Dimensions of the Julia sets of rational maps with the backward contraction property

by

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Abstract. Consider a rational map $f$ on the Riemann sphere of degree at least 2 which has no parabolic periodic points. Assuming that $f$ has Rivera-Letelier’s backward contraction property with an arbitrarily large constant, we show that the upper box dimension of the Julia set $J(f)$ is equal to its hyperbolic dimension, by investigating the properties of conformal measures on the Julia set.

1. Introduction. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree $d \geq 2$ on the Riemann sphere. We are interested in the fractal properties of the Julia set $J(f)$. It is well known that in the case that $f$ is hyperbolic, all possible dimensions coincide. In [4], this result was generalized to all rational maps which satisfy a summability condition. See [11] for more historical remarks and advances in this direction.

The summability condition and the stronger Collet–Eckmann condition can be considered as non-uniform hyperbolicity conditions. As shown by J. Rivera-Letelier [9], they imply a backward contraction condition (see the definition below) which was first introduced therein.

In the following, all the distances, diameters and norms of derivatives are measured using the spherical metric and $B(z, r)$ denotes a ball of radius $r$ centered at $z$. Let $\text{Crit}(f)$ denote the set of critical points of $f$ and let

$$\text{Crit}'(f) = \text{Crit}(f) \cap J(f).$$

For every $c \in \text{Crit}(f)$ and $\delta > 0$ we denote by $\tilde{B}(c, \delta)$ the connected component of $f^{-1}(B(f(c), \delta))$ that contains $c$.

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Definition 1. Given a constant \( r > 1 \), we say that \( f \) has the backward contraction property with constant \( r \) (\( f \in BC(r) \) for short) if there exists \( \delta_0 > 0 \) such that for every \( c \in \text{Crit}'(f) \), every \( 0 < \delta \leq \delta_0 \), every integer \( n \geq 1 \) and every component \( W \) of \( f^{-n}(B(c, r\delta)) \), we have
\[
\text{dist}(W, CV(f)) \leq \delta \Rightarrow \text{diam}(W) < \delta,
\]
where \( CV(f) = f(\text{Crit}(f)) \). If \( f \in BC(r) \) for every \( r > 1 \), we will say that \( f \in BC(\infty) \).

We call a compact forward invariant subset \( X \) of \( \mathbb{C} \) hyperbolic if there exist \( C > 0 \) and \( \lambda > 1 \) such that for every \( n \geq 1 \) and every \( z \in X \),
\[
|Df^n(z)| \geq C\lambda^n.
\]
Clearly, a hyperbolic set is contained in the Julia set.

For a compact set \( X \subset \mathbb{C} \), let \( \text{HD}(X) \) denote its Hausdorff dimension. The hyperbolic dimension \( \text{HD}_{\text{hyp}}(f) \) of \( f \) is the supremum of the Hausdorff dimensions of hyperbolic subsets of \( J(f) \), i.e.
\[
\text{HD}_{\text{hyp}}(f) = \sup\{\text{HD}(X) : X \text{ is a hyperbolic subset of } J(f)\}.
\]
Clearly, \( \text{HD}_{\text{hyp}}(f) \leq \text{HD}(J(f)) \).

The main goal of this paper is to prove the following theorem.

Main Theorem. Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a rational map of degree at least 2 without parabolic periodic points. If \( f \in BC(\infty) \), then the upper box dimension \( \text{BD}(J(f)) \) of the Julia set is equal to the hyperbolic dimension of \( f \):
\[
\text{BD}(J(f)) = \text{HD}(J(f)) = \text{HD}_{\text{hyp}}(f).
\]

For the definition of the upper and lower box dimensions and the Hausdorff dimension, see [3]. Let us mention the following well-known inequality: \( \text{HD}(X) \leq \text{BD}(X) \leq \overline{\text{BD}}(X) \).

The proof of the Main Theorem is based on analyzing the regularity of conformal measures. Recall that a probability measure \( \mu \) on \( J(f) \) is said to be \( t \)-conformal for \( f \) if for every Borel set \( A \subset J(f) \) such that \( f|_A \) is injective, we have
\[
\mu(f(A)) = \int_A |f'|^t d\mu.
\]
The number \( t \) is called the exponent of the conformal measure. The minimum exponent, denoted by \( \delta_*(f) \), is the infimum of the exponents of conformal measures on the Julia set \( J(f) \):
\[
\delta_*(f) = \inf\{t : \text{there is a } t \text{-conformal measure on } J(f)\}.
\]
Conformal measures were introduced in holomorphic dynamics by Sullivan [10], who proved the existence of at least one such measure on \( J(f) \). Denker, Urbański and Przytycki (see [2, 8]) proved that for any rational map
of degree at least 2, the hyperbolic dimension is equal to the minimum exponent, i.e.

$$\delta_*(f) = \text{HD}_{\text{hyp}}(f) \leq \text{HD}(J(f)).$$

The crucial step in obtaining the Main Theorem is to prove the following theorem.

**Theorem 1.** Let $f$ be a rational map of degree $d \geq 2$ which satisfies BC($\infty$). Assume that

(*) any forward invariant compact subset of $J(f)$ containing no critical points is hyperbolic.

Let $\mu$ be a $t$-conformal measure on $J(f)$. Then for any $\alpha > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \leq \varepsilon_0$ and any $z \in J(f)$, we have

$$\frac{\mu(B(z, \varepsilon))}{\varepsilon^t} \geq \varepsilon^\alpha.$$

It is not clear if the condition (*) holds for all rational maps without parabolic periodic points and satisfying the BC($\infty$) condition. Nevertheless, as Proposition 8.1 in [9] shows, it holds if $J(f) \neq \mathbb{C}$. In the remaining case, the main theorem reduces to the statement that $\text{HD}_{\text{hyp}}(f) = 2$.

**Remark 1.** Assume furthermore that $J(f) \neq \mathbb{C}$. Then by Theorem B of [9], $J(f)$ has zero area. By Corollary 8.3 of [9], $f$ has neither Siegel disks nor Hermann rings. So by Fact 8.1 and Lemma 8.2 of [4], $\text{BD}(J(f)) = \delta_{\text{cr}}(f)$, where $\delta_{\text{cr}}(f)$ is the Poincaré exponent. Therefore, in this case, we obtain the following equalities:

$$\text{BD}(J(f)) = \text{BD}(J(f)) = \text{HD}(J(f)) = \text{HD}_{\text{hyp}}(f) = \delta_*(f) = \delta_{\text{cr}}(f).$$

2. **Background**

2.1. **Koebe distortion.** We shall use the following version of the Koebe distortion theorem that appeared in [7].

**Koebe Principle.** There exists $r(f) > 0$, depending on $f$, and for each $\varepsilon \in (0, 1)$ there exists a constant $K(\varepsilon) > 1$ such that the following property holds. Let $x \in J(f)$, $n \geq 0$ and $r \in (0, r(f))$. Suppose that $f^n : W \to B(x, r)$ is a conformal map. Then for every $z_1, z_2 \in W$ with $f^n(z_1), f^n(z_2) \in B(x, \varepsilon r)$, we have

$$\frac{|(f^n)'(z_1)|}{|(f^n)'(z_2)|} \leq K(\varepsilon).$$

Moreover, $K(\varepsilon) \to 1$ as $\varepsilon \to 0$.

2.2. **Backward contracting rational maps.** We collect a few results about rational maps satisfying the backward contraction property. These results were proved in [9].
Lemma 1 ([9, Theorem B]). Let \( f \) be a rational map of degree at least 2. Then there is a constant \( r > 1 \), only depending on the degree of \( f \), such that if \( f \) satisfies BC\((r)\), then the following properties hold:

1. If \( J(f) \neq \mathbb{C} \), then \( J(f) \) has zero Lebesgue measure.
2. If \( J(f) = \mathbb{C} \), then there is a set of full Lebesgue measure of points in \( \mathbb{C} \) whose forward orbit accumulates on a critical point of \( f \).

An open set \( V \) is called nice if \( f^n(\partial V) \cap V = \emptyset \) for all \( n \geq 0 \). A puzzle neighborhood \( V \) of \( \text{Crit}'(f) \) is a nice open set \( V = \bigcup_{c \in \text{Crit}'(f)} V_c \), where \( V_c \)'s are pairwise disjoint Jordan disks.

Lemma 2 ([9, Lemma 6.2]). Let \( f : \mathbb{C} \to \mathbb{C} \) be a rational map of degree two or more such that \( f \in \text{BC}(\infty) \). Then there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \), there exists a puzzle neighborhood \( V = \bigcup_{c \in \text{Crit}'(f)} V_c \) of \( \text{Crit}'(f) \) with \( \tilde{B}(c, \varepsilon) \subset V_c \subset \tilde{B}(c, 2\varepsilon) \).

Lemma 3 ([9, Proposition 8.1]). Let \( f \) be a rational map of degree two or more such that \( f \in \text{BC}(\infty) \) and the set \( \{ z \in \mathbb{C} : \omega(z) \cap \text{Crit}'(f) = \emptyset \} \) has positive Lebesgue measure. If \( K \subset J(f) \) is a compact and forward invariant set which contains neither critical points nor parabolic periodic points, then \( K \) is a hyperbolic set.

3. Some preparation. In what follows, let \( f : \mathbb{C} \to \mathbb{C} \) be a rational map of degree \( d \geq 2 \) without parabolic points such that \( f \in \text{BC}(\infty) \). Let \( \ell_{\text{max}} \) be the maximum of the orders of the critical points.

Given a nice set \( V \), we will say a connected set \( U \) is a critical pull back of \( V \) if there exists \( n \geq 1 \) such that \( U \) is a connected component of \( f^{-n}(V) \) and \( U \cap \text{Crit}(f) \neq \emptyset \).

For a nice set \( V \), we define \( D(V) = \{ z \in \mathbb{C} : \exists k \geq 0 \text{ such that } f^k(z) \in V \} \).

Each connected component of \( D(V) \) is called a landing domain of \( V \); for any \( z \in D(V) \), the smallest integer \( k \geq 0 \) with \( f^k(z) \in V \) is called the landing time of \( z \) into \( V \).

Proposition 2. For any \( \beta \in (0, 1/\ell_{\text{max}}) \), there exists \( C(\beta) > 0 \) such that for every \( c \in \text{Crit}'(f) \), \( n \in \mathbb{N} \) and \( \varepsilon \) sufficiently small, if \( W \) is a component of \( f^{-n}(\tilde{B}(c, \varepsilon)) \), then

\[ \text{diam}(W) \leq C(\beta)\varepsilon^\beta. \]
Proof. Fix a large number $r > 1$. By Lemma 2, for each $k \geq 0$, there exists a puzzle neighborhood $\tilde{V}_k$ of $\text{Crit}'(f)$ such that 
\[ \tilde{B}(c, \varepsilon_0/r^k) \subset \tilde{V}_k(c) \subset \tilde{B}(c, 2\varepsilon_0/r^k), \]
where $\varepsilon_0 > 0$ is a small number. By choosing $\varepsilon_0$ smaller if necessary, we may assume that any critical pull back of $\tilde{V}_{k-1}$ is contained in $\tilde{V}_k$, since $f$ satisfies BC$(2r)$. Moreover, we can find a periodic orbit $X$ with at least two points outside $V_0$. Clearly, $\text{D}(V_0) \cap X = \emptyset$.

It suffices to prove that for any $\beta \in (0, 1/\ell_{\max})$ there exists $C > 0$ such that for any landing domain $U$ of some $\tilde{V}_n$, we have
\[ \text{diam}(U) \leq Cr^{-n\beta}. \]

Fix $z \in \text{D}(\tilde{V}_n)$. For each $k = 0, 1, \ldots, n$, let $s_k$ be the landing time of $z$ into $\tilde{V}_k$ and let $U_k$ be the landing domain of $\tilde{V}_k$ which contains $z$. Then $U_k \subset U_{k-1}$. Let $U'_{k-1}$ be the component of $(f^{s_k})^{-1}(\tilde{V}_{k-1})$ containing $z$. Then
\[ U_k \subset U'_{k-1} \subset U_{k-1}. \]

Claim. $f^{s_k} : U'_{k-1} \to \tilde{V}_{k-1}$ is conformal.

Indeed, otherwise there exists $0 \leq s < s_k$ such that $W = f^s(U'_{k-1})$ contains a critical point $c'$. But as we noted above, this would imply that $W \subset \tilde{V}_k$, which contradicts the fact that $s_k$ is the landing time of $z$ into $\tilde{V}_k$.

Thus,
\[ \text{mod}(U_{k-1} \setminus U_k) \geq \text{mod}(U'_{k-1} \setminus U_k) \geq \inf_{c \in \text{Crit}'(f)} \text{mod}(\tilde{V}_{k-1}(c) \setminus \tilde{V}_k(c)). \]

For any $r \geq 4$, there exists $L(r) > 1$ such that for every $c \in \text{Crit}'(f)$, we have
\[ \text{mod}(\tilde{V}_{k-1}(c) \setminus \tilde{V}_k(c)) \geq \frac{1}{L(r)\ell_{\max}} \log r. \]

Moreover, $L(r) \to 1$ as $r \to \infty$.

Hence, by the Grötzsch inequality (see for example [5, Corollary B.5]) we have
\[ \text{mod}(U_0 \setminus U_n) \geq \sum_{k=1}^n \text{mod}(U_{k-1} \setminus U_k) \geq \sum_{k=1}^n \inf_{c} \text{mod}(\tilde{V}_{k-1}(c) \setminus \tilde{V}_k(c)) \geq \frac{1}{L(r)\ell_{\max}} n \log r. \]

Since $U_0 \cap X = \emptyset$, the diameter of $\text{C} \setminus U_0$ is bounded away from zero. It follows that $\text{diam}(U_n) \leq Cr^{-n/L(r)\ell_{\max}}$, where $C$ is a constant. The proof is complete.
Lemma 4. If the set \{z \in \overline{C} : \omega(z) \cap \text{Crit}(f) = \emptyset\} has positive Lebesgue measure, then for any \( \delta > 0 \) there exists \( \eta > 0 \) such that if \( W \) is a connected set intersecting the Julia set, and diam\((f^n(W)) < \eta \) for some \( n \geq 0 \), then diam\((W) < \delta \).

Proof. By Proposition 2, there exists a neighborhood \( V_0 \) of \text{Crit}(f) \) such that any pull back of \( V_0 \) has diameter smaller than any given number \( \delta > 0 \). Let \( V \subseteq V_0 \) be another neighborhood of \text{Crit}(f).

Define
\[
K(V) = \{z \in J(f) : f^m(z) \notin V, \ m = 0, 1, 2, \ldots\}.
\]

By Lemma 3, \( K(V) \) is a hyperbolic set of \( f \). So there exists \( m_0 \) such that for any \( z \in K(V) \) we have
\[
|((f^{m_0})'(z)) > 2.
\]

In particular, for any \( z \in K(V) \), \( f^{m_0} \) is univalent in a neighborhood of \( z \). By continuity, there exists \( \eta_0 \in (0, \text{diam}(\overline{C})) \) such that for each \( z_0 \in K(V) \), \( f^{m_0}|B(z_0, 3\eta_0) \) is univalent and the above inequality holds for all \( z \in B(z_0, 3\eta_0) \). Let
\[
U = \{z \in \overline{C} : d(z, K(V)) < \eta_0/2\}.
\]

Then if \( A \) is a connected subset of \( \overline{C} \) which intersects \( U \), then
\[
\text{diam}(f^{m_0}(A)) \geq \min(2 \text{diam}(A), \eta_0).
\]

To see this, take \( z_0 \in U \) with \( B(z_0, \eta_0/2) \cap A \neq \emptyset \). If \( A \cap B(z_0, \eta_0) \), then \( \text{diam}(f^{m_0}(A)) \geq 2 \text{diam}(A) \), and otherwise \( \text{diam}(f^{m_0}(A)) \geq \eta_0 \).

Claim. There exists \( N \) such that for every \( z \in J(f) \setminus U \), there exists \( n(z) \leq N \) such that \( f^n(z) \in V \).

Indeed, \( \{f^{-j}(V)\}_{j=0}^{\infty} \) is an open covering of the compact set \( J(f) \setminus U \), so there exists \( N \) such that
\[
\bigcup_{j=0}^{N} f^{-j}(V) \supset J(f) \setminus U.
\]

The claim is proved.

Now let \( z \in J(f) \) and \( W \supset z \) be a connected set with \( \text{diam}(f^n(W)) < \eta_0 \).

Case 1: \( f^k(W) \subseteq U \) for all \( k = 0, 1, \ldots, n-1 \). Write \( n = qm_0 + r \), \( 0 \leq r < m_0 \). By (2), we obtain
\[
\text{diam}(f^r(W)) \leq \text{diam}(f^n(W))/2^q.
\]

It follows that \( \text{diam}(W) < \delta \) provided that \( \text{diam}(f^n(W)) \) is small enough.

Case 2: There exists a largest \( k \leq n-1 \) such that \( f^k(W) \not\subseteq U \). As in Case 1, \( \text{diam}(f^{k+1}(W)) \) is small, hence \( \text{diam}(f^k(W)) \) is small. By the claim above, there exists \( s \leq N \) such that \( f^{k+s}(W) \cap V \neq \emptyset \). Provided that
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Given an open set \( \Omega \subset \mathbb{C} \) and \( z \in \Omega \), let

\[
\text{IR}(\Omega, z) = \inf_{w \in \partial \Omega} d(z, w) \quad \text{and} \quad \text{OR}(\Omega, z) = \sup_{w \in \partial \Omega} d(z, w).
\]

**Proposition 3.** Let \( f \) be a rational map of degree \( d \geq 2 \). For any \( \eta \in (0, \text{diam}(\mathbb{C})/2) \) and \( \varepsilon \in (0, \eta) \) and for any \( z \in J(f) \), there exist \( n_0 \in \mathbb{N} \cup \{0\} \) and \( \eta' \) such that:

- \( C\eta \leq \eta' \leq \eta \), where \( C = C(f) \) is a constant;
- letting \( W_{n_0} \) be the pull-back of \( B(f^{n_0}(z), \eta') \) under \( f^{n_0} \) to \( z \), we have

\[
\text{IR}(W_{n_0}, z) = \varepsilon.
\]

**Proof.** We consider the pull-back \( \hat{W}_n \) of the disk \( B(f^n(z), \eta) \) along \( \text{orb}(z) \) to \( z \). Then

\[
\text{IR}(\hat{W}_n, z) \to 0 \quad \text{as} \quad n \to \infty,
\]

for otherwise there would be a ball \( B \) centered at \( z \) such that \( \text{diam}(f^n(B)) \leq 2\eta \), which would imply that \( z \in B \subset \mathbb{C} \setminus J(f) \).

Thus there exists a positive \( n_0 \in \mathbb{N} \) such that

\[
\text{IR}(\hat{W}_{n_0}, z) \geq \varepsilon \quad \text{but} \quad \text{IR}(\hat{W}_{n_0+1}, z) < \varepsilon.
\]

Now let \( W' \) be the component of \( f^{-1}(B(f^{n_0+1}(z), \eta)) \) containing \( f^{n_0}(z) \). Then \( W' \supset B(f^{n_0}(z), \eta) \). It follows that \( \bar{\eta} := \text{IR}(W', f^{n_0}(z)) \leq \eta \). Clearly, \( \bar{\eta} \geq C\eta \), where \( C = (\max |f'|)^{-1} \).

Let \( \Omega(t) \) be the component of \( f^{-n_0}(B(f^{n_0}(z), t)) \) containing \( z \) and consider the map \( h(t) = \text{IR}(\Omega(t), z) \). Since \( h(t) \) is continuous and \( h(\eta) \geq \varepsilon \), \( h(\bar{\eta}) < \varepsilon \), there exists \( \eta' \in (\bar{\eta}, \eta] \) such that \( h(\eta') = \varepsilon \). This completes the proof. \( \blacksquare \)

**Proposition 4.** Let \( f \) be a rational map. There exists \( C > 0 \) such that for every \( z \in \mathbb{C} \) and every small neighborhood \( U \) of \( z \),

\[
\frac{\text{OR}(U, z)}{\text{IR}(U, z)} \leq C \frac{\text{OR}(f(U), f(z))}{\text{IR}(f(U), f(z))}.
\]

**Proof.** By the Koebe principle, it suffices to consider \( U \) contained in a small neighborhood of a critical point of \( f \). Since near a critical point, \( f \) behaves like a polynomial \( z \mapsto z^k \), the proposition follows easily. \( \blacksquare \)

4. **Proof of Theorem 1.** In this section, we prove Theorem 1. We shall use the following notion introduced in [1].

**Definition 2.** A sequence \( \{G_k\}_{k=0}^n \) of connected open sets is called a quasi-chain if \( f(G_k) \supseteq G_{k+1} \) for each \( 0 \leq k < n \). The order of the quasi-chain is the number of \( k \in \{0, 1, \ldots, n-1\} \) such that \( G_k \) contains a critical point.
We shall also need the following lemma related to Lemma 1.3 of [6].

**Lemma 5.** Let \( f : \mathbb{C} \to \mathbb{C} \) be a rational map of the Riemann sphere of degree at least two and let \( \mu \) be a \( t \)-conformal measure on \( J(f) \). Then there exists a constant \( C > 0 \) (depending on \( f \)) such that if \( V \) is a connected open set and \( U \) is a component of \( f^{-1}(V) \), then
\[
\frac{\mu(U)}{\text{diam}(U)^t} \geq C \frac{\mu(V)}{\text{diam}(V)^t}.
\]

**Proof.** By Lemma 1.3 in [6], we have
\[
\frac{\text{diam}(V)}{\text{diam}(U)} \geq C \sup \{|f'(z)| : z \in U\}.
\]
By the \( t \)-covariance of \( \mu \), the lemma follows easily. \( \blacksquare \)

In the following,
\[
\tilde{B}(\delta) = \bigcup_{c \in \text{Crit}'(f)} \tilde{B}(c, \delta).
\]

**Proof of Theorem 1.** Fix \( \alpha \in (0, 1) \). Let \( r > 4^{\ell_{\text{max}}} \) be a constant to be determined. Since \( f \) satisfies BC(\( r \)), there exists \( \delta_0 \) such that for any \( \delta \in (0, \delta_0) \), if \( U \) is a critical pull back of \( \tilde{B}(r\delta) \) then \( U \subset \tilde{B}(\delta) \).

By Lemma 4, there exists \( \eta > 0 \) such that any pull back of a ball of radius \( 2\eta \) has diameter less than \( \delta_0 \). By Proposition 3, for any \( \varepsilon < \eta \) there exist \( \eta' \approx \eta \) and \( n \) such that the component \( W \) of \( f^{-n}(B(f^n(z), \eta')) \) which contains \( z \) satisfies
\[
\text{IR}(W, z) = \varepsilon^{1+\alpha/2t}.
\]

Let \( W_k = f^k(W) \), \( k = 0, 1, \ldots, n \).

We shall prove that there exist constants \( C \in (0, 1) \) and \( \theta > 0 \) depending only on \( f \) such that
\[
\frac{\mu(W)}{\text{diam}(W)^t} \geq C\varepsilon^{\theta\gamma},
\]
\[
\frac{\text{OR}(W, z)}{\text{IR}(W, z)} \leq (C\varepsilon^{\theta\gamma})^{-1},
\]
where \( \gamma = 2\ell_{\text{max}}/(\log r - \ell_{\text{max}}\log 4) \).

Choosing \( r \) large enough, we have \( \theta\gamma \leq \alpha/3 \). Provided that \( \varepsilon > 0 \) is small enough, we have \( C\varepsilon^{\theta\gamma} \geq \varepsilon^{\alpha/2} \). Thus (4) implies \( \text{OR}(W, z) \leq \varepsilon \), so \( \mu(B(z, \varepsilon)) \geq \mu(W) \); together with inequality (3) we have
\[
\mu(B(z, \varepsilon)) \geq \mu(W) \geq \varepsilon^{t+\alpha},
\]
as we wished.

If \( f^n|W \) extends to a conformal map onto \( B(f^n(z), 2\eta) \) then the inequalities follow easily from the Koebe principle and the \( t \)-covariance of \( \mu \).
In order to deal with the general case, we define a quasi-chain \( \{ \hat{W}_k \}_{k=0}^n \) by the following rules: (i) \( \hat{W}_n = B(f^n(z), 2\eta) \); (ii) once \( \hat{W}_{k+1} \supset f^{k+1}(z) \) is defined, let \( \hat{W}'_k \) be the connected component of \( f^{-1}(\hat{W}_{k+1}) \) which contains \( f^k(z) \); (iii) if \( \hat{W}_k \) contains no critical point, then \( \hat{W}_k = \hat{W}'_k \), otherwise
\[
\hat{W}_k = B(f^k(z), 2 \text{diam}(\hat{W}_k)).
\]

Let \( m \) be the order the quasi-chain \( \{ \hat{W}_k \}_{k=0}^n \). Let \( n_0 = n \) and let \( n_1 > \cdots > n_m \) be all integers in \( \{0, 1, \ldots, n - 1\} \) such that \( \hat{W}_{n_i} \) contains a critical point.

**Claim.** There exists a constant \( C_0 \) depending on \( f \) such that
\[
m \leq C_0 + 2\ell_{\text{max}} \log(1/\varepsilon)/\log r_1,
\]
where \( r_1 = r/4^{\ell_{\text{max}}} \).

In fact, \( \hat{W}_{n_1} \subset \tilde{B}(\delta_0) \). By the BC\( (r) \) property, we deduce that \( \hat{W}'_{n_2} \subset \tilde{B}(\delta_0/r) \), so that \( \hat{W}_{n_2} \subset \tilde{B}(4^{\ell_{\text{max}}} \delta_0/r) \subset \tilde{B}(\delta_0) \). Repeating the procedure, we obtain
\[
\hat{W}_{n_m} \subset \tilde{B}(\delta_0/r_1^{m-1}).
\]
By Proposition 2, for any \( \beta \in (0, 1/\ell_{\text{max}}) \), there exists \( C(\beta) > 0 \) such that
\[
\text{diam}(W) \leq C(\beta)(\delta_0/r_1^{m-1})^\beta.
\]
Since \( \text{diam}(W) \geq \text{IR}(W) = \varepsilon^{1+\alpha/2}\geq \varepsilon^{1+\alpha} \), we obtain (5).

For each \( 1 \leq i \leq m \), \( f^{n_{i-1} - n_i - 1} : W_{n_i+1} \rightarrow W_{n_i-1} \) extends to a conformal map onto \( \hat{W}_{n_i-1} \). Since \( \text{mod}(\hat{W}_{n_i-1} \setminus W_{n_i-1}) \) is bounded away from zero, the Koebe principle and the \( t \)-covariance of \( \mu \) give us
\[
\frac{\mu(W_{n_i+1})}{\text{diam}(W_{n_i+1})^t} \geq C_1 \frac{\mu(W_{n_i-1})}{\text{diam}(W_{n_i-1})^t},
\]
\[
\frac{\text{OR}(W_{n_i+1}, f^{n_{i+1}}(z))}{\text{IR}(W_{n_i+1}, f^{n_{i+1}}(z))} \leq \frac{1}{C_1} \frac{\text{OR}(W_{n_i-1}, f^{n_{i-1}}(z))}{\text{IR}(W_{n_i-1}, f^{n_{i-1}}(z))},
\]
where \( C_1 \in (0, 1) \) is a universal constant. Similarly, we have
\[
\frac{\mu(W)}{\text{diam}(W)^t} \geq C_1 \frac{\mu(W_{n_m})}{\text{diam}(W_{n_m})^t},
\]
\[
\frac{\text{OR}(W, z)}{\text{IR}(W, z)} \leq \frac{1}{C_1} \frac{\text{OR}(W_{n_m}, f^{n_m}(z))}{\text{IR}(W_{n_m}, f^{n_m}(z))},
\]
By Lemma 5, we have
\[
\frac{\mu(W_{n_i})}{\text{diam}(W_{n_i})^t} \geq C_2 \frac{\mu(W_{n_i+1})}{\text{diam}(W_{n_i+1})^t},
\]
where $C_2 \in (0, 1)$ is a universal constant. By Proposition 4, we have

$$\frac{\text{OR}(W_{n_i}, f^{n_i}(z))}{\text{IR}(W_{n_i}, f^{n_i}(z))} \leq \frac{1}{C_2} \frac{\text{OR}(W_{n_i+1}, f^{n_i+1}(z))}{\text{IR}(W_{n_i+1}, f^{n_i+1}(z))}. \tag{11}$$

Combining the estimates (6), (10) and (8), we obtain

$$\frac{\mu(W)}{\text{diam}(W)^t} \geq C_1^{m+1} C_2^m \frac{\mu(B(f^{n_i}(z), \eta'))}{(2\eta')^t}. \tag{12}$$

Since $\inf_{w \in J(f)} \mu(B(w, \eta')) > 0$, it follows that

$$\frac{\mu(W)}{\text{diam}(W)^t} \geq C(C_1 C_2)^m, \tag{13}$$

where $C$ is a constant.

Combining the estimates (7), (9) and (11), we obtain

$$\frac{\text{OR}(W, z)}{\text{IR}(W, z)} \leq (C_1^{m+1} C_2^m)^{-1}. \tag{14}$$

If we let $\theta = -\log(C_1 C_2)$ and redefine the constant $C$, then (13) and (14) give us (3) and (4) respectively. This completes the proof.

5. Proof of the Main Theorem

Proof of the Main Theorem in the case $J(f) \neq \overline{\mathbb{C}}$. The following argument is similar to the proof of the well-known Frostman’s lemma (see [8]). Since $\delta_*(f) = \text{HD}_{\text{hyp}}(J(f))$, it suffices to prove that $\text{BD}(J(f)) \leq \delta_*(f) + \alpha$ for any $\alpha > 0$.

Let $\mu$ be a $\delta_*(f)$-conformal measure of $f$. By Lemma 3, $f$ satisfies the assumption ($\ast$), so that we can apply Theorem 1. Let $N(\varepsilon)$ be the minimal number of open balls with radius $\varepsilon$ needed to cover $J(f)$. For any $\varepsilon > 0$ small, $J(f)$ can be covered by a family $\{B_i\}_{i=1}^n$ of open balls of radius $\varepsilon$ with intersection multiplicity 4. For each $i$, Theorem 1 gives us

$$\mu(B_i) \geq \varepsilon^{\delta_*(f)+\alpha},$$

provided that $\varepsilon$ is small enough. Thus

$$4 \geq \sum_{i=1}^n \mu(B_i) \geq n\varepsilon^{\delta_*(f)+\alpha} \geq N(\varepsilon)\varepsilon^{\delta_*(f)+\alpha},$$

which implies that

$$\text{BD}(J(f)) = \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \leq \delta_*(f) + \alpha. \tag{\#}$$

In the case $J(f) = \overline{\mathbb{C}}$, we do not know whether the condition ($\ast$) holds. However, we are still able to obtain enough control on the conformal measures to conclude the proof.

**Proposition 5.** If $J(f) = \overline{\mathbb{C}}$, then $\delta_*(f) = 2$. 

Proof. Arguing by contradiction, assume that \( f \) has a \( t \)-conformal measure with \( t < 2 \).

Let \( r > 1 \) be a large constant. Since \( f \) has the BC(\( r \)) property, there exists an arbitrarily small \( \varepsilon > 0 \) and a puzzle neighborhood \( V = \bigcup_{c \in \text{Crit}'(f)} V_c \) of \( \text{Crit}'(f) \) such that for each \( c \in \text{Crit}'(f) \),

\[
\tilde{B}(c, \varepsilon) \subset V_c \subset \tilde{B}(c, 2\varepsilon).
\]

Let \( W_c \) be the union of all return domains of \( V \) which are not contained in \( \tilde{B}(c, \varepsilon/\sqrt{r}) \). Let \( g : \bigcup_c W_c \to V \) denote the first return map into \( V \) under iteration of \( f \). By the BC(\( r \)) property, for each component \( U \) of \( g^{-n}(V) \), \( g^n : U \to V \) is a conformal map which extends to a conformal map onto \( \tilde{B}(c, \sqrt{r}\varepsilon) \) for some \( c \in \text{Crit}'(f) \). In fact, this follows from the following observation: if \( U \) is a component of \( W_c \) with \( g(U) = V_c' \) and \( g|U = f^s|U \), then there exists a topological disk \( \hat{U} \subset \tilde{B}(c, \sqrt{r}\varepsilon) \) such that \( f^s : \hat{U} \to \tilde{B}(c', \sqrt{r}\varepsilon) \). By the Koebe principle, it follows that for each component \( U \) of \( g^{-n}(V) \), \( g^n|U \) has small distortion provided that \( r \) is large enough.

Let \( W_n^c \) be the collection of all components of \( g^{-n}(V) \) which are contained in \( V_c \) and let \( W_n^c \) be the union of these components. By Lemma 1, almost every point returns to \( V \) under iteration of \( f \), thus,

\[
\frac{\text{area}(W_1^c)}{\text{area}(V_c)} \geq 1 - \frac{\tilde{B}(c, \varepsilon/\sqrt{r})}{\tilde{B}(c, \varepsilon)} \geq 1 - \sigma(r),
\]

where \( \sigma(r) \to 0 \) as \( r \to \infty \). Note that for each component \( U \) of \( W_1^c \), \( \text{area}(U) \leq \text{area}(V_c)/2 \). Therefore, provided that \( r \) was chosen large enough, we have

\[
\sum_{U \in W_1^c} \text{area}(U)^{t/2} \geq \lambda \text{area}(V_c)^{t/2},
\]

where \( \lambda = 2^{1-t/2} > 1 \).

For each \( U \in W_n^c \), since \( g^n|U \) has small distortion and \( g^n \) maps an element of \( W_n^{c+1} \) onto an element of \( W_{c}^1 \), it follows that

\[
\sum_{W \in W_{n+1}^c, W \subset U} \text{area}(W)^{t/2} \geq \lambda_1 \text{area}(U)^{t/2},
\]

where \( \lambda_1 \in (1, \lambda) \). Therefore,

\[
\sum_{U \in W_n^c} \text{area}(U)^{t/2} \geq \lambda_1^n \text{area}(V_c)^{t/2}.
\]

By the Koebe principle and the \( t \)-covariance of \( \mu \), for each \( U \in W_n^c \), \( n = 0, 1, \ldots \), we know that \( \mu(U)/\text{area}(U)^{t/2} \) is comparable to \( \mu(V_c)/\text{area}(V_c)^{t/2} \),
where \( V_c = g^n(U) \). Thus,
\[
\sum_{U \in \mathcal{W}_1^n} \mu(U) \geq C \lambda^n_1.
\]
Letting \( n \to \infty \) implies \( \mu(J(f)) = \infty \), a contradiction. 

Proof of the Main Theorem in the case \( J(f) = \mathbb{C} \). By the previous proposition, \( \delta_*(f) = 2 \). Hence
\[
\delta_*(f) = \text{HD}_{\text{hyp}}(f) = 2 = \text{BD}(J(f)).
\]

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