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Differentiation of n-convex functions

by

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Abstract. The main result of this paper is that if f is n-convex on a measurable subset E of $\mathbb{R}$, then f is $n-2$ times differentiable, $n-2$ times Peano differentiable and the corresponding derivatives are equal, and $f^{(n-1)} = f_{(n-1)}$ except on a countable set. Moreover $f_{(n-1)}$ is approximately differentiable with approximate derivative equal to the nth approximate Peano derivative of f almost everywhere.

1. Introduction. Throughout this paper, E denotes a measurable subset of $\mathbb{R}$. In addition, because most of the results concern right or left limit points of E, we further assume that E has no isolated points. This is not much of a restriction since we are primarily concerned with a.e. statements.

Definition 1.1. Let $f : E \to \mathbb{R}$, let $n \in \mathbb{N} \cup \{0\}$ and let $x_0 < x_1 < \cdots < x_n$ be points in E. Then the nth *divided difference* of f is denoted by $[f; x_0, x_1, \ldots, x_n]$ and defined inductively. For $n = 0$, $[f; x_0] = f(x_0)$ and for $n \in \mathbb{N}$,

$$(1)\qquad [f; x_0, x_1, \ldots, x_n] = \frac{[f; x_0, x_1, \ldots, x_{n-1}] - [f; x_1, x_2, \ldots, x_n]}{x_0 - x_n}.$$

In particular $[f; x_0, x_1] = (f(x_0) - f(x_1))/(x_0 - x_1)$.

Alternatively $[f; x_0, x_1, \ldots, x_n]$ can be defined as

$$\sum_{i=0}^{n} \frac{f(x_i)}{\prod_{\ell \neq i}(x_i - x_\ell)},$$

which shows that $[f; x_0, x_1, \ldots, x_n]$ is independent of the order in which the numbers $x_0, x_1, \ldots, x_n$ are listed. (For basic properties of divided differences see [6].) So in the future if $V = \{x_0, \ldots, x_n\}$, we use $[f; V]$ for $[f; x_0, \ldots, x_n]$. We also permit the slight abuse of notation to write $[f; u, V]$ for $[f; V \cup \{u\}]$

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where $u \in E \setminus V$. As a consequence, equation (1) can be written as

$$[f;V] = \frac{[f;V\setminus\{v\}] - [f;V\setminus\{w\}]}{w-v}$$

where $w, v \in V$ with $w \neq v$, and the equivalent definition can be written as

$$[f;V] = \sum_{u\in V} \frac{f(u)}{\prod_{\substack{v\in V\\ v\neq u}}(u-v)}. \tag{2}$$

Definition 1.2. Let $f : E \to \mathbb{R}$ and let $n \in \mathbb{N}$. Then f is *n-convex* on E means $[f;V] \geq 0$ for each $V \subset E$ of cardinality $n+1$.

Consequently, f is 0-convex on E means $f \geq 0$ on E, 1-convex on E means f is increasing on E, and 2-convex means f is convex in the usual sense. It is easy to show that a polynomial of degree n is n-convex if and only if the leading coefficient of f is positive. If f is n-convex on a set E, then, as will be shown, f has a unique n-convex extension to the closure of E. Consequently, it could be assumed that E is closed, but there is no advantage in doing so.

The motivation for this paper is the article by P. S. Bullen and S. N. Mukhopadhyay ([3]) in which the following is asserted.

Theorem (Theorem 6.1 in [3]). *If f is n-convex on a measurable set $E \subset [a,b]$ on which the $(n-1)$th Peano derivative $f_{(n-1)}$ exists finitely, then both the nth approximate Peano derivative $f_{(n),\mathrm{ap}}$ of f and the approximate derivative $(f_{(n-1)})'_{\mathrm{ap}}$ of f_{n-1} exist finitely and are equal almost everywhere in E.*

In 2000, M. Laczkovich (see [4]) pointed out that the proof is not valid, but he did not determine if the assertion itself was true or false. The purpose of this paper is to show that the assertion is true.

2. Notation and other definitions. As will be seen, convexity implies some smoothness described in terms of differentiation. We recall the definition.

Definition 2.1. Let $f : E \to \mathbb{R}$, $u \in E$ and $k \in \mathbb{N}$. Then f is *k times Peano differentiable* at u means there exist numbers $f_{(0)}(u) := f(u), f_{(1)}(u), \ldots, f_{(k)}(u)$ such that

$$f(w) = \sum_{i=0}^{k} f_{(i)}(u)\frac{(w-u)^i}{i!} + o(|w-u|^k) \quad \text{as } w \xrightarrow{E} u,$$

where $w \xrightarrow{E} u$ means that $w \to u$ with $w \in E$. In this case we say that $f_{(k)}(u)$ is the kth *Peano derivative* of f at u. We say that f is k times Peano differentiable on E if f has a kth Peano derivative at each point of E. The right and left kth Peano derivatives of f are defined in an analogous fashion for the right and left limit points of E and are denoted by $f_{(k,+)}(u)$ and $f_{(k,-)}(u)$ respectively. If in the limit above we further restrict w to a set of density 1 at u, then the corresponding derivatives are called *approximate Peano derivatives* and we use $f_{(n),\mathrm{ap}}$ in place of $f_{(n)}$. Ordinary derivatives are defined in the usual way and we use $f^{(i)}$, $f^{(i,+)}$, $f^{(i,-)}$ to denote the ith, right ith, and left ith derivatives of f respectively.

Many of the results to follow concern limit points of E. For that reason we introduce the following notation.

Notation. Let $E \subset \mathbb{R}$. Then

$$E^+ = \{u \in E;\ u \text{ is a right limit point of } E \text{ and } u \neq \min E\}.$$

E^- is defined similarly.

The maximum and minimum points of E are omitted to avoid infinite limits.

The connection between convexity and differentiation will be established through the use of the next concept.

DEFINITION 2.2. Let $f : E \to \mathbb{R}$, let $V \subset E$ be finite and let $u \in E^+ \setminus V$. Then for $k \in \mathbb{N}$, $[f; u, V]^{(k,+)}$ is defined inductively by

$$[f; u, V]^{(1,+)} = \lim_{y \xrightarrow{E} u^+} [f; u, y, V]$$

if the limit exists, and for $k \geq 2$,

$$[f; u, V]^{(k,+)} = \lim_{y \xrightarrow{E} u^+} [f; u, y, V]^{(k-1,+)}$$

if the limit exists.

$[f; u, V]^{(k,-)}$ for $u \in E^-$ is defined analogously.

3. Preliminaries. In this section some preliminary results concerning $[f; u, V]^{(k,+)}$ are presented.

PROPOSITION 3.1. *Let $f : E \to \mathbb{R}$, let $\emptyset \neq V \subset E$ be finite and let $u \in E^+ \setminus V$. For $k \in \mathbb{N}$ and $w \in V$, $[f; u, V]^{(k,+)}$ exists if and only if $[f; u, V \setminus \{w\}]^{(k,+)}$ exists, and in that case*

$$[f; u, V]^{(k,+)} = \begin{cases} \dfrac{[f; u, V] - [f; u, V \setminus \{w\}]^{(1,+)}}{w - u} & \text{for } k = 1, \\[2ex] \dfrac{[f; u, V]^{(k-1,+)} - [f; u, V \setminus \{w\}]^{(k,+)}}{w - u} & \text{for } k \geq 2. \end{cases}$$

The analogous assertion with $+$ replaced by $-$ is also true.

Proof. The proof is by induction on k. For $k = 1$ the calculation

$$[f; u, V]^{(1,+)} = \lim_{y \xrightarrow{E} u^+} [f; u, y, V] = \lim_{y \xrightarrow{E} u^+} \frac{[f; u, V] - [f; u, y, V \setminus \{w\}]}{w - y}$$
$$= \frac{[f; u, V] - \lim_{y \xrightarrow{E} u^+} [f; u, y, V \setminus \{w\}]}{w - u}$$

shows that $[f; u, V]^{(1,+)}$ exists if and only if $[f; u, V \setminus \{w\}]^{(1,+)}$ exists, and

$$[f; u, V]^{(1,+)} = \frac{[f; u, V] - [f; u, V \setminus \{w\}]^{(1,+)}}{w - u}.$$

Assume the assertion is true for k. Then by definition

$$[f; u, V]^{(k+1,+)} = \lim_{y \xrightarrow{E} u^+} [f; u, y, V]^{(k,+)}$$
$$= \lim_{y \xrightarrow{E} u^+} \frac{[f; u, y, V]^{(k-1,+)} - [f; u, y, V \setminus \{w\}]^{(k,+)}}{w - u}$$
$$= \frac{[f; u, V]^{(k,+)} - \lim_{y \xrightarrow{E} u^+} [f; u, y, V \setminus \{w\}]^{(k,+)}}{w - u}$$

(where for $k = 1$, $[f; u, y, V]^{(k-1,+)}$ means $[f; u, y, V]$). Hence $[f; u, V]^{(k+1,+)}$ exists if and only if $[f; u, V \setminus \{w\}]^{(k+1,+)}$ exists, and

$$[f; u, V]^{(k+1,+)} = \frac{[f; u, V]^{(k,+)} - [f; u, V \setminus \{w\}]^{(k+1,+)}}{w - u}. \blacksquare$$

It follows from this assertion that the existence of $[f; u, V]^{(k,+)}$ does not depend on the choice of V. In particular $[f; u, V]^{(1,+)}$ exists if and only if $[f; u]^{(1,+)}$ exists. It is easy to see that if one of $[f; u]^{(1,+)}$ or $f^{(1,+)}(u)$ exists then so does the other and they are equal. Consequently, $[f; u, V]^{(1,+)}$ exists if and only if $f^{(1,+)}(u)$ exists. The corresponding assertion with $+$ replaced by $-$ is also true. An analogous assertion for $[f; u, V]^{(k,+)}$ is established in Corollary 3.3.

The previous result and the following corollary are used in the proof of Theorem 6.3 in Section 6.

COROLLARY 3.2. *Let $f : E \to \mathbb{R}$, let $V \subset E$ be finite, and let $v, w \in V$ with $v \neq w$, $u \in E^+ \setminus V$ and $k \in \mathbb{N}$. If $[f; u, V]^{(k,+)}$ exists, then*

$$[f; u, V]^{(k,+)} = \frac{[f; u, V \setminus \{v\}]^{(k,+)} - [f; u, V \setminus \{w\}]^{(k,+)}}{w - v}. \tag{3}$$

The analogous result with $+$ replaced by $-$ holds as well.

Proof. From Proposition 3.1,

$$(w - u)[f; u, V]^{(k,+)} = [f; u, V]^{(k-1,+)} - [f; u, V \setminus \{w\}]^{(k,+)} \tag{4}$$

(where as above $[f;u,V]^{(0,+)}$ denotes $[f;u,V]$), and on replacing w with v,

$$(v-u)[f;u,V]^{(k,+)} = [f;u,V]^{(k-1,+)} - [f;u,V\setminus\{v\}]^{(k,+)}. \tag{5}$$

Subtracting (5) from (4) yields the desired result. ∎

Note the similarity between (3) and the defining equation for divided differences. That observation justifies referring to $[f;u,V]^{(k,+)}$ as *generalized divided differences.*

The following equation involving divided differences is known as Newton's formula (see [6] for a proof):

$$\begin{aligned} f(w) = f(y_0) &+ \sum_{i=1}^{k-1}[f;y_0,y_1,\ldots,y_i]\prod_{\ell=0}^{i-1}(w-y_\ell) \\ &+ [f;y_0,y_1,\ldots,y_{k-1},w]\prod_{\ell=0}^{k-1}(w-y_\ell). \end{aligned}$$

This equation will be used extensively in Section 6.

Taking, in order, first $\lim_{y_1 \xrightarrow{E} y_0^+}$, followed by $\lim_{y_2 \xrightarrow{E} y_0^+}, \ldots, \lim_{y_{k-1} \xrightarrow{E} y_0^+}$ yields

$$f(w) = f(y_0) + \sum_{i=1}^{k-1}[f;y_0]^{(i,+)}(w-y_0)^i + [f;y_0,w]^{(k-1,+)}(w-y_0)^k,$$

assuming these limits exist. Finally replace y_0 by u to get

$$f(w) = f(u) + \sum_{i=1}^{k-1}[f;u]^{(i,+)}(w-u)^i + [f;u,w]^{(k-1,+)}(w-u)^k. \tag{6}$$

Similarly

$$f(w) = f(u) + \sum_{i=1}^{k-1}[f;u]^{(i,-)}(w-u)^i + [f;u,w]^{(k-1,-)}(w-u)^k.$$

This result is first used to establish the connection between these generalized divided differences and Peano derivatives.

COROLLARY 3.3. *Let $f: E\to\mathbb{R}$, $u\in E^+$ and $k\in\mathbb{N}$. Then $[f;u]^{(k,+)}$ exists if and only if f is k times Peano differentiable from the right at u and $f_{(k,+)}(u) = k![f;u]^{(k,+)}$. The assertion with $+$ replaced by $-$ is also valid.*

Proof. Suppose $[f;u]^{(k,+)}$ exists. Then by definition $[f;u]^{(i,+)}$ exists for all $i=1,\ldots,k$, and by equation (6) for $w\in E$,

$$f(w) = f(u) + \sum_{i=1}^{k-1} [f;u]^{(i,+)}(w-u)^i + [f;u,w]^{(k-1,+)}(w-u)^k$$
$$= f(u) + \sum_{i=1}^{k} [f;u]^{(i,+)}(w-u)^i + ([f;u,w]^{(k-1,+)} - [f;u]^{(k,+)})(w-u)^k$$

Since $\lim_{w \xrightarrow{E} u^+} [f;u,w]^{(k-1,+)} = [f;u]^{(k,+)}$, by definition f is k times Peano differentiable from the right at u and $f_{(k,+)}(u) = k![f;u]^{(k,+)}$.

The converse is established by induction on k. Its validity for $k = 1$ has already been established. Suppose it is true for $k-1$ and assume that f is k times Peano differentiable from the right at u. Then from (6) we get

$$f(w) = f(u) + \sum_{i=1}^{k-1} [f;u]^{(i,+)}(w-u)^i + \frac{f_{(k,+)}(u)}{k!}(w-u)^k$$
$$+ \left([f;u,w]^{(k-1,+)} - \frac{f_{(k,+)}(u)}{k!}\right)(w-u)^k.$$

By the definition of Peano differentiability, it follows that

$$[f;u]^{(k,+)} = \lim_{w \xrightarrow{E} u^+} [f;u,w]^{(k-1,+)} \quad \text{exists and equals} \quad \frac{f_{(k,+)}(u)}{k!}. \blacksquare$$

4. Basic properties of n-convex functions. This section begins with a theorem designed to help the reader better understand the geometric meaning of n-convexity. The assertion uses the notion of the Lagrange interpolation polynomials which is recalled next.

Notation. Let V be a set of n distinct points. Then the polynomial L_V that agrees with f on V is given by

$$L_V(x) = \sum_{u \in V} f(u) \frac{\prod_{v \in V, v \neq u} (x-v)}{\prod_{v \in V, v \neq u} (u-v)}.$$

The degree of the polynomial L_V is no more than $n-1$.

A geometric description of n-convexity is given in [2]. The result is that f is n-convex if and only if for each $V = \{v_1 < \cdots < v_n\}$ the graph of f is above the graph of L_V for $x > v_n$, below for $v_{n-1} < x < v_n$, above for $v_{n-2} < x < v_{n-1}$ and so on. Here we present a simpler version of that theorem whose proof can easily be used to prove the more involved version.

THEOREM 4.1. *Let $f : E \to \mathbb{R}$ and $n \in \mathbb{N}$. Then f is n-convex if and only if for each $V \subset E$ with cardinality n, $f(x) \geq L_V(x)$ for every $x \geq \max V$.*

Proof. Suppose f is n-convex. Let $V \subset E$ be of cardinality n and let $x > \max V$. By (2) for the set $V \cup \{x\}$,

$$0 \le \frac{f(x)}{\prod_{v\in V}(x-v)} + \sum_{u\in V} \frac{f(u)}{(u-x)\prod_{\substack{v\in V\\ v\neq u}}(u-v)},$$

or

$$-\frac{f(x)}{\prod_{v\in V}(x-v)} \le \sum_{u\in V} \frac{f(u)}{(u-x)\prod_{\substack{v\in V\\ v\neq u}}(u-v)}.$$

Multiplying by the negative of the denominator on the left hand side (which is positive since $x > v$ for all $v \in V$) results in

$$f(x) \ge \sum_{u\in V} f(u)\,\frac{\prod_{v\in V}(x-v)}{(x-u)\prod_{\substack{v\in V\\ v\neq u}}(u-v)} = \sum_{u\in V} f(u)\,\frac{\prod_{\substack{v\in V\\ v\neq u}}(x-v)}{\prod_{\substack{v\in V\\ v\neq u}}(u-v)} = L_V(x).$$

Reversing the above argument proves the converse. ■

In the remainder of this section we show how the generalized divided differences introduced in the previous section relate to n-convex functions. First note that a divided difference can be interpreted as a function defined on finite subsets of E.

Notation. For $n \in \mathbb{N}$ let $(n,E) = \{V \subset E;\ V \text{ has cardinality } n\}$. In particular, $(1,E) = E$.

DEFINITION 4.2. Let V and W be finite subsets of E with the same cardinality. Then $V \le W$ means that there is a permutation $h : V \to W$ such that $v \le h(v)$ for each $v \in V$.

It is not hard to see that $f : E \to \mathbb{R}$ is n-convex if and only if $[f;V] \le [f;W]$ for all pairs $V \le W \in (n,E)$ that differ by one element, say $v < w$. From this observation the following assertion is easily proved. (See Lemma 8 in [4].)

PROPOSITION 4.3. *Let $f : E \to \mathbb{R}$ and $n \in \mathbb{N}$. Then f is n-convex if and only if $[f;V] \le [f;W]$ for each pair $V \le W$ in (n,E).*

DEFINITION 4.4. Let $F : X \subset (n,E) \to \mathbb{R}$. Then F is *nondecreasing* means that if V, $W \in X$ with $V \le W$, then $F(V) \le F(W)$.

PROPOSITION 4.5. *Let $n \in \mathbb{N}$ with $n \ge 2$ and let $f : E \to \mathbb{R}$ be n-convex. Then for each $k \in \{1,\dots,n-1\}$, $V \in (n-k-1,E^+)$ and $u \in E^+ \setminus V$, $[f;u,V]^{(k,+)}$ exists and is nondecreasing on $(n-k,E^+)$.*

The analogous assertion with $+$ replaced by $-$ is also valid.

Proof. The proof is by induction on k. For $k = 1$ let $V \subset E^+$ have cardinality $n-2$ and let $u \in E^+ \setminus V$. Because f is n-convex, $[f;u,y,V]$ is nondecreasing as a function of $y \in E$. Thus $[f;u,V]^{(1,+)}$ exists. Now let $V, W \in (n-2, E^+)$ with $V \le W$ and let $u, v \in E^+$ with $u \le v$. If $u < v$, then for $u < y_1 < v < y_2$ with $y_1, y_2 \in E$, because f is n-convex, $[f; u, y_1, V] \le [f; v, y_2, W]$ and hence $[f; u, V]^{(1,+)} \le [f; v, W]^{(1,+)}$. If $u = v$, then clearly $[f; u, V]^{(1,+)} \le [f; v, W]^{(1,+)}$, which verifies the assertion for $k = 1$.

Assume $k \le n-2$ and the assertion is true for $k-1$. Let $V \in (n-k-1, E^+)$ and $u \in E^+$. Then by the induction assumption, $[f, u, y, V]^{(k-1,+)}$ exists and is nondecreasing as a function of y. Thus $[f; u, V]^{(k,+)}$ exists. Now let $V, W \in (n-k-1, E^+)$ with $V \le W$ and let $u, v \in E^+$ with $u \le v$. If $u < v$, then for $u < y_1 < v < y_2$ with $y_1, y_2 \in E$, by the induction assumption, $[f; u, y_1, V]^{(k-1,+)} \le [f; v, y_2, W]^{(k-1,+)}$ and hence $[f; u, V]^{(k,+)} \le [f; v, W]^{(k,+)}$. If $u = v$, then clearly $[f; u, V]^{(k,+)} \le [f; v, W]^{(k,+)}$, which completes the proof. ∎

Corollary 4.6. *Let $n \in \mathbb{N}$ with $n \ge 2$ and let $f : E \to \mathbb{R}$ be n-convex. Then for each $k \in \{1, \ldots, n-1\}$ and for each $u \in E^+$, $[f; u]^{(k,+)}$ exists. Similarly for $u \in E^-$, $[f; u]^{(k,-)}$ exists.*

This assertion follows easily from Propositions 4.5 and 3.1.

5. Extendibility of n-convex functions. Let $f : E \to \mathbb{R}$. If f is 0-convex (resp. 1-convex), then clearly f has a 0-convex (resp. 1-convex) extension to the closure of E with $\inf E$ and $\sup E$ omitted. This set will be denoted by $\tilde{E}$. These extensions need not be unique. However, it is well known that if f is 2-convex, it has a unique 2-convex extension to $\tilde{E}$. The next result proves the same assertion for n-convex functions with $n \ge 2$. We will first state the following lemma due to Miklos Laczkovich (see [5, Lemma 1]).

Lemma 5.1. *Let $f : E \to \mathbb{R}$ be n-convex with $n \ge 2$. Then f is Lipschitz on $I \cap E$ for every compact subinterval of* $(\inf E, \sup E)$.

Theorem 5.2. *Let $f : E \to \mathbb{R}$ be n-convex with $n \ge 2$. Then f has a unique n-convex extension to $\tilde{E}$.*

Proof. By Lemma 5.1 if $c \in \tilde{E}$ is a limit point not in E we can define $f(c)$ as $\lim_{x \xrightarrow{E} c} f(x)$. It is easy to check that this extended f is locally Lipschitz on $\tilde{E}$. If we know that $[f; V] \ge 0$ for $V \subset E$, the continuity of the extension of f to $\tilde{E}$ and (2) easily show that $[f; V] \ge 0$ for $V \subset \tilde{E}$. Thus f is n-convex on $\tilde{E}$. The uniqueness also follows from Lemma 5.1, since any n-convex extension of f to $\tilde{E}$ would have to be locally Lipschitz on $\tilde{E}$. ∎

For $n = 0, 1, 2$ an n-convex function defined on E can be further extended to an interval with endpoints $\inf E$ and $\sup E$ on which the extension is n-convex. The example to follow shows that 3-convex functions need not have such an extension, which is important because many of the results in the next section are known for n-convex functions defined on an interval where they are proved using facts not valid on measurable sets. For example, in [2] Bullen showed that the kth ordinary and kth Peano derivatives of an n-convex function, $1 \le k < n-1$, are equivalent using results of Oliver [7]. He also showed that the kth derivative of an n-convex function is $(n-k)$-convex using an integration technique. For finite sets nonextendable 3-convex functions are easy to find, as was pointed out by Miklos Laczkovich, who offered the following example. Let the set E consist of the numbers -3, -2, -1, 1, 2, 3 and let

$$f(x) = \begin{cases} 0 & \text{if } x = -3, -2, -1, \\ 1 & \text{if } x = 1, \\ 6 & \text{if } x = 2, \\ c \in (14, 15) & \text{if } x = 3. \end{cases}$$

Then f is strictly 3-convex on E, but f has no extension to $E \cup \{0\}$ as a 3-convex function. However, for infinite sets, such examples are somewhat more difficult.

Example 5.3. *Let*

$$f(x) = \begin{cases} 0 & \textit{if } x \le 0, \\ x(x-1) & \textit{if } x \ge 1. \end{cases}$$

Then f is 3-convex on its domain, but cannot be extended to a 3-convex function on $\mathbb{R}$.

Proof. We use the criterion in Theorem 4.1. We must test every set of three elements $a < b < c$ with a, b, c in the domain of f. If $a < b < c \le 0$ the corresponding Lagrange polynomial is identically 0 and hence $f(x)$ exceeds it for every $x \ge c$. If $a < b \le 0$ and $1 \le c$, then the Lagrange parabola has its vertex at $(a+b)/2$ and hence is increasing on $[b, \infty)$. Thus it has a second intersection with $x^2 - x$ between b and 0. It cannot have a third intersection with $x^2 - x$, for it would then be $x^2 - x$. Therefore it lies below $x^2 - x$ for $x > c$. Next assume that $a < 0 < 1 < b < c$. If the graph of the Lagrange parabola is above $x^2 - x$ on $[1, b]$, then it must intersect $x^2 - x$ for a third time and be identical with $x^2 - x$. So it lies below $x^2 - x$ on $[1, b]$ and hence above on $[b, c]$ and below on $[c, \infty)$. Finally if $1 < a < b < c$, then the Lagrange parabola agrees with $x^2 - x$. Therefore f is 3-convex on its domain.

Suppose f has a 3-convex extension to $\mathbb{R}$, denoted also by f. First we assert that f must lie below $x^2 - x$ on $[0, 1]$, for if there is an $x \in [0, 1]$ with $f(x) > x^2 - x$, then the Lagrange parabola through $(x, f(x)), (b, f(b))$ and $(c, f(c))$ with $1 < b < c$ would lie above f on $[c, \infty)$ contrary to the assumption that f is 3-convex. Consequently, $f^{(1,+)}(0) \le -1$. However, as will be seen from Theorem 6.3 below, a 3-convex function is differentiable everywhere. Hence $f'(0) = 0$. Thus f has no 3-convex extension to $\mathbb{R}$. ∎

6. Convexity and differentiation

PROPOSITION 6.1. *Let $f : E \to \mathbb{R}$ and $V \subset E^+$. Suppose $f^{(1,+)}(u)$ exists for each $u \in V$. Then $[f; u, V \setminus \{u\}]^{(1,+)}$ exists for each $u \in V$ and*

$$[f^{(1,+)}; V] = \sum_{u\in V} [f; u, V \setminus \{u\}]^{(1,+)}.$$

The analogous result holds when $+$ is replaced by $-$.

Proof. The proof is by induction on the cardinality of V. If V consists of one element, say u, then the result becomes $[f^{(1,+)}; u] = f^{(1,+)}(u)$, which is true by definition.

Assume the assertion holds for any set with cardinality k and let V have cardinality $k+1$. First $[f; u, V \setminus \{u\}]^{(1,+)}$ exists for each $u \in V$ by Proposition 3.1. Choose $v, w \in V$ with $v \neq w$. Then

$$\begin{aligned}
[f^{(1,+)}; V] &= \frac{[f^{(1,+)}; V \setminus \{v\}] - [f^{(1,+)}; V \setminus \{w\}]}{w - v} \\
&= \frac{\sum_{u\in V\setminus\{v\}} [f; u, V \setminus \{u, v\}]^{(1,+)} - \sum_{u\in V\setminus\{w\}} [f; u, V \setminus \{u, w\}]^{(1,+)}}{w - v} \\
&\qquad\qquad \text{(by the induction hypothesis)} \\
&= \frac{[f; w, V \setminus \{v, w\}]^{(1,+)} - [f; v, V \setminus \{v, w\}]^{(1,+)}}{w - v} \\
&\quad + \sum_{u\in V\setminus\{v,w\}} \frac{[f; u, V \setminus \{u, v\}]^{(1,+)} - [f; u, V \setminus \{u, w\}]^{(1,+)}}{w - v} \\
&= A + \sum_{u\in V\setminus\{v,w\}} B_u.
\end{aligned}$$

A direct application of Corollary 3.2 yields $B_u = [f; u, V \setminus \{u\}]^{(1,+)}$.

Note that $[f; V] = [f; w, V \setminus \{w\}] = [f; v, V \setminus \{v\}]$. Hence

$$\begin{aligned}
A &= \frac{[f; w, V \setminus \{v, w\}]^{(1,+)} - [f; w, V \setminus \{w\}]}{w - v} \\
&\quad + \frac{[f; v, V \setminus \{v\}] - [f; v, V \setminus \{v, w\}]^{(1,+)}}{w - v} \\
&= \frac{-[f; w, V \setminus \{w\}]^{(1,+)}(v - w) + [f, v, V \setminus \{v\}]^{(1,+)}(w - v)}{w - v} \\
&\qquad\qquad \text{(by Proposition 3.1)} \\
&= [f, v, V \setminus \{v\}]^{(1,+)} + [f; w, V \setminus \{w\}]^{(1,+)}.
\end{aligned}$$

Thus

$$[f^{(1,+)}; V] = \sum_{u \in V} [f; u, V \setminus \{u\}]^{(1,+)}. \ \blacksquare$$

COROLLARY 6.2. *Let $n \in \mathbb{N}$ and let $f : E \to \mathbb{R}$ be n-convex. Suppose $f^{(1,+)}$ exists on E^+. Then $f^{(1,+)}$ is $(n-1)$-convex on E^+. The analogous assertion for $f^{(1,-)}$ is also valid.*

Proof. Let $V \subset E^+$ have cardinality $n-1$. Then by Proposition 6.1,

$$[f^{(1,+)}; V] = \sum_{u \in V} [f; u, V \setminus \{u\}]^{(1,+)}.$$

Because f is n-convex, $[f; u, V \setminus \{u\}]^{(1,+)}$ is nondecreasing for each $u \in V$. Thus $[f^{(1,+)}; V]$ is nondecreasing, proving that $f^{(1,+)}$ is $(n-1)$-convex. $\blacksquare$

The next theorem, which is central to the results of this paper, establishes the Peano differentiability of n-convex functions.

THEOREM 6.3. *Let $n \in \mathbb{N}$ with $n \geq 3$, let $f : E \to \mathbb{R}$ be n-convex and let $u \in E^+$. Then f is $n-2$ times Peano differentiable at u and $f_{(i)}(u) = i![f; u]^{(i,+)}$ for each $i = 1, \ldots, n-2$. In addition, f is $n-1$ times Peano right differentiable at $u \in E^+$ and $f_{(n-1,+)}(u) = (n-1)![f; u]^{(n-1,+)}$ The corresponding assertion with $+$ replaced by $-$ is also true. Furthermore for $u \in E^+ \cap E^-$, $f_{(i,+)}(u) = f_{(i,-)}(u)$ for $i = i, \ldots, n-2$ and $f_{(n-1,-)}(u) \leq f_{(n-1,+)}(u)$.*

Proof. Letting $k = n-1$ in (6) yields

$$f(w) = f(u) + \sum_{i=1}^{n-2} [f; u]^{(i,+)} (w-u)^i + [f; u, w]^{(n-2,+)} (w-u)^{n-1}.$$

By Proposition 4.5 all of the limiting objects in the above formula exist because f is n-convex and in addition, the function $[f; u, w]^{(n-2,+)}$ is nondecreasing in w and hence bounded near u. (Here is where $u \neq \min E$ or $\max E$ is used.) Thus $\lim_{w \xrightarrow{E} u} [f; u, w]^{(n-2,+)} (w-u) = 0$. Consequently, f is $n-2$ times Peano differentiable at u with $f_{(i)}(u) = i![f; u]^{(i,+)}$ for $i = 1, \ldots, n-2$. Similarly, $f_{(i)}(u) = i![f; u]^{(i,-)}$ for $u \in E^-$ and $i = 1, \ldots, n-2$. By the uniqueness of Peano derivatives, $f_{(i,+)}(u) = f_{(i,-)}(u)$ for $u \in E^+ \cap E^-$.

Now let $k = n$ in (6). In this case the existence of the last term, $[f; u, w]^{(n-1,+)}$, does not follow immediately. However, by Proposition 3.1, $[f; u, w]^{(n-1,+)}$ exists if and only if $[f; u]^{(n-1,+)}$ exists, and this latter limit does exist because f is n-convex. Thus

$$f(w) = f(u) + \sum_{i=1}^{n-1} [f; u]^{(i,+)} (w-u)^i + [f; u, w]^{(n-1,+)} (w-u)^n$$

$$\begin{aligned}
&= f(u) + \sum_{i=1}^{n-2} \frac{f_{(i)}(u)}{i!}(w-u)^i + [f;u]^{(n-1,+)}(w-u)^{n-1} \\
&\quad + [f;u,w]^{(n-1,+)}(w-u)^n \\
&= f(u) + \sum_{i=1}^{n-2} \frac{f_{(i)}(u)}{i!}(w-u)^i + [f;u]^{(n-1,+)}(w-u)^{n-1} \\
&\quad + ([f;u,w]^{(n-2,+)} - [f;u]^{(n-1,+)})(w-u)^{n-1}.
\end{aligned}$$

by Proposition 3.1. Because $\lim_{w \xrightarrow{E} u^+} [f;u,w]^{(n-2,+)} = [f;u]^{(n-1,+)}$, it follows that the last term in the above equation is $o((w-u)^{n-1})$; that is, f is $n-1$ times Peano differentiable from the right at u and $f_{(n-1,+)}(u) = (n-1)![f;u]^{(n-1,+)}$. Similarly $f_{(n-1,-)}(u) = (n-1)![f;u]^{(n-1,-)}$ for $u \in E^-$.

Finally for $u \in E^+ \cap E^-$ since f is n-convex, for $x_i \in E$ with $x_i < u$ and $y_i \in E$ with $y_i > u$ for $i = 1, \dots, n-1$, we have $[f;u,x_1,\dots,x_{n-1}] \le [f;u,y_1,\dots,y_{n-1}]$. Consequently, $[f;u]^{(n-1,-)} \le [f;u]^{(n-1,+)}$ and hence $f_{(n-1,-)}(u) \le f_{(n-1,+)}(u)$. ∎

Corollary 6.4. *Let $n \in \mathbb{N}$ with $n \ge 3$ and let $f : E \to \mathbb{R}$ be n-convex. Then $f^{(1)} = f_{(1)}$ is $(n-1)$-convex on $E^+ \cap E^-$. Consequently, $f^{(i)}$ is $(n-i)$-convex for $i = 1, \dots, n-2$.*

This assertion follows immediately from Corollary 6.2 and the preceding theorem.

Next we investigate the relationship between the Peano and ordinary derivatives of an n-convex function.

Theorem 6.5. *Let $n \in \mathbb{N}$ with $n \ge 3$ and let $f : E \to \mathbb{R}$ be n-convex. Then $f^{(1)}$ is $n-3$ times Peano differentiable on $E \setminus \{\inf E, \sup E\}$ and $(f^{(1)})_{(i)} = f_{(i+1)}$ for $i = 1, \dots, n-3$. Moreover $(f^{(1)})_{(n-2,+)}(u) \ge f_{(n-1,+)}(u)$ for $u \in E^+$ and $(f^{(1)})_{(n-2,-)}(u) \le f_{(n-1,-)}(u)$ for $u \in E^-$.*

Proof. Start from equation (6) with $k = n-2$ to obtain

$$f(w) = f(u) + \sum_{i=1}^{n-3} \frac{f_{(i)}(u)}{i!}(w-u)^i + [f;u,w]^{(n-3,+)}(w-u)^{n-2}.$$

Now assume $w \in E^+ \cup E^-$ and differentiate with respect to w to get

$$\begin{aligned}
f_{(1)}(w) &= \sum_{i=1}^{n-3} \frac{f_{(i)}(u)}{(i-1)!}(w-u)^{i-1} + \frac{d}{dw}([f;u,w]^{(n-3,+)}(w-u)^{n-2}) \\
&= \sum_{i=0}^{n-4} \frac{f_{(i+1)}(u)}{i!}(w-u)^i + [f;u,w]^{(n-3,+)}(n-2)(w-u)^{n-3}
\end{aligned}$$

$$+ (w-u)^{n-2}\,\frac{d}{dw}[f;u,w]^{(n-3,+)}$$
$$= \sum_{i=0}^{n-4} \frac{f_{(i+1)}(u)}{i!}(w-u)^i + A + B.$$

First we handle A. By Proposition 3.1,

$$[f;u,w]^{(n-3,+)} = [f;u]^{(n-2,+)} + (w-u)[f;u,w]^{(n-2,+)}.$$

Thus

$$A = \frac{f_{(n-2)}(u)}{(n-3)!}(w-u)^{n-3} + [f;u,w]^{(n-2,+)}(n-2)(w-u)^{n-2}.$$

Because f is n-convex, $[f;u,w]^{(n-2,+)}$ is nondecreasing in w and hence bounded near u. Thus

$$A = \frac{f_{(n-2)}(u)}{(n-3)!}(w-u)^{n-3} + o((w-u)^{n-3}).$$

Next we turn to B. We have

$$\frac{d}{dw}[f;u,w]^{(n-3,+)} = \lim_{v \overset{E}{\to} w} \frac{[f;u,v]^{(n-3,+)} - [f;u,w]^{(n-3,+)}}{v-w}$$
$$= \lim_{v \overset{E}{\to} w} [f;u,v,w]^{(n-3,+)} \qquad \text{(by Corollary 3.2)}.$$

Because f in n-convex, $[f;u,v,w]^{(n-3,+)}$ is nondecreasing in u,v,w. It follows that $\lim_{v \overset{E}{\to} w}[f;u,v,w]^{(n-3,+)}$ exists and is nondecreasing in u and w. Consequently, it is bounded near u; that is, $B = o((w-u)^{n-3})$. Therefore

$$f_{(1)}(w) = \sum_{i=0}^{n-3} \frac{f_{(i+1)}(u)}{i!}(w-u)^i + o((w-u)^{n-3}),$$

completing the first part of the proof.

To prove the inequality $(f^{(1)})_{(n-2,+)}(u) \geq f_{(n-1,+)}(u)$ we must estimate $[f^{(1,+)};u]^{(n-2,+)}$. By definition

$$[f^{(1,+)};u]^{(n-2,+)} = \lim_{y_{n-2} \overset{E}{\to} u^+} \cdots \lim_{y_1 \overset{E}{\to} u^+} [f^{(1,+)};u,y_1,\ldots,y_{n-2}].$$

By Proposition 6.1,

$$[f^{(1,+)};u,y_1,\ldots,y_{n-2}] = [f;u,y_1,\ldots,y_{n-2}]^{(1,+)}$$
$$+ \sum_{i=1}^{n-2}[f;y_i,u,y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{n-2}]^{(1,+)}.$$

Because for every $u < w < y_i$ we have

$$L := [f;y_i,u,y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{n-2}]^{(1,+)}$$
$$\geq [f;y_i,u,w,y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_{n-2}],$$

taking $w \to u$ we obtain

$$L \geq [f; u, y_i, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n-2}]^{(1,+)}.$$

Thus

$$\begin{aligned}[f^{(1,+)}; u, y_1, \ldots, y_{n-2}] &\geq [f; u, y_1, \ldots, y_{n-2}]^{(1,+)} + \sum_{i=1}^{n-2} [f; u, y_1 \ldots, y_{n-2}]^{(1,+)} \\ &= (n-1)[f; u, y_1, \ldots, y_{n-2}]^{(1,+)}.\end{aligned}$$

Consequently, taking limits results in $[f^{(1,+)}; u]^{(n-2,+)} \geq (n-1)[f; u]^{(n-1,+)}$. According to Corollary 3.3,

$$\begin{aligned}(f^{(1,+)})_{(n-2,+)}(u) &= (n-2)![f^{(1,+)}; u]^{(n-2,+)} \\ &\geq (n-2)!(n-1)[f; u]^{(n-1,+)} = f_{(n-1,+)}(u).\end{aligned}$$

A similar argument proves $(f^{(1,-)})_{(n-2,-)}(u) \leq f_{(n-1,-)}(u)$ for $u \in E^-$. ∎

Next we recall an assertion about convex (2-convex) functions.

THEOREM 6.6. *Let $f : E \to \mathbb{R}$ be convex. Then $f^{(1)}$ exists for all but at most countably many $u \in E$.*

Proof. Both $f^{(1,+)}$ and $f^{(1,-)}$ are nondecreasing on E^+ and E^- respectively and hence are continuous except at countably many points of $E^+ \cap E^-$. It is easy to see that $f^{(1,-)}(u) \leq f^{(1,+)}(u)$ for $u \in E^+ \cap E^-$. Moreover $f^{(1,+)}(u_1) \leq f^{(1,-)}(u_2)$ for $u_1 \in E^+$ and $u_2 \in E^-$ with $u_1 < u_2$. Consequently, at a point $u \in E^+ \cap E^-$ of continuity of $f^{(1,-)}$ the function $f^{(1,+)}$ is continuous and $f^{(1,+)}(u) = f^{(1,-)}(u)$. Finally $E \setminus (E^+ \cap E^-)$ is countable. ∎

COROLLARY 6.7. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $f : E \to \mathbb{R}$ be n-convex. Then $f^{(i)}(u) = f_{(i)}(u)$ for $u \in E^+ \cap E^-$ and $i = 1, \ldots, n-2$. Also $f^{(n-1)}(u) = f_{(n-1)}(u)$ except for countably many $u \in E$.*

Proof. The proof is by induction on n. For $n = 2$ the assertion is true by Theorem 6.6 and because first order Peano and ordinary derivatives are equal.

Assume the assertion is true for $n-1$. By Corollary 6.4, $f^{(1)}$ is $(n-1)$-convex. So by the induction hypothesis $f^{(i+1)}(u) = (f^{(1)})^{(i)}(u) = (f^{(1)})_{(i)}(u)$ for $u \in E^+ \cup E^-$ and $i = 1, \ldots, n-3$. By Theorem 6.5, $(f^{(1)})_{(i)}(u) = f_{(i+1)}(u)$. Thus $f^{(i)}(u) = f_{(i)}(u)$ for $i = 2, \ldots, n-2$. The equality holds for $i = 1$ again because first order Peano and ordinary derivatives are equal.

By the inequality

$$(f^{(1)})_{(n-2,-)}(u) \leq f_{(n-1,-)}(u) \leq f_{(n-1,+)}(u) \leq (f^{(1)})_{(n-2,+)}(u),$$

which follows from Theorem 6.5, $f_{(n-1)}(u)$ exists if $(f^{(1)})_{(n-2)}(u)$ exist, and in that case $f_{(n-1)}(u) = (f^{(1)})_{(n-2)}(u)$. By the induction hypothesis, for all but countably many $u \in E$, $(f^{(1)})_{(n-2)}(u) = (f^{(1)})^{(n-2)}(u) = f^{(n-1)}(u)$. Thus $f_{(n-1)}(u) = f^{(n-1)}(u)$ for all but countably many $u \in E$. ∎

Finally we investigate the approximate differentiability of n-convex functions. We begin by reminding the reader of the pertinent definition.

DEFINITION 6.8. Let $f : E \to \mathbb{R}$ be Lebesgue measurable, and let $u \in E$ and $n \in \mathbb{N}$. Then f is *n times approximately Peano bounded* at u means there are numbers $f_i(u)$ for $i = 1, \dots, n-1$ such that

$$f(u+h) = f(u) + \sum_{i=1}^{n-1} \frac{h^i}{i!} f_i(u) + M_u(h)h^n$$

where $M_u(h)$ remains bounded (not necessarily uniformly in u) as $h \to 0$ through a set of *density* 1 *at* $h = 0$. The set A has density 1 at u (equivalently, u is a *point of density* of A) if A is Lebesgue measurable and

$$\lim_{r \to 0} \frac{m(A \cap (u-r, u+r))}{2r} = 1.$$

PROPOSITION 6.9. *Let $n \in \mathbb{N}$ and let $f : E \to \mathbb{R}$ be n-convex. Set*

$$B^+ = \{u \in E^+;\ \limsup_{w \xrightarrow{E} u^+} [f; u, w]^{(n-1,+)} = \infty\},$$
$$B^- = \{u \in E^-;\ \limsup_{w \xrightarrow{E} u^-} [f; u, w]^{(n-1,+)} = \infty\}.$$

Then $m(B^+) = m(B^-) = 0$.

Proof. Let $a < b \in E^+$. It suffices to show that $m(B^+ \cap (a,b)) = 0$. Fix $K \in \mathbb{N}$ and let $O = \{[u,w];\ [f; u, w]^{(n-1,+)} > K\}$. By Proposition 3.1,

$$[f; u, w]^{(n-1,+)} = \frac{[f; u, w]^{(n-2,+)} - [f; u]^{(n-1,+)}}{w - u}.$$

If $[u, w] \in O$, then

$$\frac{[f; u, w]^{(n-2,+)} - [f; u]^{(n-1,+)}}{K} > w - u.$$

Let $u \in B^+$ and let $\delta > 0$. Then there is $w \in (u, u+\delta)$ such that $[u, w] \in O$. Thus O is a Vitali cover of $B^+ \cap (a,b)$ and by the Vitali Covering Theorem, for each $\epsilon > 0$ there are finitely many pairwise disjoint intervals, $\{[u_i, w_i]\}_{i=1}^k$, covering all of $B^+ \cap (a,b)$ except for a set of measure $< \epsilon$. Assuming, as we may, that $u_1 < \dots < u_k$, and that $w_k \le b$ we get

$$m(B^+ \cap (a,b)) - \epsilon \le \sum_{i=1}^k (w_i - u_i) \le \sum_{i=1}^k \frac{[f; u_i, w_i]^{(n-2,+)} - [f; u_i]^{(n-1,+)}}{K}.$$

Since $[f;u,w]^{(n-2,+)}$ is nondecreasing, $[f;u_i,w_i]^{(n-2,+)} \le [f;u_{i+1}]^{(n-1,+)}$ for $i=1,\dots,k-1$ and $[f;u_k,w_k]^{(n-2,+)]} \le [f;b]^{(n-1,+)}$, we obtain

$$m(B^+ \cap (a,b)) - \epsilon \le \frac{[f;b]^{(n-1,+)} - [f;u_1]^{(n-1,+)}}{K} \le \frac{f_{(n-1,+)}(b) - f_{(n-1,+)}(a)}{K(n-1)!}.$$

Since K and ϵ are arbitrary, $m(B^+ \cap (a,b)) = 0$. Similarly one can prove that $m(B^- \cap (a,b)) = 0$. ∎

COROLLARY 6.10. *Let $n \in \mathbb{N}$ and let $f: E \to \mathbb{R}$ be n-convex. Then f is n times approximately Peano bounded a.e. on E.*

Proof. Let B^+ and B^- be as in Proposition 6.9. Then $m(B^+ \cup B^-) = 0$. Let $A' = \{u \in E;\ f_{(n-1)}(u) \text{ exists}\}$. By Corollary 6.7, $m(E \setminus A') = 0$. Let $u \in A' \setminus (B^+ \cup B^-)$. Combining (6) (applied with $k = n-1$) with (3) (applied to the term $[f;u,w]^{(n-2,+)}$) for $y, w \in E$ we get

$$f(w) = f(u) + \sum_{i=1}^{n-2} [f;u]^{(i,+)}(w-u)^i + [f;u,y]^{(n-2,+)}(w-u)^{n-1} \\ + [f;u,y,w]^{(n-2,+)}(w-y)(w-u)^{n-1}.$$

Letting $y \xrightarrow{E} u$ yields

$$f(w) = f(u) + \sum_{i=1}^{n-1} \frac{f_{(i)}(u)}{i!}(w-u)^i + \lim_{y \xrightarrow{E} u} [f;u,y,w]^{(n-2,+)}(w-u)^n.$$

The remaining limit must exist and is $[f;u,w]^{(n-1,+)}$. By Proposition 6.9, it follows that $[f;u,w]^{(n-1,+)}$ is bounded for w near u. Hence f is n times approximately Peano bounded at u. ∎

We now employ the following result about approximately Peano bounded functions, which is a special case of the main theorem from [1].

THEOREM 6.11. *Let $n \in \mathbb{N}$ and let $f: E \to \mathbb{R}$ be n times approximately Peano bounded on E. Then for every $\epsilon > 0$ there is a perfect set $P \subset E$ and a C^n function g such that $m(E \setminus P) < \epsilon$ and $f = g$ on P.*

Finally we are ready to prove the main result of the paper.

COROLLARY 6.12. *Let $n \in \mathbb{N}$ and let $f: E \to \mathbb{R}$ be n-convex. Then f is n times approximately Peano differentiable almost everywhere on E. Moreover $f_{(n-1)}$ is approximately differentiable with $(f_{(n-1)})'_{\mathrm{ap}} = f_{(n),\,\mathrm{ap}}$.*

Proof. Let $E' \subset E$ be such that every point of E' is a density point of E' and such that f is n times approximately Peano bounded on E'. For n-convex functions Corollary 6.10 shows that there is an E' such that $m(E') = m(E)$. By Theorem 6.11 for every $\epsilon > 0$ there is a perfect set $P \subset E'$ and a C^n function g such that $m(E' \setminus P) < \epsilon$ and $f = g$ on P. Since

$(g^{(n-1)})' = g^{(n)}$ everywhere, from the Taylor formula for g it follows that $f_{(n),\,\mathrm{ap}}$ exists and equals $(f_{(n-1)})'_{\mathrm{ap}}$ at every density point of P. Since ϵ is arbitrary, the same is true almost everywhere on E. ∎

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