FUNDAMENTA
MATHEMATICAE
198 (2008)

Hurewicz–Serre theorem in extension theory

by

M. Cencelj (Ljubljana), **J. Dydak** (Knoxville, TN),
A. Mitra (Knoxville, TN), and **A. Vavpetič** (Ljubljana)

Abstract. The paper is devoted to generalizations of the Cencelj–Dranishnikov theorems relating extension properties of nilpotent CW complexes to their homology groups. Here are the main results of the paper:

THEOREM 0.1. *Let L be a nilpotent CW complex and F the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If X is a metrizable space such that $X\tau K(H_k(L),k)$ for all $k \geq 1$, then $X\tau K(\pi_k(F),k)$ and $X\tau K(\pi_k(L),k)$ for all $k \geq 2$.*

THEOREM 0.2. *Let X be a metrizable space such that $\dim(X) < \infty$ or $X \in ANR$. Suppose L is a nilpotent CW complex. If $X\tau SP(L)$, then $X\tau L$ in the following cases*:

(a) *$H_1(L)$ is finitely generated.*
(b) *$H_1(L)$ is a torsion group.*

1. Introduction. For basic facts on nilpotent groups see [17]. Localization of nilpotent groups is discussed in [13].

Recall that $X\tau L$ and $L \in AE(X)$ are shortcuts to the statement: L is an absolute extensor of X (see [7]). The following geometric result, repeatedly used in the paper, is a consequence of Theorem 1.9 in [9]: if F is the homotopy fiber of $L \to K$ and $X\tau F$, then $X\tau L$ is equivalent to $X\tau K$.

Given a metrizable space X and a connected CW complex L consider the following conditions:

(1) $X\tau L$.
(2) $X\tau K(\pi_n(L),n)$ for all $n \geq 1$.
(3) $X\tau K(H_n(L),n)$ for all $n \geq 1$.

2000 *Mathematics Subject Classification*: Primary 54F45; Secondary 55M10, 54C65.

Key words and phrases: extension dimension, cohomological dimension, absolute extensor, nilpotent groups.

Supported in part by the Slovenian–USA research grant BI-US/05-06/002 and the ARRS research project No. J1-6128-0101-04. The second-named author was partially supported by Grant No. 2004047 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

It is well-known that (1) implies (3) as proved by Dranishnikov [6] for X compact and for arbitrary X in [11, Theorem 3.4]. The difficulty in generalizing results in cohomological dimension theory from compact spaces to arbitrary metrizable spaces usually lies in the fact that the First Bockstein Theorem does not hold for metric spaces.

By a *Hurewicz–Serre Theorem in Extension Theory* we mean any result showing (3) implies (2). However, in practice we are really interested in arriving at (1) (see [4]).

Here is the main problem we are interested in:

PROBLEM 1.1. Suppose X is a metrizable space such that $X\tau K(H_n(L), n)$ for all $n \geq 1$. If L is nilpotent, does $X\tau K(\pi_n(L), n)$ hold for all $n \geq 1$?

Even a specialized version of 1.1 is open:

PROBLEM 1.2. Suppose L is a nilpotent CW complex. If X is a metrizable space such that $X\tau K(H_k(L), k)$ for all $k \geq 1$, does $X\tau K(\pi_1(L), 1)$ hold?

Notice that it is not sufficient to assume $X\tau K(H_1(L), 1)$ in Problem 1.2. Namely, take the group G from [18] whose abelianization is $\mathbb{Q}\oplus\mathbb{Q}$ and whose commutator group is $\mathbb{Z}/p^\infty$. Pick a compactum X so that $\dim_{\mathbb{Q}}(X) = 1$ and $\dim_{\mathbb{Z}/p^\infty}(X)=2$. The complex $L=K(G,1)$ is nilpotent and $X\tau K(H_1(L), 1)$, but $X\tau L$ does not hold. Indeed, as $\pi_1(L)$ is not $\bar{p}$-local and $H_1(L;\mathbb{Z}/p^\infty) = 0$, Lemma 5.4 says $H_2(L;\mathbb{Z}/p^\infty) \neq 0$, which means $H_2(L)/\mathrm{Tor}(H_2(L))$ is not p-divisible. If $X\tau L$, then $X\tau K(H_2(L), 2)$ and $\dim_{\mathbb{Z}_{(p)}}(X) \leq 2$, and that, in combination with $\dim_{\mathbb{Q}}(X) = 1$, implies $\dim_{\mathbb{Z}/p^\infty}(X) \leq 1$, a contradiction. See more in [2] about the First Bockstein Theorem for nilpotent groups.

However, if $H_1(L)$ is a torsion group, then the answer to Problem 1.2 is positive.

LEMMA 1.3. *Suppose N is a nilpotent group. If $X\tau K(\mathrm{Ab}(N), 1)$ for some metrizable space X and $\mathrm{Ab}(N)$ is a torsion group, then $X\tau K(N, 1)$.*

Proof. We use induction on the nilpotency class n of N. Let $\Gamma^n N = \Gamma^n$. Notice N/Γ^n is a nilpotent group of class $n-1$ whose abelianization is an image of $\mathrm{Ab}(N)$. Thus $X\tau K(N/\Gamma^n, 1)$. The epimorphism

$$\bigotimes^n \mathrm{Ab}\, N \to \Gamma^n N = \Gamma^n$$

implies $X\tau K(\Gamma^n, 1)$, so the fact that N is a central extension

$$1 \to \Gamma^n \to N \to N/\Gamma^n \to 1$$

concludes the proof. ∎

For the sake of completeness let us show (2) is always stronger than (3).

PROPOSITION 1.4. *Suppose X is a metrizable space and L is a connected CW complex. If $X\tau K(\pi_n(L), n)$ for all $n \geq 1$, then $X\tau K(H_n(L), n)$ for all $n \geq 1$.*

Proof. Let L_n be the CW complex obtained from L by killing all homotopy groups higher than n. Since L_n is obtained from L by attaching k-cells for $k > n+1$, $H_n(L_n) = H_n(L)$. Also, $X\tau L_n$, because $X\tau K(\pi_i(L_n), i)$ holds for all i and only finitely many homotopy groups of L_n are non-trivial (see Theorem G of [10]). Therefore $X\tau K(H_n(L_n), n)$. ■

Also, (1) is always stronger than (2) provided the issue of the fundamental group is avoided.

PROPOSITION 1.5. *Suppose X is a metrizable space and L is a connected CW complex. If $X\tau K(\pi_1(L), 1)$ and $X\tau L$, then $X\tau K(\pi_n(L), n)$ for all $n \geq 1$.*

Proof. Notice that the homotopy fiber of the covering projection $\widetilde{L} \to L$ is $K(\pi_1(L), 1)$. Therefore $X\tau\widetilde{L}$ and $X\tau K(H_n(\widetilde{L}), n)$ for all n. By Theorem F of [10] (see also Theorem 4.2 of this paper) one has $X\tau K(\pi_n(\widetilde{L}), n) = K(\pi_n(L), n)$ for all $n \geq 2$. ■

DEFINITION 1.6. X is called a *Knoxville space* if it is metrizable and for any connected CW complex L the conditions $X\tau K(\pi_n(L), n)$ for all $n \geq 1$ imply $X\tau L$.

PROBLEM 1.7. Characterize Knoxville spaces.

It follows from Theorem G of [10] that any finite-dimensional X or any $X \in ANR$ is a Knoxville space. Also, it is easy to see that any countable union of closed Knoxville subspaces is a Knoxville subspace.

2. Properties of the homotopy fiber of $L \to SP(L)$. Notice that condition (3) of Section 1 is equivalent to $X\tau SP(L)$ as $SP(L)$ is the weak product of $K(H_n(L), n)$ for all $n \geq 1$ according to the famous theorem of Dold and Thom [5]. Since we are interested in deriving $X\tau L$, it makes sense to ponder the stronger condition $X\tau F$, where F is the homotopy fiber of the inclusion $L \to SP(L)$. That is the main idea of the whole paper and in this section we concentrate on basic properties of F and its homotopy groups.

PROPOSITION 2.1. *Suppose L is a CW complex and F is the homotopy fiber of the inclusion i of L into its infinite symmetric product $SP(L)$. If L is nilpotent, then F is nilpotent.*

Proof. The homotopy sequence

$$\cdots \to \pi_n(F) \xrightarrow{j_*} \pi_n(L) \xrightarrow{i_*} \pi_n(SP(L)) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots$$

of the fibration $F \xrightarrow{j} L \xrightarrow{i} SP(L)$ is a sequence of $\pi_1(L)$-modules [15, Proposition $8^{\mathrm{bis}}.2$]. For the action of $\pi_1(L)$ on $\pi_n(F)$ which is described in the proof of [15, Proposition $8^{\mathrm{bis}}.2$] we have $g \cdot \alpha = j_*(g) \cdot \alpha$ for $\alpha \in \pi_n(F)$ and $g \in \pi_1(F)$.

Let I_F and I_L be the augmentation ideals of the group rings $\mathbb{Z}[\pi_1(F)]$ and $\mathbb{Z}[\pi_1(L)]$, respectively. Because L is a nilpotent, there is an integer c such that $(I_L)^c \pi_n(L) = 0$. Let $\eta \in (I_F)^c$ and $\alpha \in \pi_n(F)$. Then $j_*(\eta\alpha) = j_*(\eta) j_*(\alpha) = 0$, because $j_*(\eta) \in (I_L)^c$. Thus there exists $\beta \in \pi_{n+1}(SP(L))$ such that $\partial\beta = \eta\alpha$. Let $g \in \pi_1(F)$. Then $j_*(g) - 1 \in I_L$ and

$$\partial((j_*(g) - 1)\beta) = (j_*(g) - 1)\partial\beta = (j_*(g) - 1)\eta\alpha = (g - 1)\eta\alpha.$$

The action of $\pi_1(L)$ on $\pi_n(SP(L))$ is defined as $l\gamma = i_*(l)\gamma$ for $l \in \pi_1(L)$ and $\gamma \in \pi_n(SP(L))$. Hence

$$(j_*(g) - 1)\beta = (i_* j_*(g) - 1)\beta = (1 - 1)\beta = 0,$$

therefore $(g - 1)\eta\alpha = 0$. This shows that $(I_F)^{c+1}\pi_n(F) = 0$, so the space F is nilpotent. ∎

Proposition 2.2. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into $SP(L)$. If $\mathcal{P}$ is a set of primes such that $H_k(L)$ is a $\mathcal{P}$-torsion group for all $k \le n$, where $n \ge 1$ is given, then $\pi_k(F)$ is a $\mathcal{P}$-torsion group for all $1 \le k \le n+1$.*

Proof. Let $\mathcal{P}'$ be the complement of $\mathcal{P}$ in the set of all primes. Consider the localization $L_{(\mathcal{P}')}$ of L at $\mathcal{P}'$. Notice that $L_{(\mathcal{P}')}$ is n-connected, so the Hurewicz homomorphism $\phi_k : \pi_k(L_{(\mathcal{P}')}) \to H_k(L_{(\mathcal{P}')})$ is an isomorphism for $k = n + 1$ and an epimorphism for $k = n + 2$. Let us split the exact sequence $\cdots \to \pi_k(F) \to \pi_k(L) \to H_k(L) \to \cdots$ into $\cdots \to \pi_2(F) \to \pi_2(L) \to H_2(L) \to A \to 0$ and $1 \to A \to \pi_1(F) \to B \to 1$, where B is the commutator subgroup of $\pi_1(L)$. Localizing the first sequence at $\mathcal{P}'$ yields A being a $\mathcal{P}$-torsion group and $\pi_k(F)$ being $\mathcal{P}$-torsion for $2 \le k \le n+1$. Since B is $\mathcal{P}$-torsion, the assertion follows. ∎

Corollary 2.3. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into $SP(L)$. If $n > 1$ is such that $H_k(L)$ is a torsion group for all $k < n$, then for any metrizable space X the conditions $X \tau K(H_k(L), k)$ for all $k \le n$ imply $X \tau K(\pi_k(F), k)$ for all $1 \le k \le n$.*

Proof. The case $k = 1$ is taken care of by Lemma 1.3. If the p-torsion of $\pi_k(F)$ is not trivial, then Proposition 2.2 implies that the p-torsion of $H_m(L)$ is not trivial for some $m < k$. Hence $X \tau K(\mathbb{Z}/p^\infty, m)$ and $X \tau K(\mathbb{Z}/p, m+1)$. This implies $X \tau K(G, k)$ for all G in the Bockstein basis of $\pi_k(F)$, resulting in $X \tau K(\pi_k(F), k)$. ∎

3. Homotopy groups with coefficients. Given a countable Abelian group G consider a pointed compactum $P_n(G)$ whose integral cohomology is concentrated in dimension n and equals G. The nth homotopy group $\pi_n(L;G)$ of a pointed CW complex L is defined in [16] to be the set $[P_n(G),L]$ of pointed homotopy classes from $P_n(G)$ to L. If $P_{n-1}(G)$ exists (which is always true if $n>2$ or G is torsion free and $n\ge 2$), then $P_n(G)$ could be taken as the suspension $\Sigma P_{n-1}(G)$ of $P_{n-1}(G)$ with the resulting group structure on $\pi_n(L,G)$.

If one puts $D=P_2(G)$ (or $D=P_1(G)$ if G is torsion-free), then one can analyze homotopy groups of $L^D=\mathrm{Map}(D,L)$ and realize that $\pi_n(L^D)=\pi_{n+2}(L;G)$ (respectively, $\pi_n(L^D)=\pi_{n+1}(L;G)$). Therefore, given a Hurewicz fibration $F\to E\to B$, one concludes there is a long exact sequence $\cdots\to\pi_n(F;G)\to\pi_n(E;G)\to\pi_n(G;G)\to\pi_{n-1}(F;G)\to\cdots$ (see [16] for the special case of $G=\mathbb{Z}/m$) because $F^D\to E^D\to B^D$ is a Serre fibration.

In the special case of $G=\mathbb{Z}/m$ one can pick the Moore space $D=M(\mathbb{Z}/m,1)$ for $P_2(G)$. In that case one has a Serre fibration (that follows from the Homotopy Extension Theorem) $\mathrm{Map}(S^2,L)\to\mathrm{Map}(D,L)\to\mathrm{Map}(S^1,L)$ where S^1 is the 1-skeleton of D and $S^2=D/S^1$. The map $\mathrm{Map}(D,L)\to\mathrm{Map}(S^1,L)$ is simply restriction induced. Since the boundary homomorphism $\pi_{n+1}(B)\to\pi_n(F)$ in that case amounts to multiplication by m from $\pi_{n+1}(\mathrm{Map}(S^1,L))=\pi_{n+2}(L)$ to $\pi_n(\mathrm{Map}(S^2,L))=\pi_{n+2}(L)$, one concludes the following (see [16] for another way of deriving an equivalent result):

LEMMA 3.1. *Let $D=M(\mathbb{Z}/m,1)$ for some $m\ge 2$. For each pointed CW complex L and each $n\ge 0$ one has a natural exact sequence*

$$0\to\pi_{n+2}(L)\otimes\mathbb{Z}/m\to\pi_n(L^D)\to\pi_{n+1}(L)*\mathbb{Z}/m\to 0,$$

where $\pi_1(L)*\mathbb{Z}/m$ is $\{x\in\pi_1(L)\mid x^m=1\}$.

We are interested in homotopy groups with coefficients in $\mathbb{Z}/p^\infty$, the direct limit of $\mathbb{Z}/p\to\mathbb{Z}/p^2\to\cdots$. Notice that one can construct $P_2(\mathbb{Z}/p^\infty)$ as the inverse limit of $\cdots\to M(\mathbb{Z}/p^{n+1},1)\to M(\mathbb{Z}/p^n,1)\to\cdots\to M(\mathbb{Z}/p,1)$ which can be viewed as $M(\widehat{Z}_p,1)$, the Moore space for the p-adic integers $\widehat{Z}_p$ in terms of Steenrod homology. In that case Lemma 3.1 becomes

LEMMA 3.2. *Let p be prime. For each pointed CW complex L and each $n\ge 0$ one has a natural exact sequence*

$$0\to\pi_{n+2}(L)\otimes\mathbb{Z}/p^\infty\to\pi_{n+2}(L;\mathbb{Z}/p^\infty)\to\pi_{n+1}(L)*\mathbb{Z}/p^\infty\to 0,$$

where $\pi_1(L)*\mathbb{Z}/p^\infty$ is $\{x\in\pi_1(L)\mid x^{p^k}=1$ for some $k\ge 1\}$.

As a consequence of Lemmas 3.1, 3.2, and the Dold–Thom Theorem [5] $(\pi_n(SP(L)=H_n(L))$, one can deduce that $\pi_n(SP(L);G)=H_n(L;G)$ for all n and $G=\mathbb{Z}/p$ or $G=\mathbb{Z}/p^\infty$.

Proposition 3.3. *Suppose L is a nilpotent CW complex whose fundamental group is $\bar{p}$-local for some prime p. Let F be the homotopy fiber of the inclusion $i : L \to SP(L)$. If $H_k(L;\mathbb{Z}/p) = 0$ for $k \le n$, where $n \ge 1$, then $\pi_k(F;\mathbb{Z}/p) = 0$ for $2 \le k \le n+1$ and $\pi_k(L;\mathbb{Z}/p) = 0$ for $2 \le k \le n$.*

Proof. Let $\widetilde{L}$ be the universal cover of L and let $\pi : \widetilde{L} \to L$ be the covering projection. Recall that every nilpotent CW complex L has the p-completion L_p with the map $L \to L_p$ inducing isomorphisms of all homology groups with coefficients in $\mathbb{Z}/p$ such that one has a natural exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_n(L)) \to \pi_n(L_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(L)) \to 0$$

for all $n \ge 1$ (see [12, Theorem 3.7 on p. 416]). By Lemmas 5.2 and 5.3 one has $\mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_1(L)) = \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_1(L)) = 0$, so the induced map $\widehat{\widetilde{L}}_p \to \widehat{L}_p$ is a homotopy equivalence. Therefore it induces isomorphism of homology mod p and π induces isomorphism of homology mod p. However, $H_2(\widetilde{L};\mathbb{Z}/p) = \pi_2(\widetilde{L};\mathbb{Z}/p) = \pi_2(L;\mathbb{Z}/p)$ and $\pi_3(L) \to H_3(\widetilde{L})$ is an epimorphism resulting in $\pi_3(L;\mathbb{Z}/p) \to H_3(L;\mathbb{Z}/p)$ being an epimorphism. By exactness of mod p groups of a fibration one gets $\pi_2(F;\mathbb{Z}/p) = 0$. That proves the assertion for $n = 1$.

If $n > 1$ then we apply the mod p Hurewicz Theorem of [16] to $\widetilde{L}$ to conclude that $\pi_{n+1}(\widetilde{L};\mathbb{Z}/p) \to H_{n+1}(\widetilde{L};\mathbb{Z}/p)$ is an isomorphism and that $\pi_{n+2}(\widetilde{L};\mathbb{Z}/p) \to H_{n+2}(\widetilde{L};\mathbb{Z}/p)$ is an epimorphism. Hence, $\pi_{n+1}(L;\mathbb{Z}/p) \to H_{n+1}(L;\mathbb{Z}/p)$ is an isomorphism and $\pi_{n+2}(L;\mathbb{Z}/p) \to H_{n+2}(L;\mathbb{Z}/p)$ is an epimorphism. Thus $\pi_{n+1}(F;\mathbb{Z}/p) = 0$. ∎

Corollary 3.4. *Suppose L is a nilpotent CW complex whose fundamental group is $\bar{p}$-local for some prime p. Let F be the homotopy fiber of the inclusion $i : L \to SP(L)$. If $H_k(L;\mathbb{Z}/p^\infty) = 0$ for $k \le n$, where $n \ge 2$, then $\pi_k(L;\mathbb{Z}/p^\infty) = \pi_k(F;\mathbb{Z}/p^\infty) = 0$ for $2 \le k \le n$.*

Proof. Case 1: $n > 2$. Notice $H_k(L;\mathbb{Z}/p) = 0$ for $k \le n-1$, resulting in $\pi_k(F;\mathbb{Z}/p) = 0$ for $2 \le k \le n$. Hence $\pi_k(F;\mathbb{Z}/p^\infty) = 0$ for $2 \le k \le n$ and from an exact sequence we get $\pi_k(L,\mathbb{Z}/p^\infty) = 0$ for $2 \le k \le n$.

Case 2: $n = 2$. Let $\widetilde{L}$ be the universal cover of L and let $\pi : \widetilde{L} \to L$ be the covering projection. Notice that the induced map $\widehat{\widetilde{L}}_p \to \widehat{L}_p$ is a homotopy equivalence. Therefore it induces isomorphism of homology mod p and π induces isomorphism of homology mod p. However, $H_2(\widetilde{L};\mathbb{Z}/p^\infty) = \pi_2(L;\mathbb{Z}/p^\infty)$ and $\pi_3(L) \to H_3(\widetilde{L})$ is an epimorphism resulting in $\pi_3(L;\mathbb{Z}/p^\infty) \to H_3(L;\mathbb{Z}/p^\infty)$ being an epimorphism. By exactness of mod p groups of a fibration one gets $\pi_2(F;\mathbb{Z}/p^\infty) = \pi_2(L;\mathbb{Z}/p^\infty) = 0$. ∎

Proposition 3.5. *Suppose L is a nilpotent CW complex, F is the homotopy fiber of the inclusion $i : L \to SP(L)$ of L into its infinite symmetric* *product, and* $n \geq 2$. *If* $H_k(L;\mathbb{Z}_{(\mathcal{P})}) = 0$ *for* $k \leq n$, *then* $\pi_k(F;\mathbb{Z}_{(\mathcal{P})}) = 0$ *for* $2 \leq k \leq n+1$ *and* $\pi_k(L;\mathbb{Z}_{(\mathcal{P})}) = 0$ *for* $2 \leq k \leq n$.

Proof. Let $\mathcal{P}'$ be the complement of $\mathcal{P}$ in the set of all primes. If $H_k(L;\mathbb{Z}_{(\mathcal{P})}) = 0$ for $k \leq n$, then $H_1(L)$ is a $\mathcal{P}'$-torsion group resulting in $\pi_1(L)$ being a $\mathcal{P}'$-torsion group. Let $L_{(\mathcal{P})}$ be the $\mathcal{P}$-localization of L. It is n-connected, so by the classical Hurewicz Theorem $\pi_k(L_{(\mathcal{P})}) \to H_k(L_{(\mathcal{P})})$ is an isomorphism for $k \leq n+1$ and an epimorphism for $k = n+2$. That corresponds to $\pi_k(L;\mathbb{Z}_{(\mathcal{P})}) \to H_k(L;\mathbb{Z}_{(\mathcal{P})})$ being an isomorphism for $k \leq n+1$ and an epimorphism for $k = n+2$. In view of exactness of $\cdots \to \pi_k(F;\mathbb{Z}_{(\mathcal{P})}) \to \pi_k(L;\mathbb{Z}_{(\mathcal{P})}) \to H_k(L;\mathbb{Z}_{(\mathcal{P})}) \to \cdots$, the assertion follows. ∎

4. Main results

LEMMA 4.1. *Suppose* X *is a metrizable space*, G *is an Abelian group*, *and* $n \geq 1$. *If* $\dim_G(X) > n$, *then one of the following conditions holds*:

(a) $\dim_{\mathbb{Q}}(X) \geq n+1$ *and* G *is not a torsion group.*
(b) *There is a prime* p *such that* $G \otimes \mathbb{Z}/p^\infty \neq 0$ *and* $\dim_{\mathbb{Q}}(X) \leq n$, $\dim_{\mathbb{Z}/p^\infty}(X) \geq n$.
(c) *There is a prime* p *such that* G *is* p*-divisible*, $\mathrm{Tor}_p(G) \neq 0$, *and* $\dim_{\mathbb{Z}/p^\infty}(X) \geq n+1$.
(d) *There is a prime* p *such that* $\mathrm{Tor}_p(G)$ *is not* p*-divisible and* $\dim_{\mathbb{Z}/p}(X) \geq n+1$.

Proof. Let $\dim_G(X) = m$. Suppose none of (a)–(d) holds. According to part (b) of Theorem B of [10] one has $m = \dim_{G/\mathrm{Tor}(G)}(X)$ or $m = \dim_{\mathrm{Tor}(G)}(X)$. If $\dim_{\mathrm{Tor}(G)}(X) \geq n+1$, then, according to part (a) of Theorem B of [10], there is a prime p such that either $\mathrm{Tor}(G)$ is p-divisible, $\mathrm{Tor}_p(G) \neq 0$, and $\dim_{\mathrm{Tor}(G)}(X) = \dim_{\mathbb{Z}/p^\infty}(X)$ (in which case (c) holds), or $\mathrm{Tor}(G)$ is not p-divisible, $\mathrm{Tor}_p(G) \neq 0$, and $\dim_{\mathrm{Tor}(G)}(X) = \dim_{\mathbb{Z}/p}(X)$ (in which case (d) holds). Therefore $m = \dim_{G/\mathrm{Tor}(G)}(X)$ and $\dim_{\mathrm{Tor}(G)}(X) \leq n$. In particular, G is not a torsion group, so $\dim_{\mathbb{Q}}(X) \leq n$ as (a) fails to hold.

Consider $\mathcal{P} = \{p \mid G \otimes \mathbb{Z}/p^\infty \neq 0\}$, the set of primes p such that $G/\mathrm{Tor}(G)$ is not p-divisible. It is shown in [10] (part (f) of Theorem B) that $\dim_{\mathbb{Z}_{(\mathcal{P})}}(X) \geq \dim_{G/\mathrm{Tor}(G)}(X)$, so $\dim_{\mathbb{Z}_{(\mathcal{P})}}(X) \geq m$. As (b) does not hold, it follows that $\dim_{\mathbb{Z}/p^\infty}(X) \leq n-1$ for all $p \in \mathcal{P}$. From the exact sequence

$$0 \to \mathbb{Z}_{(\mathcal{P})} \to \mathbb{Q} \to \bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p^\infty \to 0$$

one concludes that the homotopy fiber of $K(\mathbb{Z}_{(\mathcal{P})}, m-1) \to K(\mathbb{Q}, m-1)$ is $K(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p^\infty, m-2)$. Since $m-2 \geq n-1$, $X\tau K(\bigoplus_{p\in\mathcal{P}} \mathbb{Z}/p^\infty, m-2)$, which implies $X\tau K(\mathbb{Z}_{(\mathcal{P})}, m-1)$ as $X\tau K(\mathbb{Q}, m-1)$. Thus $\dim_{\mathbb{Z}_{(\mathcal{P})}}(X) \le m-1$, a contradiction. ∎

THEOREM 4.2. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into $SP(L)$. If X is a metrizable space such that $X\tau K(H_k(L), k)$ for all $k \ge 1$, then $X\tau K(\pi_k(F), k)$ and $X\tau K(\pi_k(L), k)$ for all $k \ge 2$.*

Proof. Suppose $n \ge 2$ is the smallest k such that $X\tau K(\pi_k(F), k)$ fails (similar argument in case $X\tau K(\pi_k(L), k)$ fails). By Lemma 4.1 one of the following cases holds for $G = \pi_n(F)$:

(a) $\dim_{\mathbb{Q}}(X) \ge n+1$ and G is not a torsion group.
(b) There is a prime p such that $G \otimes \mathbb{Z}/p^\infty \ne 0$ and $\dim_{\mathbb{Q}}(X) \le n$, $\dim_{\mathbb{Z}/p^\infty}(X) \ge n$.
(c) There is a prime p such that G is p-divisible, $\mathrm{Tor}_p(G) \ne 0$, and $\dim_{\mathbb{Z}/p^\infty}(X) \ge n+1$.
(d) There is a prime p such that $\mathrm{Tor}_p(G)$ is not p-divisible and $\dim_{\mathbb{Z}/p}(X) \ge n+1$.

CASE 1: $\dim_{\mathbb{Q}}(X) \le n-1$. Now only (b)–(d) are possible. Let p be the prime from one of those cases. Notice $H_k(L)$ is $\bar{p}$-local for $k \le n-1$ as otherwise $\dim_{\mathbb{Z}/p^\infty}(X) \le n-1$ and $\dim_{\mathbb{Z}/p}(X) \le n$ so none of (b)-(d) would be valid. Another observation is $H_n(L) \otimes \mathbb{Z}/p^\infty = 0$. Indeed, otherwise $H_n(L)/\mathrm{Tor}(H_n(L))$ is not p-divisible, in which case [10, part (d) of Theorem B] implies $\dim_{\widehat{\mathbb{Z}}_p}(X) \le n$ as $\dim_{H_n(L)}(X) \le n$. Therefore $\dim_{\mathbb{Z}_{(p)}}(X) \le n$ (see part (e) of Theorem B in [10]) and $\dim_{\mathbb{Z}/p^\infty}(X) \le \max(\dim_{\mathbb{Q}}(X), \dim_{\mathbb{Z}_{(p)}}(X) - 1) \le n-1$, a contradiction.

$\pi_1(L)$ is $\bar{p}$-local by Lemma 5.4 and $G \otimes \mathbb{Z}/p^\infty = 0$ by Corollary 3.4. That means (b) is not possible. If $H_n(L)$ is not p-divisible, then $\dim_{\mathbb{Z}/p}(X) \le n$ and neither (c) nor (d) would be possible. Thus $H_k(L; \mathbb{Z}/p) = 0$ for $k \le n$, resulting in G being p-divisible by Proposition 3.3. That means only (c) is possible. In addition, $\mathrm{Tor}_p(H_n(L)) = 0$. Also $H_{n+1}(L) \otimes \mathbb{Z}/p^\infty = 0$ (otherwise $\dim_{\mathbb{Z}_{(p)}}(X) \le n+1$ and $\dim_{\mathbb{Z}/p^\infty}(X) \le \max(\dim_{\mathbb{Q}}(X), \dim_{\mathbb{Z}_{(p)}}(X) - 1) \le n$). Thus $H_k(L; \mathbb{Z}/p^\infty) = 0$ for $k \le n+1$. By Corollary 3.4, $\mathrm{Tor}_p(G) = 0$, a contradiction.

CASE 2: $\dim_{\mathbb{Q}}(X) > n-1$. By Proposition 2.2 the group G is $\mathcal{P}$-torsion such that $\dim_{\mathbb{Z}/p^\infty}(X) \le n-1$ for all $p \in \mathcal{P}$, which implies $\dim_G(X) \le n$, a contradiction. ∎

COROLLARY 4.3. *Suppose L is a nilpotent CW complex such that $\pi_n(L) = \pi_{n+1}(L) = 0$ for some $n \ge 1$. If $X\tau SP(L)$ for some metrizable space X, then $X\tau K(H_{n+1}(L), n)$.*

Proof. If $n = 1$, then $H_{n+1}(L) = 0$, so assume $n \geq 2$. Notice that $\pi_n(F) = H_{n+1}(L)$, where F is the homotopy fiber of $i : L \to SP(L)$. ∎

LEMMA 4.4. *Suppose L is a nilpotent CW complex and F is the homotopy fiber of the inclusion i of L into $SP(L)$. If $X\tau K(H_1(L), 1)$ for some metrizable space X and $H_1(L)$ is finitely generated, then $X\tau K(\pi_1(F), 1)$.*

Proof. If $H_1(L)$ is finitely generated and non-torsion, then X is at most 1-dimensional, in which case $X\tau L$ for all connected CW complexes. Therefore assume $H_1(L)$ is a torsion group and (see Proposition 2.2) there is an exact sequence $1 \to A \to \pi_1(F) \to B \to 1$ such that A and B are $\mathcal{P}$-torsion groups, where $\mathcal{P} = \{p \mid \mathrm{Tor}_p(H_1(L)) \neq 0\}$. Notice $H_1(L)$ does not contain $\mathbb{Z}/p^\infty$ for any prime p, so $X\tau K(A, 1)$ and $X\tau K(B, 1)$, which implies $X\tau K(\pi_1(F), 1)$. ∎

THEOREM 4.5. *Let X be a metrizable space such that* $\dim(X) < \infty$ *or $X \in ANR$. Suppose L is a nilpotent CW complex. If $X\tau SP(L)$, then $X\tau L$ in the following cases*:

(a) *$H_1(L)$ is finitely generated.*
(b) *$H_1(L)$ is a torsion group.*

Proof. (a) By Theorem 4.2 and Lemma 4.4 one concludes $X\tau K(\pi_n(F), n)$ for all $n \geq 1$. Theorem G of [10] gives $X\tau F$, which implies $X\tau L$.

(b) By Theorem 4.2 and Lemma 1.3 one concludes $X\tau K(\pi_n(L), n)$ for all $n \geq 1$. Theorem G of [10] yields $X\tau L$. ∎

5. Appendix

LEMMA 5.1. *Suppose N is a nilpotent group and p is a prime. Then* $\mathrm{Ab}(N)$ *is p-divisible if and only if N is p-divisible.*

Proof. If N is p-divisible clearly so is its abelianization. We will prove the converse by induction on the nilpotency class n of N. Let $\Gamma^n N = \Gamma^n$. Notice N/Γ^n is a nilpotent group of class $n - 1$ whose abelianization is p-divisible. Thus the group itself is p-divisible. The epimorphism

$$\bigotimes^n \mathrm{Ab}(N) \to \Gamma^n N = \Gamma^n$$

implies Γ^n is p-divisible, so the fact that N is a central extension

$$1 \to \Gamma^n \to N \to N/\Gamma^n \to 1$$

concludes the proof. ∎

LEMMA 5.2. *Suppose N is a nilpotent group and p is a prime. The following conditions are equivalent*:

(a) $\mathrm{Ext}(\mathbb{Z}/p^\infty, N) = 0$,
(b) $\mathrm{Ext}(\mathbb{Z}/p^\infty, N)$ *is p-divisible*,
(c) *N is p-divisible.*

Proof. (a)$\Rightarrow$(b) is obvious. For (c)$\Rightarrow$(a) let N be p-divisible. Then so is its abelianization and Proposition 3 of [3] implies $\mathrm{Ext}(\mathbb{Z}/p^\infty, N) = 0$.

(b)$\Rightarrow$(c). If N is not p-divisible, neither is its abelianization by Lemma 5.1. Therefore, by Proposition 3 of [3], $\mathrm{Ext}(\mathbb{Z}_{p^\infty}, \mathrm{Ab}(N))$ is not p-divisible. Then the six-term exact sequence of Hom and Ext [1, p. 170] implies that $\mathrm{Ext}(\mathbb{Z}/p^\infty, N)$ is not p-divisible. ∎

LEMMA 5.3. *Suppose G is a nilpotent group and p is a prime. The following conditions are equivalent*:

(a) $\mathrm{Hom}(\mathbb{Z}/p^\infty, G) = 0$,
(b) $\mathrm{Hom}(\mathbb{Z}/p^\infty, G)$ *is p-divisible*,
(c) $\mathrm{Hom}(\mathbb{Z}/p^\infty, G) \otimes \mathbb{Z}/p^\infty = 0$,
(d) *G does not contain $\mathbb{Z}/p^\infty$.*

Proof. Note that albeit Bousfield and Kan [1] defined Hom as a space, they showed that it is also the set of the respective homomorphisms.

(a)$\Rightarrow$(b) and (b)$\Rightarrow$(c) are obvious.

(c)$\Rightarrow$(b). Notice the p-torsion of $\mathrm{Hom}(\mathbb{Z}/p^\infty, G)$ is trivial. Indeed, if $i : \mathbb{Z}/p^\infty \to G$ and $i^p = 1$, then for any $a \in \mathbb{Z}/p^\infty$ we find $b \in \mathbb{Z}/p^\infty$ satisfying $b^p = a$. Now, $i(a) = i(b^p) = i^p(b) = 1$. If an Abelian group A has no p-torsion and $A \otimes \mathbb{Z}/p^\infty = 0$, then A is p-divisible.

(b)$\Rightarrow$(d). Suppose $i : \mathbb{Z}/p^\infty \to G$ is a monomorphism. Given $a \in \mathbb{Z}/p^\infty$ find $k \geq 1$ such that $a^{p^k} = 1$ and choose $\phi : \mathbb{Z}/p^\infty \to G$ so that $i = \phi^{p^k}$. Now $i(a) = \phi^{p^k}(a) = (\phi(a))^{p^k} = \phi(a^{p^k}) = \phi(1) = 1$, a contradiction.

(d)$\Rightarrow$(a). Given a non-trivial $i : \mathbb{Z}/p^\infty \to G$ its image is a direct sum of copies of $\mathbb{Z}/p^\infty$, a contradiction. ∎

LEMMA 5.4. *Suppose L is a nilpotent CW complex and p is a prime. If $H_1(L; \mathbb{Z}/p^\infty) = H_2(L; \mathbb{Z}/p^\infty) = 0$, then $\pi_1(L)$ is $\bar{p}$-local.*

Proof. In view of $H_2(L; \mathbb{Z}/p^\infty) = 0$, $H_1(L)$ has trivial p-torsion and $H_1(L; \mathbb{Z}/p^\infty) = 0$ implies $H_1(L)$ is p-divisible. Hence so is $\pi_1(L)$ (see Lemma 5.1). Consider the p-completion $\widehat{L}_p$ of L. As $\pi_1(\widehat{L}_p) = \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_1(L)) = 0$ and $H_2(\widehat{L}_p; \mathbb{Z}/p^\infty) = H_2(L; \mathbb{Z}/p^\infty) = 0$ one gets $\pi_2(\widehat{L}_p) \otimes \mathbb{Z}/p^\infty = 0$ by the Hurewicz Theorem. The exact sequence

$$0 \to \mathrm{Ext}(\mathbb{Z}/p^\infty, \pi_2(L)) \to \pi_2(\widehat{L}_p) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_1(L)) \to 0$$

implies $\mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_1(L)) \otimes \mathbb{Z}/p^\infty = 0$. By Lemma 5.3, $\pi_1(L)$ is $\bar{p}$-local. ∎

References

[1] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localizations*, 2nd corrected printing, Lecture Notes in Math. 304, Springer, Berlin, 1987.

[2] M. Cencelj, J. Dydak, A. Mitra, and A. Vavpetič, *Bockstein theorem for nilpotent groups*, preprint.
[3] M. Cencelj and A. N. Dranishnikov, *Extension of maps to nilpotent spaces II*, Topology Appl. 124 (2002), 77–83.
[4] —, —, *Extension of maps to nilpotent spaces III*, ibid. 153 (2005), 208–212.
[5] A. Dold und R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. 67 (1958), 239–281.
[6] A. N. Dranishnikov, *Extension of mappings into CW complexes*, Mat. USSR-Sb. 74 (1993), 47–56.
[7] —, *Cohomological Dimension Theory of Compact Metric Spaces*, Topology Atlas, 1999.
[8] A. N. Dranishnikov and J. Dydak, *Extension dimension and extension types*, Proc. Steklov Inst. Math. 212 (1996), 55–88.
[9] —, —, *Extension theory of separable metrizable spaces with applications to dimension theory*, Trans. Amer. Math. Soc. 353 (2000), 133–156.
[10] J. Dydak, *Cohomological dimension and metrizable spaces*, ibid. 337 (1993), 219–234.
[11] —, *Cohomological dimension and metrizable spaces*, ibid. 348 (1996), 1647–1661.
[12] P. G. Goerss and J. F. Jardine, *Simplicial Homotopy Theory*, Birkhäuser, Basel, 1999.
[13] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, North-Holland, Amsterdam, and American Elsevier, New York, 1975.
[14] V. I. Kuz'minov, *Homological dimension theory*, Russian Math. Surveys 23 (1968), no. 5, 1–45.
[15] J. McCleary, *A User's Guide to Spectral Sequences*, 2nd ed., Cambridge Stud. Adv. Math. 58, Cambridge Univ. Press, 2001.
[16] J. A. Neisendorfer, *Primary homotopy theory*, Mem. Amer. Math. Soc. 232 (1980).
[17] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer, New York, 1993.
[18] R. B. Warfield, *Nilpotent groups*, Lecture Notes in Math. 513, Springer, Berlin, 1976.

IMFM
Univerza v Ljubljani
Jadranska ulica 19
SI-1111 Ljubljana, Slovenija
E-mail: matija.cencelj@guest.arnes.si

University of Tennessee
Knoxville, TN 37996, U.S.A.
E-mail: dydak@math.utk.edu
ajmitra@math.utk.edu

Fakulteta za Matematiko in Fiziko
Univerza v Ljubljani
Jadranska ulica 19
SI-1111 Ljubljana, Slovenija
E-mail: ales.vavpetic@fmf.uni-lj.si

Received 31 March 2006