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On the Hausdorff dimension of ultrametric subsets in $\mathbb{R}^n$

by

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Abstract. For every $\varepsilon > 0$, any subset of $\mathbb{R}^n$ with Hausdorff dimension larger than $(1-\varepsilon)n$ must have ultrametric distortion larger than $1/(4\varepsilon)$.

We prove the following theorem.

THEOREM 1. *For every $D > 1$, every $n \in \mathbb{N}$, and every norm $\|\cdot\|$ on $\mathbb{R}^n$, any subset $S \subset \mathbb{R}^n$ having ultrametric distortion at most D must have Hausdorff dimension at most*

$$\left(1 - \frac{1}{2(D+1)}\right)n.$$

An *ultrametric space* (X,ρ) is a metric space satisfying

$$\rho(x,y) \le \max\{\rho(x,z), \rho(y,z)\}$$

for all $x,y,z \in X$. The *ultrametric distortion of a metric space* (X,d), written $c_{\mathrm{UM}}(X,d)$, is the infimum over D such that there exists an ultrametric ρ on X satisfying

$$d(x,y) \le \rho(x,y) \le D \cdot d(x,y)$$

for all $x,y \in X$. The *Euclidean distortion* $c_2(X,d)$ of (X,d) is defined similarly with respect to Hilbertian metrics over X. The *diameter* of a metric space (X,d) is given by

$$\operatorname{diam}(X) = \sup_{x,y\in X} d(x,y).$$

The *α-Hausdorff content* of a metric space (X,d) is defined as

$$\mathcal{C}^\alpha(X) = \inf\Big\{\sum_{i\in\mathbb{N}} \operatorname{diam}(A_i)^\alpha : \bigcup_{i\in\mathbb{N}} A_i \supseteq X\Big\},$$

and the *Hausdorff dimension* of X is $\dim_{\mathrm{H}}(X) = \inf\{\alpha > 0 : \ \mathcal{C}^\alpha(X) = 0\}$.

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Theorem 1 proves that the Euclidean spaces $\mathbb{R}^n$ form (asymptotically) tight examples to the following Dvoretzky-type theorem for Hausdorff dimension from [4].

THEOREM 2 ([4]). *For every* $\varepsilon \in (0,1)$, *every locally compact metric space* (X,d) *contains a subset* $S \subseteq X$ *having ultrametric distortion at most* $9/\varepsilon$, *while having Hausdorff dimension at least* $(1-\varepsilon)\dim_{\mathrm{H}}(X)$.

Since separable ultrametrics embed isometrically in Hilbert space [5], Theorem 2 is also true if one replaces "ultrametric distortion" with "Euclidean distortion." Of course, for this (weaker) Euclidean version of Theorem 2, Euclidean spaces cannot serve as tight examples. Tight examples for the Euclidean version of Theorem 2 are constructed in [4]; those spaces are stronger than $\mathbb{R}^n$ in the current context, but being "fractals" based on expander graphs, they are also more exotic.

Previously, Luosto [3] proved a qualitative result along the lines of Theorem 1: Any subset $S \subseteq \mathbb{R}^n$ of the n-dimensional Euclidean space which has finite ultrametric distortion must have $\dim_{\mathrm{A}}(S) < n$, where $\dim_{\mathrm{A}}(S)$ is the Assouad dimension of S (note that $\dim_{\mathrm{A}}(X) \geq \dim_{\mathrm{H}}(X)$ for every metric space X). Luosto's proof gives only a weak quantitative bound on the Assouad dimension, namely, $\dim_{\mathrm{A}}(S) \leq (1 - c/(2Dn)^n)n$ for some universal constant $c > 0$. The proof of Theorem 1 presented here is sufficiently flexible to derive a stronger version of Theorem 1, with Assouad dimension replacing Hausdorff dimension; see Remark 6. This variant of Theorem 1 is an asymptotically tight quantitative version of Luosto's theorem.

It is not clear whether the constant $1 - 1/(2(D+1))$ in Theorem 1 is close to optimal when D is large. However, it is clear that Theorem 1 does not give meaningful estimates when $D > 1$ is small. Luosto [3] observed that the Hausdorff dimension of subsets $S \subset \mathbb{R}^n$ must approach 0 as their ultrametric distortion approaches 1, i.e., for every $\delta > 0$ there exists $\varepsilon > 0$ such that if $c_{\mathrm{UM}}(S) < 1+\varepsilon$, then $\dim_{\mathrm{H}}(S) < \delta$. On the other hand, we have the following proposition.

PROPOSITION 3. *For every* $\varepsilon \in [0,1/4]$ *and* $n \in \mathbb{N}$, *there exists* $S \subset \mathbb{R}^n$ *for which* $c_{\mathrm{UM}}(S) \leq 1+3\varepsilon$ *and* $\dim_{\mathrm{H}}(S) \geq \frac{c\varepsilon^2}{\log(1/\varepsilon)} n$ *for some universal* $c > 0$.

Sketch of proof. The argument is similar to [1, Lemma 8]. Take a binary code in $C \subset \{0,1\}^n$ of size $2^{c\varepsilon^2 n}$ in which all pairwise Hamming distances are in the range $[(1-\varepsilon)n/2, (1+\varepsilon)n/2]$. The set $S \subset [0,1]^n$ is defined as

$$S = \Big\{ \sum_{i=0}^{\infty} (1-\varepsilon)\varepsilon^i x_i : x_i \in C \Big\}. \ \blacksquare$$

This property of $\mathbb{R}^n$ is qualitatively different from general metric spaces, where there is an example of a compact metric space X for which $\dim_{\mathrm{H}}(X) = \infty$, but for every subset $S \subset X$, if $c_{\mathrm{UM}}(S) < 2$, then $\dim_{\mathrm{H}}(S) = 0$; see [4, 2].

Proof of Theorem 1. Fix $D > 1$, $n \in \mathbb{N}$ and a norm $\|\cdot\|$ on $\mathbb{R}^n$. Denote by $B^o(r) = \{x \in \mathbb{R}^n : \|x\| < r\}$ the open ball of radius r around the origin. For subsets $A, B \subset \mathbb{R}^n$ we denote the *Minkowski sum* of A and B by $A + B = \{a + b : a \in A,\ b \in B\}$, and for measurable sets A, we use $|A|$ for the n-dimensional Lebesgue measure of A.

Claim 4. *Let (X, d) a metric space that embeds in an ultrametric space with distortion at most D, and let $x_0, \ldots, x_m \in X$. Then* $\max_i d(x_i, x_{i-1}) \geq d(x_0, x_m)/D$.

Proof. Let ρ be an ultrametric on X such that $d \leq \rho \leq D \cdot d$. We claim that $\max_i \rho(x_i, x_{i-1}) \geq \rho(x_0, x_m)$. Indeed, by induction

$$\begin{aligned}\rho(x_0, x_m) &\leq \max\{\rho(x_0, x_1), \rho(x_1, x_m)\}\\ &\leq \max\{\rho(x_0, x_1), \rho(x_1, x_2), \rho(x_2, x_m)\} \leq \cdots\\ &\leq \max\{\rho(x_0, x_1), \rho(x_1, x_2), \ldots, \rho(x_{m-1}, x_m)\}.\end{aligned}$$

Hence,

$$d(x_0, x_m) \leq \rho(x_0, x_m) \leq \max_i \rho(x_{i-1}, x_i) \leq D \cdot \max_i d(x_{i-1}, x_i). \ \blacksquare$$

Claim 5. *Let $S \subset \mathbb{R}^n$ be a subset that embeds in an ultrametric space with distortion D. If C is a path-connected subset of $S + B^o(r)$, then* $\mathrm{diam}(C) \leq 2(D + 1)r$.

Proof. Suppose for the sake of contradiction that $\mathrm{diam}(C) > 2(D+1)r$. Fix $\eta > 0$, and let $a_0, a_1 \in C$ be such that $\|a_0 - a_1\| > 2(D + 1)r$. Since C is path-connected, there exists a continuous path $a : [0, 1] \to C$ such that $a(0) = a_0$ and $a(1) = a_1$. Define $b : [0, 1] \to S$, where $b(t) \in S$ is a point in S such that $\|b(t) - a(t)\| \leq r$. From the continuity of a there exists a sequence of points $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $\|a(t_i) - a(t_{i-1})\| \leq \eta$ for every $i \in \{1, \ldots, m\}$. Hence $\|b(t_0) - b(t_m)\| > 2(D + 1)r - 2r$, and for every $i \in \{1, \ldots, m\}$, $\|b(t_i) - b(t_{i-1})\| \leq 2r + 2\eta$. But from Claim 4,

$$2r + 2\eta \geq \max_i \|b(t_i) - b(t_{i-1})\| \geq \frac{\|b(t_0) - b(t_m)\|}{D}.$$

Since the above is true for any $\eta > 0$, we conclude that

$$2r \geq \frac{\|b(t_0) - b(t_m)\|}{D} > 2r,$$

contradicting our initial assumption. $\blacksquare$

Proof of Theorem 1. Suppose that $c_{\mathrm{UM}}(S) \le D$. We may assume without loss of generality that $S \subseteq B^o(1)$ (since one can find a countable subset $\mathcal{N} \subset \mathbb{R}^n$ such that $\bigcup_{x\in\mathcal{N}}((x + B^o(1)) \cap S) = S$, and for any countable collect of subsets $\{A_x\}_{x\in\mathcal{N}}$ we have $\dim_{\mathrm{H}}(\bigcup_{x\in\mathcal{N}} A_x) = \sup_{x\in\mathcal{N}} \dim_{\mathrm{H}}(A_x)$). Fix $\delta > 0$, fix $r > 0$, and fix a path-component $C \subset S + B^o(e^\delta r)$ of $S + B^o(e^\delta r)$. Note that C is an open subset. By Claim 5, $\mathrm{diam}(C) \le 2(D+1)e^\delta r$, and hence $|C| \le |B^o(2(D+1)e^\delta r)|$, which means that

$$|B^o((e^\delta - 1)r)| \ge \left(\frac{e^\delta - 1}{2(D+1)e^\delta}\right)^n |C|.$$

Let $A = (S \cap C) + B^o(r)$. Observe that $C = A + B^o((e^\delta - 1)r)$, and that A, $B^o((e^\delta - 1)r)$, and C are bounded and open. By the Brunn–Minkowski inequality,

$$|A|^{1/n} \le |C|^{1/n} - |B^o((e^\delta - 1)r)|^{1/n} \le \left(1 - \frac{e^\delta - 1}{2(D+1)e^\delta}\right)|C|^{1/n}. \tag{1}$$

Since the path-components of $S+B^o(e^\delta r)$ are open, and constitute a pairwise disjoint cover of $S + B^o(e^\delta r)$, by summing the nth power of (1) over the path-components of $S + B^o(e^\delta r)$, we obtain

$$|S + B^o(r)| \le \left(1 - \frac{e^\delta - 1}{2(D+1)e^\delta}\right)^n |S + B^o(e^\delta r)|. \tag{2}$$

Fix

$$\alpha > \left(1 + \delta^{-1} \log\left(1 - \frac{e^\delta - 1}{2(D+1)e^\delta}\right)\right)n.$$

We will prove that $\mathcal{C}^\alpha(S) = 0$ by constructing a sequence of covers of S. The jth cover of S is the set of path-components of $S + B^o(e^{-\delta j})$. Let $\beta = |B^o(1)| > 0$. Note that $S + B^o(1) \subset B^o(2)$, and hence $|S + B^o(e^{\delta 0})| \le 2^n\beta$. By inductively applying (2), we obtain

$$|S + B^o(e^{-\delta j})| \le \left(1 - \frac{e^\delta - 1}{2(D+1)e^\delta}\right)^{jn} 2^n\beta.$$

On the other hand, each path-component of $S + B^o(e^{-\delta j})$ has a volume at least $|B^o(e^{-\delta j})| = e^{-\delta jn}\beta$. Therefore, $S + B^o(e^{-\delta j})$ has at most

$$2^n \left(e^\delta \left(1 - \frac{e^\delta - 1}{2(D+1)e^\delta}\right)\right)^{jn}$$

path-components, and by Claim 5 each of the components has diameter at most $2(D+1)e^{-\delta j}$. Hence,

$$\begin{aligned}\mathcal{C}^\alpha(S) &\le 2^n\left(e^\delta\left(1-\frac{e^\delta-1}{2(D+1)e^\delta}\right)\right)^{jn}\cdot(2(D+1)e^{-\delta j})^\alpha\\ &\le 2^n(4D)^\alpha\cdot\left(\frac{e^\delta\left(1-\frac{e^\delta-1}{2(D+1)e^\delta}\right)}{e^{\delta\alpha/n}}\right)^{jn}\xrightarrow[j\to\infty]{}0,\end{aligned}$$

and therefore $\dim_{\mathrm{H}}(S)\le\alpha$. Since the preceding bound is true for every $\delta>0$ and every

$$\alpha>\left(1+\delta^{-1}\log\left(1-\frac{e^\delta-1}{2(D+1)e^\delta}\right)\right)n,$$

we conclude that

$$\begin{aligned}\dim_{\mathrm{H}}(S) &\le \lim_{\delta\to 0^+}\left(1+\delta^{-1}\log\left(1-\frac{e^\delta-1}{2(D+1)e^\delta}\right)\right)n\\ &=\left(1-\frac{1}{2(D+1)}\right)n. \ \blacksquare\end{aligned}$$

Remark 6. One may obtain the same conclusion with Assouad dimension replacing Hausdorff dimension. This follows from the fact that we have a uniform bound on the diameter of the elements in our cover at every step; hence the same sequence of covers shows that

$$\dim_{\mathrm{A}}(S)\le\left(1-\frac{1}{2(D+1)}\right)n$$

as well. We leave verification as an exercise for the interested reader.

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