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A locally commutative free group acting on the plane

by

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Abstract. The purpose of this paper is to prove the existence of a free subgroup of the group of all affine transformations on the plane with determinant 1 such that the action of the subgroup is locally commutative.

Introduction. We say that a group G of transformations is *locally commutative* if $\gamma\gamma' = \gamma'\gamma$ for any $\gamma, \gamma' \in G$ such that $\gamma(\vec{z}) = \vec{z} = \gamma'(\vec{z})$ for some point $\vec{z}$. We will study the group $SA_2(\mathbb{R})$ of affine area-preserving and orientation-preserving transformations on the plane $\mathbb{R}^2$, that is, transformations of the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix},$$

where $ad - bc = 1$. We will prove that there exists a free non-abelian locally commutative subgroup of $SA_2(\mathbb{R})$. More precisely:

THEOREM. *For every transcendental real number θ, the affine transformations*

$$u : \vec{z} \mapsto \begin{pmatrix} 2 & \theta \\ 0 & \frac{1}{2} \end{pmatrix} \vec{z} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \textit{and} \quad v : \vec{z} \mapsto \begin{pmatrix} 2 & 0 \\ \theta & \frac{1}{2} \end{pmatrix} \vec{z} + \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

are free generators of a group of transformations which is locally commutative.

By this Theorem (see the last section of our paper) the set of n-tuples $(\alpha_0, \ldots, \alpha_{n-1})$ in $(SA_2(\mathbb{R}))^n$ which fail to be free generators of a free locally commutative subgroup is a subset of a countable union of proper algebraic subsets. It follows of course that this union is meager and of measure 0. This has an additional consequence (for the proof see [M_1], for related general results see [M_2] and [K]):

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Corollary. *There exists a set* $F \subseteqq SA_2(\mathbb{R})$ *which generates freely a locally commutative subgroup such that for every open non-empty subset* V *of* $SA_2(\mathbb{R})$, $F \cap V$ *contains a perfect set* (*that is*, *non-empty*, *closed and dense in itself*).

Remark. Of course F has power continuum. However, by a well known theorem of Banach, F or even any Borel set of free generators must be meager and of measure 0.

These results are motivated by a long line of research starting with Klein, Fricke, and Hausdorff and more specifically von Neumann [V] (see also [W, Th. 7.3], [MW, M_3, L, S_0]). He proved that every bounded subset S of $\mathbb{R}^2$ with non-empty interior is $SA_2(\mathbb{R})$-paradoxical, that is, for some positive integers N and N' there are pairwise disjoint subsets $S_0, \ldots, S_{N+N'-1}$ of S and $\gamma_0, \ldots, \gamma_{N+N'-1} \in SA_2(\mathbb{R})$ such that

$$S = \gamma_0(S_0) \cup \ldots \cup \gamma_{N-1}(S_{N-1}) = \gamma_N(S_N) \cup \ldots \cup \gamma_{N+N'-1}(S_{N+N'-1}).$$

Von Neumann's proof requires only the existence of a free non-abelian subgroup of $SA_2(\mathbb{R})$. But other results require local commutativity. For example the above corollary implies that there exists a set $E \subseteqq \mathbb{R}^2$ such that for every countable set $D \subseteqq \mathbb{R}^2$ there exists a $\gamma \in SA_2(\mathbb{R})$ such that

$$\gamma(E) = E \bigtriangleup D,$$

where $\bigtriangleup$ denotes the symmetric difference of sets (for the proof see [M_0]). Also, it implies a very general theorem about the existence of partitions of $\mathbb{R}^2$ into disjoint sets S_j $(j \in I)$ satisfying certain systems of equations of the form

$$\alpha\Big(\bigcup_{j\in I_\alpha} S_j\Big) = \bigcup_{j'\in I'_\alpha} S_{j'},$$

where α runs over our set F of free generators of a locally commutative group, and $\emptyset \neq I_\alpha, I'_\alpha \subsetneq I$. The only requirement is that the consequences of this system obtained by complementation of both sides of an equation and by transitivity do not yield an equation of the form $\gamma(S) = \mathbb{R}^2 \setminus S$ (for the proof see [W, Corollary 4.12]). A special case is a partition of $\mathbb{R}^2$ into four sets S_0, S_1, S_2, and S_3 satisfying

$$u(S_1) = S_1 \cup S_2 \cup S_3, \qquad v(S_3) = S_0 \cup S_1 \cup S_3,$$

where u and v are as in our Theorem. Thus, each of the sets $S_0 \cup S_1$ and $S_2 \cup S_3$ yields the whole plane $\mathbb{R}^2$ by decomposition into two pieces and transformation of one of them. For more applications similar to the above, see [W].

Finally let us add that, in connection with the above mentioned theorem of von Neumann, M. Laczkovich [L] proved that $]0,1] \times]0,1]$ is $SL_2(\mathbb{R})$-paradoxical, where $SL_2(\mathbb{R})$ is the subgroup of $SA_2(\mathbb{R})$ consisting of linear transformations. And the author has shown in [S_0] that a subgroup of $SA_2(\mathbb{R})$ is freely generated by the transformations

$$\zeta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ 9 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$
$$\eta\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 94 & 39 \\ 147 & 61 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

and this subgroup acts without fixed points on $\mathbb{Z}^2$ (that is, $\gamma(\vec{z}) \neq \vec{z}$ for all $\vec{z} \in \mathbb{Z}^2$ and γ different from the identity).

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Proof of Theorem. The first part of the proof, beginning with Lemma 0 and up to Proposition 0, will be parallel to an argument in [S_1]; however, the calculations are rather different. The remainder of the proof takes advantage of Lemmas 3–5, Proposition 0 and Theorem 3 of [S_1]. Then we need a new Lemma 2 which is used in the final argument, the proof of local commutativity.

Let u and v be elements of $SA_2(\mathbb{R})$, i.e.,

$$u : \vec{z} \mapsto A\vec{z} + \vec{p}, \qquad v : \vec{z} \mapsto B\vec{z} + \vec{q},$$

where A and B are 2×2 matrices with determinant 1 and $\vec{p}$ and $\vec{q}$ are vectors in $\mathbb{R}^2$.

LEMMA 0. *Let*

$$w = u^{k_0} v^{l_0} u^{k_1} v^{l_1} \ldots u^{k_{m-1}} v^{l_{m-1}},$$

where m is a positive integer and $k_0, l_0, k_1, l_1, \ldots, k_{m-1}, l_{m-1}$ are non-zero integers. If the matrices

$$\mathrm{id} - A, \qquad \mathrm{id} - B, \qquad \mathrm{id} - A^{k_0} B^{l_0} A^{k_1} B^{l_1} \ldots A^{k_{m-1}} B^{l_{m-1}}$$

are invertible, *then the vector*

$$(\mathrm{id} - A^{k_0} B^{l_0} A^{k_1} B^{l_1} \ldots A^{k_{m-1}} B^{l_{m-1}})^{-1} \cdot (\mathrm{id} - A^{k_0} + A^{k_0} B^{l_0} - A^{k_0} B^{l_0} A^{k_1} + \ldots$$
$$+ A^{k_0} B^{l_0} \ldots B^{l_{m-2}} - A^{k_0} B^{l_0} \ldots B^{l_{m-2}} A^{k_{m-1}}) \cdot ((\mathrm{id} - A)^{-1}\vec{p} - (\mathrm{id} - B)^{-1}\vec{q})$$
$$+ (\mathrm{id} - B)^{-1}\vec{q}$$

is the unique fixed point of the transformation w.

Proof. We have the equalities

$$u^k(\vec{z}) = A^k\vec{z} + \frac{\mathrm{id} - A^k}{\mathrm{id} - A} \cdot \vec{p}, \qquad v^l(\vec{z}) = B^l\vec{z} + \frac{\mathrm{id} - B^l}{\mathrm{id} - B} \cdot \vec{q}.$$

Hence

$$
\begin{aligned}
&w(\vec{z})\\
&= A^{k_0}\Bigg(B^{l_0}\Bigg(\dots\Bigg(A^{k_{m-1}}\Bigg(B^{l_{m-1}}\vec{z} + \frac{\mathrm{id}-B^{l_{m-1}}}{\mathrm{id}-B}\cdot\vec{q}\Bigg) + \frac{\mathrm{id}-A^{k_{m-1}}}{\mathrm{id}-A}\cdot\vec{p}\Bigg)\dots\Bigg)\\
&\quad + \frac{\mathrm{id}-B^{l_0}}{\mathrm{id}-B}\cdot\vec{q}\Bigg) + \frac{\mathrm{id}-A^{k_0}}{\mathrm{id}-A}\vec{p}\\
&= A^{k_0}B^{l_0}\dots B^{l_{m-1}}\vec{z} + ((\mathrm{id}-A^{k_0}) + A^{k_0}B^{l_0}(\mathrm{id}-A^{k_1}) + \dots\\
&\quad + A^{k_0}B^{l_0}\dots B^{l_{m-2}}(\mathrm{id}-A^{k_{m-1}}))\cdot(\mathrm{id}-A)^{-1}\cdot\vec{p}\\
&\quad + (A^{k_0}(\mathrm{id}-B^{l_0}) + A^{k_0}B^{l_0}A^{k_1}(\mathrm{id}-B^{l_1}) + \dots\\
&\quad + A^{k_0}B^{l_0}\dots A^{k_{m-1}}(\mathrm{id}-B^{l_{m-1}}))\cdot(\mathrm{id}-B)^{-1}\cdot\vec{q}\\
&= A^{k_0}B^{l_0}\dots B^{l_{m-1}}\vec{z} + (\mathrm{id}-A^{k_0} + A^{k_0}B^{l_0} - A^{k_0}B^{l_0}A^{k_1} + \dots\\
&\quad + A^{k_0}B^{l_0}\dots B^{l_{m-2}} - A^{k_0}B^{l_0}\dots A^{k_{m-1}})\cdot((\mathrm{id}-A)^{-1}\vec{p} - (\mathrm{id}-B)^{-1}\vec{q})\\
&\quad + (\mathrm{id}-A^{k_0}B^{l_0}\dots B^{l_{m-1}})\cdot(\mathrm{id}-B)^{-1}\cdot\vec{q},
\end{aligned}
$$

which implies immediately the conclusion of the lemma. ∎

From now on, let A, $\vec{p}$, B, $\vec{q}$ be as in the Theorem, i.e.,

$$
A = \begin{pmatrix} 2 & \frac{3}{2}\tau \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \vec{p} = \begin{pmatrix} -1 \\ 0 \end{pmatrix},
$$

$$
B = \begin{pmatrix} 2 & 0 \\ \frac{3}{2}\tau & \frac{1}{2} \end{pmatrix}, \quad \vec{q} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix},
$$

where $\tau = \frac{2}{3}\theta$.

LEMMA 1. *Let w be a non-empty reduced word in u and v. Then the diagonal elements of the matrix part in w are polynomials in $\mathbb{Q}[\tau^2]$ and other elements are polynomials of the form $\tau\cdot f$, where $f\in\mathbb{Q}[\tau^2]$. Moreover*:

- *if $w = u^{k_0}v^{l_0}\dots v^{l_{m-1}}$ $(m\ge 1)$ then the highest terms of the matrix part of w are*

$$
\begin{aligned}
&A^{k_0}B^{l_0}\dots B^{l_{m-1}}\\
&\qquad = \begin{pmatrix} \dots + t(k_0)t(l_0)\dots t(l_{m-1})\tau^{2m} & \dots + t(k_0)t(l_0)\dots t(k_{m-1})2^{-l_{m-1}}\tau^{2m-1} \\ \dots + 2^{-k_0}t(l_0)\dots t(l_{m-1})\tau^{2m-1} & \dots + 2^{-k_0}t(l_0)\dots t(k_{m-1})2^{-l_{m-1}}\tau^{2m-2} \end{pmatrix},
\end{aligned}
$$

- *if $w = u^{k_0}v^{l_0}\dots u^{k_m}$ $(m\ge 0)$ then the highest terms of the matrix part of w are*

$$A^{k_0}B^{l_0}\dots A^{k_m}$$
$$= \begin{cases} \begin{pmatrix} \dots + t(k_0)t(l_0)\dots t(l_{m-1})2^{k_m}\tau^{2m} & \dots + t(k_0)t(l_0)\dots t(k_m)\tau^{2m+1} \\ \dots + 2^{-k_0}t(l_0)\dots t(l_{m-1})2^{k_m}\tau^{2m-1} & \dots + 2^{-k_0}t(l_0)\dots t(k_m)\tau^{2m} \end{pmatrix} & (\textit{if } m \geq 1), \\ \begin{pmatrix} 2^{k_0} & t(k_0)\tau \\ 0 & 2^{-k_0} \end{pmatrix} & (\textit{if } m = 0), \end{cases}$$

• *if* $w = v^{l_0}u^{k_1}\dots u^{k_m}$ $(m \geq 1)$ *then the highest terms of the matrix part of* w *are*

$$B^{l_0}A^{k_1}\dots A^{k_m}$$
$$= \begin{pmatrix} \dots + 2^{l_0}t(k_1)\dots t(l_{m-1})2^{k_m}\tau^{2m-2} & \dots + 2^{l_0}t(k_1)\dots t(k_m)\tau^{2m-1} \\ \dots + t(l_0)t(k_1)\dots t(l_{m-1})2^{k_m}\tau^{2m-1} & \dots + t(l_0)t(k_1)\dots t(k_m)\tau^{2m} \end{pmatrix},$$

• *if* $w = v^{l_0}u^{k_1}\dots v^{l_m}$ $(m \geq 0)$ *then the highest terms of the matrix part of* w *are*

$$B^{l_0}A^{k_1}\dots B^{l_m}$$
$$= \begin{cases} \begin{pmatrix} \dots + 2^{l_0}t(k_1)\dots t(l_m)\tau^{2m} & \dots + 2^{l_0}t(k_1)\dots t(k_m)2^{-l_m}\tau^{2m-1} \\ \dots + t(l_0)t(k_1)\dots t(l_m)\tau^{2m+1} & \dots + t(l_0)t(k_1)\dots t(k_m)2^{-l_m}\tau^{2m} \end{pmatrix} & (\textit{if } m \geq 1), \\ \begin{pmatrix} 2^{l_0} & 0 \\ t(l_0)\tau & 2^{-l_0} \end{pmatrix} & (\textit{if } m = 0), \end{cases}$$

where $t(k) = 2^k - 2^{-k}$.

Proof. This lemma is shown inductively by simple calculations (the proof is similar to the proof of Lemma 1 in [S_1], but the calculations are simpler). ■

Lemma 1 implies immediately the following facts:

$$\begin{aligned} \det(\mathrm{id} - A^{k_0}) &= 2 - \mathrm{tr}(A^{k_0}) = -(2^{k_0} - 1)(1 - 2^{-k_0}) \neq 0, \\ \det(\mathrm{id} - B^{l_0}) &= 2 - \mathrm{tr}(B^{l_0}) = -(2^{l_0} - 1)(1 - 2^{-l_0}) \neq 0, \\ \det(\mathrm{id} - A^{k_0}B^{l_0}\dots B^{l_{m-1}}) &= 2 - \mathrm{tr}(A^{k_0}B^{l_0}\dots B^{l_{m-1}}) \\ &= \dots - t(k_0)\dots t(l_{m-1})\tau^{2m} \neq 0. \end{aligned}$$

In other words, each word $w = u^{k_0}v^{l_0}\dots v^{l_{m-1}}$ satisfies the assumption of Lemma 0. Hence the fixed point of w is unique. The fixed point of a power of u or of v is also unique, so the fixed point of each non-empty reduced word is unique. So the first part of our Theorem follows: *The group* F_2 *generated by* u *and* v *is a free group of rank* 2. Therefore we can define an equivalence relation on all non-empty reduced words in u and v,

$$w \sim w' \underset{\text{def}}{\iff} w \text{ and } w' \text{ have a common fixed point in } \mathbb{R}^2,$$

and the following Proposition 0 of [S_1] is true:

Proposition 0. *For non-zero integers k and l, we have*

$$w^k \sim w'^l \ \Leftrightarrow\ w \sim w' \ \Leftrightarrow\ \overline{w}w\overline{w}^{-1} \sim \overline{w}w'\overline{w}^{-1}.$$

Furthermore, if $w^{-1} \neq w'$, we have

$$w \sim w'w \ \Leftrightarrow\ w \sim w' \ \Leftrightarrow\ w \sim ww'.$$

Let us divide cyclically reduced words of F_2 into six different types according to their first and last letters up to inversions:

$$(*) \qquad \begin{array}{ccc} u\ldots u, & u^{-1}\ldots v^{-1}, & u^{-1}\ldots v, \\ v\ldots v, & u\ldots v^{-1}, & u\ldots v. \end{array}$$

Using Lemmas 3–5 and Theorem 3 of [S$_1$], we see that in order to prove that the action of F_2 on $\mathbb{R}^2$ is locally commutative it is enough to show that words of distinct types $(*)$ have no common fixed points. From now on, our proof diverges from that of [S$_1$]. Consider the following four types:

$$(\diamond) \qquad \begin{array}{cc} u^{-1}\ldots v^{-1}, & u^{-1}\ldots v, \\ u\ldots v^{-1}, & u\ldots v. \end{array}$$

Lemma 2. *If any two reduced words of distinct types of $(\diamond)$ have no common fixed points in $\mathbb{R}^2$, then the same is true for reduced words of distinct types of $(*)$.*

Proof. Let $\not\sim$ denote the negation of $\sim$. We will argue by induction relative to the length of reduced words. For $w = u\ldots u$ and $w' = v\ldots v$, from the assumption of this lemma, we have $w^{-1}w' \not\sim ww'$. Thus $w \not\sim w'$ (since if $w \sim w'$, then from Proposition 0 we have $w^{-1}w' \sim w' \sim ww'$, a contradiction). For $w = u\ldots u$ and $w'' = u^{\varepsilon}\ldots v^{\delta}$ with $\varepsilon, \delta \in \{-1, 1\}$, let $\widetilde{w} = w^{-\varepsilon}w''$. If $w^{\varepsilon}u^{\varepsilon} \subseteq w''$ (where $w_0 \subseteq w_1$ means w_0 is an initial segment of w_1, i.e., we can represent w_1 in the form $w_0\overline{w}$ without cancellation), then we can reduce $w \not\sim w''$ to $w \not\sim \widetilde{w}$ with shorter length. So, applying the inductive assumption, we can assume that $w^{\varepsilon}u^{\varepsilon} \not\subseteq w''$. If $w^{\varepsilon} \subseteq w''$, i.e., $w^{\varepsilon}v^{-1} \subseteq w''$ or $w^{\varepsilon}v \subseteq w''$, then $w^{-\varepsilon}\widetilde{w} \not\sim w''$. If $w^{\varepsilon} \supseteq w''u^{\varepsilon}$, then we can reduce the relation $w \not\sim w''$ to $\widetilde{w}^{-\varepsilon} \not\sim w''$ with shorter length. So we can assume $w^{\varepsilon} \not\supseteq w''u^{\varepsilon}$. If $w^{\varepsilon} \supseteq w''$, i.e., $w^{\varepsilon} \supseteq w''u^{-\varepsilon}$ or $w^{\varepsilon} \supseteq w''v^{\delta}$, then $\widetilde{w}w'' \not\sim w''$. Otherwise, i.e., if neither $w^{\varepsilon} \subseteq w''$ nor $w^{\varepsilon} \supseteq w''$, we have $\widetilde{w} \not\sim w''$. For $w' = v\ldots v$ and $w'' = u^{\varepsilon}\ldots v^{\delta}$, the proof is similar. ∎

Now we will prove the second part of the Theorem, which states that: *The group F_2 acts on $\mathbb{R}^2$ in a locally commutative way.* By Lemma 2 this follows immediately from the following:

Lemma 3. *Words of distinct types $(\diamond)$ have no common fixed points.*

Proof. For $w = u^{k_0}v^{l_0}\ldots v^{l_{m-1}}$, we define

$$\begin{aligned}\vec{r}(w) &= (2-\mathrm{tr}(A^{k_0}B^{l_0}\dots B^{l_{m-1}}))\cdot((\text{the fixed point of } w)-(\mathrm{id}-B)^{-1}\cdot\vec{q})\\ &= (\mathrm{id}-(A^{k_0}B^{l_0}\dots B^{l_{m-1}})^{-1})\cdot(\mathrm{id}-A^{k_0}+A^{k_0}B^{l_0}-A^{k_0}B^{l_0}A^{k_1}+\dots\\ &\quad +A^{k_0}B^{l_0}\dots B^{l_{m-2}}-A^{k_0}B^{l_0}\dots A^{k_{m-1}})\cdot((\mathrm{id}-A)^{-1}\vec{p}-(\mathrm{id}-B)^{-1}\vec{q})\\ &= (\mathrm{id}-A^{k_0}+A^{k_0}B^{l_0}-A^{k_0}B^{l_0}A^{k_1}+\dots\\ &\quad +A^{k_0}B^{l_0}\dots B^{l_{m-2}}-A^{k_0}B^{l_0}\dots A^{k_{m-1}}\\ &\quad -B^{-l_{m-1}}A^{-k_{m-1}}\dots A^{-k_0}+B^{-l_{m-1}}A^{-k_{m-1}}\dots B^{-l_0}-\dots\\ &\quad -B^{-l_{m-1}}A^{-k_{m-1}}+B^{-l_{m-1}})\cdot((\mathrm{id}-A)^{-1}\vec{p}-(\mathrm{id}-B)^{-1}\vec{q}),\end{aligned}$$

where the second equality follows from Lemma 0. Since

$$2-\mathrm{tr}(A^{k_0}B^{l_0}\dots B^{l_{m-1}})\in\mathbb{Q}[\tau^2],$$

it is enough to show that for two reduced words w and w' of distinct types $(\diamond)$, the vectors $\vec{r}(w)$ and $\vec{r}(w')$ are linearly independent over $\mathbb{Q}(\tau^2)$. For $w=u^{k_0}v^{l_0}\dots v^{l_{m-1}}$, we have

$$\begin{aligned}\vec{r}(w) &= (\dots+A^{k_0}B^{l_0}\dots B^{l_{m-2}}-A^{k_0}B^{l_0}\dots A^{k_{m-1}}\\ &\qquad -B^{-l_{m-1}}A^{-k_{m-1}}\dots A^{-k_0}+B^{-l_{m-1}}A^{-k_{m-1}}\dots B^{-l_0}-\dots)\\ &\qquad \cdot((\mathrm{id}-A)^{-1}\vec{p}-(\mathrm{id}-B)^{-1}\vec{q})\\ &= \Bigg(\dots+\begin{pmatrix}\dots+t(k_0)\dots t(l_{m-2})\tau^{2m-2} & \dots\\ \dots & \dots\end{pmatrix}\\ &\quad -\begin{pmatrix}\dots+t(k_0)\dots t(l_{m-2})2^{k_{m-1}}\tau^{2m-2} & \dots+t(k_0)\dots t(k_{m-1})\tau^{2m-1}\\ \dots & \dots\end{pmatrix}\\ &\quad -\begin{pmatrix}\dots+2^{-l_{m-1}}t(-k_{m-1})\dots\cdot t(-l_0)2^{-k_0}\tau^{2m-2} & \dots+2^{-l_{m-1}}t(-k_{m-1})\dots\cdot t(-k_0)\tau^{2m-1}\\ \dots+t(-l_{m-1})\dots t(-l_0)2^{-k_0}\tau^{2m-1} & \dots+t(-l_{m-1})\dots t(-k_0)\tau^{2m}\end{pmatrix}\\ &\quad +\begin{pmatrix}\dots+2^{-l_{m-1}}t(-k_{m-1})\dots t(-l_0)\tau^{2m-2} & \dots\\ \dots+t(-l_{m-1})\dots t(-l_0)\tau^{2m-1} & \dots\end{pmatrix}-\dots\Bigg)\cdot\begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{cases}\dots+t(l_0)\dots t(l_{m-2})\tau^{2m-2}\\ \quad\cdot\begin{pmatrix}t(k_0)(1-2^{k_{m-1}})+(-2^{-k_0}+1)t(k_{m-1})2^{-l_{m-1}}\\ \qquad +t(k_0)t(k_{m-1})(-1+2^{-l_{m-1}})\tau\\ (2^{-k_0}-1)t(k_{m-1})t(l_{m-1})\tau-t(k_0)t(k_{m-1})t(l_{m-1})\tau^2\end{pmatrix} & (\text{if } m\geq 2),\\ \dots+\begin{pmatrix}(1-2^{k_0})+(-2^{-k_0}+1)2^{-l_0}+t(k_0)(-1+2^{-l_0})\tau\\ (2^{-k_0}-1)t(l_0)\tau-t(k_0)t(l_0)\tau^2\end{pmatrix} & (\text{if } m=1),\end{cases}\end{aligned}$$

where $t(-k) = -t(k)$ implies the last equality. Let P, Q and R be non-zero rational numbers such that

$$\vec{r}(w) = \begin{pmatrix} \ldots + Q\tau^{2m-1} \\ \ldots + P\tau^{2m-1} + R\tau^{2m} \end{pmatrix}.$$

Then $R/P = 2^{k_0} + 1$ and $R/Q = 2^{l_{m-1}} + 1$. On the other hand, if

$$\begin{pmatrix} \ldots + Q\tau^{2m-1} \\ \ldots + P\tau^{2m-1} + R\tau^{2m} \end{pmatrix}, \quad \begin{pmatrix} \ldots + Q'\tau^{2m'-1} \\ \ldots + P'\tau^{2m'-1} + R'\tau^{2m'} \end{pmatrix}$$

are linearly dependent over $\mathbb{Q}(\tau^2)$ then

$$P : Q : R = P' : Q' : R'.$$

Hence, for

$$w = u^{k_0} v^{l_0} \ldots v^{l_{m-1}} \quad \text{and} \quad w' = u^{k'_0} v^{l'_0} \ldots v^{l'_{m'-1}},$$

if $k_0 \neq k'_0$ or $l_{m-1} \neq l'_{m'-1}$ then $\vec{r}(w)$ and $\vec{r}(w')$ are linearly independent over $\mathbb{Q}(\tau^2)$. ■

By the remarks preceding Proposition 0 and Lemma 3 this concludes the proof of the Theorem.

Proof of the Corollary. It remains to show a proposition stated in the Introduction prior to the Corollary.

PROPOSITION 1. *The set of n-tuples $(\alpha_0, \ldots, \alpha_{n-1})$ in $(SA_2(\mathbb{R}))^n$ which fail to be free generators of a free locally commutative subgroup is a subset of a countable union of proper algebraic subsets.*

Proof. First we mention that $v, u^{-1}vu, u^{-2}vu^2, \ldots$ generate a free subgroup of infinite rank. Let w be a non-empty reduced word in the letters $X_0^{-1}, \ldots, X_{n-1}^{-1}, X_0, \ldots, X_{n-1}$. For $\alpha_0, \ldots, \alpha_{n-1}$ in $SA_2(\mathbb{R})$, the transformation $w(\alpha_0, \ldots, \alpha_{n-1})$ of $SA_2(\mathbb{R})$ is the value of the map $(SA_2(\mathbb{R}))^n \to SA_2(\mathbb{R})$ defined by the word w (e.g., $w(\alpha_0, \alpha_1, \alpha_2) = \alpha_0^{-1}\alpha_1^{-1}\alpha_0^2\alpha_2^3$ for $w = X_0^{-1}X_1^{-1}X_0^2X_2^3$). For each w, the set

$$(\natural) \quad \{(\alpha_0, \ldots, \alpha_{n-1}) \in (SA_2(\mathbb{R}))^n : w(\alpha_0, \ldots, \alpha_{n-1}) = \text{id in } SA_2(\mathbb{R})\}$$

is a proper algebraic subset of $(SA_2(\mathbb{R}))^n$. In particular, by the Theorem, the n-tuple $(v, u^{-1}vu, \ldots, u^{-n+1}vu^{n-1})$ does not belong to it.

Let w, w' be two words such that $ww' \neq w'w$. If $w(\alpha_0, \ldots, \alpha_{n-1})$ and $w'(\alpha_0, \ldots, \alpha_{n-1})$ have a common fixed point $\vec{z}$, then

$$\vec{\xi}_{w(\alpha_0,\ldots,\alpha_{n-1})} = (\text{id} - M_{w(\alpha_0,\ldots,\alpha_{n-1})})\vec{z},$$
$$\vec{\xi}_{w'(\alpha_0,\ldots,\alpha_{n-1})} = (\text{id} - M_{w'(\alpha_0,\ldots,\alpha_{n-1})})\vec{z},$$

where M_γ is the linear part and $\vec{\xi}_\gamma$ is the translation part of γ. So we have

$$\det(\mathrm{id}-M_{w'(\alpha_0,\dots,\alpha_{n-1})})\,\mathrm{adj}(\mathrm{id}-M_{w(\alpha_0,\dots,\alpha_{n-1})})\vec{\xi}_{w(\alpha_0,\dots,\alpha_{n-1})}$$
$$= \det(\mathrm{id}-M_{w'(\alpha_0,\dots,\alpha_{n-1})})\,\mathrm{adj}(\mathrm{id}-M_{w(\alpha_0,\dots,\alpha_{n-1})})(\mathrm{id}-M_{w(\alpha_0,\dots,\alpha_{n-1})})\vec{z}$$
$$= \det(\mathrm{id}-M_{w'(\alpha_0,\dots,\alpha_{n-1})})\det(\mathrm{id}-M_{w(\alpha_0,\dots,\alpha_{n-1})})\vec{z}$$
$$= \det(\mathrm{id}-M_{w(\alpha_0,\dots,\alpha_{n-1})})\,\mathrm{adj}(\mathrm{id}-M_{w'(\alpha_0,\dots,\alpha_{n-1})})(\mathrm{id}-M_{w'(\alpha_0,\dots,\alpha_{n-1})})\vec{z}$$
$$= \det(\mathrm{id}-M_{w(\alpha_0,\dots,\alpha_{n-1})})\,\mathrm{adj}(\mathrm{id}-M_{w'(\alpha_0,\dots,\alpha_{n-1})})\vec{\xi}_{w'(\alpha_0,\dots,\alpha_{n-1})},$$

where

$$\mathrm{adj}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Hence the set

$$(\sharp)\qquad \{(\alpha_0,\dots,\alpha_{n-1}) \in (SA_2(\mathbb{R}))^n : w(\alpha_0,\dots,\alpha_{n-1}) \text{ and } w'(\alpha_0,\dots,\alpha_{n-1}) \text{ have a common fixed point in } \mathbb{R}^2\}$$

is contained in a proper algebraic subset (proper, since by the Theorem, $(v, u^{-1}vu, \dots, u^{-n+1}vu^{n-1})$ does not belong to it). So, the union of all sets represented by ($\natural$) and ($\sharp$),

$$\{(\alpha_0,\dots,\alpha_{n-1}) \in (SA_2(\mathbb{R}))^n : \alpha_0,\dots,\alpha_{n-1} \text{ do not generate a free group which acts on } \mathbb{R}^2 \text{ in a locally commutative way}\},$$

is a subset of a countable union of proper algebraic subsets of $(SA_2(\mathbb{R}))^n$. ∎

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