On the non-extendibility of strongness and supercompactness through strong compactness

by

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Abstract. If $\kappa$ is either supercompact or strong and $\delta < \kappa$ is $\alpha$ strong or $\alpha$ supercompact for every $\alpha < \kappa$, then it is known $\delta$ must be (fully) strong or supercompact. We show this is not necessarily the case if $\kappa$ is strongly compact.

1. Introduction and preliminaries. A well-known result of Magidor [16] states that if $\kappa$ is supercompact and $\delta < \kappa$ is supercompact for all $\alpha < \kappa$, then $\delta$ is supercompact. Indeed, the following is true.

Lemma 1.1 (Folklore). If $\kappa$ is a strong cardinal and $\delta < \kappa$ is either $\alpha$ strong, $\alpha$ strongly compact, or $\alpha$ supercompact for every $\alpha < \kappa$, then $\delta$ must be (fully) strong, strongly compact, or supercompact.

Proof. Let $\lambda > \kappa$ be a cardinal so that $\lambda = \beth_{\lambda}$, and let $\gamma = \beth_{\gamma}(\lambda)$. Take $j : V \rightarrow M$ as an elementary embedding witnessing the $\gamma$ strongness of $\kappa$. Since $V \models \text{"}\delta \text{ is either } \alpha \text{ strong, } \alpha \text{ strongly compact, or } \alpha \text{ supercompact for every } \alpha < \kappa \text{"}$ and $\delta < \kappa$, $M \models \text{"}j(\delta) = \delta \text{ is either } \alpha \text{ strong, } \alpha \text{ strongly compact, or } \alpha \text{ supercompact for every } \alpha < j(\kappa) \text{"}$. In particular, because $j(\kappa) > \gamma > \lambda$, $M \models \text{"}\delta \text{ is either } \lambda \text{ strong, } \lambda \text{ strongly compact, or } \lambda \text{ supercompact"}$ as well. Since $\lambda$ may be chosen arbitrarily large, this proves Lemma 1.1. \(\blacksquare\)

We observe that Lemma 1.1 has a local version. Specifically, if $\kappa$ is measurable and $\delta < \kappa$ is either $\alpha$ strong, $\alpha$ strongly compact, or $\alpha$ supercompact

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for \( \alpha < \kappa \), then \( \delta \) is either \( \kappa + 1 \) strong, \( \kappa \) strongly compact, or \( \kappa \) supercompact. The proof is essentially the same as the one given above, with \( j \) replaced by an elementary embedding witnessing \( \kappa \)'s measurability, and the observation that the \( \kappa \) closure of \( M \) with respect to \( V \) is enough to ensure that \( \delta \) is either \( \kappa + 1 \) strong, \( \kappa \) strongly compact, or \( \kappa \) supercompact in \( V \).

Key to the proof of Lemma 1.1 is the fact that the inner model \( M \) contains a large chunk of the universe \( V \), something which will be true if \( \kappa \) is either supercompact or, more weakly, strong. It is not necessarily the case, however, that if \( \kappa \) is only strongly compact, then there is an elementary embedding witnessing any degree of strong compactness into an inner model \( M \) containing any more of \( V \) than \( V_{\kappa+1} \). Thus, we can ask the following question: If \( \kappa \) is a non-supercompact strongly compact cardinal and \( \delta < \kappa \) is either \( \alpha \) supercompact or \( \alpha \) strong for every \( \alpha < \kappa \), then must \( \delta \) be either (fully) supercompact or strong? Note that by a theorem of Di Prisco [7], the answer to the analogue of this question if \( \kappa \) is strongly compact for every \( \alpha < \kappa \) is yes.

The purpose of this paper is to show that the answer to the above question is no. Specifically, we prove the following two theorems.

**Theorem 1.** Suppose \( V \models \text{“} ZFC + \kappa_1 < \kappa_2 \text{ are supercompact} \text{”} \). There is then a partial ordering \( P \in V \) so that \( V^P \models \text{“} ZFC + \kappa_1 \text{ is strongly compact but not supercompact} + \kappa_1 \text{ is } \alpha \text{ supercompact for every } \alpha < \kappa_2 + \kappa_1 \text{ is not supercompact} \text{”} \).

**Theorem 2.** Suppose \( V \models \text{“} ZFC + \kappa \text{ is supercompact} \text{”} \). There is then a partial ordering \( P \in V \) and a strong cardinal \( \delta < \kappa \) so that \( V^P \models \text{“} ZFC + \kappa \text{ is strongly compact but not supercompact} + \delta \text{ is } \alpha \text{ strong for every } \alpha < \kappa + \delta \text{ is not strong} \text{”} \).

Before giving the proofs of Theorems 1 and 2, we briefly mention some preliminary information. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. For \( \alpha < \beta \) ordinals, \( [\alpha, \beta] \), \( (\alpha, \beta] \), \( (\alpha, \beta) \), and \( (\alpha, \beta] \) are as in standard interval notation.

When forcing, \( q \geq p \) will mean that \( q \) is stronger than \( p \). If \( G \) is \( V \)-generic over \( P \), we will use both \( V[G] \) and \( V^P \) to indicate the universe obtained by forcing with \( P \). If \( x \in V[G] \), then \( \dot{x} \) will be a term in \( V \) for \( x \). We may, from time to time, confuse terms with the sets they denote and write \( x \) when we actually mean \( \dot{x} \), especially when \( x \) is some variant of the generic set \( G \), or \( x \) is in the ground model \( V \).

If \( \kappa \) is a cardinal and \( P \) is a partial ordering, \( P \) is \( \kappa \)-directed closed if for every cardinal \( \delta < \kappa \) and every directed set \( \langle p_\alpha : \alpha < \delta \rangle \) of elements of \( P \) (where \( \langle p_\alpha : \alpha < \delta \rangle \) is directed if any two elements \( p_\theta \) and \( p_\nu \) have a common upper bound of the form \( p_\sigma \)) there is an upper bound \( p \in P \).
\(\mathbb{P}\) is \(\kappa\)-strategically closed if in the two-person game in which the players construct an increasing sequence \(\langle p_\alpha : \alpha \leq \kappa \rangle\), where player I plays odd stages and player II plays even and limit stages (choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued. Note that if \(\mathbb{P}\) is \(\kappa\)-strategically closed and \(f : \kappa \rightarrow V\) is a function in \(V^\mathbb{P}\), then \(f \in V\). \(\mathbb{P}\) is \(<\kappa\)-strategically closed if in the two-person game in which the players construct an increasing sequence \(\langle p_\alpha : \alpha < \kappa \rangle\), where player I plays odd stages and player II plays even and limit stages (again choosing the trivial condition at stage 0), player II has a strategy which ensures the game can always be continued.

Suppose \(\kappa < \lambda\) are regular cardinals. A partial ordering \(\mathbb{P}_{\kappa,\lambda}\) that will be used in this paper is the partial ordering for adding a non-reflecting stationary set of ordinals of cofinality \(\kappa\) to \(\lambda\). Specifically, \(\mathbb{P}_{\kappa,\lambda} = \{s : s\) is a bounded subset of \(\lambda\) consisting of ordinals of cofinality \(\kappa\) so that for every \(\alpha < \lambda\), \(s \cap \alpha\) is non-stationary in \(\alpha\}\}, ordered by end-extension. Two things which can be shown (see [5] or [2]) are that \(\mathbb{P}_{\kappa,\lambda}\) is \(\delta\)-strategically closed for every \(\delta < \lambda\), and if \(G\) is \(V\)-generic over \(\mathbb{P}_{\kappa,\lambda}\), in \(V[G]\), a non-reflecting stationary set \(S = S[G] = \bigcup \{S_p : p \in G\} \subseteq \lambda\) of ordinals of cofinality \(\kappa\) has been introduced. It is also virtually immediate that \(\mathbb{P}_{\kappa,\lambda}\) is \(\kappa\)-directed closed.

We mention that we are assuming familiarity with the large cardinal notions of measurability, strongness, strong compactness, and supercompactness. Interested readers may consult [12] for further details. Also, unlike [12], we will say that the cardinal \(\kappa\) is \(\lambda\) strong for \(\lambda > \kappa\) if there is \(j : V \rightarrow M\) an elementary embedding having critical point \(\kappa\) so that \(j(\kappa) > |V_\lambda|\) and \(V_\lambda \subseteq M\). As always, \(\kappa\) is strong if \(\kappa\) is \(\lambda\) strong for every \(\lambda > \kappa\).

### 2. The proof of Theorem 1

Let \(V \models \text{"ZFC + } \kappa_1 < \kappa_2 \text{ are supercompact"}\). Without loss of generality, by first using an iteration of Laver’s partial ordering of [13] (such as the one given in [1]) to force \(\kappa_i\) for \(i = 1, 2\) to have its supercompactness indestructible under \(\kappa_i\)-directed closed forcing, then employing an Easton support iteration to add to every measurable cardinal \(\delta > \kappa_2\) a non-reflecting stationary set of ordinals of cofinality \(\kappa_2\), and then forcing with a \(\kappa_1\)-directed closed partial ordering to ensure GCH holds at and above \(\kappa_1\), we may also assume that \(V \models \text{"No cardinal } \lambda > \kappa_2 \text{ is measurable + } \kappa_1\text{’s supercompactness is indestructible under } \kappa_1\text{-directed closed forcing + } 2^\delta = \delta^+ \text{ for every cardinal } \delta \geq \kappa_1\"\). The fact that no cardinal above \(\kappa_2\) is measurable in \(V\) follows from the Gap Forcing Theorem of [10] and [11].

Take now \(\mathbb{P}_0\) as the Easton support iteration of length \(\kappa_2\) which adds, to every measurable cardinal \(\delta \in (\kappa_1, \kappa_2)\), a non-reflecting stationary set of ordinals of cofinality \(\kappa_1\). \(\mathbb{P}_0\) can be defined so as to have cardinality \(\kappa_2\).

Since \(V \models \text{"No cardinal } \lambda > \kappa_2 \text{ is measurable + } 2^\delta = \delta^+ \text{ for every cardinal}\)
\[ \delta \geq \kappa_1 \], a theorem of Magidor (whose proof is given in Theorem 2 of [3]) tells us that \( V_0^{\P_0} \models \) “There are no measurable cardinals in the interval \((\kappa_1, \kappa_2) + \kappa_2\) is strongly compact”. It then immediately follows that \( V_0^{\P_0} \models \) “\( \kappa_2 \) is not \( 2^{\kappa_2} = \kappa_2^+ \) supercompact”. Further, since \( \P_0 \), by its definition, is \( \kappa_1 \)-directed closed, \( V_0^{\P_0} \models \) “\( \kappa_1 \) is supercompact”.

Work in \( V_0 = V_0^{\P_0} \). For the remainder of this paper, for \( \alpha \) an arbitrary ordinal, let \( \lambda_\alpha \) be the least measurable cardinal above \( \alpha \). Since \( V_0 \models \) “\( \kappa_1 \) is supercompact + \( \kappa_2 \) is the least measurable cardinal above \( \kappa_1 \)”, by reflection, \( A = \{ \delta < \kappa_1 : \delta \text{ is } \lambda_\delta \text{ supercompact} \} \) is unbounded in \( \kappa_1 \). Therefore, we may define \( \P_1 \) in \( V_0 \) as the Easton support iteration of length \( \kappa_1 \) which first adds a Cohen subset of \( \omega \) and then adds, to every \( \delta \in A \), a non-reflecting stationary set of ordinals of cofinality \( \omega \). In analogy to the definition of \( \P_0 \), \( \P_1 \) can be defined so as to have cardinality \( \kappa_1 \).

**Lemma 2.1.** \( V_1 = V_0^{\P_1} \models \) “\( \kappa_1 \) is \( \alpha \) supercompact for every \( \alpha < \kappa_2 \)”.

**Proof.** Let \( \eta < \kappa_2 \) be an arbitrary inaccessible cardinal in the interval \((\kappa_1, \kappa_2) \), and let \( j : V_0 \rightarrow M \) be an elementary embedding witnessing the \( \eta \) supercompactness of \( \kappa_1 \) so that \( M \models \) “\( \kappa_1 \) is not \( \eta \) supercompact”. Since \( \eta \) is below the least measurable cardinal above \( \kappa_1 \), \( M \models \) “\( \kappa_1 \) is not \( \lambda_{\kappa_1} \) supercompact”. This means \( j(\P_1) = \P_1 * \bar{Q} \), where \( \bar{Q} \) is a term for a partial ordering that does not add a non-reflecting stationary set of ordinals of cofinality \( \omega \) to \( \kappa_1 \), and the least \( M \)-cardinal above \( \kappa_1 \) to which \( \bar{Q} \) is forced to add a non-reflecting stationary set of ordinals of cofinality \( \omega \) must also be above \( \eta \).

Let \( G_0 \) be \( V_0 \)-generic over \( \P_1 \), and let \( H \) be \( V_0[G_0] \)-generic over \( \bar{Q} \). Standard arguments show that \( M[G_0] \) remains \( \eta \) closed with respect to \( V_0[G_0] \). Further, \( j''G_0 \subseteq G_0 * H \). This means that in \( V_0[G_0][H] \), \( j \) lifts to \( j : V_0[G_0] \rightarrow M[G_0][H] \). By its definition, the closure properties of \( M[G_0] \), and the last sentence of the preceding paragraph, \( H \) is \( V_0[G_0] \)-generic over a partial ordering which is \( \eta \)-strategically closed in both \( V_0[G_0] \) and \( M[G_0] \). Therefore, \( V_0[G_0] \models \) “\( \kappa_1 \) is \( \alpha \) supercompact for every \( \alpha < \eta \)”. Since \( \eta \) was chosen as an arbitrary inaccessible cardinal in the interval \((\kappa_1, \kappa_2) \), this proves Lemma 2.1. \( \Box \)

We remark that by the observation made immediately following the proof of Lemma 1.1, Lemma 2.1 actually shows that \( \kappa_1 \) is \( \kappa_2 \) supercompact in \( V_1 \).

**Lemma 2.2.** \( V_1 = V_0^{\P_1} \models \) “\( \kappa_1 \) is not \( 2^{\kappa_2} = \kappa_2^+ \) supercompact”.

**Proof.** We begin by noting that \( V_1 = V_0^{\P_1} \models \) “\( \kappa_2 \) is the least measurable cardinal above \( \kappa_1 + \kappa_2 \) is strongly compact but is not \( 2^{\kappa_2} = \kappa_2^+ \) supercompact”. This follows by the fact \( \kappa_2 \) is both the least measurable and least strongly compact cardinal above \( \kappa_1 \) in \( V_0 \), the fact that \( \P_1 \) has cardinality \( \kappa_1 < \kappa_2 \) in \( V_0 \), and the Lévy–Solovay results [14].
Work in $V_0$. For any $\alpha$, write $P_1 = Q_0 \ast Q_1$, where $Q_0$ adds non-reflecting stationary sets of ordinals of cofinality $\omega$ to cardinals at most $\alpha$, and $Q_1$ is a term for the rest of $P_1$. Since $|Q_0| \leq 2^\alpha < \lambda_\alpha$, the results of [14] and the fact $\Vdash_{Q_0} \"Q_1 is $\lambda_\alpha$-strategically closed\" together imply that $(\lambda_\alpha)^{V_0} = (\lambda_\alpha)^{V_1}$.

Write $P_1 = P_0 \ast P''$, where $|P'| = \omega$ and $\Vdash_{P'} \"P'' is $\kappa_1$-strategically closed\". In Hamkins’ terminology of [9], [10], and [11], $P_1$ “admits a gap at $\kappa_1\”, so by the Gap Forcing Theorem of [10] and [11], any cardinal $\delta$ which is $\lambda_\delta$ supercompact in $V_1$ had to have been $\lambda_\delta$ supercompact in $V_0$. Since by its definition, forcing with $P_1$ over $V_0$ destroys the weak compactness of any cardinal $\delta < \kappa_1$ that was $\lambda_\delta$ supercompact in $V_0$, the preceding sentence implies that $V_1 = V_0^{P_1} \models \"No cardinal $\delta < \kappa_1$ is $\lambda_\delta$ supercompact\". This immediately implies that $V_1 \models \"\kappa_1$ is not $2^{\kappa_2} = \kappa_2^+ \) supercompact\", since otherwise, by choosing $k : V_1 \rightarrow N$ as an elementary embedding witnessing the $2^{\kappa_2}$ supercompactness of $\kappa_1$ and reflecting the fact that $N \models \"\kappa_1$ is $\kappa_2$ supercompact and $\kappa_2$ is the least measurable cardinal above $\kappa_1\”, we would infer that $\{ \delta < \kappa_1 : \delta$ is $\lambda_\delta$ supercompact $\}$ is unbounded in $\kappa_1$ in $V_1$. This proves Lemma 2.2.

By defining $P = P_0 \ast \dot{P}_1$, Lemmas 2.1 and 2.2 complete the proof of Theorem 1.

We conclude Section 2 with some observations. It is possible to change the definition of $P_1$ so as to ensure $\kappa_1$ will satisfy a greater degree of supercompactness in $V_1$. If, e.g., we modify the definition of $P_1$ so that we add non-reflecting stationary sets of ordinals of cofinality $\omega$ to every cardinal $\delta < \kappa_1$ which is $\beth_\delta(\lambda_\delta)$ supercompact (and by the supercompactness of $\kappa_1$, there are unboundedly in $\kappa_1$ many such cardinals), then in $V_1$, $\kappa_1$ will be $\beth_{\kappa_1}(\kappa_2)$ supercompact but not $2^{\beth_{\kappa_1}(\kappa_2)} = (\beth_{\kappa_1}(\kappa_2))^+$ supercompact. However, due to the restrictions on the proof of Theorem 2 of [3], we need to know that $V \models \"No cardinal $\lambda > \kappa_2$ is measurable\”. No such restrictions, however, are required in the proof of Theorem 2 of this paper, which we give below.

3. The proof of Theorem 2. Let $V \models \"ZFC + $\kappa$ is supercompact\”. By Lemma 2.1 of [4] and the succeeding remark, we know that $\{ \delta < \kappa : \delta$ is a strong cardinal $\}$ is unbounded in $\kappa$. Without loss of generality, by first forcing GCH, then choosing a strong cardinal $\delta < \kappa$, and then forcing with Gitik and Shelah’s indestructibility partial ordering of [8] (which can be defined so as to have cardinality $\delta$), we may further assume that $V \models \"GCH holds for cardinals at and above $\delta + \delta$ is a strong cardinal whose strongness is indestructible under forcing with an iteration of Prikry forcing as defined by Magidor in [15] which adds Prikry sequences to cardinals above $\delta\"$. 
Take now $\mathbb{P}_0$ as Magidor’s iterated Prikry forcing of $[15]$ which adds, to every measurable cardinal $\gamma \in (\delta, \kappa)$, a Prikry sequence. By the indestructibility properties of $V$ and Magidor’s work of $[15]$, $V^\mathbb{P}_0 = V_0 \vDash \text{GCH holds for cardinals at and above } \delta + \delta \text{ is a strong cardinal + } \kappa \text{ is strongly compact + There are no measurable cardinals in the interval } (\delta, \kappa)$. As in the proof of Theorem 1, $V_0 \vDash \text{“}\kappa \text{ is not } 2^\kappa = \kappa^+ \text{ supercompact”}$. Work in $V_0$. Since $V_0 \vDash \text{“}\delta \text{ is strong + } \kappa \text{ is the least measurable cardinal above } \delta\text{”, by reflection, } B = \{ \gamma < \delta : \gamma \text{ is } \lambda\gamma \text{ strong} \} \text{ is unbounded in } \delta$. Therefore, in analogy to the proof of Theorem 1, we may define $\mathbb{P}_1$ in $V_0$ as the Easton support iteration which begins by adding a Cohen subset of $\omega$ and then adds, to every $\gamma \in B$, a non-reflecting stationary set of ordinals of cofinality $\omega$. As in the proof of Theorem 1, $\mathbb{P}_1$ can be defined so as to have cardinality $\kappa$. By the preceding paragraph, this has as an immediate consequence that in $V_1$, GCH holds for cardinals at and above $\delta$.

**Lemma 3.1.** $V_0^{\mathbb{P}_1} = V_1 \vDash \text{“}\delta \text{ is } \alpha \text{ strong for every } \alpha < \kappa\text{”}$. **Proof.** The proof is very similar to the proof of Lemma 2.5 of [4]. We use the notation and terminology from the introductory section of [6]. Fix $\eta > \delta$, $\eta < \kappa$ an inaccessible cardinal which is not also a Mahlo cardinal. Let $j : V_0 \to M$ be an elementary embedding witnessing the $\eta + 1$ strength of $\delta$ generated by a $(\delta, \eta + 1)$-extender of width $\delta$ so that $M \vDash \text{“}\delta \text{ is not } \eta + 1 \text{ strong”}$, and let $i : V_0 \to N$ be the elementary embedding witnessing the measurability of $\delta$ generated by the normal ultrafilter $\mathcal{U} = \{ x \subseteq \delta : \delta \in j(x) \}$. We then have the commutative diagram

\[
\begin{array}{ccc}
V_0 & \xrightarrow{j} & M \\
\downarrow{i} & & \downarrow{k} \\
N
\end{array}
\]

where $j = k \circ i$ and the critical point of $k$ is above $\delta$.

Since $\eta$ is below the least measurable cardinal above $\delta$ and $\eta$ is not a Mahlo cardinal, $M \vDash \text{“There are no measurable cardinals in the interval } (\delta, \eta) + \delta \text{ is not } \lambda_\delta \text{ strong”}$. Define $\varrho$ to be the least cardinal in $M$ above $\delta$ which is $\lambda_\varrho$ strong. By the next to last sentence, we can now infer that $\varrho > \eta$.

Define $f : \delta \to \delta$ as $f(\alpha) = \text{The least inaccessible cardinal above } \lambda_\alpha$. By our choice of $\eta$ and the preceding paragraph, $\delta < \eta < j(f)(\delta) < \varrho$. Observe that $\varrho$ is also the least $M$-cardinal above $\delta$ to which $j(\mathbb{P}_1)$ adds a non-reflecting stationary set of ordinals of cofinality $\omega$.

Note now that $M = \{ j(g)(a) : a \in [\eta^+]^{< \omega}, \text{dom}(g) = [\delta]^{\alpha}, \ g : [\delta]^{\alpha} \to V_0 \} = \{ k(i(g))(a) : a \in [\eta^+]^{< \omega}, \text{dom}(g) = [\delta]^{\alpha}, \ g : [\delta]^{\alpha} \to V_0 \}$. By defining $\gamma = i(f)(\delta)$, we have $k(\gamma) = k(i(f)(\delta)) = j(f)(\delta) > \eta^+$. This means $j(g)(a) =$
\(k(i(g))(a) = k(i(g)\langle \gamma \rangle^a)(a)\), i.e., \(M = \{k(h)(a) : a \in [\eta^+]<\omega, h \in N, \dom(h) = [\gamma]^a, h : [\gamma]^a \to N\}\). By elementariness, we must have \(N \vDash \"\delta \) is not \(\lambda_\delta \) strong and \(\delta < \gamma = i(f)(\delta) < \delta_0 = \) The least cardinal \(\zeta \) in \(N\) above \(\delta \) which is \(\lambda_\zeta \) strong = The least cardinal to which \(i(\mathbb{P}_1) - \delta \) adds a non-reflecting stationary set of ordinals of cofinality \(\omega\)”, since \(M \vDash \"k(\delta) = \delta \) is not \(\lambda_\delta \) strong and \(k(\delta) = \delta < k(\gamma) = k(i(f)(\delta)) = j(f)(\delta) < k(\delta_0) = \vartheta\)”. Therefore, \(k\) can be assumed to be generated by an \(N\)-extender of width \(\gamma \in (\delta, \delta_0)\).

Write \(i(\mathbb{P}_1) = \mathbb{P}_1 \ast \check{Q}_0\), where \(\check{Q}_0\) is a term for the portion of \(i(\mathbb{P}_1)\) adding non-reflecting stationary sets of ordinals of cofinality \(\omega\) to \(N\)-cardinals in the interval \([\delta, i(\delta)]\). Since \(N \vDash \"\delta \) is not \(\lambda_\delta \) strong\”, \(\check{Q}_0\) is actually a term for a partial ordering adding non-reflecting stationary sets of ordinals of cofinality \(\omega\) to \(N\)-cardinals in the interval \((\delta, i(\delta))\), or more precisely, to \(N\)-cardinals in the interval \([\delta_0, i(\delta))\).

Let \(G_0\) be \(V_0\)-generic over \(\mathbb{P}_1\). By the definition of \(\mathbb{P}_1\) and the fact GCH holds in \(V_0\) for cardinals at and above \(\delta\), \(N[G_0] \vDash \"|\check{Q}_0| = i(\delta) + |2^{\check{Q}_0}| = i(\delta^+) = (i(\delta))^+\). As \(N\) is an ultrapower via a normal measure over \(\delta\), this means \(V_0 \vDash \"|i(\delta))^+| = \delta^+\", so we can let \(<D_\alpha : \alpha < \delta^+\rangle \in V_0[G_0]\) be an enumeration of the dense open subsets of \(\check{Q}_0\) present in \(N[G_0]\). For the purposes of the argument to be given below, we also assume that \(<D_\alpha : \alpha < \delta^+\rangle\) has been defined so that for every dense open subset \(D \subseteq \check{Q}_0\) found in \(N[G_0]\), for some odd ordinal \(\beta + 1, D = D_{\beta+1}\). Further, since \(V_0 \vDash \"|\mathbb{P}_1| = \delta\", standard arguments show that \(N[G_0]\) remains \(\delta\) closed with respect to \(V_0[G_0]\). Therefore, as \(N[G_0] \vDash \"\check{Q}_0\) is \(\prec\delta^+\)-strategically closed\”, this fact is true in \(V_0[G_0]\) as well.

We can now construct an \(N[G_0]\)-generic object, \(G^*_1\), in \(V_0[G_0]\) as follows. Players I and II play a game of length \(\delta^+\). The initial pair of moves is generated by player II choosing the trivial condition \(q_0\) and player I responding by choosing \(q_1 \in D_1\). Then, at an even stage \(\alpha + 2, \) player II picks \(q_{\alpha+2} \geq q_{\alpha+1}\) by using some fixed strategy \(S\), where \(q_{\alpha+1}\) was chosen by player I to be so that \(q_{\alpha+1} \in D_{\alpha+1}\) and \(q_{\alpha+1} \geq q_\alpha\). If \(\alpha\) is a limit ordinal, player II uses \(S\) to pick \(q_\alpha\) extending each \(q_\beta\) for \(\beta < \alpha\). By the \(\prec\delta^+\)-strategic closure of \(\check{Q}_0\) in both \(N[G_0]\) and \(V[G_0]\), the sequence \(<q_\alpha : \alpha < \delta^+\rangle\) as just described exists. By construction, \(G^*_1 = \{p \in \check{Q}_0 : \exists \alpha < \delta^+ [q_\alpha \geq p]\}\) is our \(N[G_0]\)-generic object over \(\check{Q}_0\). Since \(i''G_0 \subseteq G_0 \ast G^*_1\), \(i\) lifts to \(i : V_0[G_0] \to N[G_0][G^*_1]\), and since \(k''G_0 = G_0\) and \(k(\delta) = \delta, k\) lifts to \(k : N[G_0] \to M[G_0]\). By Fact 3 of Section 1.2.2 of [6], \(k : N[G_0] \to M[G_0]\) can also be assumed to be generated by an extender of width \(\gamma \in (\delta, \delta_0)\).

In analogy to the above, write \(j(\mathbb{P}_1) = \mathbb{P}_1 \ast \check{Q}_1\). By the last sentence of the preceding paragraph and the fact \(\delta_0\) is the least \(N\)-cardinal to which \(\check{Q}_0\) is forced to add a non-reflecting stationary set of ordinals of cofinality \(\omega\), we can use Fact 2 of Section 1.2.2 of [6] to infer that \(H = \{p \in Q_1 : \)
\[ \exists q \in k''G_1 \ [q \geq p] \] is \( M[G_0] \)-generic over \( k(Q_0) = Q_1 \). Thus, \( k \) lifts to \( k : N[G_0][G_1^*] \to M[G_0][H] \), and we get the new commutative diagram

\[
\begin{array}{ccc}
V_0[G_0] & \xrightarrow{j} & M[G_0][H] \\
\downarrow{i} & \downarrow{k} & \\
N[G_0][G_1^*] & \end{array}
\]

Since \( \eta > \eta \), the \( M \)-cardinals to which \( Q_1 \) is forced to add non-reflecting stationary sets of ordinals of cofinality \( \omega \) lie in the interval \( (\eta^+, j(\delta)) \). Therefore, as \( V_{\eta+1} \subseteq M \), \( V_{\eta+1}[G_0] \subseteq M[G_0] \), and as \( Q_1 \) adds non-reflecting stationary sets of ordinals of cofinality \( \omega \) to certain inaccessible \( M \)-cardinals in the interval \( (\eta^+, j(\delta)) \), \( V_{\eta+1}[G_0] \) is the set of all sets of rank below \( \eta + 1 \) in \( M[G_0][H] \). Hence, \( j \) is an \( \eta + 1 \) strong embedding. Since \( \eta \) was an arbitrary non-Mahlo inaccessible cardinal below \( \kappa \), this proves Lemma 3.1. \( \blacksquare \)

We remark that by the observation made immediately following the proof of Lemma 1.1, Lemma 3.1 actually shows that \( \delta \) is \( \kappa + 1 \) strong in \( V_1 \).

**Lemma 3.2.** \( V_1 \models \text{“} \delta \text{ is not } \kappa + 2 \text{ strong”} \).

**Proof.** We argue in analogy to the proof of Lemma 2.2. We again begin by noting that \( V_1 = V_0^{P_1} \models \text{“} \kappa \text{ is the least measurable cardinal above } \delta + \kappa \text{ is strongly compact but is not } 2^\kappa = \kappa^+ \text{ supercompact”} \). This follows by the fact \( \kappa \) is both the least measurable and least strongly compact cardinal above \( \delta \) in \( V_0 \), the fact that \( P_1 \) has cardinality \( \delta < \kappa \) in \( V_0 \), and the results of [14].

Work in \( V_0 \). As in the proof of Lemma 2.2, for any ordinal \( \alpha \), \( (\lambda_\alpha)^{V_0} = (\lambda_\alpha)^{V_1} \). Also, we can once more write \( P_1 = P' \ast \hat{P}' \), where \( |P'| = \omega \) and \( \models P' \text{ is } \kappa_1 \text{-strategically closed} \). As before, \( P_1 \) “admits a gap at \( \kappa_1 \)”, so by the Gap Forcing Theorem of [10] and [11], any cardinal \( \zeta \) which is \( \lambda_\zeta \) strong in \( V_1 \) had to have been \( \lambda_\zeta \) strong in \( V_0 \). Since by its definition, forcing with \( P_1 \) over \( V_0 \) destroys the weak compactness of any cardinal \( \zeta < \delta \) that was \( \lambda_\zeta \) strong in \( V_0 \), the preceding sentence implies that \( V_1 = V_0^{P_1} \models \text{“} \text{No cardinal } \zeta < \delta \text{ is } \lambda_\zeta \text{ strong”} \). This immediately implies that \( V_1 \models \text{“} \delta \text{ is not } \kappa + 2 \text{ strong”} \), since otherwise, by choosing \( \ell : V_1 \to M^* \) as an elementary embedding witnessing the \( \kappa + 2 \) strongness of \( \delta \) and reflecting the fact that \( M^* \models \text{“} \delta \text{ is } \kappa \text{ strong and } \kappa \text{ is the least measurable cardinal above } \delta \text{”} \), we would infer that \( \{ \zeta < \delta : \zeta \text{ is } \lambda_\zeta \text{ strong} \} \) is unbounded in \( \delta \) in \( V_1 \). This proves Lemma 3.2. \( \blacksquare \)

By defining \( P = P_0 \ast \hat{P}_1 \), Lemmas 3.1 and 3.2 complete the proof of Theorem 2. \( \blacksquare \)
We conclude Section 3 and this paper with several observations. First, as the referee has essentially indicated, if $V \models \text{"ZFC + GCH + } \delta < \kappa \text{ are so that } \delta \text{ is strong and } \kappa \text{ is strongly compact"}$, then we may force over $V$ with the partial ordering $\mathbb{P}$ as just defined in order to obtain the conclusions of Theorem 2. In addition, as before, it is possible to change the definition of $\mathbb{P}_1$ so as to ensure $\delta$ will satisfy a greater degree of strongness in $V_1$. If, e.g., we change the definition of $\mathbb{P}_1$ so that we add non-reflecting stationary sets of ordinals of cofinality $\omega$ to every cardinal $\zeta < \delta$ which is $\beth_\delta(\lambda_\zeta)$ strong (and by the strongness of $\delta$, there are unboundedly in $\delta$ many such cardinals), then in $V_1$, $\delta$ will be $\beth_\delta(\kappa)$ strong but not $\beth_\delta(\kappa) + 1$ strong. Also, since Magidor’s proof from [15] that iterated Prikry forcing preserves the strong compactness of $\kappa$ is valid regardless of the large cardinal structure of the universe above $\kappa$, unlike Theorem 1, there is no need to do an initial forcing to ensure that $V \models \text{"No cardinal } \lambda > \kappa \text{ is measurable"}$.

Finally, we note that under the same hypotheses as in Theorem 1, i.e., that $V \models \text{"ZFC + } \kappa_1 < \kappa_2 \text{ are supercompact"}$, it is possible to modify the definition of the partial ordering $\mathbb{P}$ of Theorem 1 so that $V^\mathbb{P} \models \text{"ZFC + } \kappa_2 \text{ is strongly compact but not supercompact + } \kappa_1 \text{ is } \alpha \text{ supercompact for every } \alpha < \kappa_2 + \kappa_1 \text{ is not supercompact + } \kappa_1 \text{ is strong"}$. To do this, we observe that Lemma 2.1 of [4] and the succeeding remark actually imply that if $j : V \rightarrow M$ is an elementary embedding witnessing (at least) the $2^{\lambda_{\kappa_1}}$ supercompactness of $\kappa_1$, then $M \models \text{"}\kappa_1 \text{ is a strong cardinal and } \kappa_1 \text{ is } \lambda_{\kappa_1} \text{ supercompact"}$, meaning that $A = \{ \delta < \kappa_1 : \delta \text{ is a strong cardinal and } \delta \text{ is } \lambda_\delta \text{ supercompact} \}$ is unbounded in $\kappa_1$. Therefore, if $\mathbb{P}_0$ is as in the definition given in the proof of Theorem 1, $V_0 = V^{\mathbb{P}_0}$, and $\mathbb{P}_1$ is defined in $V_0$ as the Easton support iteration of length $\kappa_1$ which first adds a Cohen subset of $\omega$ and then adds, to every $\delta \in A$, a non-reflecting stationary set of ordinals of cofinality $\omega$, the exact same arguments as before show that $V^\mathbb{P} \models \text{"ZFC + } \kappa_2 \text{ is strongly compact but not supercompact + } \kappa_1 \text{ is } \alpha \text{ supercompact for every } \alpha < \kappa_2 \text{"}$. If in the proof of Lemma 2.2, we replace the property \"$\delta$ is $\lambda_\delta$ supercompact\" with \"$\delta$ is a strong cardinal and $\delta$ is $\lambda_\delta$ supercompact\", then the same proof as given in Lemma 2.2 remains valid and shows $V^\mathbb{P} \models \text{"}\kappa_1 \text{ is not } 2^{\kappa_2} = \kappa_2^+ \text{ supercompact"}$. Further, if we choose $\lambda > \kappa_2$ as any cardinal so that $\lambda = \aleph_\lambda = \beth_\lambda$ and $j : V_0 \rightarrow M$ as an elementary embedding witnessing the $\lambda$ strongness of $\kappa_1$ so that $M \models \text{"}\kappa_1 \text{ is not } \lambda \text{ strong"}$, then either the argument given in the proof of Lemma 2.5 of [4] or the one in the proof of Lemma 3.1 shows that $V_0^{\mathbb{P}_1} = V^\mathbb{P} \models \text{"}\kappa_1 \text{ is } \lambda \text{ strong"}$. Since $\lambda$ may be chosen arbitrarily large, this means that $V^\mathbb{P} \models \text{"}\kappa_1 \text{ is } \lambda \text{ strong"}$. And, in analogy to what was mentioned in the concluding remarks of Section 2, it is possible to change the definition of $\mathbb{P}_1$ to ensure that $\kappa_1$ witnesses a greater degree of supercompactness in $V^\mathbb{P}$, assuming that the cardinals to which non-reflecting stationary sets of ordinals of cofinality $\omega$ are added are also strong.
References


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