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Potential isomorphism and semi-proper trees

by

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Abstract. We study a notion of potential isomorphism, where two structures are said to be potentially isomorphic if they are isomorphic in some generic extension that preserves stationary sets and does not add new sets of cardinality less than the cardinality of the models. We introduce the notion of weakly semi-proper trees, and note that there is a strong connection between the existence of potentially isomorphic models for a given complete theory and the existence of weakly semi-proper trees.

We show that the existence of weakly semi-proper trees is consistent relative to ZFC by proving the existence of weakly semi-proper trees under certain cardinal arithmetic assumptions. We also prove the consistency of the non-existence of weakly semi-proper trees assuming the consistency of some large cardinals.

Introduction. Two structures are said to be *potentially isomorphic* if they are isomorphic in some extension of the universe in which they reside. Different notions of potential isomorphism arise as restrictions are placed on the method to extend the universe. Nadel and Stavi [13] considered generic extensions in which there are no new subsets of cardinality less than κ, where κ is the cardinality of the models. They used some cardinal arithmetic assumptions on κ to show the existence of a pair of non-isomorphic but potentially isomorphic models. This kind of result can be interpreted as a non-structure theorem for the theory of the models in question.

In [6] these studies were continued, with an emphasis on classification theory. One of the results obtained there concerning the notion introduced in [13] is:

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THEOREM 1. *Let T be a countable first order theory and let $\kappa = \kappa^{\aleph_0}$ be a regular cardinal. The theory T is unclassifiable if and only if there exists a pair of non-isomorphic but potentially isomorphic models of T of cardinality κ^+.*

A theory is said to be *unclassifiable* if it is unsuperstable or has either the dimensional order property (DOP) or the omitting types order property (OTOP).

Baldwin, Laskowski, and Shelah [1, 11] studied another notion by considering isomorphism in extensions by ccc forcing notions, which allows changes in the universe that affect small substructures of the models in question. They showed that even classifiable theories may have a pair of non-isomorphic models that are potentially isomorphic in this sense.

We must have some restrictions on how cardinals can be collapsed in the extensions, because otherwise potential isomorphism will be reduced to $L_{\infty\omega}$-equivalence. But one may consider weakening the requirement that the extension must be generic. Such notions are studied in [4], and it is shown there that this kind of notions are not always decidable. By a *cardinal preserving extension* of L we mean a transitive model of ZFC that contains all ordinals, is contained in a set-generic extension of V, and has the same cardinals as L. For a tree $T \in L$ on $(\omega_1)^L$, let C_T denote the set of all the trees $T' \in L$ on $(\omega_1)^L$ that are isomorphic to T in some cardinal preserving extension of L. The following was proved in [4]:

THEOREM 2. *Assume $0^\sharp$ exists. There exists a tree $T \in L$ on $(\omega_1)^L$ such that C_T is equiconstructible with $0^\sharp$.*

The topic of this paper is a very strong notion of potential isomorphism. We consider generic extensions that preserve stationary subsets of the cardinality of the models and do not add new sets of cardinality less than the cardinality of the models. To investigate this notion of potential isomorphism is natural since Theorem 1 was proved in [6] by coding a stationary set S into a pair of models, which are then forced isomorphic by killing S.

A (λ,κ)-*tree* is a tree with the properties that every branch has length less than κ and every element has less than λ immediate successors. Thus a (λ,κ)-tree has height at most κ. Bearing some of the forthcoming proofs in mind it is worth noting that the cardinality of a (λ^+,κ)-tree is at most $\lambda^{<\kappa}$.

We say that a (λ,κ)-tree T is *weakly semi-proper* if there exists a forcing notion P that adds a κ-branch to T, but preserves stationary subsets of κ and adds no sets of cardinality less than κ. If T itself, regarded as a forcing notion, has the properties of P mentioned above, then we say that T is *strongly semi-proper* or just *semi-proper*.

The following fact has led us to questions concerning the existence of weakly semi-proper (κ^+,κ)-trees (for simplicity we consider only countable theories):

THEOREM 3. *Assume that κ is uncountable and $\kappa^{<\kappa}=\kappa$. The following statements are equivalent*:

(i) *There exists a weakly semi-proper (κ^+,κ)-tree.*

(ii) *There exists a pair of non-isomorphic structures of size κ that can be made isomorphic by forcing, without adding new sets of cardinality less than κ or destroying stationary subsets of κ.*

(iii) *Statement* (ii) *strengthened with the requirement that the structures can be chosen to be models of any complete countable theory T such that either*

1. *T is unstable*,
2. *T has DOP, $\kappa>(c_{\mathrm{r}})^+$, and $\xi^{c_{\mathrm{r}}}<\kappa$ for every $\xi<\kappa$, where c_{r} is the smallest regular cardinal not less than the continuum*, *or*
3. *T is superstable with DOP or OTOP.*

Proof. (ii) implies (i). Suppose that two non-isomorphic structures $\mathfrak{A}$ and $\mathfrak{B}$ of size κ can be forced to be isomorphic without killing stationary sets or adding new subsets of cardinality less than κ. Assume that κ is the universe of both structures. Let P denote the set of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ of cardinality less than κ. Let T_α denote the set

$$\{f\in P:\alpha\subseteq\operatorname{dom} f\cap\operatorname{ran} f,\ f[\kappa\setminus\alpha]\cup f^{-1}[\kappa\setminus\alpha]\subseteq\alpha\}$$

and let $T=\bigcup_{\alpha<\kappa}T_\alpha$ ordered by inclusion. We shall prove that T is a (κ^+,κ)-tree and that any forcing notion that makes $\mathfrak{A}$ and $\mathfrak{B}$ isomorphic without adding bounded subsets of κ adds a κ-branch to T.

It is straightforward to check that T is indeed a tree. Since $\kappa^{<\kappa}=\kappa$, the cardinality of P is κ. Therefore every node in T has at most κ immediate successors. The union of a κ-branch would clearly be an isomorphism, so T cannot have κ-branches. Finally suppose that f is an isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$ in a generic extension. If there are no new bounded subsets of κ in the extension, then the function $(f|\alpha)\cup(f^{-1}|\alpha)^{-1}$ is in T_α for every $\alpha<\kappa$ and it follows that $\mathscr{P}(f)\cap T$ is a κ-branch through T in the generic extension.

(i) implies (ii). The proof of Lemma 7.13 of [6] is essentially the proof of this implication. It relies on results of [9] and [8]. ∎

Suslin trees are semi-proper $(\aleph_2,\aleph_1)$-trees, and are in fact used in that role in the proof of Lemma 7.13 of [6], but in this paper we shall see that semi-proper trees exist under much weaker assumptions than Suslin trees.

The following theorem summarises the rest of the results of this paper except for some minor observations and strengthenings:

THEOREM 4. (a) *It is consistent relative to a supercompact cardinal that there are no weakly semi-proper $(\infty, \aleph_1)$-trees.*

(b) (Gregory) *If $2^{\aleph_0} < 2^{\aleph_1}$ then there exists a semi-proper $(\aleph_2, \aleph_1)$-tree.*

(c) *It is consistent relative to a weakly compact cardinal that there are no weakly semi-proper $(\aleph_3, \aleph_2)$-trees.*

(d) *Under GCH there exists a semi-proper $(\kappa^{++}, \kappa^{+})$-tree for every infinite successor cardinal κ.*

(e) *For any regular $\kappa > \aleph_1$ there exists a semi-proper $((2^{\kappa})^{+}, \kappa)$-tree.*

Clause (c) is proved in Section 2, (b) and (d) are proved in Section 3, and Section 4 constitutes the proof of (e). Clause (a) follows from the observation that under Martin's maximum (the semi-proper forcing axiom) there exist no weakly semi-proper $(\infty, \aleph_1)$-trees. Feng [3] has made a similar observation concerning strongly semi-proper $(\infty, \aleph_1)$-trees.

On the necessity of a weakly compact cardinal in clause (c) we want to state the following:

CONJECTURE 1. *Let $\kappa > \aleph_1$ be a regular cardinal. If there are no semi-proper (κ^{+}, κ)-trees then κ is weakly compact in L.*

1. Preliminaries and notation. Let A be a set of ordinals. The set of ordinals α such that $\sup(A \cap \alpha) = \alpha$ (the accumulation points of A) is denoted by $\operatorname{acc}^{+} A$; moreover, $\operatorname{acc} A = \operatorname{acc}^{+} A \cap A$ and $\operatorname{nacc} A = A \setminus \operatorname{acc} A$. For infinite cardinals κ and μ we let S^{κ}_{μ} denote the set $\{\alpha \in \operatorname{acc} \kappa : \operatorname{cf} \alpha = \lambda\}$. NS_{κ} denotes the ideal of non-stationary subsets of κ.

We say that a tree T is *splitting* if it has unique limits and if every node of T has at least two immediate successors. If T is splitting and for every $x \in T$ and $\alpha < \operatorname{ht} T$ there exists an element $y \in T$ such that $x <_T y$ and $\operatorname{ht} y \geq \alpha$, then we say that T is *normal.* Let κ be regular and uncountable and let T be a normal tree of height κ. If forcing with T adds a new set of cardinality less than κ, then κ becomes singular in the generic extension. Thus if forcing with T preserves stationary subsets of κ, then no new sets of cardinality less than κ are added.

In forcing arguments we follow the convention that $p \leq q$ means "p is stronger than q". Our upward growing trees get inverted, often without explicit mention, as soon as forcing with the tree in question is discussed.

2. A consistency result. We say that a tree T is an *α-representation* (of a tree) if the domain of T is the ordinal α and $x <_T y$ implies $x < y$ for all $x, y \in T$. Note that under the assumption $\kappa^{<\kappa} = \kappa$, every (κ^{+}, κ)-tree of height κ is isomorphic to a κ-representation.

LEMMA 1. *If κ is a regular uncountable cardinal, T is a κ-representation of a (κ^+,κ)-tree and the set*

$$S = \{\alpha < \kappa : T \cap \alpha \textit{ has no } \alpha\textit{-branch}\}$$

is stationary, then T is not weakly semi-proper.

Proof. Suppose that P is a forcing notion and $\dot{B}$ is a P-name for a κ-branch through T. Let $\dot{C}$ be a P-name that satisfies

$$\Vdash \text{“}\dot{C} = \{\alpha < \kappa : \mathrm{ot}(\dot{B} \cap \alpha) = \alpha\}\text{”}.$$

Assuming that κ remains regular in the generic extension by P, we get

$$\Vdash \text{“}\dot{C} \text{ is club and } \dot{C} \cap \check{S} = \emptyset\text{”}.$$

Thus P necessarily kills a stationary set, which shows that T cannot be weakly semi-proper. ■

Let κ be weakly compact. There exist (κ^+,κ)-trees that receive κ-branches when used as forcing notions. An example is $T(\{\alpha < \kappa : \mathrm{cf}\,\alpha \neq \alpha\})$, where $T(A)$ denotes (see e.g. [16]) the set of closed bounded subsets of A ordered by end extension. However, the lemma above yields the following:

COROLLARY 1. *If κ is weakly compact then weakly semi-proper (κ^+,κ)-trees do not exist.*

Proof. Let T be a κ-representation for a (κ^+,κ)-tree. The fact that T has no cofinal branches can be expressed as a Π^1_1-statement in the structure $\langle V_\kappa, \in, T\rangle$. For regular $\alpha < \kappa$ the same Π^1_1-statement interpreted in $\langle V_\alpha, \in, T \cap V_\alpha\rangle$ expresses the fact that $T \cap \alpha$ has no α-branches. Given this Π^1_1-statement, the corollary immediately follows from Lemma 1 by Π^1_1-reflection. ■

We shall now give the definition of a forcing notion that was introduced by Mitchell [12]. Let κ be a weakly compact cardinal. Let P be the classical forcing notion for adding κ Cohen reals. In other words P is the set of finite partial functions from κ to 2, ordered by reverse inclusion. Let $B(P)$ be the complete boolean algebra associated with P. For $s \subseteq P$ we shall use the notation b_s for the regular open cover (see e.g. Jech [10, Lemma 17.2]) of s, so that we have $B(P) = \{b_s : s \subseteq P\}$.

Let $P_\alpha = \{p \in P : p|\alpha = p\}$ and $B_\alpha = \{b_s : s \subseteq P_\alpha\}$. Then B_α is isomorphic to $B(P_\alpha)$. A partial function $f : \kappa \to B(P)$ is *acceptable* if $|f| < \aleph_1$ and $f(\gamma) \in B_{\gamma+\omega}$ for every $\gamma < \kappa$. We let A denote the set of all acceptable functions. Given a P-generic set G, we define a forcing notion Q in $V[G]$ as follows: For every $f \in A$, where A is regarded as an element of V, let $\overline{f}$ denote the characteristic function of $\{\gamma \in \mathrm{dom}\, f : f(\gamma) \cap G \neq \emptyset\}$. Then let Q be $\{\overline{f} : f \in A\}$ ordered by reverse inclusion. With $\dot{Q}$ being a P-name for Q, we finally let R be the two-step iteration $P * \dot{Q}$. We shall also refer to

R as the *Mitchell forcing*. The model V^R obtained by assuming GCH and then forcing with R will be called the *Mitchell model*. In the notation of [12] our R is isomorphic to $R_2(\aleph_0, \aleph_1, \kappa)$.

Let $Q_\alpha = \{\bar{f} \in Q : \bar{f}|\alpha = \bar{f}\}$ and let $R_\alpha = P_\alpha * \dot{Q}_\alpha$ where the ordering of Q_α is reverse inclusion. Thus $R_\kappa = R$. For any R-generic set G, we let G_α denote the set $G \cap R_\alpha$. We shall need the following results from [12]:

LEMMA 2 (Mitchell). *Assume that GCH holds.*

(a) *Suppose that α is a limit ordinal in κ and G is an R-generic set. Then G_α is R_α-generic.*

(b) *Suppose that* $\operatorname{cf}\gamma > \omega$ *and f is a function $\gamma \to V$ in V^R. If $f|\zeta \in V^{R_\alpha}$ for every $\zeta < \gamma$ then $f \in V^{R_\alpha}$.*

(c) *R has the κ-cc.*

(d) *In V^R, $2^{\aleph_1} = \kappa = \aleph_2$.*

PROPOSITION 1. *In the Mitchell model there are no weakly semi-proper $(\aleph_3, \aleph_2)$-trees.*

Proof. Let R_κ be the Mitchell forcing notion and let $\dot{T}$ be an R_κ-name for an arbitrary $(\aleph_3, \aleph_2)$-tree. By clause (d) of Lemma 2 we can assume that $\dot{T}$ is a name for an ω_2-representation and by Lemma 1 it is then enough to prove that

$$\Vdash_{R_\kappa} \text{“}\{\alpha < \omega_2 : \dot{T} \cap \alpha \text{ has no } \alpha\text{-branch}\} \text{ is stationary”}.$$

Since R_κ is κ-cc and therefore does not destroy stationary sets, it is sufficient to find a stationary set $S \subseteq \kappa$ in the ground model such that

$$\Vdash_{R_\kappa} \text{“}\dot{T} \cap \alpha \text{ has no } \alpha\text{-branch when } \alpha \in \check{S}\text{”}. \tag{1}$$

We shall use Π^1_1-reflection to find a stationary set S satisfying (1). To be able to capture various facts about forcing using Π^1_1-statements in a structure like $\langle V_\kappa, \in, R_\kappa, \dot{T}\rangle$ we need to make some assumptions on the names used. The name $\dot{T}$ can be assumed to be a subset of $(\kappa \times \kappa) \times R_\kappa$ where we identify ordinals with their canonical names. Furthermore we can assume that for every $(\alpha, \beta) \in \kappa \times \kappa$ the set

$$A_{(\alpha,\beta)} = \{p \in R_\kappa : ((\alpha, \beta), p) \in \dot{T}\}$$

is a maximal antichain of the set consisting of all conditions p with the property $p \Vdash (\alpha, \beta) \in \dot{T}$. Then for any $q \in R_\kappa$, $q \Vdash (\alpha, \beta) \notin \dot{T}$ if and only if $\{p \in A_{(\alpha,\beta)} : p \| q\}$ is empty. An arbitrary name for a subset of $\dot{T}$ can be thought of as a name for a subset of κ and then there always exists an equivalent name that is a subset of $\kappa \times R_\kappa$ and has similar properties to $\dot{T}$ above. For such a name $\dot{B}$ for a subset of $\dot{T}$ the statement

$$\Vdash_{R_\kappa} \text{“}\dot{B} \text{ is a } \kappa\text{-branch through } \dot{T}\text{”}$$

can be expressed by a first order sentence in the structure $\langle V_\kappa, \in, R_\kappa, \dot{T}, \dot{B}\rangle$. Let us call a name like $\dot{T}$ or $\dot{B}$ *normal* for the rest of the proof. Normality of a name is also a first order property of the structure mentioned above.

For inaccessible cardinals $\alpha < \kappa$ we have $R_\kappa \cap V_\alpha = R_\alpha$, and if we let $\dot{T}_\alpha = \dot{T} \cap V_\alpha$ and $\dot{B}_\alpha = \dot{B} \cap V_\alpha$ then $\dot{T}_\alpha$ and $\dot{B}_\alpha$ are R_α-names. So there is a Π^1_1-sentence σ such that for every inaccessible $\alpha \le \kappa$, $\langle V_\alpha, \in, R_\alpha, T_\alpha\rangle \models \sigma$ if and only if $\dot{T}_\alpha$ is normal and

$$\Vdash_{R_\alpha} \text{“}\dot{T}_\alpha \text{ has no } \alpha\text{-branch.”} \tag{2}$$

Furthermore there exists a club subset D of κ such that

$$(\dot{T}_\alpha)_{G_\alpha} = \dot{T}_G \cap \alpha \tag{3}$$

for every $\alpha \in D$ and every R_κ-generic set G. Let S be a stationary set of ordinals such that (2) and (3) hold for every $\alpha \in S$. By clause (b) of Lemma 2 it now follows that S satisfies (1). ∎

3. Using weak diamond principles. In this section we shall freely use some of the results presented in [7] about the ideal $I[\lambda]$ and the κ-club game on a subset of λ, although we shall not always stick to the notation used there. The *κ-club game* on $S \subseteq \lambda$ is played by players I and II as follows: The game lasts for κ rounds. On round ξ player I first picks an ordinal $\alpha_\xi < \lambda$ that is greater than all the ordinals played on earlier rounds. Then player II picks an ordinal β_ξ such that $\alpha_\xi < \beta_\xi < \lambda$. If the supremum of the ordinals picked during the entire game is an element of S, then player II wins the game. Otherwise player I wins the game. The game characterisation of the κ-club filter on λ is the following statement: If player II has a winning strategy in the κ-club game on $S \subseteq \lambda$ then there exists a set $C \subseteq S$ which is κ-club in λ.

A subset U of a tree T is called a *μ-fan* of T if there exists a sequence $(\delta_\xi : \xi < \mu)$ and an indexed family $(x_f : f \in {}^{<\mu}2)$ such that:

(1) $U = \{x_f : f \in {}^{<\mu}2\}$,
(2) $(\delta_\xi : \xi < \mu)$ is strictly increasing and continuous,
(3) $\mathrm{ht}_T\, x_f = \delta_{\mathrm{dom}\, f}$ for every $f \in {}^{<\mu}2$,
(4) $\inf_T\{x_{f\frown(0)}, x_{f\frown(1)}\} = f_x$ for every $f \in {}^{<\mu}2$.

We say that T is *μ-fan closed* if T is μ-closed as a forcing notion, and for every μ-fan U of T there exists an element $x \in T$ that extends one of the cofinal branches in U.

LEMMA 3. *Suppose that $\mu^{<\mu} = \mu$ and $\kappa = \mu^+$. Then every splitting μ-fan closed (∞, κ)-tree is semi-proper.*

Proof. It is straightforward to prove by induction that a splitting μ-fan closed (∞, κ)-tree must be a normal tree of height κ. By normality, forcing with the tree must produce a κ-branch. Thus it only remains to prove that stationary sets are preserved.

Let P be an inverted normal μ-fan closed tree of height κ, let $S \subseteq \kappa$, and let $\dot{C}$ be a P-name such that

$$\Vdash \text{"}\dot{C} \text{ is club and } \dot{C} \cap \check{S} = \emptyset\text{"}.$$

Because $I[\kappa]$ is improper by our assumptions, the game characterisation of the μ-club filter on κ holds. We shall finish the proof by showing that player II has a winning strategy in the μ-club game on the complement of S. This will be enough since we can assume that $S \subseteq S^\kappa_\mu$. The strategy can be described as follows. At round ξ in the game, player I has picked α_ξ and player II should now answer with $\beta_\xi > \alpha_\xi$. But before fixing β_ξ we pick a set $\{p_f : f \in {}^\xi 2\}$ of conditions in P and a set $\{\gamma_f : f \in {}^\xi 2\}$ of ordinals such that the following holds for every f and g in ${}^\xi 2$:

(5) $p_f \le p_{f|\nu}$ for every $\nu < \xi$.
(6) If $f \ne g$ then $\operatorname{ht} p_f = \operatorname{ht} p_g$ and if $\xi = \nu + 1$ then $\sup\{p_f, p_g\} = p_{f|\nu}$.
(7) If ξ is a limit ordinal then $\operatorname{ht} p_f = \sup_{\nu<\xi} \operatorname{ht} p_{f|\nu}$.
(8) $\operatorname{ht} p_f > \gamma_h$ for every $h \in \bigcup_{\nu<\xi} {}^\nu 2$.
(9) $\gamma_f > \alpha_\xi$ and if ξ is a successor ordinal then $p_f \Vdash \gamma_f \in \dot{C}$.

Then we put $\beta_\xi = \sup\{\gamma_f \cup \operatorname{ht} p_f : f \in {}^\xi 2\}$ if ξ is a successor ordinal and $\beta_\xi = \alpha_\xi + 1$ otherwise. Let $\alpha = \sup_{\xi<\mu} \alpha_\xi$. Since $\{p_f : f \in \bigcup_{\xi<\mu} {}^\xi 2\}$ is a μ-fan, there exists a function $f : \mu \to 2$ and a condition p such that $p \le p_{f|\xi}$ for every $\xi < \mu$. Now $p \Vdash \alpha \in \dot{C}$, which implies that $\alpha \notin S$. ∎

The combinatorial principle called *weak diamond* defined in [2] is equivalent to $2^{\aleph_0} < 2^{\aleph_1}$. The tree construction in the proof below is essentially due to Gregory [5]. The proof is considerably shortened by the use of the weak diamond principle of [2], which is implicitly proved in Gregory's construction.

PROPOSITION 2 (Gregory). *If* $2^{\aleph_0} < 2^{\aleph_1}$ *then there exists a semi-proper* $(\aleph_2, \aleph_1)$*-tree.*

Proof. We can recursively define a function $F : {}^{<\omega_1}2 \to 2$ with the following property: Every $\aleph_0$-fan of ${}^{<\omega_1}2$ has two cofinal branches such that if x and y are the unions of these branches then $F(x) \ne F(y)$. By the weak diamond principle there exists a function $g : \omega_1 \to 2$ such that $\{\alpha < \omega_1 : F(f|\alpha) = g(\alpha)\}$ is stationary for every $f : \omega_1 \to 2$. Clearly

$$T = \{f \in {}^{<\omega_1}2 : F(f|\alpha) \ne g(\alpha) \text{ for all } \alpha \in \operatorname{acc}^+(\operatorname{dom} f)\}$$

is a splitting $(\aleph_2, \aleph_1)$-tree. The function F was constructed in such a way that T is guaranteed to be $\aleph_0$-fan closed. Then T is a semi-proper $(\aleph_2, \aleph_1)$-tree by Lemma 3. ∎

Let E be a stationary subset of κ^+ where κ is some infinite cardinal. For $\delta \in E$, let $\eta_\delta : \operatorname{cf}\delta \to \delta$ be an increasing continuous function with limit δ. We let $\Phi(\eta_\delta : \delta \in E)$ denote the following combinatorial principle: There exists a sequence $(d_\delta : \delta \in E)$ where each d_δ is a function $\operatorname{cf}\delta \to \delta$ such that for any function $h : \kappa^+ \to 2$, there is a stationary set of ordinals $\delta \in E$ satisfying

$$\{i < \operatorname{cf}\delta : d_\delta(i) = h(\eta_\delta(i))\} \text{ is stationary in } \operatorname{cf}\delta.$$

The sequence $(d_\delta : \delta \in E)$ can be referred to as a *weak diamond* sequence.

We shall use the following result by Shelah [15, Appendix, Theorem 3.6]:

LEMMA 4. *If $\kappa = \kappa^{<\kappa}$ and $\kappa = 2^\theta$ for some cardinal θ, then $\Phi(\eta_\delta : \delta \in S^{\kappa^+}_\kappa)$ holds for any sequence $(\eta_\delta : \delta \in S^{\kappa^+}_\kappa)$ as defined above.*

PROPOSITION 3. *If $\kappa = \theta^+ = 2^\theta$ for some cardinal θ then there exists a semi-proper (κ^{++}, κ^+)-tree.*

Proof. Let $E = S^{\kappa^+}_\kappa$, fix $(\eta_\delta : \delta \in E)$, and let $(d_\delta : \delta \in E)$ be a weak diamond sequence given by $\Phi(\eta_\delta : \delta \in E)$. We claim that

$$T = \{f \in {}^{<\kappa^+}2 : \forall \delta \in E \cap \operatorname{acc}^+(\operatorname{dom} f)(\{i < \kappa : d_\delta(i) = f(\eta_\delta(i))\} \in \mathrm{NS}_\kappa)\}$$

is the required tree. Clearly T is a splitting (κ^{++}, κ^+)-tree. By Lemma 3 it then suffices to prove that T is κ-fan closed.

It is immediate from the definition that T is κ-closed. Let U a κ-fan of T and suppose that $(x_f : f \in {}^{<\kappa}2)$ and the sequence $(\delta_\xi : \xi < \kappa)$ satisfy conditions (1)–(4). Let $\delta = \sup_{\xi<\kappa} \delta_\xi$. By (4) we may assume without loss of generality that

$$x_f(\delta_\xi) = f(\xi) \quad \text{ for all } \xi < \kappa \text{ and } f : \xi + 1 \to 2.$$

Now we make use of the fact that $\{\delta_\xi : \xi < \kappa\} \cap \operatorname{ran} \eta_\delta$ is a club subset of δ. Define a function $f : \kappa \to 2$ by letting $f(\nu) = 1 - d_\delta(i)$ whenever $\eta_\delta(i) = \delta_\nu$. Now $\bigcup_{\xi<\kappa} x_{f|\xi}$ is in T, which shows that T is κ-fan closed. ∎

4. Semi-proper trees in ZFC. This entire section constitutes the proof of clause (e) of Theorem 4. For convenience we restate the result.

PROPOSITION 4. *For any regular $\kappa > \aleph_1$ there exists a semi-proper $((2^\kappa)^+, \kappa)$-tree.*

In most of the arguments in this section the assumption $\kappa > \aleph_1$ could be replaced by $\kappa \geq \aleph_1$. But at the end of the proof of Lemma 6 one needs to pick an ordinal $\delta < \kappa$ which is not the limit of a certain ω-sequence. This is accomplished by letting $\operatorname{cf}\delta > \omega$ and Lemma 5 is formulated with this in mind. Thus with $\kappa \geq \aleph_1$ one would get a slightly weaker version of Lemma 5 that would not suffice for Lemma 6. Recall that clause (a) of Theorem 4 indicates that the assumption $\kappa > \aleph_1$ is necessary.

We shall first define a tree T as a subtree of $\bigcup_{\alpha<\kappa} {}^{\alpha+1}\mathscr{P}(\kappa)$ ordered by inclusion. T will be a semi-proper $((2^\kappa)^+,\kappa)$-tree unless it has a κ-branch. If T has a κ-branch we shall use this branch to construct another tree that meets the requirements. In fact this second tree will be a semi-proper (κ^+,κ)-tree.

The first tree. For functions $p:\alpha+1\to\mathscr{P}(\kappa)$ we shall use the following notation. The ordinal α is denoted by $\alpha(p)$. For every $\beta\le\alpha$,

$$u_\beta=\begin{cases} p(\beta) & \text{if } p(\beta) \text{ is a closed subset of } \beta,\\ \emptyset & \text{otherwise,}\end{cases}$$

$$S_\beta=\begin{cases} p(\beta) & \text{if } p(\beta) \text{ is stationary in } \kappa,\\ \kappa & \text{otherwise.}\end{cases}$$

We write u^p_β and S^p_β for u_β and S_β, respectively, if p is not clear from the context.

We let $p\in T$ if and only if the following conditions hold whenever $\gamma<\beta\le\alpha$:

(1) If u_β is empty then $S_\beta=p(\beta)$.
(2) If $\gamma\in u_\beta$ then $u_\gamma=u_\beta\cap\gamma$.
(3) If β is a limit ordinal then u_β is unbounded in β.
(4) If $\gamma\in u_\beta$ and γ is a limit then $\gamma\notin S_{\min u_\gamma}$.

We shall now prove that forcing with T does not destroy stationary subsets of κ. Let S be a stationary set, let $p\in T$, and let $\dot C$ be a name that is forced by p to be club in κ. We construct a condition $q\le p$ such that $q\Vdash \dot C\cap\check S\ne\emptyset$. By induction on $i<\kappa$ we continue for as long as possible to pick conditions p_i and ordinals α_i such that the following holds when p_i and α_i have been defined for every $i<\zeta$:

(5) $S^{p_0}_{\alpha_0}=S$.
(6) $(p_i:i<\zeta)$ is decreasing and $p_0\le p$.
(7) $(\alpha_i:i<\zeta)$ is increasing and continuous.
(8) $p_{i+1}\Vdash \dot C\cap(\alpha_{i+1}\setminus\alpha_i)\ne\emptyset$.
(9) $\alpha(p_i)\ge\alpha_i$ (alternatively $\alpha(p_i)=\alpha_i$) and $u^{p_i}_{\alpha_i}=\{\alpha_j:j<i\}$.
(10) If α_i is a limit then i is a limit and $\alpha_i\notin S$.

We shall drop the superscripts on $u^{p_i}_\beta$ and $S^{p_i}_\beta$ because condition (6) makes them obsolete. Clearly we can put $p_0=p^\frown(S)$ and $\alpha_0=\alpha(p)+1$. We shall now check that appropriate p_{i+1} and α_{i+1} can always be picked once the preceding conditions and ordinals have been successfully defined. First pick $q\le p_i$ and $\gamma\ge\alpha_i$ such that $q\Vdash\gamma\in\dot C$. Then let $\alpha_{i+1}=\max\{\alpha(q),\gamma\}+1$. Now we shall define $p_{i+1}:\alpha_{i+1}+1\to\mathscr{P}(\kappa)$ by fixing u_β and S_β for ordinals β such that $\alpha(q)<\beta\le\alpha_{i+1}$. Let $u_{\alpha_{i+1}}=\{\alpha_j:j<i+1\}$ and if $\alpha_{i+1}>\alpha(q)+1$, let $S_{\alpha(q)+1}=\kappa\setminus\alpha_{i+1}$. Finally fill the possible gap by letting $u_\beta=\beta\setminus(\alpha(q)+1)$ for those ordinals β that satisfy $\alpha(q)+1<\beta<\alpha_{i+1}$.

Now suppose that we are about to pick p_i where i is a limit. By (7) we must have $\alpha_i = \bigcup_{j<i} \alpha_j$ in this situation. The only possible way to define $p_i(\alpha_i)$ is to let $u_{\alpha_i} = \{\alpha_j : j < i\}$. Let $q = (\bigcup_{j<i} p_j)^\frown(u_{\alpha_i})$. If α_i happens to be in the complement of S, we can make the induction go on by putting $p_i = q$. But if $\alpha_i \in S$ we are done with the proof because, in any case, $q \Vdash \alpha_i \in \dot{C}$. The latter must happen sooner or later because otherwise we finally have $S \cap \operatorname{acc}\{\alpha_i : i < \kappa\} = \emptyset$, contradicting the assumption that S is stationary.

The proof that T is normal is similar to the successor step in the construction above. If T does not have cofinal branches then the proposition is proved. Let us now assume that T has a cofinal branch and construct another tree that has the required properties.

The second tree. The cofinal branch through T gives us two sequences $(u_\beta : \beta < \kappa)$ and $(S_\beta : \beta < \kappa)$ such that u_β is a closed subset of β and S_β is stationary in κ for every $\beta < \kappa$ and the conditions (2)–(4) hold. For every $\alpha < \kappa$ let

$$S^*_\alpha = \{\beta < \kappa : \alpha \in u_\beta\} \tag{11}$$

and let E_α be a club subset of κ such that $S^*_\alpha \cap E_\alpha = \emptyset$ whenever S^*_α is non-stationary. Let E be the diagonal intersection $\{\beta < \kappa : \beta \in \bigcap_{\alpha<\beta} E_\alpha\}$. It is now easy to verify that if $\beta \in E$ then S^*_α is stationary for every $\alpha \in u_\beta$.

LEMMA 5. *There exist ordinals $\alpha(*)$ and $\beta(*)$ such that $\alpha(*) < \beta(*) < \kappa$, $S^*_{\alpha(*)}$ and $S^*_{\beta(*)} \cap S^\kappa_{\omega_1}$ are stationary in κ, and $S^*_{\alpha(*)} \cap S^*_{\beta(*)} = \emptyset$.*

Proof. First we shall find limit ordinals $\alpha, \beta \in E$ such that $\alpha < \beta$ and $\alpha \notin u_\beta$. Let α be a limit ordinal in E and let $\beta > \alpha$ be a limit ordinal in $E \cap S_{\min u_\alpha}$. Let $\gamma > \beta$ be a limit ordinal in E. If $\alpha \in u_\beta$ then $\beta \notin u_\gamma$ so the required ordinals can be picked by replacing, if necessary, α and β by β and γ respectively.

Fix $\alpha(*) \in u_\alpha$ such that $\alpha(*) > \bigcup(u_\beta \cap \alpha)$ and let $\beta(*) = \min(u_\beta \setminus \alpha)$. From what was noted above about E it is now clear that $S^*_{\alpha(*)}$ and $S^*_{\beta(*)}$ are both stationary and disjoint from each other. We shall now prove that $S^*_{\beta(*)} \cap S^\kappa_{\omega_1}$ can be assumed to be stationary. Suppose that C is a club such that $S^*_{\beta(*)} \cap S^\kappa_{\omega_1} \cap C = \emptyset$. Define a function $f : S^\kappa_{\omega_1} \setminus \beta(*) \to \kappa$ by $f(\gamma) = \min(u_\gamma \setminus \beta(*))$. By Fodor's lemma there exists a stationary set $S \subseteq S^\kappa_{\omega_1} \cap C$ and an ordinal $\delta(*)$ such that $f[S] = \{\delta(*)\}$. Now $S^*_{\delta(*)} \cap S^\kappa_{\omega_1}$ is stationary because it has S as a subset. We must have $\beta(*) \notin u_{\delta(*)}$ and $\beta(*) < \delta(*)$ because $\beta(*) \in u_{\delta(*)}$ or $\beta(*) = \delta(*)$ would imply that $S^*_{\delta(*)} \subseteq S^*_{\beta(*)}$, which contradicts the assumption that $S^*_{\beta(*)} \cap S^\kappa_{\omega_1}$ is non-stationary. But this means that $S^*_{\beta(*)}$ and $S^*_{\delta(*)}$ are disjoint and could thus serve as replacements for $S^*_{\alpha(*)}$ and $S^*_{\beta(*)}$ respectively. ∎

Fix ordinals $\alpha(*)$ and $\beta(*)$ with the properties stated in the last lemma. Next we shall construct a "club guessing" sequence that can be used in tree constructions in a similar way to the weak diamond principles presented in Section 3. For sets u and E of ordinals

$$\operatorname{drop}(u, E) = \{\sup(E \cap \alpha) : \alpha \in u,\ \alpha > \min E\}.$$

One can think of $\operatorname{drop}(u, E)$ as the result of "dropping" u onto E. (In [14], $\operatorname{drop}(u, E)$ is denoted by $\mathrm{g}\ell(u, E)$ where $\mathrm{g}\ell$ stands for "glue".) Some of the fundamental properties of drop that are needed below can be summarised as follows: If E is closed then $\operatorname{drop}(u, E) \subseteq E$. If u is a club subset of some limit ordinal δ and $E \cap \delta$ is club in δ then $\operatorname{drop}(u, E)$ is club in δ and $\operatorname{acc}(\operatorname{drop}(u, E)) \subseteq \operatorname{acc} u \cap \operatorname{acc} E$.

Lemma 6. *There exists a club* $E^* \subseteq \operatorname{acc}\kappa$ *and a sequence* $(C_\delta : \delta \in S^*_{\beta(*)} \cap \operatorname{acc} E^*)$ *such that*:

(12) C_δ *is club in* δ.

(13) $C_\delta \cap S^*_{\alpha(*)} \subseteq \operatorname{nacc} C_\delta$.

(14) *For any club* $E' \subseteq E^*$ *the set*

$$\{\delta \in S^*_{\beta(*)} \cap \operatorname{acc} E^* : \delta = \sup(E' \cap \operatorname{nacc} C_\delta \cap S^*_{\alpha(*)})\}$$

is stationary in κ.

(15) $\delta' \in u_\delta \cap S^*_{\beta(*)} \cap \operatorname{acc} E^*$ *implies* $C_{\delta'} = C_\delta \cap \delta'$.

Proof. Let $E_0 = \operatorname{acc}\kappa$ and let $C^0_\delta = \operatorname{drop}(u_\delta, E_0)$ for every $\delta \in S^*_{\beta(*)} \cap \operatorname{acc} E_0$. By recursion on n we define club sets E_n and sequences $(C^n_\delta : \delta \in S^*_{\beta(*)} \cap \operatorname{acc} E_n)$ such that $E_{n+1} \subseteq \operatorname{acc} E_n$,

$$\delta > \sup(E_{n+1} \cap \operatorname{nacc} C^n_\delta \cap S^*_{\alpha(*)}) \quad \text{for all } \delta \in S^*_{\beta(*)} \cap E_{n+1}, \tag{16}$$

and C^{n+1}_δ is defined by

$$C^{n+1}_\delta = C^n_\delta \cup \bigcup_\beta \operatorname{drop}(u_\beta, E_{n+1}) \setminus \gamma^n_\delta(\beta) \tag{17}$$

where the large union is taken over all $\beta \in (\operatorname{nacc} C^n_\delta) \setminus (S^*_{\alpha(*)} \cap E_{n+1})$ and

$$\gamma^n_\delta(\beta) = \begin{cases} \max((C^n_\delta \cap \beta) \cup \{0\}) & \text{if } \sup(E_{n+1} \cap \beta) = \beta, \\ \max((E_{n+1} \cap \beta) \cup \{0\}) & \text{otherwise.} \end{cases} \tag{18}$$

We claim that for some $n < \omega$ there exists no club $E_{n+1} \subseteq E_n$ satisfying (16), and that when this happens the sets $C_\delta = C^n_\delta$ and the set $E^* = E_n$ satisfy the conditions of the lemma.

In fact it is straightforward to check that conditions (12), (13), and (15) hold for every $n < \omega$ even if we drop the requirement (16) and just pick any club $E_{n+1} \subseteq \operatorname{acc} E_n$ during the construction. To see by induction that (12) and (13) hold, let $(\alpha_i : i < \zeta)$ be a strictly increasing sequence of ordinals in C^{n+1}_δ such that $\alpha = \sup_{i<\zeta} \alpha_i$ is a limit ordinal and $\alpha \leq \min(C^n_\delta \setminus \alpha_0)$. We shall verify that $\alpha \in C^{n+1}_\delta \setminus S^*_{\alpha(*)}$. Let β be the least ordinal in $(\operatorname{nacc} C^n_\delta) \setminus (S^*_{\alpha(*)} \cap E_{n+1})$ not less than α. Without loss of generality we may assume that $\{\alpha_i : 0 < i < \zeta\} = C^{n+1}_\delta \cap (\alpha_0, \alpha)$. Then

$$\{\alpha_i : 0 < i < \zeta\} = \operatorname{drop}(u_\beta, E_{n+1}) \cap (\alpha_0, \alpha)$$

by (18) and the fact that $\alpha \in E_{n+1}$ and $\beta \in C^n_\delta$.

First suppose that $\alpha \notin C^n_\delta$. Then $\alpha \in \operatorname{acc} u_\beta \cap \operatorname{acc} E_{n+1}$, which gives us $\alpha \in C^{n+1}_\delta$. If $\beta \in S^*_{\alpha(*)}$ then $\beta \notin E_{n+1}$ and it follows that $\gamma^n_\delta(\beta) \geq \alpha$, which contradicts the fact that $C^{n+1}_\delta \cap (\alpha_0, \alpha) \neq \emptyset$. Thus $\beta \notin S^*_{\alpha(*)}$, which implies that $u_\beta \cap S^*_{\alpha(*)} = \emptyset$ and thereby that $\alpha \notin S^*_{\alpha(*)}$. In the other case, where we have $\alpha \in C^n_\delta$, we only need to check that $\alpha \notin S^*_{\alpha(*)}$. But this is almost immediate since if $\alpha \in S^*_{\alpha(*)}$ we must have $\beta > \alpha$, which again implies the contradictory inequality $\gamma^n_\delta(\beta) \geq \alpha$.

For condition (15) in the case $n = 0$ we use (2) and note that $\delta' \in E_0$ and (3) gives $\operatorname{drop}(u_\delta, E_0) \cap \delta' = \operatorname{drop}(u_\delta \cap \delta', E_0)$. In the induction step $\delta' \in E_{n+1} \cap u_\delta$ implies $\delta' \in C^0_\delta \subseteq C^n_\delta$ by (2) and (3). Thus $\gamma^n_\delta(\beta) \geq \delta'$ for every $\beta > \delta'$, which clearly suffices.

It is also straightforward to see that (14) will hold when we reach a point where no club $E_{n+1} \subseteq \operatorname{acc} E_n$ satisfies (16). We shall now derive a contradiction from the assumption that (16) holds for every $n < \omega$. Let $E^\omega = \bigcap_{n<\omega} E_n$ and pick

$$\delta \in \operatorname{acc}^+(E^\omega \cap S^*_{\alpha(*)}) \cap S^*_{\beta(*)} \cap S^\kappa_{\omega_1}.$$

Let $\gamma_n = \sup(E_{n+1} \cap \operatorname{nacc} C^n_\delta \cap S^*_{\alpha(*)})$ and $\gamma = \sup_{n<\omega} \gamma_n$. Because $\delta \in S^\kappa_{\omega_1}$ we have $\operatorname{cf} \delta > \omega$ and thus by (16) and the fact that $\delta \in E^\omega \cap S^*_{\beta(*)}$ we have $\gamma < \delta$. Pick $\alpha \in E^\omega \cap S^*_{\alpha(*)}$ such that $\gamma < \alpha < \delta$ and let $\beta_n = \min(C^n_\delta \setminus \alpha)$ for every $n < \omega$. Clearly $\alpha \notin \operatorname{nacc} C^n_\delta$ and by (13) it then follows that $\alpha \notin C^n_\delta$. Thus $\beta_n > \alpha$. Because $\beta_n > \gamma$ we have $\beta_n \notin E_{n+1} \cap S^*_{\alpha(*)}$ and by (17) and (18) it then follows that $\beta_{n+1} < \beta_n$. This is a contradiction since $n < \omega$ was arbitrary. ∎

Fix a sequence $(C_\delta : \delta \in S^*_{\beta(*)} \cap \operatorname{acc} E^*)$ that satisfies the conditions of the lemma above. Let R_0 be the tree consisting of all closed bounded subsets of κ ordered by end extension and consider the subtree

$$R = \{c \in R_0 : \delta > \sup(c \cap \operatorname{nacc} C_\delta \cap S^*_{\alpha(*)}) \text{ for all } \delta \in S^*_{\beta(*)} \cap \operatorname{acc} E^*\}.$$

Note that intersecting with $S^*_{\alpha(*)}$ is not essential in the definition of R. As far as the argument that follows is concerned, $S^*_{\alpha(*)}$ could be dropped from the definition, or more exactly, replaced by any set that contains $S^*_{\alpha(*)}$. Condition (13) is essential however. We shall show that R is a semi-proper

(κ^+, κ)-tree. We start by noting that R cannot have κ-branches by condition (14). Also, for every $c \in R$ and $\alpha < \kappa$ there exists a condition $d \in R$ such that $d \leq c$ and $\max d > \alpha$. If R does not collapse κ, it then follows that forcing with R adds a κ-branch. We finish the proof of Proposition 4 by showing that R does not kill stationary sets.

Let S be an arbitrary stationary subset of κ, let $\dot{C}$ be an R-name for a club, and let $c \in R$. We shall find a condition $c^+ \leq c$ such that $c^+ \Vdash \dot{C} \cap \check{S} \neq \emptyset$.

Fix an increasing continuous sequence $(M_\eta : \eta < \kappa)$ of elementary submodels of H_χ, where χ is some large enough regular cardinal, such that $|M_\eta| < \kappa$,

$$M_{\eta+1} \cap \kappa \in S^*_{\alpha(*)}, \tag{19}$$

and $(M_\nu : \nu \leq \eta) \in M_{\eta+1}$ for all $\eta < \kappa$, and S, R, $\dot{C}$, $\alpha(*)$, $\beta(*)$, and the sequences $(u_\beta : \beta < \kappa)$ and $(C_\delta : \delta \in S^*_{\beta(*)} \cap E^*)$ are elements of M_0. Pick a limit ordinal $\delta(*) \in S \cap \operatorname{acc} E^*$ such that $M_{\delta(*)} \cap \kappa = \delta(*)$.

The rest of the proof is divided into two cases. In the first case we assume that $\delta(*) \notin S^*_{\beta(*)}$. By (2) and (11) it follows from this assumption that

$$u_{\delta(*)} \cap S^*_{\beta(*)} = \emptyset. \tag{20}$$

We shall define a decreasing sequence $(c_i : i < \zeta)$ of conditions in R simultaneously with an increasing sequence $(\alpha_i : i < \zeta)$ of ordinals such that $c_0 = c$, $\sup_{i<\zeta} \alpha_i = \delta(*)$ and the following conditions hold for every $i < \zeta$:

(21) $c_i \in M_{\delta(*)}$ and $\alpha_i < \delta(*)$,

(22) $\alpha_{i+1} \geq \max c_i$ and $c_{i+1} \Vdash \alpha_{i+1} \in \dot{C}$,

(23) $\max c_{i+1} > \min(u_{\delta(*)} \setminus \alpha_i)$.

We shall also assume that all the choices done during the construction are made using a choice function that is in $M_{\delta(*)}$. The length ζ of the sequence will be determined during the construction. The successor steps in the construction are straightforward and present no problems.

Now suppose that we are about to pick c_i and α_i where i is a limit ordinal. Let $\gamma = \sup_{j<i} \max c_j$. If $\gamma = \delta(*)$ we put $\zeta = i$ and the construction is successfully completed. Thus assume that $\gamma < \delta(*)$. Clearly the only things we have to show now is that

$$\bigcup_{j<i} c_j \cup \{\gamma\} \in R \tag{24}$$

and $(c_j : j < i) \in M_{\delta(*)}$. By condition (23), $\gamma \in u_{\delta(*)}$, which by (20) implies that $\gamma \notin S^*_{\beta(*)}$ and this takes care of (24). Because the sequence $(u_\beta : \beta < \kappa)$ is in $M_{\delta(*)}$ we also have $u_\gamma \in M_{\delta(*)}$. But since $u_\gamma = u_{\delta(*)} \cap \gamma$ and the choice function being used is in $M_{\delta(*)}$, we could obtain the same sequences

$(c_j : j < i)$ and $(\alpha_j : j < i)$ arguing in $M_{\delta(*)}$, if we replace $u_{\delta(*)}$ by u_γ in condition (23). Thus $(c_j : j < i) \in M_{\delta(*)}$. Having completed the construction we just need to put $c^+ = \bigcup_{i<\zeta} c_i \cup \{\delta(*)\}$ and note that $c^+ \Vdash \delta(*) \in \dot{C}$.

We shall now deal with the other case, where we have $\delta(*) \in S^*_{\beta(*)}$. We shall reconstruct the sequences $(c_i : i < \zeta)$ and $(\alpha_i : i < \zeta)$ in a slightly different way. We keep conditions (21) and (22) but replace (23) by the conditions

$$\max c_{i+1} > \max\{\min(u_{\delta(*)} \setminus \alpha_i), \min(E^* \setminus \alpha_i)\}, \tag{25}$$

$$c_i \cap \operatorname{nacc} C_{\delta(*)} = c_0 \cap \operatorname{nacc} C_{\delta(*)}, \tag{26}$$

and require that $\alpha_0 \geq \beta(*)$. We first deal with the successor step since now it requires some work. Suppose that c_i and α_i are defined. Let η be the least ordinal in $\delta(*)$ such that c_i and the ordinal $\max\{\min(u_{\delta(*)} \setminus \alpha_i), \min(E^* \setminus \alpha_i)\}$ are elements of M_η and let $\gamma = \sup(\operatorname{nacc} C_{\delta(*)} \cap M_{\eta+1})$. By (12), (13), and (19), $\gamma \in \kappa \cap M_{\eta+1}$. Then pick c_{i+1} and α_{i+1} in $M_{\eta+1}$ such that $c_{i+1} \leq c_i \cup \{\gamma + 1\}$ and conditions (22) and (25) are satisfied. In this way $c_{i+1} \cap \operatorname{nacc} C_{\delta(*)} = c_i \cap \operatorname{nacc} C_{\delta(*)}$, which takes care of (26).

Suppose then that i is a limit ordinal and $\gamma = \sup_{j<i} \max c_j < \delta(*)$. Because $\gamma \in u_{\delta(*)}$ and $\gamma > \beta(*)$ we have $\gamma \in S^*_{\beta(*)}$ by the assumption $\delta(*) \in S^*_{\beta(*)}$. Furthermore $\gamma \in \operatorname{acc} E^*$ and therefore $C_\gamma = C_{\delta(*)} \cap \gamma$ by (15). From now on the argument is very similar to the limit step in the case $\delta(*) \notin S^*_{\beta(*)}$. One difference is that $C_{\delta(*)}$ and C_γ now play the role of $u_{\delta(*)}$ and u_γ in the previous argument. We also have to note that the required initial segment of the sequence $(M_i : i < \delta(*))$ is in $M_{\delta(*)}$. Of course (20) does not hold now but instead condition (26) is designed to make (24) come true. This also applies in the final limit step where we again put $c^+ = \bigcup_{i<\zeta} c_i \cup \{\delta(*)\}$. We have found the required condition c^+, which concludes the proof of Proposition 4.

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