $F_n \not\subset B$, which is a contradiction. ∎

Let $\mathcal{F}$ be a filter on $\mathbb{N}$, X be a set and let (Z,d) be a metric space. Recall that a sequence $f_n : X \to Z$, $n \in \mathbb{N}$, is said to be $\mathcal{F}$*-convergent* if there is a function $f : X \to Z$ such that for every $x \in X$ and $\varepsilon > 0$, $\{n \in \mathbb{N} : d(f_n(x), f(x)) < \varepsilon\} \in \mathcal{F}$. Note that f is uniquely determined. We shall call f the $\mathcal{F}$*-limit* of $(f_n)_{n\in\mathbb{N}}$ and denote it by $\lim_{\mathcal{F}} f_n$.

Proposition 5.5. *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then the following are equivalent:*

(a) *$\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets.*

(b) *For any set X and a metric space (Z,d), every $\mathcal{F}$-convergent sequence $f_n : X \to Z$, $n \in \mathbb{N}$, satisfies the following condition: for every $\varepsilon > 0$ and $(A_x)_{x\in X} \subset \mathcal{F}$ one can assign a finite set $\psi(x) \subset A_x$ to each $x \in X$ so that*
$$d\Big(\lim_{\mathcal{F}} f_n(x), \lim_{\mathcal{F}} f_n(y)\Big) < \varepsilon$$
whenever $(x,y) \in V_y \times V_x$, where
$$V_x = \bigcap_{n\in\psi(x)} \{y \in X : d(f_n(x), f_n(y)) < \varepsilon/3\} \quad (x \in X).$$

(c) *The characteristic function $\mathbb{1}_{\mathcal{F}} : \mathcal{F} \cup \mathcal{F}^* \to \{0,1\}$ is weakly separated, where $\mathcal{F} \cup \mathcal{F}^*$ is viewed as a subspace of the Cantor space $2^{\mathbb{N}}$ and $\{0,1\}$ is discrete.*

Proof. To show that (a) implies (b), let $f_n : X \to Z$, $n \in \mathbb{N}$, be an $\mathcal{F}$-convergent sequence and write $f = \lim_{\mathcal{F}} f_n$. By Lemma 5.3, choose a function $\phi : \mathcal{F} \to [\mathbb{N}]^{<\omega}$ so that $[\phi(A) \cup \phi(B)] \cap A \cap B \neq \emptyset$ for every

$A, B \in \mathcal{F}$. Without loss of generality we assume that $\phi(A) \subset A$ for every $A \in \mathcal{F}$. Let $(A_x)_{x\in X} \subset \mathcal{F}$ and $\varepsilon > 0$. For $x \in X$, put $\psi(x) = \phi(A_x \cap I_x)$, where $I_x = \{n \in \mathbb{N} : d(f_n(x), f(x)) < \varepsilon/3\}$, and as in condition (b), set

$$V_x = \bigcap_{k\in\psi(x)} \{y \in X : d(f_k(x), f_k(y)) < \varepsilon/3\}.$$

Let $y \in V_x$ and $x \in V_y$. Choose $k \in \psi(x) \cup \psi(y)$ such that $k \in A_x \cap I_x \cap A_y \cap I_y$. Assume that $k \in \psi(x)$. Then $d(f(x), f_k(x)) < \varepsilon/3$. Since $y \in V_x$ and $k \in I_y$, we have $d(f_k(x), f_k(y)) < \varepsilon/3$ and $d(f_k(y), f(y)) < \varepsilon/3$. Therefore $d(f(x), f(y)) < \varepsilon$.

To show that (b) implies (c), consider the $\mathcal{F}$-convergent sequence $f_n : \mathcal{F} \cup \mathcal{F}^* \to \{0,1\}$ defined by $f_n(A) = \mathbb{1}_A(n)$, $n \in \mathbb{N}$, and notice that $\lim_{\mathcal{F}} f_n = \mathbb{1}_{\mathcal{F}}$. Since the f_n's are continuous, the collection $(V_x)_{x\in X}$ in (b) is a neighbourhood assignment of X. Hence, $\mathbb{1}_{\mathcal{F}}$ is weakly separated.

Before we turn to (c)⇒(a), we present a simple argument showing that (b) implies (a). Let $X = \mathcal{F} \cup \mathcal{F}^*$, considered as a subspace of $2^{\mathbb{N}}$. As above, for each $n \in \mathbb{N}$ define $f_n : X \to \{0,1\}$ by $f_n(x) = 1$ if and only if $n \in x$. For $x \in X$, put $A_x = x$ if $x \in \mathcal{F}$ and $A_x = \mathbb{N} \setminus x$ if $x \in \mathcal{F}^*$. Since $\lim_{\mathcal{F}} f_n = \mathbb{1}_{\mathcal{F}}$, it follows from (b) that for each $x \in X$ there is a finite set $\psi(x) \subset A_x$ so that $\mathbb{1}_{\mathcal{F}}(x) = \mathbb{1}_{\mathcal{F}}(y)$ whenever $x \in V_y$ and $y \in V_x$, where for each $z \in X$,

$$V_z = \bigcap_{n\in\psi(z)} \{t \in X : f_n(t) = f_n(z)\}.$$

For each $x \in X$, let $\phi(x) = \psi(x) \cup \psi(\mathbb{N} \setminus x)$. Now, by Lemma 5.3, it suffices to verify that $(\phi(x) \cup \phi(y)) \cap x \cap y \neq \emptyset$. Let $x, y \in \mathcal{F}$. Since $x \notin V_{\mathbb{N}\setminus y}$ or $\mathbb{N} \setminus y \notin V_x$, there is $k \in \phi(x) \cup \phi(y)$ such that $f_k(x) \neq f_k(\mathbb{N} \setminus y)$. Since $\phi(x) \cup \phi(y) \subset x \cup y$, it follows that $k \in x \cap y$.

Let us prove that (c) implies (a). Write $X = \mathcal{F} \cup \mathcal{F}^*$. Since $\mathbb{1}_{\mathcal{F}} : X \to \{0,1\}$ is weakly separated, for each $x \in X$ there is $n_x \in \mathbb{N}$ such that

$$(*) \qquad \forall x, y \in \mathcal{F},\ \exists k \leq n_x \vee n_{\mathbb{N}\setminus y} \text{ such that } \mathbb{1}_x(k) \neq \mathbb{1}_{\mathbb{N}\setminus y}(k).$$

To simplify the proof we may assume that $n_x = n_{\mathbb{N}\setminus x}$ for every $x \in \mathcal{F}$. Write $\{n_x : x \in \mathcal{F}\} = \{n_l : l \in \mathbb{N}\}$ and put

$$\mathcal{I}_{(l,j)} = \big\{x \cap \{0, \ldots, n_l \vee n_j\} : x \in \mathcal{F} \text{ and } n_x = n_j\big\}.$$

Let

$$\mathcal{U} = \bigcup_{l\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} \bigcap_{A\in\mathcal{I}_{(l,j)}} \big\{x \in \{0,1\}^{\mathbb{N}} : \exists k \leq n_l \vee n_j \text{ such that } 1_x(k) = 1_A(k)\big\}.$$

Clearly, $\mathcal{U}$ is an F_σ-set of $2^{\mathbb{N}}$. Let us verify that $\mathcal{F} \subset \mathcal{U}$ and $\mathcal{U} \cap \mathcal{F}^* = \emptyset$. Let $x \in \mathcal{F}$. Choose l such that $n_l = n_x$ and let $A \in \mathcal{I}_{(l,j)}$ for some $j \in \mathbb{N}$. Then there is $y \in \mathcal{F}$ such that $n_y = n_j$ and $A = y \cap [0, n_l \vee n_j]$. By $(*)$, there is $k \leq n_l \vee n_j$ such that $\mathbb{1}_x(k) \neq \mathbb{1}_{\mathbb{N}\setminus y}(k)$, equivalently, $\mathbb{1}_x(k) = \mathbb{1}_A(k)$.

This shows that $x \in \mathcal{U}$. Now, let $y \in \mathcal{F}^*$ and $l \in \mathbb{N}$. Choose $x \in \mathcal{F}$ such that $n_x = n_l$ and consider the integer j for which $n_{\mathbb{N}\setminus y} = n_j$. Then the set $A = (\mathbb{N}\setminus y) \cap \{0, \dots, n_l \vee n_j\}$ belongs to $\mathcal{I}_{(l,j)}$ and for every $k \le n_l \vee n_j$, $\mathbb{1}_y(k) \ne \mathbb{1}_A(k)$. Thus $y \notin \mathcal{U}$. It follows from Lemma 5.4 that $\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets. ∎

The abstract form of 5.5(b) allows us to apply Proposition 5.5 to sequences of functions which are not necessarily continuous.

Remark 5.6. For a function $f : X \to Z$, where X is a space and Z is a metric space, let $\operatorname{osc}(f, x)$ stand for the oscillation of f at $x \in X$. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ which is ω-diagonalizable by $\mathcal{F}$-universal sets. Let $f_n : X \to Z$, $n \in \mathbb{N}$, be an $\mathcal{F}$-convergent sequence of functions such that for each $x \in X$ and $\varepsilon > 0$, the set $\{n \in \mathbb{N} : \operatorname{osc}(f_n, x) < \varepsilon\}$ belongs to $\mathcal{F}$. Then, by Proposition 5.5, $\lim_{\mathcal{F}} f_n$ is weakly separated. Hence, if for example X is hereditarily Baire and monotonically semistratifiable, then $\lim_{\mathcal{F}} f_n$ has the PCP (see Theorem 4.1).

Following [23], a collection $(U_i)_{i\in I}$ of subsets of the space X is said to be *strongly discrete* if there is a discrete collection $(O_i)_{i\in I}$ of open subsets of X such that $\overline{U}_i \subset O_i$ for every $i \in I$. A function $f : X \to Z$ is said to be *strongly σ-discrete* [23] if f has a base $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ such that for each $n \in \mathbb{N}$ the collection $\mathcal{U}_n$ is strongly discrete.

Proposition 5.7. *For any filter $\mathcal{F}$ on $\mathbb{N}$, the following are equivalent:*

1. *$\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets.*
2. *For every $\mathcal{F}$-convergent sequence $f_n : X \to Z$, $n \in \mathbb{N}$, of continuous functions, where X is a space and Z is a metric space, the function $\lim_{\mathcal{F}} f_n$ is F_σ-measurable and strongly σ-discrete.*

Proof. To show that (1) implies (2), let $\mathcal{Z}_k = \{F_k^n : n \in \mathbb{N}\}$, $k \in \mathbb{N}$, be a sequence satisfying (c) of Lemma 5.3. Write $f = \lim_{\mathcal{F}} f_n$. For every open set $G \subset Z$, let

$$\psi(G) = \bigcup_{k\in\mathbb{N}} \bigcap_{i\in\mathbb{N}} \bigcup_{n\in F_k^i} f_n^{-1}(\overline{G}).$$

Clearly, $\psi(G)$ is an F_σ-set of X.

Claim. $f^{-1}(G) \subset \psi(G) \subset f^{-1}(\overline{G})$.

To prove the claim, let $x \in f^{-1}(G)$. There is $k_x \in \mathbb{N}$ such that $F_{k_x}^i \cap \{n \in \mathbb{N} : f_n(x) \in G\} \ne \emptyset$ for every $i \in \mathbb{N}$. Thus $f^{-1}(G) \subset \psi(G)$. To show the second inclusion, suppose that there is $x \in \psi(G)$ such that $f(x) \in Y \setminus \overline{G}$. Let $k_x \in \mathbb{N}$ witness that $x \in \psi(G)$. Using (i) of 5.2, there is $i \in \mathbb{N}$ such that

$F^i_{k_x} \subset \{n \in \mathbb{N} : f_n(x) \in Y \setminus \overline{G}\}$, which is impossible because $f_n(x) \in \overline{G}$ for some $n \in F^i_{k_x}$.

Having proved the Claim, let us assign a sequence $(\tilde{G}_n)_{n\in\mathbb{N}}$ of open sets to each open set $G \subset Z$, so that $G = \bigcup_{n\in\mathbb{N}} \tilde{G}_n$ and $\overline{\tilde{G}_n} \subset G$ for every $n \in \mathbb{N}$.

a) Let $G \subset Z$ be an open set. By the Claim, we have $f^{-1}(\tilde{G}_n) \subset \psi(\tilde{G}_n) \subset f^{-1}(\overline{\tilde{G}_n})$ for each $n \in \mathbb{N}$. Let $F = \bigcup_{n\in\mathbb{N}} \psi(\tilde{G}_n)$. Then F is an F_σ-set and

$$(*) \qquad f^{-1}(G) \subset \bigcup_{n\in\mathbb{N}} f^{-1}(\tilde{G}_n) \subset F \subset \bigcup_{n\in\mathbb{N}} f^{-1}(\overline{\tilde{G}_n}) \subset f^{-1}(G).$$

This proves that f is F_σ-measurable.

b) To show that f is strongly σ-discrete, we shall proceed as in the proof of [23, Proposition 1.10]. Choose a base $\bigcup_{n\in\mathbb{N}} \mathcal{M}_n$ of the metric space Z such that for each $n \in \mathbb{N}$, $\mathcal{M}_n$ is a discrete open collection [9]. Let $j, k, l \in \mathbb{N}$ and $m \in F^0_k$. Define

$$\mathcal{B}(j,k,l,m) = \Big\{ f_m^{-1}(\overline{\tilde{G}_l}) \cap \Big(\bigcap_{i\geq 1} \bigcup_{n \in F^i_k} f_n^{-1}(\overline{\tilde{G}_l}) \Big) : G \in \mathcal{M}_j \Big\}.$$

Since the function f_m is continuous and $\mathcal{M}_j$ is discrete, the collection $\mathcal{B}(j,k,l,m)$ is discrete. On the other hand, according to $(*)$, the collection $\mathcal{B} = \bigcup\{\mathcal{B}(j,k,l,m) : j,k,l \in \mathbb{N}, m \in F^0_k\}$ is a closed base for f. Hence f is strongly σ-discrete.

We turn now to the proof of (2)⇒(1). Notice first that the function $\mathbb{1}_{\mathcal{F}} : \mathcal{F} \cup \mathcal{F}^* \to \{0,1\}$ is F_σ-measurable (if and) only if $\mathcal{F}$ is separated from its dual $\mathcal{F}^*$ by an F_σ-set of $2^{\mathbb{N}}$. Since $\mathbb{1}_{\mathcal{F}}$ is the $\mathcal{F}$-limit of a sequence of continuous functions, Lemma 5.4 applies. ∎

For a space X and a filter $\mathcal{F}$ on $\mathbb{N}$, let $\mathcal{B}_{\mathcal{F}}(X)$ be the set of $\mathcal{F}$-limits of $\mathcal{F}$-convergent sequences of continuous real-valued functions defined on X. If $\mathcal{F}$ is the Fréchet filter, then $\mathcal{B}_{\mathcal{F}}(X) = \mathcal{B}_1(X)$ is the set of first Baire class functions defined on X. The following is established in [18] for Borel filters:

Corollary 5.8. *Let $\mathcal{F}$ be a filter on $\mathbb{N}$. Then the following are equivalent:*

(i) *$\mathcal{F}$ is ω-diagonalizable by $\mathcal{F}$-universal sets.*
(ii) *$\mathcal{B}_{\mathcal{F}}(X) \subset \mathcal{B}_1(X)$ for every metrizable space X.*

In view of [23, Theorem 3.7] and 5.7, the implication (i)⇒(ii) in Corollary 5.8 remains true for any normal space X, furthermore the reals can be replaced by any arcwise connected and locally arcwise connected metric space.

The following brief discussion will lead us to an open question. Let $\mathcal{F}$ be a filter on $\mathbb{N}$ such that the function $\mathbb{1}_{\mathcal{F}} : X \to \{0,1\}$ is weakly separated, where $X = \mathcal{F} \cup \mathcal{F}^*$. Then there is no Baire subspace C of X such that both

$C \cap \mathcal{F}$ and $C \cap \mathcal{F}^*$ are dense in C. Indeed, otherwise the restriction of $\mathbb{1}_{\mathcal{F}}$ to C would be cliquish (see Propositions 2.4 and 3.2), but this is impossible because $\mathcal{F}$ and $\mathcal{F}^*$ are dense in C. Consequently, Proposition 5.5 gives:

PROPOSITION 5.9. *Let $\mathcal{F}$ be a filter which is ω-diagonalizable by $\mathcal{F}$-universal sets. Then there is no Baire subspace C of $\mathcal{F} \cup \mathcal{F}^*$ such that $\mathcal{F}$ and $\mathcal{F}^*$ are dense in C.*

QUESTION 5.10. Is the converse of Proposition 5.9 true for any filter $\mathcal{F}$?

The answer is yes if $\mathcal{F}$ is analytic. Indeed, if $\mathcal{F}$ is not ω-diagonalizable by $\mathcal{F}$-universal sets, then, by Lemma 5.4, $\mathcal{F}$ is not F_σ-separated from $\mathcal{F}^*$. Now, by [16] there is a Cantor set, hence a Baire subspace C of $\mathcal{F} \cup \mathcal{F}^*$ such that $\mathcal{F}$ and $\mathcal{F}^*$ are dense in C.

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