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A conjecture on the unstable Adams spectral sequences for SO and U

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Abstract. We give a systematic account of a conjecture suggested by Mark Mahowald on the unstable Adams spectral sequences for the groups SO and U. The conjecture is related to a conjecture of Bousfield on a splitting of the E_2-term and to an algebraic spectral sequence constructed by Bousfield and Davis. We construct and realize topologically a chain complex which is conjectured to contain in its differential the structure of the unstable Adams spectral sequence for SO. A filtration of this chain complex gives rise to a spectral sequence that is conjectured to be the unstable Adams spectral sequence for SO. If the conjecture is correct, then it means that the entire unstable Adams spectral sequence for SO is available from a primary level calculation. We predict the unstable Adams filtration of the homotopy elements of SO based on the conjecture, and we give an example of how the chain complex predicts the differentials of the unstable Adams spectral sequence. Our results are also applicable to the analogous situation for the group U.

1. Introduction. In this paper, we consider the unstable Adams spectral sequence (UASS) of the group SO at the prime 2. In particular, we give a systematic account of a conjecture suggested by Mark Mahowald concerning the calculation of the differentials in this spectral sequence. We give a geometric realization of the conjecture in the form of a tower with the 2-completion of SO as inverse limit. Our tower comes equipped with a map from the destabilization of the stable Adams tower for the infinite delooping of SO. We use this map and theorems of Bousfield on h_0-towers in unstable Ext to predict the Adams filtrations of the unstable homotopy of SO. Our results are equally valid for the group U, and thus differentials and unstable filtrations can be predicted for this group as well. Of course, the homotopy of SO and U is well known by Bott periodicity, and what is of interest is the workings of the UASS, not the end result.

Before we describe our results and conjectures, we establish some notation. We work entirely at the prime 2, all cohomology will be taken with mod 2 coefficients, and all spaces will be taken to be completed at 2 as

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appropriate. Let A be the mod 2 Steenrod algebra, let $\underline{\mathbf{U}}$ be the category of unstable A-modules, and let $\underline{\mathbf{K}}$ be the category of unstable A-algebras. There is a functor $U : \underline{\mathbf{U}} \to \underline{\mathbf{K}}$, described by Massey and Peterson [M-P], which takes the free unstable A-algebra on an unstable A-module. This functor is left adjoint to the forgetful functor from unstable A-algebras to unstable A-modules.

In general, the unstable Adams spectral sequence for a space X has the form

$$E_2^{s,t} = \operatorname{Ext}^s_{\underline{\mathbf{K}}}(H^*X, H^*S^t) \Rightarrow \pi_{t-s}X,$$

where Ext is a derived functor in the nonabelian category $\underline{\mathbf{K}}$. However, for a space X with the property that $H^*X \cong U(N)$ for some $N \in \underline{\mathbf{U}}$, the unstable Adams spectral sequence has the form

$$E_2^{s,t} = \operatorname{Ext}^s_{\underline{\mathbf{U}}}(N, \Sigma^t \mathbb{F}_2) \Rightarrow \pi_{t-s}X.$$

We will follow the stable notation and write $\operatorname{Ext}^{s,t}_{\underline{\mathbf{U}}}(N, \mathbb{F}_2)$ for $\operatorname{Ext}^s_{\underline{\mathbf{U}}}(N, \Sigma^t \mathbb{F}_2)$.

We will be discussing the unstable Adams spectral sequence for the special orthogonal group SO and indicating modifications to be made to the discussion for the unitary group U. Let $M_\infty = \overline{H}^* RP^\infty$, with nonzero elements x_i in dimension i and A-action $\operatorname{Sq}^j x_i = \binom{i}{j} x_{i+j}$; then $H^*SO \cong U(M_\infty)$. Hence the unstable Adams spectral sequence for SO takes the form

$$\operatorname{Ext}^{s,t}_{\underline{\mathbf{U}}}(M_\infty, \mathbb{F}_2) \Rightarrow \pi_{t-s}SO.$$

Let $\alpha(i)$ be the number of ones in the dyadic expansion of i, and filter M_∞ by $M_n = \{x_i \mid \alpha(i) \le n\}$. This filtration leads to a spectral sequence converging to the E_2-term of the UASS:

$$\operatorname{Ext}^{*,*}_{\underline{\mathbf{U}}}(M_n/M_{n-1}, \mathbb{F}_2) \Rightarrow \operatorname{Ext}^{*,*}_{\underline{\mathbf{U}}}(M_\infty, \mathbb{F}_2).$$

It is a conjecture of Bousfield from the 1970s that this spectral sequence collapses, giving

$$\operatorname{Ext}^{*,*}_{\underline{\mathbf{U}}}(M_\infty, \mathbb{F}_2) \cong \bigoplus_n \operatorname{Ext}^{*,*}_{\underline{\mathbf{U}}}(M_n/M_{n-1}, \mathbb{F}_2).$$

A similar conjecture for the E_2-term of the UASS for the group U arises from the fact that if we take $M_\infty = \overline{H}^* \Sigma CP^\infty_+$, then $H^*U \cong U(M_\infty)$, and in this case also, M_∞ can be filtered by dyadic expansion of the dimension of the elements.

In this paper, we use the destabilization of the stable Adams resolution of the connective *so* spectrum to construct a chain complex whose constituent parts are minimal resolutions of the filtration quotients M_n/M_{n-1}. When realized topologically using the Massey–Peterson theorem [M-P], this chain complex gives a tower of spaces whose inverse limit is SO (2-completed), and whose homotopy spectral sequence collapses at E_2. The E_1-term of the homotopy spectral sequence is $\bigoplus \operatorname{Ext}^{*,*}_{\underline{\mathbf{U}}}(M_n/M_{n-1}, \mathbb{F}_2)$, a very large vector space, while $E_2 = E_\infty$ is the associated graded to $\pi_* SO$, a rather small vector space ($\pi_i SO \cong \mathbb{Z}$ for $i \equiv 3 \bmod 4$, and $\mathbb{Z}/2$ for $i \equiv 0$ or $1 \bmod 8$). Hence the spectral sequence has a very complicated d_1, but the complete calculation of d_1 is part of the computation of the chain complex, a primary level calculation. The conjecture suggested by Mahowald (Conjecture 5.1) is that in a certain precise sense, this d_1 differential contains all the differentials in the UASS. Because the tower comes equipped with a map from a modified Postnikov tower for SO, it is possible to use theorems of Bousfield on unstable Ext to predict where the homotopy of SO is represented, and this, in turn, allows a prediction of the unstable Adams filtration of those elements. It is the hope of the author that in the future it will be possible to manipulate this tower by an elaboration of methods of [Lesh] to prove Conjecture 5.1. Extensive knowledge of differentials in the UASS for SO would allow the computation of differentials in other unstable Adams spectral sequences by naturality. For example, it should be possible to recover a form of Hopf invariant one from the model's calculation of the UASS for SO.

The splitting conjecture of Bousfield was discussed and an algebraic model for the UASS for U and SO constructed in [B-D]. However, the model was considered strictly on an algebraic level and was not realized topologically. Although the author believes that the spectral sequence of [B-D] is the same as that of the current work, the advantages of the model described here seem to be the following. First, the construction of the model is essentially formal, and very similar to the standard construction of the spectral sequence converging to the derived functors of a composite functor. All of the differentials can be calculated by a primary level calculation that is a strictly mechanical process. Second, the model comes equipped with a topological realization. It seems that in order to prove that the model actually *does* give the UASS, it will be necessary to have such a realization.

The rest of this paper is organized as follows. In Section 2, we give some background on the stable and destabilized Postnikov towers of so, as well as some algebraic preliminaries. In Section 3, we construct a tower of spaces and an associated chain complex that models the UASS for SO. In Section 4, we study the homotopical properties of the tower. Finally, in Section 5 we use theorems of Bousfield to predict the unstable Adams filtration of elements of $\pi_* SO$, we give a counterexample to a conjecture of [B-D], we draw some conclusions about what may be necessary to prove Conjecture 5.1, and we give an example of a differential in the UASS that is predicted by our methods.

2. Preliminaries. In this section, we review algebraic properties of the category of unstable A-modules, we recall the Massey–Peterson theorem, and we consider the cohomology of the stages of the destabilized Adams tower of *so*.

We begin by reviewing properties of the algebraic looping functor $\Omega : \underline{\mathbf{U}} \to \underline{\mathbf{U}}$ and its iterates. (See also [M-P].) The functor $\Omega : \underline{\mathbf{U}} \to \underline{\mathbf{U}}$ is the left adjoint to the suspension functor $\Sigma : \underline{\mathbf{U}} \to \underline{\mathbf{U}}$. Given an unstable A-module M, the module ΩM can be calculated as the largest unstable quotient of the desuspension of M:

$$\Omega M \equiv (\Sigma^{-1} M)/(\Sigma^{-1} \operatorname{Sq}_0 M),$$

where $\operatorname{Sq}_0 x = \operatorname{Sq}^{|x|} x$. The functor Ω is not exact, but it can have at most one nonzero derived functor, which we denote by Ω_1^1. The module $\Omega_1^1 M$ can be expressed as a regrading of the kernel of Sq_0 on M. In particular, if Sq_0 acts freely on M, then $\Omega_1^1 M = 0$. We write Ω^n for the n-fold iterate of Ω, and we write Ω_j^n for the jth derived functor of Ω^n. There is a composite functor spectral sequence (the Singer spectral sequence) $\Omega_i^s \Omega_j^t M \Rightarrow \Omega_{i+j}^{s+t} M$ which allows us to calculate derived functors of Ω^n inductively. For any unstable module M, $\Omega_j^n M = 0$ for $j > n$.

We will also need the following routine lemma.

LEMMA 2.1. *Let $g : N_1 \to N_2$ be a map of unstable A-modules. If* $\operatorname{im}(g)$ *is* Sq_0*-free, then the natural map* $\Omega \ker(g) \to \ker(\Omega g)$ *is a monomorphism. If in addition N_2 is* Sq_0*-free, then there is a short exact sequence*

$$0 \to \Omega \ker(g) \to \ker(\Omega g) \to \Omega_1^1 \operatorname{cok}(g) \to 0.$$

Proof. The map Ωg factors as $\Omega N_1 \to \Omega \operatorname{im}(g) \to \Omega N_2$, and since Ω is right exact, $\Omega N_1 \to \Omega \operatorname{im}(g)$ is an epimorphism. Thus there is a short exact sequence

$$0 \to \ker[\Omega N_1 \to \Omega \operatorname{im}(g)] \to \ker(\Omega g) \to \ker[\Omega \operatorname{im}(g) \to \Omega N_2] \to 0. \tag{2.1}$$

To calculate the left-hand term, observe that the short exact sequence

$$0 \to \ker(g) \to N_1 \to \operatorname{im}(g) \to 0$$

gives rise to an exact sequence

$$\Omega_1^1 \operatorname{im}(g) \to \Omega \ker(g) \to \Omega N_1 \to \Omega \operatorname{im}(g) \to 0.$$

Thus if $\Omega_1^1 \operatorname{im}(g) = 0$, then $\ker[\Omega N_1 \to \Omega \operatorname{im}(g)] \cong \Omega \ker(g)$, proving that $\Omega \ker(g)$ injects into $\ker(\Omega g)$.

Consider the right-hand term of (2.1). The short exact sequence

$$0 \to \operatorname{im}(g) \to N_2 \to \operatorname{cok}(g) \to 0$$

gives rise to a long exact sequence

$$0 \to \Omega_1^1 \operatorname{im}(g) \to \Omega_1^1 N_2 \to \Omega_1^1 \operatorname{cok}(g) \to \Omega \operatorname{im}(g) \to \Omega N_2 \to \Omega \operatorname{cok}(g) \to 0.$$

If $\Omega_1^1 N_2 = 0$, then $\ker[\Omega \operatorname{im}(g) \to \Omega N_2] \cong \Omega_1^1 \operatorname{cok}(g)$. ∎

REMARK 2.2. Suppose that M is an unstable A-module, that N_1 and N_2 are unstable projective A-modules, and that we are given a map $M \to \Omega \ker(N_1 \to N_2)$. Then we can consider the composition

$$M \to \Omega \ker(N_1 \to N_2) \hookrightarrow \ker(\Omega N_1 \to \Omega N_2) \hookrightarrow \Omega N_1,$$

and so $\ker[M \to \Omega \ker(N_1 \to N_2)] = \ker(M \to \Omega N_1)$. We will use this remark frequently in Section 3.

Going in the opposite direction from looping, we define a "delooping" on free modules. If we write $F(n)$ for the free unstable A-module on a single generator in dimension n, then we define $BF(n) = F(n+1)$. Given a free unstable A-module P, we write BP for the free unstable A-module whose generators are one dimension higher than those of P, and we see that $\Omega BP \cong P$. Note that "delooping" is not a functor on $\underline{\mathbf{U}}$, because given a map $g : P_1 \to P_0$, there is no canonical choice of map $Bg : BP_1 \to BP_0$ with $\Omega Bg = g$. In most cases where we will use this notation, P will itself be an iterated looping, and BP will simply mean one fewer loops: $P = \Omega^i N$ and $BP = \Omega^{i-1} N$.

We remind the reader of the content of the Massey–Peterson theorem, which we will need to use repeatedly. Essentially, this theorem says that under favorable conditions, the Serre spectral sequence for a fibration behaves much like the long exact sequence in cohomology for a stable cofibration.

Because $H^*K(\mathbb{Z}/2, n) \cong U(F(n))$, if P is a projective unstable A-module we write KP for the Eilenberg–MacLane space with $H^*KP \cong U(P)$.

DEFINITION 2.3. We call a map $X \to KP$ *Massey–Peterson* if the following hold:

(a) There is an unstable A-module M with $H^*X \cong U(M)$.

(b) There is a map $f : P \to M$ that induces the map on cohomology. That is, $H^*KP \to H^*X$ is $U(f)$.

(c) $\operatorname{im}(H^*KP \to H^*X)$ is contained in a polynomial subalgebra of H^*X.

(d) X is simple and of finite type.

We think of the topological map $X \to KP$ as realizing f, and by abuse of notation we call the topological map f as well. If Y is the homotopy fiber of a Massey–Peterson map $f : X \to KP$, then the Massey–Peterson theorem says that $H^*Y \cong U(N)$, where there is a short exact sequence (the *fundamental sequence* of f)

$$0 \to \operatorname{cok}(f) \to N \to \Omega \ker(f) \to 0.$$

The short exact sequence does not, in general, split as A-modules, although $U(N)$ does split as an algebra into the tensor product of $U(\operatorname{cok}(f))$ and $U(\Omega \ker(f))$.

We begin our discussion of SO by describing the stable Postnikov tower of so, which is very close to its stable Adams resolution ([1]). We know $H^*so \cong \Sigma A/A\,\mathrm{Sq}^3$, and if we write $\bar{A} = A/A\,\mathrm{Sq}^1$, the stable Postnikov tower of so realizes the acyclic complex of stable A-modules

$$\ldots \to \Sigma^{13}A \to \Sigma^{11}A \to \Sigma^9\bar{A} \to \Sigma^4\bar{A} \to \Sigma A. \tag{2.2}$$

Each term is monogenic and the differentials run cyclically through the list Sq^2, Sq^2, $\overline{\mathrm{Sq}^5}$, $\overline{\mathrm{Sq}^3}$. Only the fact that $\bar{A}$ is not projective keeps this chain complex from being the Adams resolution. Next we destabilize the stable Postnikov tower for the spectrum so by taking the zero space of the infinite loop spectrum at each level of the tower. We obtain the unstable Postnikov tower for SO, a tower of spaces $\{X_n\}$ (Figure 1) with very nice cohomological properties summarized in the following lemma. (Recall that M_n is the nth filtration of $M_\infty \equiv \overline{H}^*RP^\infty$ by dyadic expansion.)

LEMMA 2.4 ([Long]). (1) $\operatorname{holim}_{\leftarrow n} X_n \simeq SO$.
(2) *The k-invariants in the tower $\{X_n\}$ are Massey–Peterson maps.*
(3) $\ker(H^*X_n \to H^*X_{n+1}) = \ker(H^*X_n \to H^*SO)$.
(4) $\operatorname{im}(H^*X_n \to H^*X_{n+1}) \cong \operatorname{im}(H^*X_n \to H^*SO) \cong U(M_n)$.

$$\begin{tikzcd}
 & SO \arrow[d] & \\
 & \vdots & \\
K(F(8)) \arrow[r] & X_4 \arrow[r] \arrow[d] & K(F(10)) \\
K(\bar{F}(7)) \arrow[r] & X_3 \arrow[r] \arrow[d] & K(F(9)) \\
K(\bar{F}(3)) \arrow[r] & X_2 \arrow[r] \arrow[d] & K(\bar{F}(8)) \\
K(F(1)) \arrow[r] & X_1 \arrow[r] \arrow[d] & K(\bar{F}(4)) \\
 & * \arrow[r] & K(F(2))
\end{tikzcd}$$

Fig. 1. The Postnikov tower for SO

([1]) An appropriate reference for the remainder of the section is [Long].

However, we will be interested in the destabilization, not of the Postnikov tower for so, but of the Adams tower. The only difference this introduces is that instead of having only one homotopy group in each dimension, we have to introduce the copies of the integers one $\mathbb{Z}/2$ at a time (building up the completion $\mathbb{Z}_2^\wedge$). To do this, take a projective resolution of each term in (2.2), take the total complex, and destabilize. The realization of this projective chain complex will have the form of Figure 2. An exercise in homological algebra shows that the tower has the same cohomological properties as those of the Postnikov tower which were summarized in Lemma 2.4:

$$\begin{tikzcd}
 & SO_2^\wedge \arrow[d] & \\
 & \vdots & \\
K(F(8) \oplus F(7) \oplus F(3)) \arrow[r, "i_4"] & Y_4 \arrow[r, "k_4"] \arrow[d] & K(F(10) \oplus F(8) \oplus F(4)) \\
K(F(7) \oplus F(3)) \arrow[r, "i_3"] & Y_3 \arrow[r, "k_3"] \arrow[d] & K(F(9) \oplus F(8) \oplus F(4)) \\
K(F(3)) \arrow[r, "i_2"] & Y_2 \arrow[r, "k_2"] \arrow[d] & K(F(8) \oplus F(4)) \\
K(F(1)) \arrow[r, "i_1"] & Y_1 \arrow[r, "k_1"] \arrow[d] & K(F(4)) \\
 & * \arrow[r] & K(F(2))
\end{tikzcd}$$

Fig. 2. The destabilized Adams tower for SO

LEMMA 2.5. (1) $\underleftarrow{\mathrm{holim}}_n Y_n \simeq SO_2^\wedge$.

(2) *There is an unstable A-module Z_n with $H^*Y_n \cong U(Z_n)$, and k_n is a Massey–Peterson map.*

(3) $\ker(H^*Y_n \to H^*Y_{n+1}) = \ker(H^*Y_n \to H^*SO)$.

(4) $\mathrm{im}(H^*Y_n \to H^*Y_{n+1}) \cong \mathrm{im}(H^*Y_n \to H^*SO) \cong U(M_n)$.

REMARK 2.6. (1) For the reader interested in carrying out this calculation in detail, we note that the issues are the same as those laid out in the proofs of Proposition 4.1 and Proposition 4.3.

(2) Let P_n be the unstable projective such that KP_n is the homotopy fiber of $Y_n \to Y_{n-1}$. Thus $P_1 = F(1)$, $P_2 = F(3)$, $P_3 = F(7) \oplus F(3)$, etc. It

is a consequence of Lemma 2.5(4) that

$$\frac{\Omega \ker(BP_n \to P_{n-1})}{\operatorname{im}(BP_{n+1} \to P_n)} \cong M_n/M_{n-1}.$$

(3) The filtration quotients M_n/M_{n-1} have been calculated in terms of generators and relations [Massey]:

$$M_n/M_{n-1} \cong F(2^n - 1)/\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^{n-2}}.$$

3. A chain complex model for the UASS. In this section, we use $\{Y_n\}$, the destabilized Adams tower of *so*, to construct a tower $\{E_n\}$ that also has $SO_2^\wedge$ as its inverse limit, but that involves in its k-invariants the unstable resolutions of the filtration quotients M_n/M_{n-1}. The tower $\{E_n\}$ will come equipped with a map $\{Y_n\} \to \{E_n\}$, which will allow us to calculate where the homotopy of SO is represented in the homotopy spectral sequence of $\{E_n\}$. This in turn will allow us in Section 5 to make predictions about unstable Adams filtrations in the homotopy of SO.

We need a considerable amount of notation. Choose a minimal projective $\underline{\mathbf{U}}$-resolution D_*^n of M_n/M_{n-1}. The tower we are going to build will have the form

$$\begin{array}{ccccc} & & E_{n+1} & & \\ & & \downarrow & & \\ K(D_0^n \oplus \ldots \oplus \Omega^{n-1} D_{n-1}^1) & \longrightarrow & E_n & \longrightarrow & KB(D_0^{n+1} \oplus \ldots \oplus \Omega^n D_n^1) \end{array}$$

Note that D_*^n will make its first appearance at the nth stage of the tower. Because the module D_i^n appears in the tower as $\Omega^i D_i^n$, we avoid excessive loops in our notation by letting $C_i^n = \Omega^i D_i^n$ and $BC_i^n = \Omega^{i-1} D_i^n$. We write $L_n = \bigoplus_{i=1}^n C_{n-i}^i$, and our tower will have the form

$$\begin{array}{ccccc} & & E_{n+1} & & \\ & & \downarrow & & \\ KL_n & \longrightarrow & E_n & \longrightarrow & KBL_{n+1} \end{array}$$

We define the following filtration, along with similar filtrations of BL_n and ΩL_n:

$$F_{-j} L_n = \bigoplus_{i=j}^{n} C_{n-i}^i.$$

Thus $C_0^n = F_{-n} L_n \subseteq F_{-(n-1)} L_n \subseteq \ldots \subseteq F_{-1} L_n = L_n$.

The tower of spaces $\{E_n\}$ that we construct in this section has the following properties. Recall from Lemma 2.5 that Z_n is the unstable A-module such that $H^* Y_n \cong U(Z_n)$, and from Remark 2.6 that P_n is the unstable projective such that Y_n is the homotopy fiber of a map $Y_{n-1} \to KBP_n$.

(1) There exists an unstable A-module F_{n-1} with $H^*E_{n-1} \cong U(F_{n-1})$, and E_n is the homotopy fiber of a Massey–Peterson map $E_{n-1} \to KBL_n$.

(2) There are commuting diagrams of Massey–Peterson maps

$$\begin{array}{ccccc} KP_{n-1} & \longrightarrow & Y_{n-1} & \longrightarrow & KBP_n \\ \downarrow & & \downarrow & & \downarrow \\ KL_{n-1} & \longrightarrow & E_{n-1} & \longrightarrow & KBL_n \end{array}$$

induced by commuting diagrams of unstable A-modules

$$\begin{array}{ccccc} BL_n & \longrightarrow & F_{n-1} & \longrightarrow & L_{n-1} \\ {\scriptstyle h_n}\downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega h_{n-1}} \\ BP_n & \longrightarrow & Z_{n-1} & \longrightarrow & P_{n-1} \end{array}$$

(3) $\ker(BL_n \to F_{n-1}) = \ker(BL_n \to L_{n-1})$.

(4) $\operatorname{cok}(BL_n \to F_{n-1}) \to \operatorname{cok}(BP_n \to Z_{n-1})$ is an isomorphism.

(5) Algebraic properties of the map f_n are described in detail below.

Property (3) is analogous to Lemma 2.5(3); both say that the k-invariants do not kill any cohomology that comes from lower down in the tower. Property (4) is related to Lemma 2.5(4), and arranges for the towers $\{E_n\}$ and $\{Y_n\}$ to give the same filtration of H^*SO.

To describe the last set of properties we recall that by the Massey–Peterson theorem, if E_{n-1} is the fiber of a Massey–Peterson map $E_{n-2} \to KBL_{n-1}$, then the fundamental sequence for E_{n-1} is

$$0 \to \operatorname{cok}(BL_{n-1} \to F_{n-2}) \to F_{n-1} \to \Omega \ker(BL_{n-1} \to F_{n-2}) \to 0,$$

where the right-hand term is the contribution of the fiber, KL_{n-1}, to H^*E_{n-1}. The next space, E_n, will be the fiber of a Massey–Peterson map $E_{n-1} \to KBL_n$, and our last requirement is on the composition of the k-invariants, $KL_{n-1} \to E_{n-1} \to KBL_n$. Let f_n denote the composite $BL_n \to F_{n-1} \to \Omega \ker(BL_{n-1} \to F_{n-2}) = \Omega \ker(BL_{n-1} \to L_{n-2})$. The final requirement on the tower $\{E_n\}$ is detailed below:

(5) f_n has the following algebraic properties:

(a) f_n is filtration preserving.

(b) For $1 \le j \le n$, on $F_{-j}/F_{-(j+1)}$ the map $E_0(f_n)$ is the map

$$BC^j_{n-j} \to \Omega \ker(BC^j_{n-j-1} \to C^j_{n-j-2})$$

that comes from looping down the differential in the resolution $D^j_* \to M_j/M_{j-1}$.

(c) $E_0(\ker(f_n)) \cong \ker(E_0(f_n))$.

We will use Remark 2.2 freely throughout this section. In particular, Remark 2.2 together with requirement (5) tells us that the associated graded of $\ker(f_n)$ is $F_{-j}/F_{-(j+1)}(\ker(f_n)) \cong \ker(BC^j_{n-j} \to C^j_{n-j-1})$.

The construction of $\{E_n\}$ is inductive. For the first stage we observe that $P_1 = L_1 = C^1_0$, and we define $L_1 \to P_1$ to be the identity map. Thus $Y_1 = KP_1 = KL_1 = E_1$, and the requirements are certainly satisfied in this case. Observe that $P_1 = Z_1 = F_1 = L_1$.

At the next stage, $L_2 = C^2_0 \oplus C^1_1$; we want a commuting diagram

$$\begin{array}{ccc} BL_2 & \longrightarrow & L_1 = F_1 \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle =} \\ BP_2 & \longrightarrow & P_1 = Z_1 \end{array} \qquad \text{i.e.} \qquad \begin{array}{ccc} B(C^2_0 \oplus C^1_1) & \longrightarrow & C^1_0 \\ {\scriptstyle h_2}\downarrow & & \downarrow{\scriptstyle =} \\ BP_2 & \longrightarrow & P_1 \end{array}$$

We define $BC^2_0 \to C^1_0$ to be zero, and $BC^1_1 \to C^1_0$ by the differential for C^1_*. The composite $BC^1_1 \to C^1_0 = P_1 \to \operatorname{cok}(BP_2 \to P_1) \cong M_1$ is zero because $BC^1_1 \to C^1_0 \to M_1$ begins a resolution, and so the composite $BC^1_1 \to C^1_0 \to P_1$ factors through BP_2. We use this factoring to define $h_2 : BL_2 \to BP_2$ on the factor BC^1_1. To define h_2 on the factor BC^2_0, choose a class $x_2 \in \ker(BP_2 \to P_1)$ that, when looped, gives the generator of the quotient $\Omega \ker(BP_2 \to P_1)/\operatorname{im}(BP_3 \to P_2) \cong M_2/M_1$. This gives us the desired commuting diagram above. If we look at the topological realization

$$\begin{array}{ccc} Y_1 & \longrightarrow & KBP_2 \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & KBL_2 \end{array}$$

the properties required for $E_1 \to KBL_2$ are easily verified by inspection, and we take homotopy fibers in the diagram to obtain the space E_2 together with a map $Y_2 \to E_2$ and maps of fundamental sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{cok}(BL_2 \to L_1) & \longrightarrow & F_2 & \longrightarrow & \Omega \ker(BL_2 \to L_1) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{cok}(BP_2 \to P_1) & \longrightarrow & Z_2 & \longrightarrow & \Omega \ker(BP_2 \to P_1) & \longrightarrow & 0 \end{array}$$

For an inductive hypothesis, we assume that for $i \le n$ we have defined spaces E_i and maps f_i satisfying the required conditions, and we seek to define E_{n+1}. Thus we have maps $BP_{n+1} \to Z_n$ and $F_n \to Z_n$ induced by $Y_n \to KBP_{n+1}$ and $Y_n \to E_n$, respectively. We need to define a commuting diagram

$$\begin{array}{ccc} BL_{n+1} & \longrightarrow & F_n \\ {\scriptstyle h_{n+1}}\downarrow & & \downarrow \\ BP_{n+1} & \longrightarrow & Z_n \end{array}$$

and verify that when we realize it by a diagram of spaces

$$\begin{array}{ccc} Y_n & \longrightarrow & KBP_{n+1} \\ \big\downarrow & & \big\downarrow \\ E_n & \longrightarrow & KBL_{n+1} \end{array}$$

taking horizontal fibers gives rise to a space E_{n+1} and a map $Y_{n+1} \to E_{n+1}$ that satisfy the inductive hypotheses.

Consider the ladder of fundamental sequences for Y_n and E_n:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{cok}(BL_n \to F_{n-1}) & \longrightarrow & F_n & \longrightarrow & \Omega \ker(BL_n \to F_{n-1}) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \operatorname{cok}(BP_n \to Z_{n-1}) & \longrightarrow & Z_n & \longrightarrow & \Omega \ker(BP_n \to Z_{n-1}) & \longrightarrow & 0 \end{array} \tag{3.1}$$

Lemma 2.5(4) yields $\Omega \ker(BP_n \to Z_{n-1}) = \Omega \ker(BP_n \to P_{n-1})$, and by the inductive hypothesis $\Omega \ker(BL_n \to F_{n-1}) = \Omega \ker(BL_n \to L_{n-1})$. Our strategy is to construct a commuting diagram

$$\begin{array}{ccc} BL_{n+1} & \longrightarrow & \Omega \ker(BL_n \to L_{n-1}) \\ {\scriptstyle h_{n+1}}\big\downarrow & & {\scriptstyle \Omega h_n}\big\downarrow \\ BP_{n+1} & \longrightarrow & \Omega \ker(BP_n \to P_{n-1}). \end{array} \tag{3.2}$$

This will give a map of BL_{n+1} into the right-hand term of the top fundamental sequence in (3.1), and then we will lift to F_n using projectivity of BL_{n+1}. We will make the construction in such a way that Ωh_n induces an isomorphism between the cokernel of $BL_{n+1} \to \Omega \ker(BL_n \to L_{n-1})$ and the cokernel of $BP_{n+1} \to \Omega \ker(BP_n \to P_{n-1})$, which we know to be M_n/M_{n-1}. This will lead to condition (4) for the tower $\{E_n\}$.

To construct diagram (3.2), we compute $\Omega \ker(BL_n \to L_{n-1})$. From inductive hypothesis (5), we know the associated graded of $\ker(BL_n \to L_{n-1})$, and since Ω commutes with cokernels, we know that $\Omega \ker(BL_n \to L_{n-1})$ has associated graded

$$\begin{aligned} F_{-j}/F_{-(j+1)} &\cong \Omega \ker[BC^j_{n-j} \to \Omega \ker(BC^j_{n-j-1} \to C^j_{n-j-2})] \\ &= \Omega \ker(BC^j_{n-j} \to C^j_{n-j-1}). \end{aligned}$$

We first define a filtration preserving map $g_{n+1} : BL_{n+1} \to \Omega \ker(BL_n \to L_{n-1})$ as follows. On the lowest filtration, $F_{-(n+1)} = BC_0^{n+1}$, let g_{n+1} be zero. In filtration $-j$, let $g_{n+1} : BC^j_{n-j+1} \to \Omega \ker(BL_n \to L_{n-1})$ lift the natural map

$$BC^j_{n-j+1} \to \Omega \ker(BC^j_{n-j} \to C^j_{n-j-1}) = F_{-j}/F_{-(j+1)}(\Omega \ker(BL_n \to L_{n-1}))$$

to $F_{-j}(\Omega \ker(BL_n \to L_{n-1}))$. Note that $F_{-n}(\Omega \ker(BL_n \to L_{n-1})) = C_0^n$ splits off from $\Omega \ker(BL_n \to L_{n-1})$. We can take $g_{n+1} : \bigoplus_{j=1}^{n-1} BC_{n+1-j}^j \to C_0^n$ to be zero, and the only factor on which $g_{n+1} : BL_{n+1} \to C_0^n$ is nonzero is BC_1^n.

LEMMA 3.1. *g_{n+1} is filtration preserving and* $\operatorname{cok}(g_{n+1}) \cong M_n/M_{n-1}$.

Proof. g_{n+1} is filtration preserving by its construction. To calculate the cokernel, we first consider the cokernel on the level of the associated graded. For $j \geq 1$, in filtration $F_{-j}/F_{-(j+1)}$ we have

$$BC_{n-j+1}^j \to \Omega \ker(BC_{n-j}^j \to C_{n-j-1}^j),$$

that is,

$$\Omega^{n-j} D_{n-j+1}^j \to \Omega \ker(\Omega^{n-j-1} D_{n-j}^j \to \Omega^{n-j-1} D_{n-j-1}^j).$$

By definition, $D_*^j \to M_j/M_{j-1}$ is a resolution, and so for $j < n$ the homology at the middle of the three-term sequence $\Omega^{n-j-1} D_{n-j+1}^j \to \Omega^{n-j-1} D_{n-j}^j \to \Omega^{n-j-1} D_{n-j-1}^j$ is $\Omega_{n-j}^{n-j-1} M_j/M_{j-1}$, which we know is zero since $n-j > n-j-1$. Hence the map

$$\Omega^{n-j-1} D_{n-j+1}^j \to \ker(\Omega^{n-j-1} D_{n-j}^j \to \Omega^{n-j-1} D_{n-j-1}^j)$$

is a surjection. Looping preserves surjections, and hence

$$BC_{n-j+1}^j \to \Omega \ker(BC_{n-j}^j \to C_{n-j-1}^j)$$

is a surjection.

Thus the cokernel of $E_0(g_{n+1})$ is zero on $F_{-j}/F_{-(j+1)}$ for $j < n$. Consider $j = n$: on F_{-n} we have defined g_{n+1} to be the differential $BC_1^n \to C_0^n$, whose cokernel is M_n/M_{n-1}. Since we have taken g_{n+1} to be zero from higher filtrations into F_{-n}, we find that $\operatorname{cok}(g_{n+1}) \cong M_n/M_{n-1}$ as desired. ∎

Recall that the cokernel of $BP_{n+1} \to \Omega \ker(BP_n \to P_{n-1})$ is M_n/M_{n-1} (Remark 2.6). To get diagram (3.2), we must have a map $f_{n+1} : BL_{n+1} \to \Omega \ker(BL_n \to L_{n-1})$ whose cokernel is M_n/M_{n-1} and whose composition with Ωh_n factors through BP_{n+1}. So far, we have a map $g_{n+1} : BL_{n+1} \to \Omega \ker(BL_n \to L_{n-1})$ whose cokernel is M_n/M_{n-1}, but the composition of g_{n+1} with Ωh_n does not necessarily factor through BP_{n+1}. To adjust g_{n+1}, consider the composite

$$\begin{aligned}\bigoplus_{j=1}^{n-1} BC_{n-j+1}^j \hookrightarrow BL_{n+1} &\xrightarrow{g_{n+1}} \Omega \ker(BL_n \to L_{n-1}) \\ &\xrightarrow{\Omega h_n} \Omega \ker(BP_n \to P_{n-1}) \to M_n/M_{n-1}.\end{aligned}$$

Choose a lift of the composite across the epimorphism $C_0^n \to M_n/M_{n-1}$. We define $f_{n+1} : BL_{n+1} \to \Omega \ker(BL_n \to L_{n-1})$ as the sum of g_{n+1} with the lift $\bigoplus_{j=1}^{n-1} BC_{n-j+1}^j \to C_0^n$. Observe that f_{n+1} is the same as g_{n+1} on the factors BC_0^{n+1} and BC_1^n of BL_{n+1}, and further, the adjustment added to g_{n+1} to obtain f_{n+1} strictly lowers filtration; thus f_{n+1} and g_{n+1} induce the same map on the associated graded. By construction, $\Omega h_n \circ f_{n+1} : \bigoplus_{j=1}^n BC_{n-j+1}^j \to \Omega \ker(BP_n \to P_{n-1})$ composes to zero in M_n/M_{n-1}, and so $\Omega h_n \circ f_{n+1}$ factors through BP_{n+1}. We define $h_{n+1} : BL_{n+1} \to BP_{n+1}$ to be the sum of this factoring with a map $BC_0^{n+1} \to BP_{n+1}$ that hits a class x_{n+1} whose looping generates $\Omega \ker(BP_{n+1} \to P_n)/\mathrm{im}(BP_{n+2} \to P_{n+1}) \cong M_{n+1}/M_n$.

LEMMA 3.2. *The commuting diagram*

$$\begin{array}{ccc} BL_{n+1} & \xrightarrow{f_{n+1}} & \Omega \ker(BL_n \to L_{n-1}) \\ {\scriptstyle h_{n+1}}\downarrow & & \downarrow{\scriptstyle \Omega h_n} \\ BP_{n+1} & \xrightarrow{d_{n+1}} & \Omega \ker(BP_n \to P_{n-1}) \end{array}$$

induces an isomorphism

$$\mathrm{cok}(f_{n+1}) \cong \mathrm{cok}(d_{n+1}).$$

Proof. By the construction of $h_n : BL_n \to BP_n$ at the previous stage,

$$\Omega \ker(BL_n \to L_{n-1}) \to \mathrm{cok}(d_{n+1}) \cong M_n/M_{n-1}$$

is an epimorphism. On the other hand, the cokernel of $E_0(f_{n+1})$ is M_n/M_{n-1} in filtration $-n$ and zero in higher filtrations, and so Ωh_n induces an isomorphism $\mathrm{cok}(f_{n+1}) \cong \mathrm{cok}(d_{n+1})$. ∎

COROLLARY 3.3. $E_0(\ker f_{n+1}) \cong \ker(E_0(f_{n+1}))$.

Proof. The result follows from the proof of the preceding lemma, since we established that $E_0(\mathrm{cok}\, f_{n+1}) \cong \mathrm{cok}(E_0(f_{n+1}))$. ∎

We are ready to define the k-invariant that takes us from E_n to E_{n+1}. Let k_{n+1} be a lift of f_{n+1} across the epimorphism $F_n \to \Omega \ker(BL_n \to L_{n-1})$ that comes from the fundamental sequence for E_n.

LEMMA 3.4. *k_{n+1} can be chosen to give a commuting diagram*

$$\begin{array}{ccc} BL_{n+1} & \xrightarrow{k_{n+1}} & F_n \\ {\scriptstyle h_{n+1}}\downarrow & & \downarrow \\ BP_{n+1} & \longrightarrow & Z_n \end{array}$$

Proof. The choice of the lift k_{n+1} can be adjusted if necessary by a routine diagram chase. Use the ladder of fundamental sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{cok}(BL_n \to F_{n-1}) & \longrightarrow & F_n & \longrightarrow & \Omega\ker(BL_n \to F_{n-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{cok}(BP_n \to Z_{n-1}) & \longrightarrow & Z_n & \longrightarrow & \Omega\ker(BP_n \to Z_{n-1}) & \longrightarrow & 0 \end{array}$$

in which the left vertical arrow is an isomorphism by induction, and the commuting diagram

$$\begin{array}{ccc} BL_{n+1} & \xrightarrow{f_{n+1}} & \Omega\ker(BL_n \to L_{n-1}) = \Omega\ker(BL_n \to F_{n-1}) \\ {\scriptstyle h_{n+1}}\downarrow & & {\scriptstyle \Omega h_n}\downarrow \\ BP_{n+1} & \xrightarrow{d_{n+1}} & \Omega\ker(BP_n \to P_{n-1}) = \Omega\ker(BP_n \to Z_{n-1}). \end{array} \blacksquare$$

The remaining task for this section is the verification of the inductive hypotheses. Let

$$\begin{array}{ccc} Y_n & \longrightarrow & KBP_{n+1} \\ \downarrow & & \downarrow \\ E_n & \longrightarrow & KBL_{n+1} \end{array} \tag{3.3}$$

be a homotopy commutative diagram of spaces that realizes the commutative diagram of Lemma 3.4, let E_{n+1} be the homotopy fiber of $E_n \to KBL_{n+1}$, and let $Y_{n+1} \to E_{n+1}$ be the map between the homotopy fibers. By construction, $E_n \to KBL_{n+1}$ is a Massey–Peterson map, because the image of $BL_{n+1} \to F_n$ injects to $\Omega\ker(BL_n \to L_{n-1}) \subseteq L_n$, and thus is Sq_0-free. The commuting square (3.3) is a map between Massey–Peterson maps by construction, and thus we get the first two inductive hypotheses immediately.

LEMMA 3.5. $\ker(k_{n+1}) = \ker(f_{n+1})$.

Proof. f_{n+1} is the top composite in the commuting diagram

$$\begin{array}{ccccc} BL_{n+1} & \xrightarrow{k_{n+1}} & F_n & \longrightarrow & \Omega\ker(BL_n \to L_{n-1}) \\ {\scriptstyle h_{n+1}}\downarrow & & \downarrow & & {\scriptstyle \Omega h_n}\downarrow \\ BP_{n+1} & \longrightarrow & Z_n & \longrightarrow & \Omega\ker(BP_n \to P_{n-1}). \end{array}$$

Certainly $\ker(k_{n+1}) \subseteq \ker(f_{n+1})$. Suppose $x \in \ker(f_{n+1})$; then

$$h_{n+1}(x) \in \ker[BP_{n+1} \to \Omega\ker(BP_n \to P_{n-1})] = \ker[BP_{n+1} \to Z_n]$$

by Lemma 2.5(4). Thus $k_{n+1}(x) \in \ker(F_n \to Z_n)$. However, by inductive hypothesis (4) and the ladder (3.1) of fundamental sequences for Y_n and E_n, $\ker[F_n \to \Omega\ker(BL_n \to L_{n-1})] \cong \ker[Z_n \to \Omega\ker(BP_n \to P_{n-1})]$. Thus $k_{n+1}(x) = 0$. $\blacksquare$

LEMMA 3.6. $\operatorname{cok}(BL_{n+1} \to F_n) \cong \operatorname{cok}(BP_{n+1} \to Z_n)$.

Proof. Apply the Snake Lemma to the ladder of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & BP_{n+1} & \longrightarrow & BP_{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{cok}(BP_n \to Z_{n-1}) & \longrightarrow & Z_n & \longrightarrow & \Omega \ker(BP_n \to P_{n-1}) & \longrightarrow & 0 \end{array}$$

By Lemma 2.5, $\ker(BP_{n+1} \to Z_n) \cong \ker[BP_{n+1} \to \Omega \ker(BP_n \to P_{n-1})]$, and so the cokernels of the vertical maps form a short exact sequence. The same reasoning applied to BL_{n+1} and the fundamental sequence for E_n gives a commuting ladder of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{cok}(BL_n \to F_{n-1}) & \longrightarrow & \operatorname{cok}(BL_{n+1} \to F_n) & \longrightarrow & \operatorname{cok}(f_{n+1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{cok}(BP_n \to Z_{n-1}) & \longrightarrow & \operatorname{cok}(BP_{n+1} \longrightarrow Z_n) & \to & \operatorname{cok}(d_{n+1}) & \longrightarrow & 0 \end{array}$$

The leftmost column is an isomorphism by the inductive hypothesis and the right-hand column is an isomorphism by Lemma 3.2. ∎

COROLLARY 3.7. *The natural map $\varinjlim_n F_n \to \varinjlim_n Z_n$ is an isomorphism.*

Proof. Consider

$$\begin{array}{ccc} F_n & \longrightarrow & Z_n \\ \downarrow & & \downarrow \\ F_{n+1} & \longrightarrow & Z_{n+1} \\ \downarrow & & \downarrow \\ \varinjlim_n F_n & \longrightarrow & \varinjlim_n Z_n \end{array}$$

By the preceding lemma, $\operatorname{im}(F_n \to F_{n+1}) \cong \operatorname{im}(Z_n \to Z_{n+1})$, and by Lemma 2.5, $\operatorname{im}(Z_n \to Z_{n+1}) \cong \operatorname{im}(Z_n \to Z_{n+j})$ for $j > 1$. The corollary follows. ∎

4. Homotopical properties of $\{E_n\}$. In this section we give the homotopical and homological properties of the tower $\{E_n\}$. We prove that it has inverse limit $SO_2^\wedge$ and that its homotopy spectral sequence collapses at the E_2-term. Notation is continued from Section 3.

PROPOSITION 4.1. *The map of towers $\{Y_n\} \to \{E_n\}$ induces a homotopy equivalence on the homotopy inverse limits.*

Proof. We already know from Corollary 3.7 that the map of towers induces an isomorphism $\varinjlim_n H^*E_n \to \varinjlim_n H^*Y_n$. Although cohomology is not in general well related to inverse limits, an application of [Lannes, Lemme 3.2.3] tells us that in our situation,

$$H^* \operatorname{holim}_{\overleftarrow{n}} Y_n \cong \lim_{\overrightarrow{n}} H^* Y_n \quad \text{and} \quad H^* \operatorname{holim}_{\overleftarrow{n}} E_n \cong \lim_{\overrightarrow{n}} H^* E_n.$$

The essential ingredients that allow the use of Lannes's lemma are:

(1) For all n, the spaces Y_n and E_n are connected and have mod 2 cohomology that is finite in each dimension.

(2) The towers of groups $\{\pi_1 Y_n\}$ and $\{\pi_1 E_n\}$ are constant.

(3) The towers of groups $\{H_1 Y_n\}$ and $\{H_1 E_n\}$ are constant.

The proposition then follows by observing that $\operatorname{holim}_{\overleftarrow{n}} Y_n \to \operatorname{holim}_{\overleftarrow{n}} E_n$ is a mod 2 cohomology isomorphism, and the source and target are each 2-complete, being built from mod 2 Eilenberg–MacLane spaces by fibrations. ■

COROLLARY 4.2. $\operatorname{holim}_{\overleftarrow{n}} E_n \simeq SO_2^\wedge$.

Our next goal is Corollary 4.5, in which we prove that the homotopy spectral sequence of $\{E_n\}$ collapses at the E_2-term. This follows by using a homological argument to show that the map $\{Y_n\} \to \{E_n\}$ induces an isomorphism at E_2 of the homotopy spectral sequences, and then observing that the homotopy spectral sequence of $\{Y_n\}$ does in fact collapse at E_2. The following proposition performs the main technical calculation.

PROPOSITION 4.3. *The following ladder gives a homology isomorphism at the middle term*:

$$\begin{tikzcd} BL_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "h_{n+1}"'] & L_n \arrow[r, "\Omega f_n"] \arrow[d, "\Omega h_n"'] & \Omega L_{n-1} \arrow[d, "\Omega^2 h_{n-1}"'] \\ BP_{n+1} \arrow[r, "d_{n+1}"] & P_n \arrow[r, "\Omega d_n"] & \Omega P_{n-1} \end{tikzcd}$$

That is, Ωh_n *induces an isomorphism*

$$\frac{\ker(\Omega f_n)}{\operatorname{im}(f_{n+1})} \cong \frac{\ker(\Omega d_n)}{\operatorname{im}(d_{n+1})}.$$

Proof. The proof is inductive. For $n = 1$, we take $P_0 = L_0 = 0$ and the result is easily established by direct calculation. Suppose that the proposition is true for

$$\begin{tikzcd} BL_n \arrow[r] \arrow[d] & L_{n-1} \arrow[r] \arrow[d] & \Omega L_{n-2} \arrow[d] \\ BP_n \arrow[r] & P_{n-1} \arrow[r] & \Omega P_{n-2} \end{tikzcd}$$

and consider the next stage. By Lemma 3.2, we already know that

$$\frac{\Omega \ker(BL_n \to L_{n-1})}{\operatorname{im}(BL_{n+1} \to L_n)} \cong \frac{\Omega \ker(BP_n \to P_{n-1})}{\operatorname{im}(BP_{n+1} \to P_n)}.$$

Let $i_L : \Omega\ker(BL_n \to L_{n-1}) \to \ker(L_n \to \Omega L_{n-1})$ be the natural map $\Omega\ker(f_n) \to \ker(\Omega f_n)$, let $\bar{i}_L$ be the induced map on cokernels, and consider the diagram of exact sequences

$$\begin{array}{ccccccc}
BL_{n+1} & \xrightarrow{f_{n+1}} & \Omega\ker(BL_n \to L_{n-1}) & \longrightarrow & \dfrac{\Omega\ker(BL_n \to L_{n-1})}{\operatorname{im}(BL_{n+1} \to L_n)} & \longrightarrow & 0 \\
{\scriptstyle =}\downarrow & & {\scriptstyle i_L}\downarrow & & {\scriptstyle \bar{i}_L}\downarrow & & \\
BL_{n+1} & \longrightarrow & \ker(L_n \to \Omega L_{n-1}) & \longrightarrow & \dfrac{\ker(L_n \to \Omega L_{n-1})}{\operatorname{im}(BL_{n+1} \to L_n)} & \longrightarrow & 0
\end{array}$$

By Lemma 2.1 and the Snake Lemma, i_L and $\bar{i}_L$ are monomorphisms and $\operatorname{cok}(\bar{i}_L) \cong \operatorname{cok}(i_L) \cong \Omega_1^1 \operatorname{cok}(BL_n \to L_{n-1})$. The same argument with $i_P : \Omega\ker(BP_n \to P_{n-1}) \to \ker(P_n \to \Omega P_{n-1})$ and the corresponding map of cokernels, $\bar{i}_P$, shows that $\bar{i}_P$ is a monomorphism and $\operatorname{cok}(\bar{i}_P) \cong \Omega_1^1 \operatorname{cok}(BP_n \to P_{n-1})$. Consider the diagram

$$\begin{array}{ccc}
\dfrac{\Omega\ker(BL_n \to L_{n-1})}{\operatorname{im}(BL_{n+1} \to L_n)} & \xrightarrow{\cong} & \dfrac{\Omega\ker(BP_n \to P_{n-1})}{\operatorname{im}(BP_{n+1} \to P_n)} \\
{\scriptstyle \bar{i}_L}\downarrow & & {\scriptstyle \bar{i}_P}\downarrow \\
\dfrac{\ker(L_n \to \Omega L_{n-1})}{\operatorname{im}(BL_{n+1} \to L_n)} & \longrightarrow & \dfrac{\ker(P_n \to \Omega P_{n-1})}{\operatorname{im}(BP_{n+1} \to P_n)}
\end{array}$$

We already know that the top row is an isomorphism. Since $\bar{i}_L$ and $\bar{i}_P$ are monomorphisms, the corollary will be established by the Five Lemma if we prove that the diagram induces an isomorphism $\operatorname{cok}(\bar{i}_L) \to \operatorname{cok}(\bar{i}_P)$. Thus we must show that $\Omega_1^1 \operatorname{cok}(BL_n \to L_{n-1}) \cong \Omega_1^1 \operatorname{cok}(BP_n \to P_{n-1})$.

The three-term sequence $BL_n \to L_{n-1} \to \Omega L_{n-2}$ gives us a short exact sequence

$$\frac{\ker(L_{n-1} \to \Omega L_{n-2})}{\operatorname{im}(BL_n \to L_{n-1})} \hookrightarrow \frac{L_{n-1}}{\operatorname{im}(BL_n \to L_{n-1})} \twoheadrightarrow \frac{L_{n-1}}{\ker(L_{n-1} \to \Omega L_{n-2})}.$$

The middle term is $\operatorname{cok}(BL_n \to L_{n-1})$, and the right-hand term is Sq_0-free, because it injects into ΩL_{n-2}, which is itself Sq_0-free. This argument and a similar one applied to $BP_n \to P_{n-1} \to \Omega P_{n-2}$ give us

$$\Omega_1^1 \operatorname{cok}(BL_n \to L_{n-1}) \cong \Omega_1^1 \left[\frac{\ker(L_{n-1} \to \Omega L_{n-2})}{\operatorname{im}(BL_n \to L_{n-1})}\right],$$
$$\Omega_1^1 \operatorname{cok}(BP_n \to P_{n-1}) \cong \Omega_1^1 \left[\frac{\ker(P_{n-1} \to \Omega P_{n-2})}{\operatorname{im}(BP_n \to P_{n-1})}\right],$$

and these are isomorphic by the inductive hypothesis. ∎

Corollary 4.4. *The commuting ladder*

$$\begin{tikzcd}
BL_{n+1} \arrow[r] \arrow[d] & L_n \arrow[r] \arrow[d] & \Omega L_{n-1} \arrow[d] \\
BP_{n+1} \arrow[r] & P_n \arrow[r] & \Omega P_{n-1}
\end{tikzcd}$$

induces an isomorphism on $H^* \operatorname{Hom}_{\underline{\mathrm{U}}}(-, \Sigma^t \mathbb{F}_2)$ *for all* t *at the middle term.*

Proof. We first prove that for all n, the commuting ladder

$$\begin{tikzcd}
BL_{n+1} \arrow[r] \arrow[d] & L_n \arrow[r] \arrow[d] & \Omega L_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & \Omega^{n-1} L_1 \arrow[d] \\
BP_{n+1} \arrow[r] & P_n \arrow[r] & \Omega P_{n-1} \arrow[r] & \dots \arrow[r] & \Omega^{n-1} P_1
\end{tikzcd}$$

induces an isomorphism on the homology of the rows up to and including $L_n \to P_n$. The proof is by induction, beginning with

$$\begin{tikzcd}
BL_2 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d, "="'] & 0 \arrow[d] \\
BP_2 \arrow[r] & P_1 \arrow[r] & 0
\end{tikzcd}$$

In the case of SO, $BL_2 \to BP_2$ is an equality. In the case of U, we observe $BL_2 = BC_1^1 \oplus BC_0^2 = BP_2 \oplus BC_0^2$ where the BP_2 summand maps to BP_2 by the identity and BC_0^2 maps to L_1 by the zero map. Thus we have a base for the induction in the case of U also.

Suppose that

$$\begin{tikzcd}
BL_n \arrow[r] \arrow[d] & L_{n-1} \arrow[r] \arrow[d] & \Omega L_{n-2} \arrow[r] \arrow[d] & \dots \arrow[r] & \Omega^{n-2} L_1 \arrow[d] \\
BP_n \arrow[r] & P_{n-1} \arrow[r] & \Omega P_{n-2} \arrow[r] & \dots \arrow[r] & \Omega^{n-2} P_1
\end{tikzcd}$$

induces an isomorphism on homology up to and including $L_{n-1} \to P_{n-1}$. Applying Ω to both complexes, we find that

$$\begin{tikzcd}
L_n \arrow[r] \arrow[d] & \Omega L_{n-1} \arrow[r] \arrow[d] & \Omega^2 L_{n-2} \arrow[r] \arrow[d] & \dots \arrow[r] & \Omega^{n-1} L_1 \arrow[d] \\
P_n \arrow[r] & \Omega P_{n-1} \arrow[r] & \Omega^2 P_{n-2} \arrow[r] & \dots \arrow[r] & \Omega^{n-1} P_1
\end{tikzcd}$$

is an isomorphism on homology up to and including $\Omega L_{n-1} \to \Omega P_{n-1}$, and joining this with the result of Proposition 4.3, we find that

$$\begin{tikzcd}
BL_{n+1} \arrow[r] \arrow[d] & L_n \arrow[r] \arrow[d] & \Omega L_{n-1} \arrow[r] \arrow[d] & \dots \arrow[r] & \Omega^{n-1} L_1 \arrow[d] \\
BP_{n+1} \arrow[r] & P_n \arrow[r] & \Omega P_{n-1} \arrow[r] & \dots \arrow[r] & \Omega^{n-1} P_1
\end{tikzcd}$$

is an isomorphism on homology up to and including $L_n \to P_n$, and the induction is complete.

Assume that $t \geq 1$, since all the spaces and modules we use in this work are connected. To prove the corollary, we use the ladder

$$\begin{array}{ccccccccccc} BL_{n+2} & \longrightarrow & L_{n+1} & \longrightarrow & \Omega L_n & \longrightarrow & \Omega^2 L_{n-1} & \longrightarrow & \dots & \longrightarrow & \Omega^n L_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ BP_{n+2} & \longrightarrow & P_{n+1} & \longrightarrow & \Omega P_n & \longrightarrow & \Omega^2 P_{n-1} & \longrightarrow & \dots & \longrightarrow & \Omega^n P_1 \end{array}$$

Denote the top row of the ladder by $\mathcal{L}_*$ and the bottom row by $\mathcal{P}_*$, and let $\mathcal{C}_*$ be the mapping cone. Then $H_*\mathcal{C}_* = 0$ for $* \leq n+1$, and thus $H^* \operatorname{Hom}_{\underline{\mathrm{U}}}(\mathcal{C}_*, \Sigma^{t-1}\mathbb{F}_2) = 0$ for $* \leq n$. Therefore the ladder

$$\begin{array}{ccccc} \Omega BL_{n+1} \cong L_{n+1} & \longrightarrow & \Omega L_n & \longrightarrow & \Omega^2 L_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega BP_{n+1} \cong P_{n+1} & \longrightarrow & \Omega P_n & \longrightarrow & \Omega^2 P_{n-1} \end{array}$$

gives an isomorphism on $H^*[\operatorname{Hom}_{\underline{\mathrm{U}}}(-, \Sigma^{t-1}\mathbb{F}_2)]$ at the middle term. However, the functors Ω and Σ are adjoints, and so $\operatorname{Hom}_{\underline{\mathrm{U}}}(\Omega -, \Sigma^{t-1}\mathbb{F}_2) \cong \operatorname{Hom}_{\underline{\mathrm{U}}}(-, \Sigma^{t}\mathbb{F}_2)$, and the corollary follows. ∎

COROLLARY 4.5. *The homotopy spectral sequence of* $\{E_n\}$ *collapses at* E_2.

Proof. By Corollary 4.4, the map $\{Y_n\} \to \{E_n\}$ induces a map of homotopy spectral sequences which is an isomorphism on the E_2-term. Since the homotopy spectral sequence of $\{Y_n\}$ has no further differentials (in fact, it collapses at E_1), the homotopy spectral sequence of $\{E_n\}$ collapses at E_2. ∎

5. A model for the UASS, and some predictions and reflections. In the preceding sections, we used the resolutions of the filtration quotients M_n/M_{n-1} to construct a complicated tower $\{E_n\}$ that involves those resolutions, converges to $SO_2^{\wedge}$, and has a homotopy spectral sequence that collapses at E_2. The tower $\{E_n\}$ realizes the chain complex L_*, where the notation L_* is to be interpreted as $BL_{n+1} \to L_n \to \Omega L_{n-1}$ at the nth level. The differential of the chain complex L_* gives rise to the only nonzero differential in the homotopy spectral sequence of $\{E_n\}$, since the E_1-term is $\operatorname{Hom}_{\underline{\mathrm{U}}}(L_n, \Sigma^*\mathbb{F}_2)$ at level n, and $E_2^{n,t} \cong E_\infty^{n,t}$ (Corollary 4.5).

In this section, we describe how the complex L_* gives a model for the unstable Adams spectral sequences of SO and U, we make some predictions based on the model, and we discuss some related work of Bousfield and Davis [B-D].

5.1. *A model for the UASS.* The conjecture suggested by Mahowald is, loosely, that the differential of the chain complex L_* contains all the information on the unstable Adams spectral sequence, including all of its many nonzero differentials. We already know that $H^*[\mathrm{Hom}_{\underline{\mathbf{U}}}(L_*, \Sigma^*\mathbb{F}_2)]$ is the associated graded for the filtration of $\pi_* SO_2^\wedge$ by the destabilized Adams tower (Corollaries 4.4 and 4.5). The assertion is that it is possible to produce the UASS from the complex $\mathrm{Hom}_{\underline{\mathbf{U}}}(L_*, \Sigma^*\mathbb{F}_2)$ by a combination of filtering and regrading.

To describe the proposed model, let $\mathcal{L}^*$ be the cochain complex of graded vector spaces defined by

$$(\mathcal{L}^n)_j = \mathrm{Hom}_{\underline{\mathbf{U}}}(L_n, \Sigma^j\mathbb{F}_2),$$

and use the differential $BL_{n+1} \to L_n$ and adjointness to define $d : (\mathcal{L}^n)_j \to (\mathcal{L}^{n+1})_{j-1}$ by

$$\begin{aligned} \mathrm{Hom}_{\underline{\mathbf{U}}}(L_n, \Sigma^j\mathbb{F}_2) \to\ & \mathrm{Hom}_{\underline{\mathbf{U}}}(BL_{n+1}, \Sigma^j\mathbb{F}_2) \\ \cong\ & \mathrm{Hom}_{\underline{\mathbf{U}}}(\Omega BL_{n+1}, \Sigma^{j-1}\mathbb{F}_2) \cong \mathrm{Hom}_{\underline{\mathbf{U}}}(L_{n+1}, \Sigma^{j-1}\mathbb{F}_2). \end{aligned}$$

We filter $\mathcal{L}^n$ by

$$(F^s\mathcal{L}^n)_j = \mathrm{Hom}_{\underline{\mathbf{U}}}\left(\bigoplus_{i=s}^{n} C_i^{n-i}, \Sigma^j\mathbb{F}_2\right).$$

We have $F^0 \supseteq F^1 \supseteq F^2 \ldots$, and comparing to the construction of $BL_{n+1} \to L_n$ in Section 3, it is easy to check that the differential on $\mathcal{L}^*$ is filtration-preserving. Thus the filtration gives rise to a spectral sequence that converges to $H^*\mathcal{L}^*$, and we grade it as

$$E_1^{s,t} = \bigoplus_n \mathrm{Hom}_{\underline{\mathbf{U}}}(C_s^n, \Sigma^{t-s}\mathbb{F}_2).$$

Recall that the abutment, $H^*\mathcal{L}^*$, is the associated graded to $\pi_* SO_2^\wedge$. Also, $C_s^n = \Omega^s D_s^n$, and hence by the adjointness of Ω and Σ, we have

$$E_1^{s,t} = \bigoplus_n \mathrm{Hom}_{\underline{\mathbf{U}}}(D_s^n, \Sigma^t\mathbb{F}_2).$$

The d_1-differential is induced by the differential in the resolution $D_*^n \to M_n/M_{n-1}$, and thus the spectral sequence becomes

$$E_2^{s,t} = \bigoplus_n \mathrm{Ext}^s_{\underline{\mathbf{U}}}(M_n/M_{n-1}, \Sigma^t\mathbb{F}_2) \Rightarrow \pi_* SO_2^\wedge.$$

CONJECTURE 5.1. *The spectral sequence $E_r^{s,t}$ defined above is the UASS for SO.*

If Conjecture 5.1 is correct, then it has the consequence that all of the differentials in the unstable Adams spectral sequence can be computed from the primary level calculation of the complex L_*. In principle, this could be done indefinitely far out by computer.

COROLLARY TO CONJECTURE 5.1.

$$\mathrm{Ext}^{s,t}_{\underline{\mathbf{U}}}(M_\infty, \mathbb{F}_2) \cong \bigoplus_n \mathrm{Ext}^{s,t}_{\underline{\mathbf{U}}}(M_n/M_{n-1}, \mathbb{F}_2).$$

Proof. The left side is the E_2-term of the UASS, while the right side is the E_2-term of the model. If Conjecture 5.1 is correct, these two must be isomorphic. ∎

In fact, there is a general spectral sequence that is very close to the spectral sequence of Conjecture 5.1, namely the Grothendieck spectral sequence for the calculation of the derived functors $\mathrm{Ext}^s_A(\Sigma A/\mathrm{Sq}^3, \Sigma^t\mathbb{F}_2)$. Let D be the destabilization functor from the category of (stable) A-modules to $\underline{\mathbf{U}}$, the category of unstable A-modules. (This functor is often denoted by Ω^∞.) Because $\Sigma^t\mathbb{F}_2$ is an unstable A-module, any map to $\Sigma^t\mathbb{F}_2$ from a stable A-module factors through the destabilization. Hence the functor $\mathrm{Hom}_A(-, \Sigma^t\mathbb{F}_2)$ can be written as the composition $\mathrm{Hom}_{\underline{\mathbf{U}}}(-, \Sigma^t\mathbb{F}_2) \circ D(-)$, giving rise to a composite functor spectral sequence

$$\mathrm{Ext}^{s-r}_{\underline{\mathbf{U}}}(D_r-, \Sigma^t\mathbb{F}_2) \Rightarrow \mathrm{Ext}^s_A(-, \Sigma^t\mathbb{F}_2).$$

In the case of $\Sigma A/\mathrm{Sq}^3$, $\mathrm{Ext}^s_A(\Sigma A/\mathrm{Sq}^3, \Sigma^t\mathbb{F}_2)$ actually gives the associated graded to the stable homotopy, because there are no differentials in the stable Adams spectral sequence for infinite delooping of SO. Thus the Grothendieck spectral sequence gives a spectral sequence starting from an unstable Ext and converging to π_*SO.

The Grothendieck spectral sequence is very closely related to the spectral sequence we have constructed, but it is not quite the same. In particular, let $X = \Sigma A/\mathrm{Sq}^3$, so that we are considering the case of SO. Then it can be shown that $M_{n+1}/M_n \cong D_n\Sigma^{-n}X$, the ingredients being found in Lemma 2.5, Lemma 2.1, and the proof of Proposition 4.3. Our construction gives a spectral sequence

$$\mathrm{Ext}^{s-r}_{\underline{\mathbf{U}}}(D_r\Sigma^{-r}X, \Sigma^t\mathbb{F}_2) \Rightarrow \mathrm{Ext}^s_A(X, \Sigma^t\mathbb{F}_2).$$

However, the situation for the group U is a little different, the difference being caused by the fact that while H^*SO is the free unstable A-algebra on $\overline{H}^*RP^\infty$, which is Sq^0-free, H^*U is the free unstable A-algebra on $\Sigma\overline{H}^*CP^\infty_+$, which is not. In fact, contrary to the assertion of [B-D, Proposition 4.1], if $X \cong \Sigma\overline{A}/\Lambda_1$, where Λ_1 is the subalgebra of A generated by the Milnor primitives Q_0 and Q_1, then $D_n\Sigma^{-n}X$ is not $M_{n+1}/M_n \oplus \Sigma\mathbb{Z}/2$ but a much larger module. The problem lies not in the spectral sequence constructed in the proof of the proposition, but in the assumption that the homology being converged to is M_{n+1}/M_n.

However, a small variation can repair the problem. Let X be an A-module, and let C_* be a stable resolution of X. For $n \geq 1$, define

$$D'_r X = \frac{\Omega \ker(D\Sigma C_r \to D\Sigma C_{r-1})}{\operatorname{im}(DC_{r+1} \to DC_r)}.$$

Using methods similar to those of Proposition 4.3, one can show that the definition of $D'_r X$ is independent of the resolution used, and that the modules $D'_r X$ and $D_r X$ are different exactly when $D_{r-1}\Sigma X$ is not Sq^0-free. If we let $X = \Sigma A/\mathrm{Sq}^3$ (in the case of SO) or $X = \Sigma \overline{A}/\mathrm{Sq}^3$ (in the case of U), then for both SO and U,

$$D'_n \Sigma^{-n} X \cong M_{n+1}/M_n,$$

where the modules M_n/M_{n-1} are the filtration quotients of $\overline{H}^* RP^\infty$ (in the case of SO) or $\Sigma \overline{H}^* CP^\infty_+$ (in the case of U). The construction of the previous section gives, for a general A-module X, two spectral sequences, depending on whether we use D'_r or D_r:

$$\mathrm{Ext}^{s-r}_{\underline{\mathrm{U}}}(D'_r \Sigma^{-r} X, \Sigma^t \mathbb{F}_2) \Rightarrow \mathrm{Ext}^s_A(X, \Sigma^t \mathbb{F}_2), \tag{5.1}$$

$$\mathrm{Ext}^{s-r}_{\underline{\mathrm{U}}}(D_r \Sigma^{-r} X, \Sigma^t \mathbb{F}_2) \Rightarrow \mathrm{Ext}^s_A(X, \Sigma^t \mathbb{F}_2). \tag{5.2}$$

(The spectral sequence of Conjecture 5.1 is (5.1).) These spectral sequences can be given a construction almost exactly like that of the Grothendieck spectral sequence. Conjecture 5.1 observes that because the stable Adams spectral sequences for SO and U collapse, the target of the spectral sequence in (5.1) is actually the associated graded to the homotopy of the space. Since the E_2-term is closely related to the homology of the space, because $D'_r \Sigma^{-r} X$ is the associated graded for the cohomology of SO (or U), this variation of the Grothedieck spectral sequence could actually be the unstable Adams spectral sequence.

5.2. *Predictions.* Next we discuss some predictions that arise from Conjecture 5.1 and some empirical data that support the conjecture. The main tool in making these predictions is a vanishing theorem of Bousfield [B, Theorem 2.6] that describes the location of h_0-towers in unstable Ext by giving values of $t - s$ where towers occur, though not the value of s in which they begin. Application of Bousfield's theorem gives us the following proposition. Recall that $\alpha(n)$ denotes the number of ones in the dyadic expansion of n.

PROPOSITION 5.2.

(1) *For* $M = \overline{H}^* RP^\infty$:

(a) *The* h_0*-towers of* $\mathrm{Ext}^s_{\underline{\mathrm{U}}}(M, \Sigma^t \mathbb{F}_2)$ *are found in stem degrees satisfying* $t - s \equiv 3 \bmod 4$, *and there is exactly one* h_0*-tower in each such dimension.*

(b) *The h_0-towers of* $\mathrm{Ext}^s_{\underline{\mathrm{U}}}(M_n/M_{n-1}, \Sigma^t\mathbb{F}_2)$ *are found in stem degrees satisfying* $t-s \equiv 3 \bmod 4$ *and* $\alpha(t-s) = n$*, and there is exactly one h_0-tower in each such dimension.*

(2) *For* $M = \overline{H}^*\Sigma CP^\infty_+$:

(a) *The h_0-towers of* $\mathrm{Ext}^s_{\underline{\mathrm{U}}}(M, \Sigma^t\mathbb{F}_2)$ *are found in stem degrees satisfying* $t-s \equiv 1 \bmod 2$*, and there is exactly one h_0-tower in each such dimension.*

(b) *The h_0-towers of* $\mathrm{Ext}^s_{\underline{\mathrm{U}}}(M_n/M_{n-1}, \Sigma^t\mathbb{F}_2)$ *are found in stem degrees satisfying* $t-s \equiv 1 \bmod 2$ *and* $\alpha(t-s) = n$*, and there is exactly one h_0-tower in each such dimension.*

Proof. An easy calculation with [B, Theorem 2.6]. ∎

Remark 5.3. Proposition 5.2 says that $\bigoplus_n \mathrm{Ext}^s_{\underline{\mathrm{U}}}(M_n/M_{n-1}, \Sigma^t\mathbb{F}_2)$ has the same h_0-towers as $\mathrm{Ext}^s_{\underline{\mathrm{U}}}(M, \Sigma^t\mathbb{F}_2)$, and so Corollary to Conjecture 5.1 is correct with regard to h_0-towers.

Bousfield's theorem also gives a vanishing line above which Ext is zero except for h_0-towers. To describe his theorem as it applies to our situation, we define a function $\phi(m)$ for positive integers m as follows. Suppose that $m = 8k+i$ where $i < 8$. Then:

(1) $\phi(m) = 4k+i$ for $i = 0,1,2,3$;
(2) $\phi(m) = 4k+3$ for $i = 4,5,6$;
(3) $\phi(m) = 4k+4$ for $i = 7$.

We specialize Bousfield's theorem to our situation as follows.

Theorem 5.4 ([B, Theorem 2.6]). *Let N be an unstable A-module such that $N_i = 0$ for $i < c$, where $c \geq 5$. Then* $\mathrm{Ext}^s_{\underline{\mathrm{U}}}(N, \Sigma^t\mathbb{F}_2)$ *is free over* $\mathbb{F}_2[h_0]$ *for* $s > \phi(t-s-c)$.

This gives a vanishing line of slope 1/2 in the UASS.

We are going to use Theorem 5.4 to predict the unstable Adams filtrations of the elements of π_*SO and π_*U. From the map of towers $\{Y_n\} \to \{E_n\}$, the maps $KP_{n+1} \to KL_{n+1}$ induce on homotopy a map

$$\mathrm{Ext}^n_A(\Sigma A/\mathrm{Sq}^3, \Sigma^t\mathbb{F}_2) \to \bigoplus_{r=1}^{n} \mathrm{Ext}^{n-r+1}_{\underline{\mathrm{U}}}(M_r/M_{r-1}, \Sigma^{t-r+1}\mathbb{F}_2), \tag{5.3}$$

and this map commutes with the action of h_0. All of the elements on the left represent homotopy, and since the right-hand side is the E_2-term for the spectral sequence of Conjecture 5.1, the map tells us where the homotopy is represented in this spectral sequence, which predicts the unstable Adams filtration of π_*SO.

Consider first the case of SO. Suppose $k \equiv 3 \bmod 4$; if $k \equiv 3 \bmod 8$, define $n = (k-1)/2$, and if $k \equiv 7 \bmod 8$, define $n = (k-3)/2$. Then

$\pi_k SO \cong \mathbb{Z}$, represented by an h_0-tower in $\mathrm{Ext}_A^*(\Sigma A/\mathrm{Sq}^3, \Sigma^{*+k}\mathbb{F}_2)$ beginning in filtration $s = n$. On the right side of (5.3), the only term with an h_0-tower in dimension k is $r = \alpha(k)$ (Proposition 5.2), and so the part of (5.3) that carries the bottom element of the h_0-tower is

$$\mathrm{Ext}_A^n(\Sigma A/\mathrm{Sq}^3, \Sigma^t\mathbb{F}_2) \to \mathrm{Ext}_{\underline{\mathbf{U}}}^{n-\alpha(k)+1}(M_{\alpha(k)}/M_{\alpha(k)-1}, \Sigma^{t-\alpha(k)+1}\mathbb{F}_2).$$

Thus we obtain the following prediction.

Conjecture 5.5. *The unstable Adams filtrations of the nonzero, torsion free groups $\pi_k SO$ are $\alpha(k) - 1$ less than the stable Adams filtrations of the corresponding stems.*

When we consider the form of k mod 8 and the known stable filtrations, this conjecture predicts that $\pi_{8i+3}SO$ and $\pi_{8i+7}SO$ occur in unstable Adams filtration $4i - \alpha(i)$.

By exactly the same reasoning as above, we obtain a prediction for the case of U, where all the homotopy is torsion free.

Conjecture 5.6. *The unstable Adams filtrations of the nonzero groups $\pi_k U$ are $\alpha(k) - 1$ less than the stable Adams filtrations of the corresponding stems.*

In this case, comparing with the stable filtration gives us the prediction that $\pi_{2i+1}U$ has unstable Adams filtration $i - \alpha(i)$.

Next, we predict the unstable Adams filtration of the torsion elements of $\pi_* SO$, namely $\pi_k SO \cong \mathbb{Z}/2$ for $k \equiv 0$ or 1 mod 8. Consider first the case $k \equiv 0$ mod 8, and let $n = (1/2)k - 1$. Then $\pi_k SO$ is represented in $\mathrm{Ext}_A^n(\Sigma A/\mathrm{Sq}^3, \Sigma^{n+k}\mathbb{F}_2)$. As before, we predict the unstable Adams filtration by considering the image of this element under the map of (5.3):

$$\mathrm{Ext}_A^n(\Sigma A/\mathrm{Sq}^3, \Sigma^{n+k}\mathbb{F}_2) \to \bigoplus_{r=1}^{n} \mathrm{Ext}_{\underline{\mathbf{U}}}^{n-r+1}(M_r/M_{r-1}, \Sigma^{n+k-r+1}\mathbb{F}_2).$$

Using Theorem 5.4, we will prove that only the $r = 3$ summand has h_0-torsion elements in high enough filtration to be in the image of this map. We already know that M_1 has exactly one torsion element in Ext for $s = 0$ and nothing else, and M_2/M_1 has exactly one h_0-tower in Ext for $k = 3$, and nothing else. Suppose that $r \geq 4$, and note that M_r/M_{r-1} begins in dimension $2^r - 1$. To use Theorem 5.4 to rule out h_0-torsion elements in $\mathrm{Ext}_{\underline{\mathbf{U}}}^{n-r+1}(M_r/M_{r-1}, \Sigma^{n+k-r+1}\mathbb{F}_2)$, we must show that

$$n - r - 1 > \phi[(n + k - r - 1) - (n - r - 1) - (2^r - 1)],$$

a task that is easily accomplished using $k \equiv 0$ mod 8 and $n = (1/2)k - 1$. An almost identical calculation leads to the same conclusion if $k \equiv 1$ mod 8. This leaves the $r = 3$ summand as the only one where the torsion elements can go, and since $r = 3$ causes a filtration drop of 2 from the stable Ext, we arrive at the following prediction.

CONJECTURE 5.7. *If $k > 1$ and $\pi_k SO \cong \mathbb{Z}/2$ is represented in filtration n in the stable Adams spectral sequence, then it has filtration $n - 2$ in the unstable Adams spectral sequence.*

Thus the prediction is that for $i > 0$, $\pi_{8i} SO$ has unstable Adams filtration $4i - 3$ and $\pi_{8i+1} SO$ has unstable Adams filtration $4i - 2$.

REMARK 5.8. The author has verified the preceding conjectures as to filtration for $\pi_* SO$ up to approximately π_{50}, using charts of unstable Ext provided by R. Bruner's computer calculations. Likewise the author has verified the Corollary to Conjecture 5.1 for SO in the same range.

We close this discussion by giving an example of the calculation of a differential in the spectral sequence modeling the UASS for SO. In Figure 3, we exhibit part of the UASS for SO. We will show how to use the spectral sequence of Conjecture 5.1 to predict the first differential in the UASS for SO, which goes from $(s, t-s) = (0, 15)$ to $(s, t-s) = (2, 14)$. (This differential propagates to give differentials connecting the two lightning flashes, but we will deal only with the first differential.)

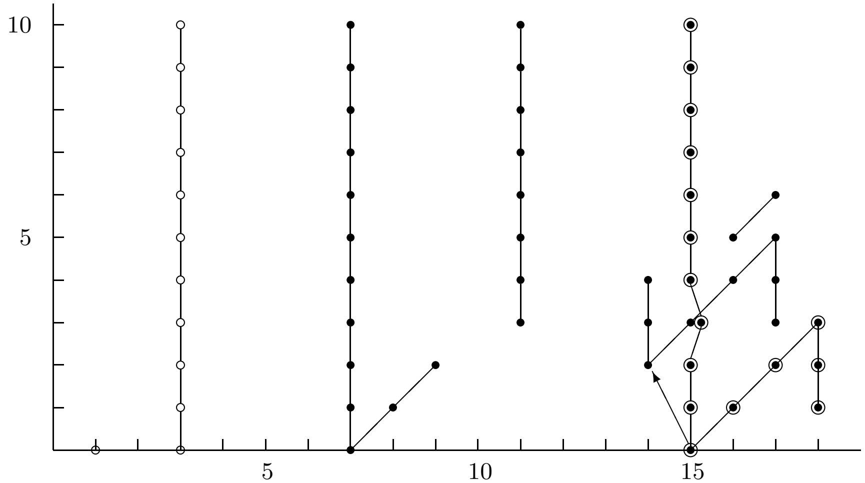

Fig. 3. The E_2-term of the UASS for SO. Elements represented by open circles arise from M_1 and M_2/M_1. Elements represented by black dots arise from M_3/M_2. Elements represented by circled dots arise from M_4/M_3.

In order to do this, we will have to calculate the first few stages of the complex L_*. In particular, we will be looking at the commuting diagram of three-term sequences

$$
\begin{tikzcd}
BL_5 \arrow[r] \arrow[d, "h_5"'] & L_4 \arrow[r] \arrow[d, "\Omega h_4"'] & \Omega L_3 \arrow[d, "\Omega^2 h_3"'] \\
BP_5 \arrow[r] & P_4 \arrow[r] & \Omega P_3
\end{tikzcd}
\tag{5.4}
$$

which is detailed in Table 1. We will need the result that $M_n/M_{n-1} \cong F(2^n - 1)/\mathrm{Sq}^1, \mathrm{Sq}^2, \dots, \mathrm{Sq}^{2^{n-2}}$ [Massey], and we remind the reader that in diagram (5.4), the top row involves resolutions of M_n/M_{n-1} for $n = 1, \dots, 5$, where the resolution of M_n/M_{n-1} is looped down $4-n$ times. When $n = 1$, $M_1 \cong F(1)$ is a projective, and has a resolution of length 1. Hence $C_i^1 = 0$ for $i > 0$. Further, $M_2/M_1 \cong \overline{F}(3)$, which is almost projective. Its projective resolution is $\ldots \to F(5) \to F(4) \to F(3)$ (each map given by Sq^1), and so all the elements contributed lie in $t-s = 3$. It turns out that this resolution does not interact with any of the other parts of L_*, corresponding to the fact that no differentials in the UASS for SO involve $t-s = 3$.

Table 1. The chain complexes of Section 3

	BL_5	$\longrightarrow$	L_4	$\longrightarrow$	ΩL_3
C_*^2 :	$F(4)$	$\mathrm{Sq}^1\,\iota_3$	$F(3)$	$\mathrm{Sq}^1\,\iota_2$	$F(2)$
C_*^3 :	$F(8)$	$\mathrm{Sq}^1\,\iota_7$	$F(7)$	$\mathrm{Sq}^1\,\iota_6$	$F(6)$
	$F(10)$	$\mathrm{Sq}^2\,\iota_8 + \mathrm{Sq}^3\,\iota_7$	$F(8)$	$\mathrm{Sq}^2\,\iota_6$	
	$F(15)$	$\mathrm{Sq}^7\,\iota_8 + \mathrm{Sq}^{4,2,1}\,\iota_8 + \mathrm{Sq}^{6,2}\,\iota_7 + \boxed{\iota_{15}}$			
C_*^4 :	$F(16)$	$\mathrm{Sq}^1\,\iota_{15}$	$F(15)$		
	$F(17)$	$\mathrm{Sq}^2\,\iota_{15}$			
	$F(19)$	$\mathrm{Sq}^4\,\iota_{15}$			
C_*^5 :	$F(32)$				

BP_5	$\longrightarrow$	P_4	$\longrightarrow$	ΩP_3
$F(4)$	$\mathrm{Sq}^1\,\iota_3$	$F(3)$	$\mathrm{Sq}^1\,\iota_2$	$F(2)$
$F(8)$	$\mathrm{Sq}^1\,\iota_7$	$F(7)$	$\mathrm{Sq}^1\,\iota_6$	$F(6)$
$F(10)$	$\mathrm{Sq}^2\,\iota_8 + \mathrm{Sq}^3\,\iota_7$	$F(8)$	$\mathrm{Sq}^2\,\iota_6$	

In Table 1, we provide all the summands of each of the terms in (5.4) and show the horizontal maps between them. In the commuting square

$$
\begin{tikzcd}
L_4 \arrow[r] \arrow[d, "\Omega h_4"'] & \Omega L_3 \arrow[d, "\Omega^2 h_3"'] \\
P_4 \arrow[r] & \Omega P_3
\end{tikzcd}
$$

$\Omega^2 h_3$ is the identity, and Ωh_4 is the identity map on the summands $F(3)$, $F(7)$, and $F(8)$. To describe Ωh_4 on the summand $F(15)$ of L_4, we recall that $\iota_{15} \in L_4$ must hit an element of P_4 that represents an A-module generator of the homology of the three-term sequence $BP_5 \to P_4 \to \Omega P_3$, and the element in question is $\mathrm{Sq}^7\,\iota_8 + \mathrm{Sq}^{4,2,1}\,\iota_8 + \mathrm{Sq}^{6,2}\,\iota_7 \in P_4$.

Now for the differential. It is predicted by the construction of the map $BL_5 \to L_4$, and it comes about because $BL_5 \to L_4$ must be defined in such a way that the composite $BL_5 \to L_4 \to P_4$ lifts across $BP_5 \to P_4$. Since there are no interactions between the filtrations in the map $L_4 \to \Omega L_3$, the map $BL_5 \to L_4$ can be constructed simply by using the differentials within the resolutions C^n_*, and then making adjustments as needed to ensure the required lifting. In terms of the construction of Section 3, this is saying that the map g_5 is just the sum of the differentials in the individual resolutions.

No corrections need to be made until we reach $F(15) \subseteq BL_5$. At this point, if no adjustments were made, the composite $BL_5 \to L_4 \to P_4$ would take the generator $\iota_{15} \in BL_5$ to $\mathrm{Sq}^7\,\iota_8 + \mathrm{Sq}^{4,2,1}\,\iota_8 + \mathrm{Sq}^{6,2}\,\iota_7 \in P_4$. Since this element generates the homology at P_4, it certainly does not lift to BP_5. Thus we add $\iota_{15} \in L_4$ to the image of $\iota_{15} \in BL_5$ (boxed for emphasis in the table). This gives a differential between adjoining filtrations in $\mathcal{L}^*$, which translates to the prediction of the nonzero d_2 differential taking $(s, t-s) = (0, 15)$ to $(s, t-s) = (2, 14)$ in the UASS of SO.

5.3. *Relation to* [B-D]. Bousfield and Davis make in [B-D] a much more general conjecture than our Conjecture 5.1. Suppose given a diagram of unstable A-modules

$$\begin{tikzcd}
F_1 \arrow[d, "f_0"] & F_2 \arrow[d, "f_1"] & F_3 \arrow[d, "f_2"] & & \\
X_0 \arrow[r, "p_0"] & X_1 \arrow[r, "p_1"] \arrow[d, "i_1"] & X_2 \arrow[r, "p_2"] \arrow[d, "i_2"] & \cdots \arrow[r] & X \\
 & \Omega F_1 & \Omega F_2 & &
\end{tikzcd}$$

satisfying the following conditions:

(1) $F_n \to X_{n-1} \to X_n \to \Omega F_n \to \Omega X_{n-1}$ is exact.

(2) F_n is a direct sum of $F(m)$'s and/or $F'(m)$'s (where $F(m)$ is a free unstable A-module on a generator of dimension m and $F'(m) = F(m)/\mathrm{Sq}^1$).

(3) $(i_n f_n)^* : \mathrm{Ext}^s_{\underline{\mathrm{U}}}(\Omega F_n, \Sigma^t \mathbb{F}_2) \to \mathrm{Ext}^s_{\underline{\mathrm{U}}}(F_{n+1}, \Sigma^t \mathbb{F}_2)$ is the zero map.

(4) $\ker(X_n \to X) = \ker(X_n \to X_{n+1})$.

(5) $X \cong \varinjlim_n (X_n)$.

Let $M_n = \mathrm{im}(X_n \to X)$.

CONJECTURE 5.9 ([B-D, Conjecture 5.1]).

$$\mathrm{Ext}^s_{\underline{\mathrm{U}}}(X, \Sigma^t \mathbb{F}_2) \cong \bigoplus_n \mathrm{Ext}^s_{\underline{\mathrm{U}}}(M_n/M_{n-1}, \Sigma^t \mathbb{F}_2).$$

However, this conjecture is false, as shown by the counterexample that follows. Consider the following tower, whose k-invariants are described below:

$$\begin{tikzcd}
K(\mathbb{Z}/2,10) \arrow[r, "i_4"] & Y_4 \arrow[d] & \\
K(\mathbb{Z}/2,8) \arrow[r, "i_3"] & Y_3 \arrow[r, "k_3"] \arrow[d] & K(\mathbb{Z}/2,10) \\
K(\mathbb{Z}/2,8) \arrow[r, "i_2"] & Y_2 \arrow[r, "k_2"] \arrow[d] & K(\mathbb{Z}/2,9) \\
K(\mathbb{Z},7) \arrow[r, "i_1"] & Y_1 \arrow[r, "k_1"] \arrow[d] & K(\mathbb{Z}/2,9) \\
& * \arrow[r] & K(\mathbb{Z},8)
\end{tikzcd}$$

Let $H^*Y_i = U(Z_i)$. The first k-invariant is $k_1 = \mathrm{Sq}^2 \iota_7$ and the second is $k_2 = 0$. For the third, let x_{10} be a class in Z_2 with $(i_2)^*(x_{10}) = \Omega \,\mathrm{Sq}^2 \iota_9 \in \Omega \ker(\mathrm{Sq}^2 : F(9) \to \bar{F}(7))$, and let x'_{10} denote its image in Z_3. Let x_8 be a class in Z_3 with $(i_3)^*(x_8) = \iota_8$, the fundamental class. Then the third k-invariant is defined by $k_3 = x'_{10} + \mathrm{Sq}^2 x_8$.

We consider Bousfield and Davis's conjecture for this situation, where the diagram is given by

$$\begin{tikzcd}
\bar{F}(8) \arrow[d] & F(9) \arrow[d, "\mathrm{Sq}^2 \iota_7"] & F(9) \arrow[d, "0"] & F(10) \arrow[d, "x'_{10}+\mathrm{Sq}^2 x_8"] & \\
0 \arrow[r] & \bar{F}(7) \arrow[r, "p_1"] \arrow[d, "i_1"] & Z_2 \arrow[r, "p_2"] \arrow[d, "i_2"] & Z_3 \arrow[r, "p_3"] \arrow[d] & Z_4 = X \arrow[d] \\
& \bar{F}(7) & F(8) & F(8) & F(9)
\end{tikzcd}$$

In particular, we consider Ext^0, so that we are really looking at A-module generators. We find that Ext^0 has nonzero groups only in the following dimensions:

(1) $\mathrm{Ext}^0_{\underline{\mathrm{U}}}(M_1, \Sigma^t \mathbb{F}_2) = \mathbb{Z}/2$ if $t = 7$.
(2) $\mathrm{Ext}^0_{\underline{\mathrm{U}}}(M_2/M_1, \Sigma^t \mathbb{F}_2) = \mathbb{Z}/2$ if $t = 10$ or 15.

(3) $\mathrm{Ext}^0_{\underline{\mathrm{U}}}(M_3/M_2, \Sigma^t\mathbb{F}_2) = \mathbb{Z}/2$ if $t = 8$.
(4) $\mathrm{Ext}^0_{\underline{\mathrm{U}}}(M_4/M_3, \Sigma^t\mathbb{F}_2) = \mathbb{Z}/2$ if $t = 12$ or 31.
(5) $\mathrm{Ext}^0_{\underline{\mathrm{U}}}(X, \Sigma^t\mathbb{F}_2) = \mathbb{Z}/2$ if $t = 7, 8, 12, 15$ and 31.

In particular, $\mathrm{Ext}^0_{\underline{\mathrm{U}}}(X, \Sigma^t\mathbb{F}_2)$ has no nonzero class for $t = 10$. In fact, $M_3/M_2 \cong F(8)/\mathrm{Sq}^2$, and in the spectral sequence for $\mathrm{Ext}^*_{\underline{\mathrm{U}}}(X, \Sigma^t\mathbb{F}_2)$ arising from the filtration of X, there is a nonzero differential

$$\mathrm{Ext}^0_{\underline{\mathrm{U}}}(M_2/M_1, \Sigma^{10}\mathbb{F}_2) \to \mathrm{Ext}^1_{\underline{\mathrm{U}}}(M_3/M_2, \Sigma^{10}\mathbb{F}_2).$$

In effect, what we have done in this example is to introduce a generator in M_2 (namely x_{10}, corresponding to $\mathrm{Sq}^2\,\iota_8$) and then to equate it with a Steenrod operation on another class at a later stage, thus eliminating it from the list of generators.

However, it is possible to revise Conjecture 5.9 to deal with this problem. The salient feature that distinguishes the situation for SO and U from the example above is that there is a stable resolution in the background. In other words, in the case of the tower $\{Y_n\}$ defined in Section 2, the tower realizes a destabilized resolution of $\Sigma A/\mathrm{Sq}^3$ or $\Sigma\bar{A}/\mathrm{Sq}^3$, whereas in the counterexample above, the tower realizes the unstable complex

$$\bar{F}(7) \xleftarrow{\mathrm{Sq}^2} F(9) \xleftarrow{0} F(10) \xleftarrow{\mathrm{Sq}^2} F(12),$$

which is certainly not the destabilization of a resolution. To reflect this, we refine Bousfield and Davis's conjecture as follows.

CONJECTURE 5.10. *Conjecture 5.9 is true if we add the hypothesis that there exist A-modules $\bar{F}_n$ and maps $\bar{d}_n : \bar{F}_{n+1} \to \bar{F}_n$ satisfying the following conditions:*

(1) *$\bar{F}_n$ is the sum of copies of A and A/Sq^1, and $\Omega^n D\bar{F}_n \cong F_n$.*
(2) *$\Omega^n D(\bar{d}_n) = i_n \circ f_n$.*
(3) *$(\bar{F}_*, \bar{d}_*)$ is a chain complex whose only nonzero homology group occurs in the lowest homological dimension.*

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