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For Coxeter groups $z^{|g|}$ is a coefficient of a uniformly bounded representation

by

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Abstract. We prove the theorem in the title by constructing an action of a Coxeter group on a product of trees.

Introduction. Recall [1] that a *Coxeter system* (Γ, W) is a group Γ with a distinguished set of generators $w_i \in W$ and relations $w_i^2 = 1 = (w_i w_j)^{m_{ij}}$, where m_{ij} is zero (and then there is no relation between w_i and w_j) or an integer ≥ 2. The Coxeter system gives rise to a function $|g|$ on Γ, defined as the minimal length of the word in w_i's representing g. We call it the *length function.* Abusing language we will refer to Coxeter systems as *Coxeter groups.*

The purpose of this paper is to prove the following:

THEOREM 1. *Let* (Γ, W) *be a finitely generated Coxeter group*, *and* $|g|$ *the* (*word*) *length function on* Γ. *Then for any complex number* z *such that* $|z| < 1$, *the function* $z^{|g|}$ *is a coefficient of a uniformly bounded representation* λ_z.

Actually we provide a slightly larger family of representations.

Theorem 1 is proved by studying cocycles and relating the natural space on which Γ acts to actions on trees, following [5, 7, 3]. The representations λ_z are deformations of the left regular representation. They depend holomorphically on z and are unitary for real z. In general they are not unitarizable: this happens if and only if the Coxeter group is a product of finite and affine Coxeter groups, that is, iff it is amenable. In the case when the natural complex on which Γ acts is essentially a tree, i.e. if Γ is virtually free, the representation λ_z is a compact perturbation of the regular representation. This is not true for higher dimensional Coxeter groups (and

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one already sees that in the right angled case), but definitely the operators $(\lambda_z - \lambda)(g)$ have small support (compared to $l^2(\Gamma)$).

THEOREM 2. *Finitely generated Coxeter groups are weakly amenable.*

This is the extension of a theorem of Alain Valette from right angled to all Coxeter groups. Since the arguments in Valette's paper can now be repeated verbatim with the help of Theorem 1, we refer the reader to [8] and skip the argument.

Acknowledgements. I am grateful to Michael Davis and Ryszard Szwarc for useful discussions. The results in this paper were obtained (but not published) independently, using a slightly different method, by Frédéric Haglund, Frédéric Paulin and Alain Valette. This paper was written after Gero Fendler's talk, where he announced that $z^{|g|}$ is a coefficient for Coxeter groups of large type. I then noticed that my old argument from [3] can be extended to general Coxeter groups.

The complex. The construction below, due to Michael Davis, is described fully in [2]. For any Coxeter group (Γ, W) there is a cell complex $C(\Gamma)$ on which Γ acts properly. It is defined as follows: cells are indexed by right cosets Γ/Γ_S, where Γ_S is a *finite* group generated by a subset S of W. A cell $[\gamma]$ is a face of $[\eta]$ if $[\gamma] \subset [\eta]$ as cosets. Its vertices correspond to elements of Γ, edges are indexed by cosets $\Gamma/\{1, s\}$, with s running over the set of generators, etc. It might be called the Cayley complex of Γ, since its 1-skeleton is the Cayley graph of (Γ, W).

The obvious action of Γ on $C(\Gamma)$ coming from the left action of Γ on itself is a reflection group action. Any reflection (i.e. an element conjugate to a generator in W) has its *mirror* of fixpoints. Any mirror is two-sided, i.e. its complement has two components. The closures of connected components of the complement of the union of all mirrors are called *chambers.*

The word length function from the introduction has a pleasant interpretation in terms of $C(\Gamma)$. It counts the number of mirrors separating the chamber F from the chamber gF, or the shortest path in the one-skeleton of $C(\Gamma)$ between vertex e and vertex g.

There are more delicate length functions $|g|_i$ which count how many times a generator from the conjugacy class of w_i occurs in the shortest word expressing g in terms of generators from W, or, equivalently, how many mirrors conjugate to the mirror of w_i lie between F and gF. It is clear that $|g| = \sum_{i=1}^n |g|_i$ where n is the number of conjugacy classes of generators.

Trees. Even though Γ often does not admit a nontrivial action on a tree, its normal torsion free subgroups do. It turns out that Γ acts on a product of *finitely many* trees.

LEMMA 1 ([4]). *Let Γ_0 be a normal torsion free subgroup of Γ. Then for each mirror H and every $\gamma_0 \in \Gamma_0$, $H \cap \gamma_0(H)$ is either H or the empty set.*

Observe that such a Γ_0, even with the additional property of being of finite index in Γ, exists by the Selberg Lemma. The proof of the lemma is just a few lines, so I repeat it.

Proof. Let h be the reflection in H, and consider the product of reflections $g = h\gamma_0 h\gamma_0^{-1}$. If $H \cap \gamma_0(H)$ is nonempty, g is a torsion element. On the other hand since Γ_0 is normal, g is in Γ_0, hence it is the identity. Thus h commutes with γ_0, and $H = \gamma_0(H)$. ■

We use Lemma 1 to construct a tree with a Γ_0-action. Let $\mathcal{H}$ be the set of orbits for the Γ_0-action on the set of all mirrors. Fix an $h \in \mathcal{H}$. Define a graph T_h as follows. Its vertices are the connected components of $C(\Gamma) - \bigcup_s \Gamma_0(s)$ where s runs through the mirrors in h. Two vertices are joined by an edge if the components are adjacent in $C(\Gamma)$, i.e. if their closures intersect. Lemma 1 guarantees that different Γ_0-conjugates of h are disjoint.

LEMMA 2 ([3]). *T_h is a tree.*

Proof. Any loop λ in T_h lifts to a path in $C(\Gamma)$ which can be closed up to a loop Λ without crossing the h-mirror. The projection of Λ is again λ. Since $C(\Gamma)$ is contractible (cf. [2]), it follows that Λ, and hence λ, is homologous to zero and thus T_h is a tree. ■

There is an obvious simplicial map from $C(\Gamma)$ to T_h, given on vertices by $g \mapsto [g]_h$, that is, mapping a chamber to the connected component of $C(\Gamma) - \bigcup_s \Gamma_0(s)$ it belongs to. Clearly this map is Γ_0-equivariant. We take the diagonal of the family $\mu : C(\Gamma) \to \prod_h T_h$ to get a Γ_0-equivariant embedding of the Davis complex into the product of trees. The beauty of this map lies in

LEMMA 3. *The map μ is a Γ-equivariant embedding.*

Proof. First notice that Γ, in fact Γ/Γ_0, acts on $\mathcal{H}$. An element g maps the tree T_h to $T_{g(h)}$ simplicially. Thus Γ acts on $\prod_h T_h$ by permuting the factors of the product. Explicitly $g(x_{h_1}, \ldots, x_{h_n}) = (gx_{g^{-1}(h_1)}, \ldots, gx_{g^{-1}(h_n)})$. Now equivariance of μ is obvious. To see that it is an embedding, notice that two vertices of $C(\Gamma)$ differ iff they are separated by some mirror, say in h; thus their images in T_h are different. ■

The map μ, being an equivariant embedding, induces an isometric equivariant embedding $\mu_* : l^2(\Gamma) \to \bigotimes_h l^2(\text{vertices}(T_h))$ of Hilbert spaces by $\mu_*(\delta_x) = \bigotimes_h \delta_{[x]_h}$. Note that we do not claim existence of maps $l^2(\Gamma) \to l^2(\text{vertices}(T_h))$.

Remark. An example of Γ_0 as above is the trivial group. Then the map given by Lemma 3 is an equivariant embedding into the (usually infinite) product of intervals. It is convenient to restrict to finite index subgroups to avoid the discussion of infinite products and tensor products. We will use this crucially in norm estimates. For finite groups the embedding is related to the action of Γ on roots.

Cocycles. Let G be a group acting on a space X, and $\pi : G \to \mathrm{GL}(V)$ be its representation on some vector space V.

Definition. A *cocycle on* (X, G) *twisted by* π is a map $c : X \times X \to \mathrm{GL}(V)$ such that

1. $c(x, x) = \mathrm{id}$,
2. $c(x, y)c(y, z) = c(x, z)$,
3. $c(gx, gy) = \pi(g)c(x, y)\pi(g)^{-1}$.

Property 2 is called the *chain rule*, property 3 *equivariance*. One should think of this definition as follows. The action of G on X and V gives rise to a “bundle” over X/G,

$$V \to X \times_G V \to X/G.$$

Here $X \times_G V = (X \times V)/G$ with the diagonal G-action on the product. A cocycle c is a parallel translation in this bundle lifted to the “covering” $X \to X/G$. The convention is that $c(x, y)$ maps the fiber over y to the fiber over x. Then we have the associated monodromy representation

$$\pi_c(g) = c(x_0, gx_0)\pi(g).$$

It is straightforward to check that it is indeed a representation of G on V. It is also called the *deformation* of π by c.

Cocycles have good functorial properties. An equivariant map $f : Y \to X$ of G-spaces pulls back a cocycle on X to one on Y by $f^*c(y_1, y_2) = c(f(y_1), f(y_2))$. Also a group homomorphism $H \to G$ turns a G-cocycle into an H-cocycle.

Crucial use will be made of the tensor product construction. Let $(X_i, G_i, V_i, \pi_i, c_i)_{i=1,\dots,n}$ be a finite family of cocycles. Then $\bigotimes_i c_i$ is a cocycle for the action of $\prod_i G_i$ on $\prod_i X_i$ twisted by the representation $\bigotimes_i \pi_i$ on $\bigotimes_i V_i$. It is explicitly given by

$$\bigotimes_i c_i\{(x_1, \dots, x_n), (y_1, \dots, y_n)\}(v_1 \otimes \dots \otimes v_n) \\ = c_1(x_1, y_1)(v_1) \otimes \dots \otimes c_1(x_n, y_n)(v_n).$$

In case some of the spaces and cocycles coincide, the tensor product cocycle has an additional symmetry. One adds to the group $\prod_i G_i$ permutations switching simultaneously factors of $\prod X_i$ and of $\bigotimes V_i$.

Cocycles on trees and their tensor products. On a tree T with a G-action we have the cocycle pv_z provided by the construction of Pimsner and Valette. In fact it is a cocycle for the full automorphism group of a tree. We will not discuss it here in detail, but rather refer to [5, 7]. Briefly, consider the space $l^2(\text{vertices}(T))$ with the obvious unitary representation π of G induced by the G-action on T. Fix complex numbers a, z with $a^2 + z^2 = 1$, and define a cocycle on adjacent vertices and on the basis of l^2 as follows (and then extend using the chain rule and linearity):

$$\mathrm{pv}_z(p,q)(\delta_r) = \begin{cases} \delta_r & \text{if } r \neq p, q, \\ a\delta_p - z\delta_q & \text{if } p = r, \\ z\delta_p + a\delta_q & \text{if } q = r. \end{cases}$$

It is a pleasant computation that pv_z is indeed a cocycle twisted by π.

Now for each $h \in \mathcal{H}$ we fix a complex number z_h and take the tensor product cocycle $\bigotimes \mathrm{pv}_{z_h}$. We want it to be twisted by the representation of Γ on $\bigotimes_h l^2(\text{vertices}(T_h))$. Actually for an arbitrary choice of z_h we will only get a cocycle twisted by the representation of Γ_0, but if we insist that the Γ/Γ_0-action should take the pv_{z_h} cocycle on T_h to $\mathrm{pv}_{z_{g(h)}}$ on $T_{g(h)}$, i.e. $z_h = z_{g(h)}$, we end up with the tensor product cocycle for Γ. This is related to the additional symmetries of tensor powers.

The orbits of Γ on $\mathcal{H}$ correspond to conjugacy classes of generators. Thus we should denote our tensor product cocycle by $\mathrm{pv}_{z_1,\ldots,z_n}$, with indices running over conjugacy classes of generators. But in computations that are coming up, we prefer to index the cocycle and representation with elements of $\mathcal{H}$, with the understanding that the parameters on conjugate mirrors are equal.

We adopt a similar convention for the monodromy representation denoted by $\pi_{z_1,\ldots,z_n}$. Specializing $z_1 = \ldots = z_n = z$ we get the cocycle pv_z and the monodromy representation π_z.

We are now interested in the matrix coefficients of $\pi_{z_1,\ldots,z_n}$. Recall that $\mu_*\delta_e$ is the image of the basis vector $\delta_e \in l^2(\Gamma)$ in $\bigotimes_h l^2(\text{vertices}(T_h))$.

Lemma 4.

$$\langle \pi_{z_1,\ldots,z_n}(g)(\mu_*(\delta_e)), \mu_*(\delta_e)\rangle = \prod_i z_i^{|g|_i}.$$

Specializing, we get

$$\langle \pi_z(g)(\mu_*(\delta_e)), \mu_*(\delta_e)\rangle = z^{|g|}.$$

To get more transparent formulas, we abuse the notation, and write g for $\mu(g)$ and δ_g for $\mu_*(\delta_g)$ in the proof of Lemma 4 below. One should remember that the computation takes place in the product of trees. The element g acts in the dual role of a group element and a vertex of $C(\Gamma)$.

Proof. We argue by induction (compare [7]). For $g = e$ there is nothing to prove. Then assume gs is an element of length $|g| + 1$. We have

$$\begin{aligned}
&\langle \pi_{z_1,\ldots,z_n}(gs)(\mu_*(\delta_e)), \mu_*(\delta_e)\rangle = \langle \mathrm{pv}_{z_1,\ldots,z_n}(e, gs)\pi(gs)(\delta_e), \delta_e\rangle \\
&\quad = \langle \mathrm{pv}_{z_1,\ldots,z_n}(e, gs)(\delta_{gs}), \delta_e\rangle = \langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\, \mathrm{pv}_{z_1,\ldots,z_n}(g, gs)(\delta_{gs}), \delta_e\rangle \\
&\quad = \langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\pi_{z_1,\ldots,z_n}(g)\, \mathrm{pv}_{z_1,\ldots,z_n}(e, s)(\delta_s), \delta_e\rangle \\
&\quad = \langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\pi_{z_1,\ldots,z_n}(g)(z_s\delta_e + a\delta_s), \delta_e\rangle \\
&\quad = \langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)(z\delta_g + a\delta_{gs}), \delta_e\rangle \\
&\quad = z_s\langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\delta_g, \delta_e\rangle + a_s\langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\delta_{gs}, \delta_e\rangle \\
&\quad = z_s\langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\pi(g)\delta_e, \delta_e\rangle \\
&\quad = z_s\langle \pi_{z_1,\ldots,z_n}(g)\delta_e, \delta_e\rangle = z_s\prod_i z^{|g|_i} = \prod_i z_i^{|gs|_i}.
\end{aligned}$$

The vanishing of the term $a_s\langle \mathrm{pv}_{z_1,\ldots,z_n}(e, g)\delta_{gs}, \delta_e\rangle$ comes from the locality of the cocycle: associate to the element

$$\mathrm{pv}_{z_1,\ldots,z_n}(e, g)\mu_*(\delta_{gs}) = \sum a^{v_1,\ldots,v_n}\delta_{v_1} \otimes \ldots, \otimes\delta_{v_n}$$

the subset in the product of trees, consisting of those $(v_1, \ldots, v_n)$ for which $a^{v_1,\ldots,v_n} \neq 0$ (its support). Each element in this subset is at distance at most $|g|$ from $\mu_*\delta_{gs}$, hence each is orthogonal to $\mu_*\delta_e$. ∎

Now we estimate norms. Here for the first time we crucially use the fact that Γ_0 is of finite index.

Lemma 5.

$$\|\pi_{z_1,\ldots,z_n}\| = \|\,\mathrm{pv}_{z_1,\ldots,z_n}\,\| = \|\bigotimes \mathrm{pv}_{z_h}\,\| \le \prod_{h\in\mathcal{H}} \frac{2|1 - z_h^2|}{1 - |z_h|}.$$

Proof. The last inequality is the estimate of the norms of cocycles coming from trees, adapted by A. Valette from [5]. The last term is a finite product. ∎

Lemma 5 finishes the proof of a slightly stronger version of Theorem 1:

Theorem 1a. *Let (Γ, W) be a finitely generated Coxeter group, and $|g|_i$ the ith (word) length function on Γ counting the occurrences of the conjugates of the ith generator in the minimal presentation of g. Then for any complex numbers z_i such that $|z_i| < 1$, the function $\prod_i z^{|g|_i}$ is a coefficient of a uniformly bounded representation $\pi_{z_1,\ldots,z_n}$.*

Final remarks. 1. Denote by $\mathcal{P}$ the map which sends δ_g to

$$\pi_{z_1,\ldots,z_n}(g)(\delta_e) = \mathrm{pv}_{z_1,\ldots,z_n}(e, g)(\delta_g).$$

It is bounded, but not unitary; in fact its inverse is unbounded. Let Λ denote the space spanned by the image of $\mathcal{P}$. A straightforward computation shows that $\mathcal{P}$ intertwines the regular representation on $l^2(\Gamma)$ with the representation $\pi_{z_1,\dots,z_n}$ restricted to Λ. We denote the latter by $\lambda_{z_1,\dots,z_n}$. Clearly it is *the* representation one should be interested in. An additional appeal of $\lambda_{z_1,\dots,z_n}$ comes from

LEMMA 6. *The representation $\lambda_{z_1,\dots,z_n}$ does not depend on the choice of the subgroup Γ_0 used in its construction.*

Proof. We only have to prove the assertion for a subgroup Γ_1 of Γ_0. Consider $\mathcal{H}_0 = \mathrm{Mirrors}/\Gamma_0$ and $\mathcal{H}_1 = \mathrm{Mirrors}/\Gamma_1$ with the obvious map $p : \mathcal{H}_1 \to \mathcal{H}_0$.

We have a map $\nu : T_{h_0} \to \prod_{h_1 \in p^{-1}(h_0)} T_{h_1}$, obtained by taking the diagonal of the family of maps $T_{h_0} \to T_{h_1}$ (each of these maps acts on vertices by sending a connected component of the complement of the h_0-mirrors to the connected component of the complement of the h_1-mirrors it belongs to). The map between trees is only Γ_1-equivariant but the diagonal is Γ_0-equivariant (and $\prod_{\mathcal{H}_0} T_h \to \prod_{\mathcal{H}_1} T_{h_1}$ is Γ-equivariant). The map ν is an embedding and induces a map $\nu_* : l^2(\mathrm{vertices}(T_{h_0})) \to \bigotimes l^2(\mathrm{vertices}(T_{h_1}))$ on l^2 spaces.

The space $\prod_{h_1 \in p^{-1}(h_0)} T_{h_1}$ carries the tensor product cocycle $\bigotimes \mathrm{pv}_z(T_{h_1})$. It is straightforward to check that it preserves the image of ν_*.

If we restrict the group from $\mathrm{Aut}(\prod T_{h_1})$ to $\mathrm{Aut}(T_h)$ and pull back by μ, the resulting cocycle is $\mathrm{pv}_z(T_h)$.

We finish the proof by observing that the map $\mu_{\Gamma_1} : C(\Gamma) \to \prod_{\mathcal{H}_1} T_{h_1}$ is the composition of μ_{Γ_0} and ν. ∎

Specializing to $z_i = 0$ one sees that $\lambda_{z_1,\dots,z_n}$ is a deformation of the regular representation of G. In general the cocycle does not preserve the subspace $\mu_*(l^2(\Gamma))$, thus the space on which $\lambda_{z_1,\dots,z_n}$ acts varies with z_i inside $\bigotimes_h l^2(\mathrm{vertices}(T_h))$.

One can write down $\lambda_{z_1,\dots,z_n}$ on $l^2(\Gamma)$ projecting it orthogonally from Λ to $l^2(\Gamma)$. The projection is bounded and its inverse is bounded since it is the map $x \mapsto x \oplus \mathcal{P}^{\perp}(x)$, where $\mathcal{P}^{\perp}$ is the composition of $\mathcal{P}$ with the projection onto the complement of $l^2(\Gamma)$.

2. Example: the symmetric group S_3. Here we have three reflections, and embedding maps into the three-cube. Each reflection inverts the sign of its coordinate and flips the others. The embedding μ is

$$e \mapsto (- - -); \quad s \mapsto (+ - -); \quad t \mapsto (- + -);$$
$$st \mapsto (+ - +); \quad ts \mapsto (- + +); \quad sts = tst \mapsto (+ + +);$$

Note that $(- - +)$ and $(+ + -)$ are not in the image of μ. The vectors

$\mathrm{pv}_z(e,g)(\delta_g)$ are

$$e \mapsto (---); \quad s \mapsto z(---) + a(+--); \quad t \mapsto z(---) + a(-+-);$$
$$st \mapsto z^2(---) + az\{(+--) + (--+)\} + a^2(+-+);$$
$$ts \mapsto z^2(---) + az\{(-+-) + (--+)\} + a^2(-++);$$
$$tst = sts \mapsto z^3(---) + az^2\{(-+-) + (--+) + (+--)\}$$
$$+ a^2 z\{(-++) + (++-) + (+-+)\} + a^3(+++).$$

With the benefit of hindsight of this example one can guess, and then prove by induction, the form of the embedding in general: Let $\gamma(g)_h$ denote the geodesic in the tree T_h running from $(e)_h$ to $(g)_h$, $p_1 = (e)_h, \ldots, p_k = (g)_h$ its vertices and ${}_{z_h}\gamma(g)_h = \sum_{i=0}^{k} z_h^{k-i} a_h^i p_i$. Then

$$\pi_{z_1,\ldots,z_n}(g)(\delta_e) = \mathrm{pv}_{z_1,\ldots,z_n}(e,g)(\delta_g) = \bigotimes_h {}_{z_h}\gamma(g)_h.$$

3. The representations we obtain are sometimes unitarizable. This happens if and only if all the trees T_h are either intervals or real lines. That in turn happens if and only if Γ is a product of finite and affine Coxeter groups, that is, iff the group is amenable.

References

[1] N. Bourbaki, *Groupes et algèbres de Lie, chapitres III–IV*, Hermann, 1968.

[2] M. Davis, *Groups generated by reflections and aspherical manifolds not covered by Euclidean space*, Ann. of Math. 117 (1983), 293–325.

[3] T. Januszkiewicz, *For right angled Coxeter groups $z^{|g|}$ is a coefficient of a uniformly bounded representation*, Proc. Amer. Math. Soc. 119 (1993), 1115–1119.

[4] J. J. Millson, *On the first Betti number of a constant negatively curved manifold*, Ann. of Math. 104 (1976), 235–247.

[5] M. Pimsner, *Cocycles on trees*, J. Operator Theory 17 (1987), 121–128.

[6] T. Pytlik and R. Szwarc, *An analytic family of uniformly bounded representations of free groups*, Acta Math. 157 (1986), 287–309.

[7] A. Valette, *Cocycles d'arbres et représentations uniformément bornées*, C. R. Acad. Sci. Paris Sér. I 310 (1990), 703–708.

[8] —, *Weak amenability of right angled Coxeter groups*, Proc. Amer. Math. Soc. 119 (1993), 1331–1334.

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