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Dimensions of the Julia sets of rational maps with the backward contraction property

by

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Abstract. Consider a rational map f on the Riemann sphere of degree at least 2 which has no parabolic periodic points. Assuming that f has Rivera-Letelier's backward contraction property with an arbitrarily large constant, we show that the upper box dimension of the Julia set $J(f)$ is equal to its hyperbolic dimension, by investigating the properties of conformal measures on the Julia set.

1. Introduction. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree $d \geq 2$ on the Riemann sphere. We are interested in the fractal properties of the Julia set $J(f)$. It is well known that in the case that f is hyperbolic, all possible dimensions coincide. In [4], this result was generalized to all rational maps which satisfy a summability condition. See [11] for more historical remarks and advances in this direction.

The summability condition and the stronger Collet–Eckmann condition can be considered as non-uniform hyperbolicity conditions. As shown by J. Rivera-Letelier [9], they imply a *backward contraction* condition (see the definition below) which was first introduced therein.

In the following, all the distances, diameters and norms of derivatives are measured using the spherical metric and $B(z, r)$ denotes a ball of radius r centered at z. Let $\mathrm{Crit}(f)$ denote the set of critical points of f and let

$$\mathrm{Crit}'(f) = \mathrm{Crit}(f) \cap J(f).$$

For every $c \in \mathrm{Crit}(f)$ and $\delta > 0$ we denote by $\widetilde{B}(c, \delta)$ the connected component of $f^{-1}(B(f(c), \delta))$ that contains c.

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DEFINITION 1. Given a constant $r > 1$, we say that f has the *backward contraction property with constant* r ($f \in \mathrm{BC}(r)$ for short) if there exists $\delta_0 > 0$ such that for every $c \in \mathrm{Crit}'(f)$, every $0 < \delta \leq \delta_0$, every integer $n \geq 1$ and every component W of $f^{-n}(\widetilde{B}(c, r\delta))$, we have

$$\mathrm{dist}(W, \mathrm{CV}(f)) \leq \delta \ \Rightarrow \ \mathrm{diam}(W) < \delta,$$

where $\mathrm{CV}(f) = f(\mathrm{Crit}(f))$. If $f \in \mathrm{BC}(r)$ for every $r > 1$, we will say that $f \in \mathrm{BC}(\infty)$.

We call a compact forward invariant subset X of $\overline{\mathbb{C}}$ *hyperbolic* if there exist $C > 0$ and $\lambda > 1$ such that for every $n \geq 1$ and every $z \in X$,

$$|Df^n(z)| \geq C\lambda^n.$$

Clearly, a hyperbolic set is contained in the Julia set.

For a compact set $X \subset \overline{\mathbb{C}}$, let $\mathrm{HD}(X)$ denote its Hausdorff dimension. The *hyperbolic dimension* $\mathrm{HD}_{\mathrm{hyp}}(f)$ of f is the supremum of the Hausdorff dimensions of hyperbolic subsets of $J(f)$, i.e.

$$\mathrm{HD}_{\mathrm{hyp}}(f) = \sup\{\mathrm{HD}(X) : X \text{ is a hyperbolic subset of } J(f)\}.$$

Clearly, $\mathrm{HD}_{\mathrm{hyp}}(f) \leq \mathrm{HD}(J(f))$.

The main goal of this paper is to prove the following theorem.

MAIN THEOREM. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree at least* 2 *without parabolic periodic points. If $f \in \mathrm{BC}(\infty)$, then the upper box dimension $\overline{\mathrm{BD}}(J(f))$ of the Julia set is equal to the hyperbolic dimension of f*:

$$\overline{\mathrm{BD}}(J(f)) = \mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f).$$

For the definition of the upper and lower box dimensions and the Hausdorff dimension, see [3]. Let us mention the following well-known inequality: $\mathrm{HD}(X) \leq \underline{\mathrm{BD}}(X) \leq \overline{\mathrm{BD}}(X)$.

The proof of the Main Theorem is based on analyzing the regularity of conformal measures. Recall that a probability measure μ on $J(f)$ is said to be *t-conformal* for f if for every Borel set $A \subset J(f)$ such that $f|_A$ is injective, we have

$$\mu(f(A)) = \int_A |f'|^t \, d\mu.$$

The number t is called the *exponent* of the conformal measure. The *minimum exponent*, denoted by $\delta_*(f)$, is the infimum of the exponents of conformal measures on the Julia set $J(f)$:

$$\delta_*(f) = \inf\{t : \text{there is a } t\text{-conformal measure on } J(f)\}.$$

Conformal measures were introduced in holomorphic dynamics by Sullivan [10], who proved the existence of at least one such measure on $J(f)$. Denker, Urbański and Przytycki (see [2, 8]) proved that for any rational map

f of degree at least 2, the hyperbolic dimension is equal to the minimum exponent, i.e.

$$\delta_*(f) = \mathrm{HD}_{\mathrm{hyp}}(f) \le \mathrm{HD}(J(f)).$$

The crucial step in obtaining the Main Theorem is to prove the following theorem.

THEOREM 1. *Let f be a rational map of degree $d \ge 2$ which satisfies* BC(∞). *Assume that*

(*) *any forward invariant compact subset of $J(f)$ containing no critical points is hyperbolic.*

Let μ be a t-conformal measure on $J(f)$. Then for any $\alpha > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \le \varepsilon_0$ and any $z \in J(f)$, we have

$$\frac{\mu(B(z,\varepsilon))}{\varepsilon^t} \ge \varepsilon^\alpha.$$

It is not clear if the condition (*) holds for all rational maps without parabolic periodic points and satisfying the BC(∞) condition. Nevertheless, as Proposition 8.1 in [9] shows, it holds if $J(f) \ne \overline{\mathbb{C}}$. In the remaining case, the main theorem reduces to the statement that $\mathrm{HD}_{\mathrm{hyp}}(f) = 2$.

REMARK 1. Assume furthermore that $J(f) \ne \overline{\mathbb{C}}$. Then by Theorem B of [9], $J(f)$ has zero area. By Corollary 8.3 of [9], f has neither Siegel disks nor Hermann rings. So by Fact 8.1 and Lemma 8.2 of [4], $\overline{\mathrm{BD}}(J(f)) = \delta_{\mathrm{cr}}(f)$, where $\delta_{\mathrm{cr}}(f)$ is the Poincaré exponent. Therefore, in this case, we obtain the following equalities:

$$\overline{\mathrm{BD}}(J(f)) = \underline{\mathrm{BD}}(J(f)) = \mathrm{HD}(J(f)) = \mathrm{HD}_{\mathrm{hyp}}(f) = \delta_*(f) = \delta_{\mathrm{cr}}(f).$$

2. Background

2.1. *Koebe distortion.* We shall use the following version of the Koebe distortion theorem that appeared in [7].

KOEBE PRINCIPLE. *There exists $r(f) > 0$, depending on f, and for each $\varepsilon \in (0,1)$ there exists a constant $K(\varepsilon) > 1$ such that the following property holds. Let $x \in J(f)$, $n \ge 0$ and $r \in (0, r(f))$. Suppose that $f^n : W \to B(x,r)$ is a conformal map. Then for every $z_1, z_2 \in W$ with $f^n(z_1), f^n(z_2) \in B(x, \varepsilon r)$, we have*

$$\frac{|(f^n)'(z_1)|}{|(f^n)'(z_2)|} \le K(\varepsilon).$$

Moreover, $K(\varepsilon) \to 1$ as $\varepsilon \to 0$.

2.2. *Backward contracting rational maps.* We collect a few results about rational maps satisfying the backward contraction property. These results were proved in [9].

LEMMA 1 ([9, Theorem B]). *Let f be a rational map of degree at least 2. Then there is a constant $r > 1$, only depending on the degree of f, such that if f satisfies* $\mathrm{BC}(r)$, *then the following properties hold*:

1. *If $J(f) \neq \overline{\mathbb{C}}$, then $J(f)$ has zero Lebesgue measure.*
2. *If $J(f) = \overline{\mathbb{C}}$, then there is a set of full Lebesgue measure of points in $\overline{\mathbb{C}}$ whose forward orbit accumulates on a critical point of f.*

An open set V is called *nice* if $f^n(\partial V) \cap V = \emptyset$ for all $n \geq 0$. A *puzzle neighborhood* V of $\mathrm{Crit}'(f)$ is a nice open set $V = \bigcup_{c \in \mathrm{Crit}'(f)} V_c$, where V_c's are pairwise disjoint Jordan disks.

LEMMA 2 ([9, Lemma 6.2]). *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree two or more such that $f \in \mathrm{BC}(\infty)$. Then there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, there exists a puzzle neighborhood $V = \bigcup_{c \in \mathrm{Crit}'(f)} V_c$ of $\mathrm{Crit}'(f)$ with*

$$\widetilde{B}(c, \varepsilon) \subset V_c \subset \widetilde{B}(c, 2\varepsilon).$$

LEMMA 3 ([9, Proposition 8.1]). *Let f be a rational map of degree two or more such that $f \in \mathrm{BC}(\infty)$ and the set*

$$\{z \in \overline{\mathbb{C}} : \omega(z) \cap \mathrm{Crit}'(f) = \emptyset\}$$

has positive Lebesgue measure. If $K \subset J(f)$ is a compact and forward invariant set which contains neither critical points nor parabolic periodic points, then K is a hyperbolic set.

3. Some preparation. In what follows, let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree $d \geq 2$ without parabolic points such that $f \in \mathrm{BC}(\infty)$. Let $\ell_{\max}$ be the maximum of the orders of the critical points.

Given a nice set V, we will say a connected set U is a *critical pull back* of V if there exists $n \geq 1$ such that U is a connected component of $f^{-n}(V)$ and $U \cap \mathrm{Crit}(f) \neq \emptyset$.

For a nice set V, we define

$$D(V) = \{z \in \overline{\mathbb{C}} : \exists k \geq 0 \text{ such that } f^k(z) \in V\}.$$

Each connected component of $D(V)$ is called a *landing domain* of V; for any $z \in D(V)$, the smallest integer $k \geq 0$ with $f^k(z) \in V$ is called the *landing time* of z into V.

PROPOSITION 2. *For any $\beta \in (0, 1/\ell_{\max})$, there exists $C(\beta) > 0$ such that for every $c \in \mathrm{Crit}'(f)$, $n \in \mathbb{N}$ and ε sufficiently small, if W is a component of $f^{-n}(\widetilde{B}(c, \varepsilon))$, then*

$$\mathrm{diam}(W) \leq C(\beta)\varepsilon^{\beta}.$$

Proof. Fix a large number $r > 1$. By Lemma 2, for each $k \geq 0$, there exists a puzzle neighborhood $\widetilde{V}_k$ of $\mathrm{Crit}'(f)$ such that

$$\widetilde{B}(c, \varepsilon_0/r^k) \subset \widetilde{V}_k(c) \subset \widetilde{B}(c, 2\varepsilon_0/r^k),$$

where $\varepsilon_0 > 0$ is a small number. By choosing ε_0 smaller if necessary, we may assume that any critical pull back of $\widetilde{V}_{k-1}$ is contained in $\widetilde{V}_k$, since f satisfies $\mathrm{BC}(2r)$. Moreover, we can find a periodic orbit X with at least two points outside V_0. Clearly, $D(V_0) \cap X = \emptyset$.

It suffices to prove that for any $\beta \in (0, 1/\ell_{\max})$ there exists $C > 0$ such that for any landing domain U of some $\widetilde{V}_n$, we have

$$\mathrm{diam}(U) \leq C r^{-n\beta}.$$

Fix $z \in D(\widetilde{V}_n)$. For each $k = 0, 1, \ldots, n$, let s_k be the landing time of z into $\widetilde{V}_k$ and let U_k be the landing domain of $\widetilde{V}_k$ which contains z. Then $U_k \subset U_{k-1}$. Let U'_{k-1} be the component of $(f^{s_k})^{-1}(\widetilde{V}_{k-1})$ containing z. Then

$$U_k \subset U'_{k-1} \subset U_{k-1}.$$

Claim. *$f^{s_k} : U'_{k-1} \to \widetilde{V}_{k-1}$ is conformal.*

Indeed, otherwise there exists $0 \leq s < s_k$ such that $W = f^s(U'_{k-1})$ contains a critical point c'. But as we noted above, this would imply that $W \subset \widetilde{V}_k$, which contradicts the fact that s_k is the landing time of z into $\widetilde{V}_k$.

Thus,

$$\mathrm{mod}(U_{k-1} \setminus U_k) \geq \mathrm{mod}(U'_{k-1} \setminus U_k) \geq \inf_{c \in \mathrm{Crit}'(f)} \mathrm{mod}(\widetilde{V}_{k-1}(c) \setminus \widetilde{V}_k(c)).$$

For any $r \geq 4$, there exists $L(r) > 1$ such that for every $c \in \mathrm{Crit}'(f)$, we have

$$\mathrm{mod}(\widetilde{V}_{k-1}(c) \setminus \widetilde{V}_k(c)) \geq \frac{1}{L(r)\ell_{\max}} \log r.$$

Moreover, $L(r) \to 1$ as $r \to \infty$.

Hence, by the Grötzsch inequality (see for example [5, Corollary B.5]) we have

$$\begin{aligned} \mathrm{mod}(U_0 \setminus U_n) &\geq \sum_{k=1}^{n} \mathrm{mod}(U_{k-1} \setminus U_k) \geq \sum_{k=1}^{n} \inf_c \mathrm{mod}(\widetilde{V}_{k-1}(c) \setminus \widetilde{V}_k(c)) \\ &\geq \frac{1}{L(r)\ell_{\max}}\, n \log r. \end{aligned}$$

Since $U_0 \cap X = \emptyset$, the diameter of $\overline{\mathbb{C}} \setminus U_0$ is bounded away from zero. It follows that $\mathrm{diam}(U_n) \leq C r^{-n/L(r)\ell_{\max}}$, where C is a constant. The proof is complete. ■

Lemma 4. *If the set* $\{z \in \overline{\mathbb{C}} : \omega(z) \cap \mathrm{Crit}'(f) = \emptyset\}$ *has positive Lebesgue measure, then for any* $\delta > 0$ *there exists* $\eta > 0$ *such that if* W *is a connected set intersecting the Julia set, and* $\mathrm{diam}(f^n(W)) < \eta$ *for some* $n \geq 0$, *then*

$$\mathrm{diam}(W) < \delta.$$

Proof. By Proposition 2, there exists a neighborhood V_0 of $\mathrm{Crit}'(f)$ such that any pull back of V_0 has diameter smaller than any given number $\delta > 0$. Let $V \Subset V_0$ be another neighborhood of $\mathrm{Crit}'(f)$.

Define

$$K(V) = \{z \in J(f) : f^m(z) \notin V,\ m = 0, 1, 2, \ldots\}.$$

By Lemma 3, $K(V)$ is a hyperbolic set of f. So there exists m_0 such that for any $z \in K(V)$ we have

$$|(f^{m_0})'(z)| > 2. \tag{1}$$

In particular, for any $z \in K(V)$, f^{m_0} is univalent in a neighborhood of z. By continuity, there exists $\eta_0 \in (0, \mathrm{diam}(\overline{\mathbb{C}}))$ such that for each $z_0 \in K(V)$, $f^{m_0}|B(z_0, 3\eta_0)$ is univalent and the above inequality holds for all $z \in B(z_0, 3\eta_0)$. Let

$$U = \{z \in \overline{\mathbb{C}} : d(z, K(V)) < \eta_0/2\}.$$

Then if A is a connected subset of $\overline{\mathbb{C}}$ which intersects U, then

$$\mathrm{diam}(f^{m_0}(A)) \geq \min(2\,\mathrm{diam}(A), \eta_0). \tag{2}$$

To see this, take $z_0 \in U$ with $B(z_0, \eta_0/2) \cap A \neq \emptyset$. If $A \subset B(z_0, \eta_0)$, then $\mathrm{diam}(f^{m_0}(A)) \geq 2\,\mathrm{diam}(A)$, and otherwise $\mathrm{diam}(f^{m_0}(A)) \geq \eta_0$.

Claim. *There exists* N *such that for every* $z \in J(f) \setminus U$, *there exists* $n(z) \leq N$ *such that* $f^{n(z)}(z) \in V$.

Indeed, $\{f^{-j}(V)\}_{j=0}^{\infty}$ is an open covering of the compact set $J(f) \setminus U$, so there exists N such that

$$\bigcup_{j=0}^{N} f^{-j}(V) \supset J(f) \setminus U.$$

The claim is proved.

Now let $z \in J(f)$ and $W \ni z$ be a connected set with $\mathrm{diam}(f^n(W)) < \eta_0$.

Case 1: $f^k(W) \subset U$ for all $k = 0, 1, \ldots, n-1$. Write $n = qm_0 + r$, $0 \leq r < m_0$. By (2), we obtain

$$\mathrm{diam}(f^r(W)) \leq \mathrm{diam}(f^n(W))/2^q.$$

It follows that $\mathrm{diam}(W) < \delta$ provided that $\mathrm{diam}(f^n(W))$ is small enough.

Case 2: There exists a largest $k \leq n-1$ such that $f^k(W) \not\subset U$. As in Case 1, $\mathrm{diam}(f^{k+1}(W))$ is small, hence $\mathrm{diam}(f^k(W))$ is small. By the claim above, there exists $s \leq N$ such that $f^{k+s}(W) \cap V \neq \emptyset$. Provided that

$\mathrm{diam}(f^n(W))$ is small enough, $\mathrm{diam}(f^{k+s}(W)) < d(\partial V, \partial V_0)$, which implies that $f^{k+s}(W) \subset V_0$, hence $\mathrm{diam}(W) < \delta$. ∎

Given an open set $\Omega \subset \mathbb{C}$ and $z \in \Omega$, let

$$\mathrm{IR}(\Omega, z) = \inf_{w \in \partial\Omega} d(z, w) \quad \text{and} \quad \mathrm{OR}(\Omega, z) = \sup_{w \in \partial\Omega} d(z, w).$$

PROPOSITION 3. *Let f be a rational map of degree $d \geq 2$. For any $\eta \in (0, \mathrm{diam}(\mathbb{C})/2)$ and $\varepsilon \in (0, \eta)$ and for any $z \in J(f)$, there exist $n_0 \in \mathbb{N} \cup \{0\}$ and η' such that*:

- *$C\eta \leq \eta' \leq \eta$, where $C = C(f)$ is a constant*;
- *letting W_{n_0} be the pull-back of $B(f^{n_0}(z), \eta')$ under f^{n_0} to z, we have*
$$\mathrm{IR}(W_{n_0}, z) = \varepsilon.$$

Proof. We consider the pull-back $\widehat{W}_n$ of the disk $B(f^n(z), \eta)$ along $\mathrm{orb}(z)$ to z. Then

$$\mathrm{IR}(\widehat{W}_n, z) \to 0 \quad \text{ as } n \to \infty,$$

for otherwise there would be a ball B centered at z such that $\mathrm{diam}(f^n(B)) \leq 2\eta$, which would imply that $z \in B \subset \mathbb{C} \setminus J(f)$.

Thus there exists a positive $n_0 \in \mathbb{N}$ such that

$$\mathrm{IR}(\widehat{W}_{n_0}, z) \geq \varepsilon \quad \text{but} \quad \mathrm{IR}(\widehat{W}_{n_0+1}, z) < \varepsilon.$$

Now let W' be the component of $f^{-1}(B(f^{n_0+1}(z), \eta))$ containing $f^{n_0}(z)$. Then $W' \not\supset B(f^{n_0}(z), \eta)$. It follows that $\overline{\eta} := \mathrm{IR}(W', f^{n_0}(z)) \leq \eta$. Clearly, $\overline{\eta} \geq C\eta$, where $C = (\max|f'|)^{-1}$.

Let $\Omega(t)$ be the component of $f^{-n_0}(B(f^{n_0}(z), t))$ containing z and consider the map $h(t) = \mathrm{IR}(\Omega(t), z)$. Since $h(t)$ is continuous and $h(\eta) \geq \varepsilon$, $h(\overline{\eta}) < \varepsilon$, there exists $\eta' \in (\overline{\eta}, \eta]$ such that $h(\eta') = \varepsilon$. This completes the proof. ∎

PROPOSITION 4. *Let f be a rational map. There exists $C > 0$ such that for every $z \in \overline{\mathbb{C}}$ and every small neighborhood U of z,*

$$\frac{\mathrm{OR}(U, z)}{\mathrm{IR}(U, z)} \leq C \frac{\mathrm{OR}(f(U), f(z))}{\mathrm{IR}(f(U), f(z))}.$$

Proof. By the Koebe principle, it suffices to consider U contained in a small neighborhood of a critical point of f. Since near a critical point, f behaves like a polynomial $z \mapsto z^k$, the proposition follows easily. ∎

4. Proof of Theorem 1. In this section, we prove Theorem 1. We shall use the following notion introduced in [1].

DEFINITION 2. A sequence $\{G_k\}_{k=0}^n$ of connected open sets is called a *quasi-chain* if $f(G_k) \supset G_{k+1}$ for each $0 \leq k < n$. The *order* of the quasi-chain is the number of $k \in \{0, 1, \ldots, n-1\}$ such that G_k contains a critical point.

We shall also need the following lemma related to Lemma 1.3 of [6].

LEMMA 5. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of the Riemann sphere of degree at least two and let μ be a t-conformal measure on $J(f)$. Then there exists a constant $C > 0$ (depending on f) such that if V is a connected open set and U is a component of $f^{-1}(V)$, then*

$$\frac{\mu(U)}{\mathrm{diam}(U)^t} \ge C \frac{\mu(V)}{\mathrm{diam}(V)^t}.$$

Proof. By Lemma 1.3 in [6], we have

$$\frac{\mathrm{diam}(V)}{\mathrm{diam}(U)} \ge C \sup\{|f'(z)| : z \in U\}.$$

By the t-covariance of μ, the lemma follows easily. ■

In the following,

$$\widetilde{B}(\delta) = \bigcup_{c \in \mathrm{Crit}'(f)} \widetilde{B}(c, \delta).$$

Proof of Theorem 1. Fix $\alpha \in (0,1)$. Let $r > 4^{\ell_{\max}}$ be a constant to be determined. Since f satisfies $\mathrm{BC}(r)$, there exists δ_0 such that for any $\delta \in (0, \delta_0)$, if U is a critical pull back of $\widetilde{B}(r\delta)$ then $U \subset \widetilde{B}(\delta)$.

By Lemma 4, there exists $\eta > 0$ such that any pull back of a ball of radius 2η has diameter less than δ_0. By Proposition 3, for any $\varepsilon < \eta$ there exist $\eta' \asymp \eta$ ($\eta' \le \eta$) and n such that the component W of $f^{-n}(B(f^n(z), \eta'))$ which contains z satisfies

$$\mathrm{IR}(W, z) = \varepsilon^{1+\alpha/2t}.$$

Let $W_k = f^k(W)$, $k = 0, 1, \ldots, n$.

We shall prove that there exist constants $C \in (0,1)$ and $\theta > 0$ depending only on f such that

$$\frac{\mu(W)}{\mathrm{diam}(W)^t} \ge C\varepsilon^{\theta\gamma}, \tag{3}$$

$$\frac{\mathrm{OR}(W, z)}{\mathrm{IR}(W, z)} \le (C\varepsilon^{\theta\gamma})^{-1}, \tag{4}$$

where $\gamma = 2\ell_{\max}/(\log r - \ell_{\max} \log 4)$.

Choosing r large enough, we have $\theta\gamma \le \alpha/3$. Provided that $\varepsilon > 0$ is small enough, we have $C\varepsilon^{\theta\gamma} \ge \varepsilon^{\alpha/2}$. Thus (4) implies $\mathrm{OR}(W, z) \le \varepsilon$, so $\mu(B(z, \varepsilon)) \ge \mu(W)$; together with inequality (3) we have

$$\mu(B(z, \varepsilon)) \ge \mu(W) \ge \varepsilon^{t+\alpha},$$

as we wished.

If $f^n|W$ extends to a conformal map onto $B(f^n(z), 2\eta)$ then the inequalities follow easily from the Koebe principle and the t-covariance of μ.

In order to deal with the general case, we define a quasi-chain $\{\widehat{W}_k\}_{k=0}^n$ by the following rules: (i) $\widehat{W}_n = B(f^n(z), 2\eta)$; (ii) once $\widehat{W}_{k+1} \ni f^{k+1}(z)$ is defined, let $\widehat{W}'_k$ be the connected component of $f^{-1}(\widehat{W}_{k+1})$ which contains $f^k(z)$; (iii) if $\widehat{W}'_k$ contains no critical point, then $\widehat{W}_k = \widehat{W}'_k$, otherwise

$$\widehat{W}_k = B(f^k(z), 2\,\mathrm{diam}(\widehat{W}'_k)).$$

Let m be the order the quasi-chain $\{\widehat{W}_k\}_{k=0}^n$. Let $n_0 = n$ and let $n_1 > \cdots > n_m$ be all integers in $\{0, 1, \ldots, n-1\}$ such that $\widehat{W}_{n_i}$ contains a critical point.

Claim. *There exists a constant C_0 depending on f such that*

$$m \le C_0 + 2\ell_{\max} \log(1/\varepsilon)/\log r_1, \tag{5}$$

where $r_1 = r/4^{\ell_{\max}}$.

In fact, $\widehat{W}_{n_1} \subset \widetilde{B}(\delta_0)$. By the BC($r$) property, we deduce that $\widehat{W}'_{n_2} \subset \widetilde{B}(\delta_0/r)$, so that $\widehat{W}_{n_2} \subset \widetilde{B}(4^{\ell_{\max}}\delta_0/r) \subset \widetilde{B}(\delta_0)$. Repeating the procedure, we obtain

$$\widehat{W}_{n_m} \subset \widetilde{B}(\delta_0/r_1^{m-1}).$$

By Proposition 2, for any $\beta \in (0, 1/\ell_{\max})$, there exists $C(\beta) > 0$ such that

$$\mathrm{diam}(W) \le C(\beta)(\delta_0/r_1^{m-1})^\beta.$$

Since $\mathrm{diam}(W) \ge \mathrm{IR}(W) = \varepsilon^{1+\alpha/2t} \ge \varepsilon^{1+\alpha}$, we obtain (5).

For each $1 \le i \le m$, $f^{n_{i-1}-n_i-1} : W_{n_i+1} \to W_{n_{i-1}}$ extends to a conformal map onto $\widehat{W}_{n_{i-1}}$. Since $\mathrm{mod}(\widehat{W}_{n_{i-1}} \setminus W_{n_{i-1}})$ is bounded away from zero, the Koebe principle and the t-covariance of μ give us

$$\frac{\mu(W_{n_i+1})}{\mathrm{diam}(W_{n_i+1})^t} \ge C_1 \frac{\mu(W_{n_{i-1}})}{\mathrm{diam}(W_{n_{i-1}})^t}, \tag{6}$$

$$\frac{\mathrm{OR}(W_{n_i+1}, f^{n_i+1}(z))}{\mathrm{IR}(W_{n_i+1}, f^{n_i+1}(z))} \le \frac{1}{C_1} \frac{\mathrm{OR}(W_{n_{i-1}}, f^{n_{i-1}}(z))}{\mathrm{IR}(W_{n_{i-1}}, f^{n_{i-1}}(z))}, \tag{7}$$

where $C_1 \in (0,1)$ is a universal constant. Similarly, we have

$$\frac{\mu(W)}{\mathrm{diam}(W)^t} \ge C_1 \frac{\mu(W_{n_m})}{\mathrm{diam}(W_{n_m})^t}, \tag{8}$$

$$\frac{\mathrm{OR}(W, z)}{\mathrm{IR}(W, z)} \le \frac{1}{C_1} \frac{\mathrm{OR}(W_{n_m}, f^{n_m}(z))}{\mathrm{IR}(W_{n_m}, f^{n_m}(z))}. \tag{9}$$

By Lemma 5, we have

$$\frac{\mu(W_{n_i})}{\mathrm{diam}(W_{n_i})^t} \ge C_2 \frac{\mu(W_{n_i+1})}{\mathrm{diam}(W_{n_i+1})^t}, \tag{10}$$

where $C_2 \in (0,1)$ is a universal constant. By Proposition 4, we have

$$\frac{\mathrm{OR}(W_{n_i}, f^{n_i}(z))}{\mathrm{IR}(W_{n_i}, f^{n_i}(z))} \le \frac{1}{C_2} \frac{\mathrm{OR}(W_{n_i+1}, f^{n_i+1}(z))}{\mathrm{IR}(W_{n_i+1}, f^{n_i+1}(z))}. \tag{11}$$

Combining the estimates (6), (10) and (8), we obtain

$$\frac{\mu(W)}{\mathrm{diam}(W)^t} \ge C_1^{m+1} C_2^m \frac{\mu(B(f^n(z), \eta'))}{(2\eta')^t}. \tag{12}$$

Since $\inf_{w\in J(f)} \mu(B(w,\eta')) > 0$, it follows that

$$\frac{\mu(W)}{\mathrm{diam}(W)^t} \ge C(C_1C_2)^m, \tag{13}$$

where C is a constant.

Combining the estimates (7), (9) and (11), we obtain

$$\frac{\mathrm{OR}(W,z)}{\mathrm{IR}(W,z)} \le (C_1^{m+1}C_2^m)^{-1}. \tag{14}$$

If we let $\theta = -\log(C_1C_2)$ and redefine the constant C, then (13) and (14) give us (3) and (4) respectively. This completes the proof. ∎

5. Proof of the Main Theorem

Proof of the Main Theorem in the case $J(f) \neq \overline{\mathbb{C}}$. The following argument is similar to the proof of the well-known Frostman's lemma (see [8]). Since $\delta_*(f) = \mathrm{HD}_{\mathrm{hyp}}(J(f))$, it suffices to prove that $\overline{\mathrm{BD}}(J(f)) \le \delta_*(f) + \alpha$ for any $\alpha > 0$.

Let μ be a $\delta_*(f)$-conformal measure of f. By Lemma 3, f satisfies the assumption $(*)$, so that we can apply Theorem 1. Let $N(\varepsilon)$ be the minimal number of open balls with radius ε needed to cover $J(f)$. For any $\varepsilon > 0$ small, $J(f)$ can be covered by a family $\{B_i\}_{i=1}^n$ of open balls of radius ε with intersection multiplicity 4. For each i, Theorem 1 gives us $\mu(B_i) \ge \varepsilon^{\delta_*(f)+\alpha}$, provided that ε is small enough. Thus

$$4 \ge \sum_{i=1}^n \mu(B_i) \ge n\varepsilon^{\delta_*(f)+\alpha} \ge N(\varepsilon)\varepsilon^{\delta_*(f)+\alpha},$$

which implies that

$$\overline{\mathrm{BD}}(J(f)) = \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \le \delta_*(f) + \alpha. \ \blacksquare$$

In the case $J(f) = \overline{\mathbb{C}}$, we do not know whether the condition $(*)$ holds. However, we are still able to obtain enough control on the conformal measures to conclude the proof.

Proposition 5. *If $J(f) = \overline{\mathbb{C}}$, then $\delta_*(f) = 2$.*

Proof. Arguing by contradiction, assume that f has a t-conformal measure with $t < 2$.

Let $r > 1$ be a large constant. Since f has the BC(r) property, there exists an arbitrarily small $\varepsilon > 0$ and a puzzle neighborhood $V = \bigcup_{c\in \mathrm{Crit}'(f)} V_c$ of $\mathrm{Crit}'(f)$ such that for each $c \in \mathrm{Crit}'(f)$,

$$\widetilde{B}(c,\varepsilon) \subset V_c \subset \widetilde{B}(c, 2\varepsilon).$$

Let W_c be the union of all return domains of V which are not contained in $\widetilde{B}(c, \varepsilon/\sqrt{r})$. Let $g : \bigcup_c W_c \to V$ denote the first return map into V under iteration of f. By the BC(r) property, for each component U of $g^{-n}(V)$, $g^n : U \to V$ is a conformal map which extends to a conformal map onto $\widetilde{B}(c, \sqrt{r}\varepsilon)$ for some $c \in \mathrm{Crit}'(f)$. In fact, this follows from the following observation: if U is a component of W_c with $g(U) = V_{c'}$ and $g|U = f^s|U$, then there exists a topological disk $\widehat{U} \subset \widetilde{B}(c, \sqrt{r}\,\varepsilon)$ such that $f^s : \widehat{U} \to \widetilde{B}(c', \sqrt{r}\,\varepsilon)$. By the Koebe principle, it follows that for each component U of $g^{-n}(V)$, $g^n|U$ has small distortion provided that r is large enough.

Let $\mathcal{W}_c^n$ be the collection of all components of $g^{-n}(V)$ which are contained in V_c and let W_c^n be the union of these components. By Lemma 1, almost every point returns to V under iteration of f, thus,

$$\frac{\mathrm{area}(W_c^1)}{\mathrm{area}(V_c)} \geq 1 - \frac{\widetilde{B}(c,\varepsilon/\sqrt{r})}{\widetilde{B}(c,\varepsilon)} \geq 1 - \sigma(r),$$

where $\sigma(r) \to 0$ as $r \to \infty$. Note that for each component U of W_c^1, $\mathrm{area}(U) \leq \mathrm{area}(V_c)/2$. Therefore, provided that r was chosen large enough, we have

$$\sum_{U\in\mathcal{W}_c^1} \mathrm{area}(U)^{t/2} \geq \lambda\,\mathrm{area}(V_c)^{t/2},$$

where $\lambda = 2^{1-t/2} > 1$.

For each $U \in \mathcal{W}_c^n$, since $g^n|U$ has small distortion and g^n maps an element of $\mathcal{W}_c^{n+1}$ onto an element of $\mathcal{W}_c^1$, it follows that

$$\sum_{W\in\mathcal{W}_c^{n+1},\, W\subset U} \mathrm{area}(W)^{t/2} \geq \lambda_1\,\mathrm{area}(U)^{t/2},$$

where $\lambda_1 \in (1, \lambda)$. Therefore,

$$\sum_{U\in\mathcal{W}_c^n} \mathrm{area}(U)^{t/2} \geq \lambda_1^n\,\mathrm{area}(V_c)^{t/2}.$$

By the Koebe principle and the t-covariance of μ, for each $U \in \mathcal{W}_c^n$, $n = 0, 1, \dots$, we know that $\mu(U)/\,\mathrm{area}(U)^{t/2}$ is comparable to $\mu(V_c)/\,\mathrm{area}(V_c)^{t/2}$, where $V_c = g^n(U)$. Thus,

$$\sum_{U \in \mathcal{W}_c^n} \mu(U) \geq C\lambda_1^n.$$

Letting $n \to \infty$ implies $\mu(J(f)) = \infty$, a contradiction. ∎

Proof of the Main Theorem in the case $J(f) = \overline{\mathbb{C}}$. By the previous proposition, $\delta_*(f) = 2$. Hence

$$\delta_*(f) = \mathrm{HD}_{\mathrm{hyp}}(f) = 2 = \overline{\mathrm{BD}}(J(f)). \ \blacksquare$$

References

[1] H. Bruin, J. Rivera-Letelier, W. X. Shen and S. van Strien. *Large derivatives, backward contraction and invariant densities for interval maps*, preprint, 2007.

[2] M. Denker and M. Urbański, *On Sullivan's conformal measures for rational maps of the Riemann sphere*, Nonlinearity 4 (1991), 365–384.

[3] K. Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, Wiley, 1990.

[4] J. Graczyk and S. Smirnov, *Non-uniform hyperbolicity in complex dynamics, I, II*, Invent. Math., to appear.

[5] J. Milnor, *Dynamics in One Complex Variable: Introductory Lectures*, Vieweg, Braunschweig, 1999.

[6] F. Przytycki, *Iterations of holomorphic Collet–Eckmann maps: conformal and invariant measures*, Trans. Amer. Math. Soc. 350 (1998), 717–742.

[7] F. Przytycki and J. Rivera-Letelier, *Statistical properties of topological Collet–Eckmann maps*, Ann. Sci. École Norm. Sup. 40 (2007), 135–178.

[8] F. Przytycki and M. Urbański, *Fractals in the Plane—the Ergodic Theory Methods*, http://www.math.unt.edu/~urbanski/.

[9] J. Rivera-Letelier, *A connecting lemma for rational maps satisfying a no growth condition*, Ergodic Theory Dynam. Systems 27 (2007), 595–636.

[10] D. Sullivan, *Conformal dynamical system*, in: Geometric Dynamics, Lecture Notes in Math. 1007, Springer, 1983, 725–752.

[11] M. Urbański, *Measures and dimensions in conformal dynamics*, Bull. Amer. Math. Soc. 40 (2003), 281–321.

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