The Suslinian number and other cardinal invariants of continua

by

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Abstract. By the Suslinian number $Sln(X)$ of a continuum $X$ we understand the smallest cardinal number $\kappa$ such that $X$ contains no disjoint family $C$ of non-degenerate subcontinua of size $|C| > \kappa$. For a compact space $X$, $Sln(X)$ is the smallest Suslinian number of a continuum which contains a homeomorphic copy of $X$. Our principal result asserts that each compact space $X$ has weight $\leq Sln(X)$ and is the limit of an inverse well-ordered spectrum of length $\leq Sln(X)$, consisting of compacta with weight $\leq Sln(X)$ and monotone bonding maps. Moreover, $w(X) \leq Sln(X)$ if no $Sln(X)$-Suslin tree exists. This implies that under the Suslin Hypothesis all Suslinian continua are metrizable, which answers a question of Daniel et al. [Canad. Math. Bull. 48 (2005)]. On the other hand, the negation of the Suslin Hypothesis is equivalent to the existence of a hereditarily separable non-metrizable Suslinian continuum. If $X$ is a continuum with $Sln(X) < 2^{\aleph_0}$, then $X$ is 1-dimensional, has rim-weight $\leq Sln(X)$ and weight $w(X) \geq Sln(X)$. Our main tool is the inequality $w(X) \leq Sln(X) \cdot w(f(X))$ holding for any light map $f : X \rightarrow Y$.

In this paper we introduce a new cardinal invariant related to the Suslinian property of continua. By a continuum we understand any compact connected Hausdorff space. Following Lelek [7], we define a continuum $X$ to be Suslinian if it contains no uncountable family of pairwise disjoint non-degenerate subcontinua. The simplest example of a Suslinian continuum is the usual interval $I = [0, 1]$. On the other hand, the existence of non-metrizable Suslinian continua is a subtle problem. The properties of such continua were considered in [1]. It was shown in [1] that each Suslinian continuum $X$ is perfectly normal, rim-metrizable, and 1-dimensional. Moreover, a locally connected Suslinian continuum has weight $\leq \omega_1$.

The simplest examples of non-metrizable Suslinian continua are Suslin lines. However this class of examples has a consistency flavor since no Suslin

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line exists in some models of ZFC (for example, in models satisfying (MA + ¬CH)). It turns out that any example of a non-metrizable locally connected Suslinian continuum necessarily has consistency nature: the existence of such a continuum is equivalent to the existence of a Suslin line (see [1]). This implies that under the Suslin Hypothesis (asserting that no Suslin line exists) each locally connected Suslinian continuum is metrizable.

It is clear that each Suslinian continuum $X$ has countable Suslin number $c(X)$. At this point we recall the definition of some known topological cardinal invariants. Given a topological space $X$ let

- $c(X) = \sup\{|U| : U$ is a disjoint family of non-empty open subsets of $X\}$ be the Suslin number of $X$;
- $l(X) = \min\{\kappa : \text{each open cover of } X \text{ contains a subcover of size } \leq \kappa\}$ be the Lindelöf number of $X$;
- $d(X) = \min\{|D| : D$ is a dense set in $X\}$ be the density of $X$;
- $hl(X) = \sup\{|l(Y) : Y \subset X\}$ be the hereditary Lindelöf number of $X$;
- $hd(X) = \sup\{|d(Y) : Y \subset X\}$ be the hereditary density of $X$;
- $w(X) = \min\{|B| : B$ is a base of the topology of $X\}$ be the weight of $X$;
- $\text{rim-w}(X) = \min\{\sup_{U \in B} w(\partial U) : B$ is a base of the topology of $X\}$ be the rim-weight of $X$.

In the context of Suslinian continua, by analogy with the Suslin number $c(X)$ it is natural to introduce a new cardinal invariant

$$\text{Sln}(X) = \sup\{|C| : C$ is a disjoint family of non-degenerate subcontinua of $X\}$$

defined for any continuum $X$ and called the Suslinian number of $X$. Thus a continuum $X$ is Suslinian if and only if $\text{Sln}(X) \leq \aleph_0$.

It is clear that $\text{Sln}(X) \leq \text{Sln}(Y)$ for any pair $X \subset Y$ of continua. It will be convenient to extend the definition of $\text{Sln}(X)$ to all Tikhonov spaces letting

$$\text{Sln}(X) = \min\{\text{Sln}(Y) : Y$ is a continuum containing $X\}$$

for a Tikhonov space $X$.

Like many other cardinal invariants the Suslinian number is monotone.

**Proposition 1.** If $X$ is a Tikhonov space and $Y$ is a subspace of $X$, then $\text{Sln}(Y) \leq \text{Sln}(X)$.

The cardinal invariant $\text{Sln}(X)$ is not trivial since it can attain any infinite value.

**Proposition 2.** $\text{Sln}(X) = c(X) = w(X) = \kappa$ for the hedgehog $X = \{(x_\alpha)_{\alpha < \kappa} : |\{\alpha < \kappa : x_\alpha \neq 0\}| \leq 1\} \subset [0, 1]^\kappa$ with $\kappa$ needles.
The Suslinian number

Note that each hedgehog is \textit{rim-finite} in the sense that it has a base of the topology consisting of sets with finite boundaries. Let us remark that a rim-finite continuum \(X\) with uncountable Suslinian number must be non-metrizable (because rim-countable metrizable continua are Suslinian, see \([7]\)).

The Suslinian number cannot increase under monotone maps. We recall that a map \(f : X \to Y\) is \textit{monotone} if \(f^{-1}(y)\) is connected for any \(y \in Y\).

\textbf{Proposition 3.} \textit{If} \(X\) and \(Y\) \textit{are compact spaces and} \(f : X \to Y\) \textit{is a surjective monotone map, then} \(\text{Sln}(Y) \leq \text{Sln}(X)\).

\textit{Proof.} Embed \(X\) in a continuum \(Z\) with \(\text{Sln}(Z) = \text{Sln}(X)\). Consider the following equivalence relation on \(Z\): \(x \sim y\) if either \(x = y\) or \(x, y \in X\) and \(f(x) = f(y)\). Let \(T = Z/\sim\) be the quotient space and \(q : Z \to T\) be the quotient map. Since all the equivalence classes are connected, the quotient map \(q\) is monotone. Since the preimage of a connected set under a monotone map is connected, \(\text{Sln}(T) \leq \text{Sln}(Z)\). It remains to observe that \(Y\) can be identified with a subspace of \(T\), which yields \(\text{Sln}(Y) \leq \text{Sln}(T) \leq \text{Sln}(Z) = \text{Sln}(X)\). For a similar argument, we also refer the readers to Theorem 2.4.13 in \([3]\). \(\blacksquare\)

\textbf{Proposition 4.} \textit{If} \(X\) \textit{is a Tikhonov space and} \(K\) \textit{is a compact subset of} \(X\), \textit{then} \(\text{Sln}(X/K) \leq \text{Sln}(X)\).

\textit{Proof.} Let \(Z\) be a continuum that contains \(X\) and has \(\text{Sln}(Z) = \text{Sln}(X)\). Since \(K\) is a compact subspace of \(X\), the quotient space \(X/K\) naturally embeds into the quotient space \(Z/K\). We claim that \(\text{Sln}(Z/K) \leq \text{Sln}(Z)\). In the opposite case we would find a disjoint family \(C\) of subcontinua in \(Z/K\) having the cardinality \(|C| > \text{Sln}(Z)\). At most one subcontinuum \(C \in C\) can contain the point \(K \in Z/K = \{K\} \cup (Z \setminus K)\). Deleting this subcontinuum from the family \(C\), if necessary, we can assume that \(\bigcup C \subset Z \setminus K\). Then \(C\) can be thought of as a disjoint family of subcontinua of \(Z\) having size \(|C| > \text{Sln}(Z)\), which contradicts the definition of \(\text{Sln}(Z)\). This contradiction witnesses that \(\text{Sln}(Z/K) \leq \text{Sln}(Z)\) and then

\[\text{Sln}(X/K) \leq \text{Sln}(Z/K) \leq \text{Sln}(Z) = \text{Sln}(X)\]. \(\blacksquare\)

Recall that a map \(f : X \to Y\) between compact Hausdorff spaces is called \textit{light} if \(f^{-1}(y)\) is zero-dimensional for each \(y \in Y\).

\textbf{Theorem 1.} \textit{If} \(X\) and \(Y\) \textit{are compact spaces and} \(f : X \to Y\) \textit{is a light map, then} \(w(X) \leq w(Y) \cdot \text{Sln}(X)\).

For the proof of this theorem we shall need two lemmas.

\textbf{Lemma 1.} \textit{For any point} \(z\) \textit{of a continuum} \(Z\) \textit{there is a family} \(\mathcal{U}\) \textit{of closed neighborhoods of} \(z\) \textit{in} \(Z\) \textit{such that} \(|\mathcal{U}| \leq \text{Sln}(Z)\) \textit{and} \(\bigcap \mathcal{U}\) \textit{is zero-dimensional.}
Proof. We shall construct a transfinite sequence \((U_\alpha)_{\alpha < \alpha_0}\) of closed neighborhoods of \(z\) and a transfinite sequence \((K_\alpha)_{\alpha < \alpha_0}\) of pairwise disjoint, non-degenerate subcontinua of \(Z\) such that \(K_\alpha \subset \bigcap_{\beta < \alpha} U_\beta\) and \(U_\alpha \cap K_\alpha = \emptyset\) for each \(\alpha < \alpha_0\).

To start the construction we choose any subcontinuum \(K_0 \subset Z \setminus \{z\}\) and take any closed neighborhood \(U_0 \subset Z\) of \(z\) missing the set \(K_0\). Then \(U_0\) is not zero-dimensional, and since \(Z\) is a continuum, we can find a subcontinuum \(K_1 \subset U_0\) not containing the point \(z\).

Suppose that for some ordinal \(\alpha\) the closed neighborhoods \(U_\beta\), \(\beta < \alpha\), of \(z\) are already selected so that \(\bigcap_{\beta < \alpha} U_\beta\) is not zero-dimensional. Choose any non-degenerate continuum \(K_\alpha \subset \bigcap_{\beta < \alpha} U_\beta \setminus \{z\}\). Then choose a closed neighborhood \(U_\alpha\) of \(z\) which is disjoint from \(K_\alpha\). Observe that when \(\beta < \alpha\), then \(K_\beta \cap U_\beta = \emptyset\) and \(K_\alpha \subset U_\beta\), whence \(K_\beta \cap K_\alpha = \emptyset\).

The construction should stop at some ordinal \(\alpha_0\) of size \(|\alpha_0| \leq \text{Sln}(Z)\). For this ordinal the intersection \(\bigcap_{\alpha < \alpha_0} U_\alpha\) is zero-dimensional. Then \(U = \{U_\alpha : \alpha < \alpha_0\}\) is the required family of closed neighborhoods of the point \(z\) in \(Z\).

Lemma 2. For any closed subset \(K\) of a continuum \(Z\) there is a family \(\mathcal{U}\) of closed neighborhoods of \(K\) such that \(|\mathcal{U}| \leq \text{Sln}(Z)\) and \(\bigcap \mathcal{U} \setminus K\) is zero-dimensional.

Proof. Consider the quotient space \(Z/K = \{K\} \cup (Z \setminus K)\) of \(Z\) by \(K\) and let \(q : Z \to Z/K\) be the quotient map. By Lemma 1, the continuum \(Z/K\) contains a family \(\mathcal{V}\) of closed neighborhoods of the point \(K \in Z/K\) such that \(|\mathcal{V}| \leq \text{Sln}(Z/K)\) and the intersection \(\bigcap \mathcal{U}\) is zero-dimensional. It is easy to see that the family \(U = \{q^{-1}(V) : V \in \mathcal{V}\}\) of closed neighborhoods of \(K\) has the desired property: it has cardinality \(|U| \leq |\mathcal{V}| \leq \text{Sln}(Z/K) \leq \text{Sln}(Z)\) and \(\bigcap \mathcal{U} \setminus K\) is zero-dimensional (being homeomorphic to \(\bigcap \mathcal{V} \setminus \{K\}\)).

Proof of Theorem. Let \(f : X \to Y\) be a light map between compact Hausdorff spaces. We need to prove that the weight of \(X\) satisfies \(w(X) \leq \kappa\) where \(\kappa = w(Y) \cdot \text{Sln}(X)\). Let \(Z\) be a continuum such that \(X \supset Z\) and \(\text{Sln}(X) = \text{Sln}(Z)\). Of course, \(\text{Sln}(Z) \leq \kappa\).

By Lemma 2, the continuum \(Z\) contains a family \(\mathcal{U}\) of closed neighborhoods of the subset \(X \subset Z\) such that \(|\mathcal{U}| \leq \text{Sln}(Z) \leq \kappa\) and \(\bigcap \mathcal{U} \setminus X\) is zero-dimensional. The family \(\mathcal{U}\) can be used to construct a map \(g : Z \to \mathbb{I}^\kappa\) such that \(X \subset g^{-1}(0) \subset \bigcap \mathcal{U}\), where \(0 = \{0\}^\kappa \in [0, 1]^\kappa = \mathbb{I}^\kappa\). It follows that \(g^{-1}(0) \setminus X \subset \bigcap \mathcal{U} \setminus X\) is zero-dimensional.

Since \(w(Y) \leq \kappa\), the space \(Y\) can be identified with a subset of the Tikhonov cube \(\mathbb{I}^\kappa\). It follows from the Tietze–Urysohn Theorem that the map \(f\) can be extended to a map \(\tilde{f} : Z \to \mathbb{I}^\kappa\). Now consider the map
\[
h = (\tilde{f}, g) : Z \to \mathbb{I}^\kappa \times \mathbb{I}^\kappa, \quad z \mapsto (\tilde{f}(z), g(z)),\]
and observe that
\[ X \subset h^{-1}(\mathbb{I}^\kappa \times 0) = g^{-1}(0) \subset \bigcap \mathcal{U}. \]

It follows that for every \( y \in \mathbb{I}^\kappa \times 0 \) the preimage \( h^{-1}(y) \) lies in the union \( f^{-1}(y) \cup (\bigcap \mathcal{U} \setminus X) \) of two zero-dimensional spaces and hence is zero-dimensional.

Since \( Z \) is a continuum, each component of a non-empty open set \( U \) contains a non-trivial subcontinuum. Consequently, \( U \) has at most \( \text{Sln}(X) \) components. Denote by \( \mathcal{C}_U \) the family of closures of components of \( U \).

Let \( \mathcal{B} \) be a base for the topology of \( h(Z) \) with \( |\mathcal{B}| \leq \kappa \). Finally consider the family \( \mathcal{C} = \bigcup_{B \in \mathcal{B}} C_{h^{-1}(B)} \) of closed subsets of \( Z \), which has size at most \( \kappa \). Because of the compactness of \( X \), the inequality \( w(X) \leq \kappa \) will follow as soon as we prove that the family \( \mathcal{C} \) separates the points of \( X \) in the sense that any two distinct points \( x, y \in X \) lie in disjoint elements \( C_x, C_y \) of the family \( \mathcal{C} \).

If \( h(x) \neq h(y) \), then we can find two basic subsets \( B_x, B_y \in \mathcal{B} \) with disjoint closures such that \( h(x) \in B_x \) and \( h(y) \in B_y \). Let \( D_x \) be the component of \( h^{-1}(B_x) \), containing the point \( x \) and \( D_y \) be the component of \( h^{-1}(B_y) \), containing the point \( y \). Then \( D_x, D_y \) are disjoint elements of \( \mathcal{C} \) separating the points \( x, y \).

Next, suppose that \( h(x) = h(y) = z \) and observe that \( z \in h(X) \subset \mathbb{I}^\kappa \times 0 \). It follows from the zero-dimensionality of \( \bigcap \mathcal{U} \setminus X \) and the inclusion \( h^{-1}(z) \subset f^{-1}(z) \cup (\bigcap \mathcal{U} \setminus X) \) that the set \( h^{-1}(z) \) is zero-dimensional. Consequently, we can find two open subsets \( O_x, O_y \subset Z \) with disjoint closures such that \( x \in O_x, y \in O_y \) and \( h^{-1}(z) \subset O_x \cup O_y \). Since the map \( h \) is closed, there is a basic neighborhood \( B_z \in \mathcal{B} \) of \( z \) such that \( h^{-1}(B_z) \subset O_x \cup O_y \). Then \( x \) and \( y \) lie in the closures of distinct components of \( h^{-1}(B_z) \). This completes the proof that the collection \( \mathcal{C} \) separates the points of \( X \). \( \blacksquare \)

The previous theorem allows us to generalize the classical monotone-light Factorization Theorem [12, 13.3] asserting that any map \( f : X \to Y \) between compact Hausdorff spaces can be represented as the (unique) composition \( \lambda \circ \mu \) of a monotone map \( \mu : X \to Z \) and a light map \( \lambda : Z \to Y \). Applying the preceding theorem and two propositions to the calculation of the weight of the space \( Z \), we conclude that \( w(Z) \leq w(Y) \cdot \text{Sln}(Z) \leq w(Y) \cdot \text{Sln}(X) \). In this way we obtain the following corollary.

**COROLLARY 1.** Let \( f : X \to Y \) be a map between compact spaces and \( f = \lambda \circ \mu \) be the monotone-light decomposition of \( f \) into a monotone surjective map \( \mu : X \to Z \) and a light map \( \lambda : Z \to Y \). Then \( w(Z) \leq w(Y) \cdot \text{Sln}(X) \) and the non-degeneracy set \( N_\mu = \{ z \in Z : |\mu^{-1}(z)| > 1 \} \) of \( \mu \) has size \( |N_\mu| \leq \text{Sln}(X) \).
In this corollary, if we assume that \( f \) is the constant map, then the space \( Z \) is the decomposition of \( X \) into its components and it is zero-dimensional. Thus we obtain the following corollary.

**Corollary 2.** Each compact Hausdorff space \( X \) admits a monotone map \( f : X \rightarrow Z \) onto a zero-dimensional space \( Z \) of weight \( w(Z) \leq \text{Sln}(X) \). In particular, each zero-dimensional compact space \( Z \) has weight \( w(Z) \leq \text{Sln}(Z) \).

As another application of Theorem 1 we prove that each Suslinian continuum \( X \) is hereditarily decomposable, that is, \( X \) contains no indecomposable subcontinuum (a continuum \( X \) is indecomposable if \( X \) cannot be written as the union of two proper non-degenerate subcontinua of \( X \)).

**Proposition 5.** If \( X \) is a Tikhonov space with \( \text{Sln}(X) \leq \aleph_0 \), then all compact zero-dimensional subspaces of \( X \) are metrizable and all subcontinua of \( X \) are decomposable.

**Proof.** If \( Z \) is a zero-dimensional compact subset of \( X \), then \( w(Z) \leq \text{Sln}(Z) \leq \text{Sln}(X) \leq \aleph_0 \) by the preceding corollary.

Now take any subcontinuum \( C \) of \( X \). Then \( \text{Sln}(C) \leq \text{Sln}(X) \leq \aleph_0 \), which means that the continuum \( C \) is Suslinian. Let \( f : C \rightarrow [0,1] \) be any non-constant map. By Theorem 1, the map \( f \) can be written as the composition \( f = \lambda \circ \mu \) of a monotone map \( \mu : C \rightarrow Z \) and a light map \( \lambda : Z \rightarrow [0,1] \) of some continuum \( Z \) with \( w(Z) \leq \text{Sln}(C) \leq \aleph_0 \). Thus, \( Z \) is a metrizable Suslinian continuum. Such a continuum is decomposable. Otherwise, since each indecomposable continuum has uncountably many composants (see [6, Theorem 7', p. 213]), we would have \( \text{Sln}(Z) > \aleph_0 \). Consequently, we can write \( Z = A \cup B \) as the sum of two properly smaller subcontinua \( A, B \subset Z \). Their preimages \( \mu^{-1}(A) \) and \( \mu^{-1}(B) \) under the monotone map \( \mu \) are proper subcontinua of \( C \) whose union equals \( C \). This means that the continuum \( C \) is decomposable.

Next we prove that the hereditary Lindelöf number of any space \( X \) is bounded from above by the Suslinian number of \( X \). For Suslinian continua this result was proved in Theorem 1 of [1].

**Theorem 2.** \( hl(X) \leq \text{Sln}(X) \) for any Tikhonov space \( X \).

**Proof.** Let \( \kappa = \text{Sln}(X) \) and \( Z \supset X \) be a continuum with \( \text{Sln}(Z) = \kappa \).

First, we prove that each singleton \( \{x_0\}, x_0 \in X \), is the intersection of \( \kappa \) many neighborhoods in \( Z \). By Lemma 1 there is a family \( \mathcal{N} \) of closed neighborhoods of \( x_0 \) in \( Z \) such that \( |\mathcal{N}| \leq \text{Sln}(Z) = \kappa \) and the intersection \( \bigcap \mathcal{N} \) is zero-dimensional. The compactum \( Y = \bigcap \mathcal{N} \), being zero-dimensional, admits a light map onto the singleton. Applying Theorem 1 we get \( w(Y) \leq \text{Sln}(Y) \leq \text{Sln}(Z) = \kappa \). Consequently, we can find a fam-
ily $\mathcal{N}'$ of neighborhoods of the point $x_0$ in $Z$ such that $Y \cap \bigcap \mathcal{N}' = \{x_0\}$ and $|\mathcal{N}'| \leq \kappa$. Then the family $\mathcal{N} \cup \mathcal{N}'$ has size $\leq \kappa$ and its intersection is $\{x_0\}$.

Now, take any subspace $A \subset X$ and let $\mathcal{U}$ be a cover of $A$ by open subsets of $Z$. Then $\bigcup \mathcal{U}$ is an open subset of $Z$ and $B = Z \setminus \bigcup \mathcal{U}$ is a closed set in $Z$. Consider the quotient space $Z/B = (Z \setminus B) \cup \{B\}$ and let $q : Z \to Z/B$ be the quotient map. Since $\text{Sln}(Z/B) \leq \text{Sln}(Z) = \kappa$, we may apply the previous reasoning to find a family $\mathcal{V}$ of open neighborhoods of the singleton $\{B\} \in Z/B$ with $\{B\} = \bigcap \mathcal{V}$ and $|\mathcal{V}| \leq \kappa$. Then $\mathcal{W} = \{q^{-1}(V) : V \in \mathcal{V}\}$ is a family of size $\leq \kappa$ with $\bigcap \mathcal{W} = B$. The complement $Z \setminus W$ of each $W \in \mathcal{W}$ is a compact subset of $Z$ which can be covered by a finite subcollection of $\mathcal{U}$. Therefore, the union $\bigcup_{W \in \mathcal{W}} (Z \setminus W) = \bigcup \mathcal{U}$ can be covered by $\leq \kappa$ elements of the cover $\mathcal{U}$.  

According to \[3, 3.12.10(l)], w(X) \leq 2^{\text{hl}(X)}$ for any compact Hausdorff space. Hence, $w(X) \leq 2^{\text{Sln}(X)}$ for any Tikhonov space. In fact, we shall prove a stronger upper bound $w(X) \leq \text{Sln}(X)^+$. 

The Generalized Suslin Hypothesis asserts that for any regular cardinal $\kappa$ there is no $\kappa$-Suslin tree, where a tree is called $\kappa$-Suslin if it has height $\kappa$ but contains no chain or antichain of length $\kappa$. We recall that the classical Suslin Hypothesis asserts that there is no $\aleph_1$-Suslin tree.

Below, for a cardinal $\kappa$, we denote by $\text{cf}(\kappa)$ the cofinality of $\kappa$ and by $\kappa^+$ the successor cardinal of $\kappa$. We identify cardinals with initial ordinals.

**Theorem 3.** Let $X$ be a Tikhonov space. Then $w(X) \leq \text{Sln}(X)^+$. Moreover, if no $\kappa^+$-Suslin tree exists for $\kappa = \text{Sln}(X)$, then $w(X) \leq \text{Sln}(X)$.

**Proof.** Let $\kappa = \text{Sln}(X)$ and embed $X$ into a continuum $K$ with $\text{Sln}(K) = \text{Sln}(X)$. Assuming that $\kappa^+ < w(X) \leq w(K)$, we can find a continuous map $f : K \to Z$ of $K$ onto a continuum $Z$ of weight $w(Z) = \kappa^+$. Moreover, we may assume that the map $f$ is monotone. Indeed, if $f$ were not monotone, then it would factorize as $f = \lambda \circ \mu$ with $\mu : K \to Z_1$ monotone and $\lambda : Z_1 \to Z$ light. Then $w(Z_1) \leq w(Z) \cdot \text{Sln}(K) = \kappa^+ \cdot \kappa = \kappa^{++}$. Now, let us see that the conditions $\text{Sln}(Z) \leq \kappa$ and $w(Z) = \kappa^{++}$ lead to a contradiction.

Express $Z$ as the inverse limit of a well-ordered transfinite spectrum $\{Z_\alpha : \alpha < \kappa^{++}\}$ consisting of continua $Z_\alpha$ with $w(Z_\alpha) \leq \kappa^+$. Let $p_\alpha : Z \to Z_\alpha$, $\alpha < \kappa^{++}$, denote the (surjective) limit projections of the spectrum.

Consider the family $\mathcal{T} = \{p_\alpha^{-1}(z) : z \in Z_\alpha, \alpha < \kappa^{++}, \dim p_\alpha^{-1}(z) > 0\}$ of point-preimages which are not zero-dimensional. Endowed with the inverse inclusion order, this family forms a tree. This tree has no chains of length more than $\kappa$. Otherwise we would obtain a strictly decreasing sequence of length $> \kappa$ consisting of closed subsets of $Z$, which is impossible as $\text{hl}(Z) \leq \text{Sln}(Z) = \kappa$. 


The tree also contains no antichain of length $> \kappa$ since otherwise we would construct a disjoint family of size $> \kappa$ consisting of components of some elements of $\mathcal{T}$. Consequently, the tree $\mathcal{T}$ has height $\leq \kappa^+$ and all levels of the tree have size $\leq \kappa$. This implies that the tree $\mathcal{T}$ contains at most $\kappa^+$ elements. Since $\kappa^+ < \kappa^{++} = \text{cf}(\kappa^{++})$, we can find an ordinal $\alpha < \kappa^{++}$ such that for any point $z \in Z_\alpha$ the preimage $p_\alpha^{-1}(z)$ is zero-dimensional. This means that the limit projection $p_\alpha : Z \to Z_\alpha$ is light. Applying Theorem 1, we get a contradiction: $w(Z) \leq w(Z_\alpha) \cdot \text{Sln}(Z) \leq \kappa^+$.

If no $\kappa^+$-Suslin tree exists, then the tree $\mathcal{T}$ constructed above is not $\kappa^+$-Suslin and thus has height $\leq \kappa$. In this case we replace the condition $w(Z) = \kappa^+$ by $w(Z) = \kappa^+$ and see that the proof above gives that $w(Z) \leq \text{Sln}(Z)$.

**Corollary 3.** If the Generalized Suslin Hypothesis holds, then $w(X) \leq \text{Sln}(X)$ for any Tikhonov space $X$.

Applying Theorem 3 to Suslinian continua, we obtain the answer to the second part of Problem 1 of [1].

**Corollary 4.** Under the Suslin Hypothesis all Suslinian continua are metrizable.

Theorem 3 allows us to describe the structure of compacta $X$ with $w(X) > \text{Sln}(X)$.

**Theorem 4.** Each compact space $X$ with $w(X) > \text{Sln}(X)$ is the inverse limit of a well-ordered spectrum $\{Z_\alpha, \pi_\alpha^\beta, \alpha \leq \beta < \text{Sln}(X)^+\}$ consisting of compacta of weight $w(Z_\alpha) \leq \text{Sln}(X)$ and monotone bonding maps $\pi_\alpha^\beta : Z_\beta \to Z_\alpha$.

**Proof.** Let $\kappa = \text{Sln}(X)$. It follows from Theorem 3 that $w(X) = \kappa^+$. Therefore, we can write $X$ as the inverse limit of a well-ordered spectrum $S = \{X_\alpha, p_\alpha^\beta, \alpha \leq \beta < \kappa^+\}$ consisting of compacta of weight $\leq \kappa$ and surjective bonding maps. Since $h\ell(X) \leq \text{Sln}(X) \leq \kappa$, this spectrum is factorizable in the sense that any continuous map $f : X \to Z$ into a compact space $Z$ of weight $w(Z) \leq \kappa$ can be written as a composition $f = f_\alpha \circ p_\alpha$ of the limit projection $p_\alpha : X \to X_\alpha$ and a continuous map $f_\alpha : X_\alpha \to Z$ for some ordinal $\alpha < \kappa^+$ (see [4, 3.1.6]).

For each ordinal $\alpha < \kappa^+$ let $p_\alpha = \lambda_\alpha \circ \mu_\alpha$ be the (unique) monotone-light decomposition of the limit projection $p_\alpha : X \to X_\alpha$ into a monotone map $\mu_\alpha : X \to Z_\alpha$ and a light map $\lambda_\alpha : Z_\alpha \to X_\alpha$. By Proposition 3, $\text{Sln}(Z_\alpha) \leq \text{Sln}(X) \leq \kappa$ and by Theorem 4, $w(Z_\alpha) \leq w(X_\alpha) \cdot \text{Sln}(Z_\alpha) \leq \kappa$. Then there is an ordinal $\xi(\alpha) > \alpha$ such that the monotone map $\mu_\alpha : X \to Z_\alpha$ factorizes through $X_{\xi(\alpha)}$ in the sense that $\mu_\alpha = \mu_{\xi(\alpha)} \circ p_{\xi(\alpha)}$ for some map $\mu_{\xi(\alpha)} : X_{\xi(\alpha)} \to Z_\alpha$. 
Thus we obtain the following commutative diagram:

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X \xrightarrow{p_{\xi(\alpha)}} X_{\xi(\alpha)} \xrightarrow{p_{\xi(\alpha)}} X_\alpha
\downarrow{\mu_\alpha} \downarrow{\mu_\alpha} \downarrow{\mu_\alpha}
\downarrow{\lambda_\alpha} \downarrow{\lambda_\alpha} \downarrow{\lambda_\alpha}
Z_\alpha
```

Let $A$ be a cofinal subset of ordinals $< \kappa^+$ such that $\xi(\alpha) < \beta$ for any $\alpha < \beta$ in $A$. For any $\alpha < \beta$ in $A$ define a bonding map $\pi^{\beta}_\alpha : Z_\beta \rightarrow Z_\alpha$ letting $\pi^{\beta}_\alpha = \mu^{\xi(\alpha)}_\alpha \circ p^{\beta}_\xi(\alpha) \circ \lambda_\beta$. We claim that the map $\pi^{\beta}_\alpha$ is monotone. This follows from the monotonicity of the map $\mu_\alpha = \mu^{\xi(\alpha)}_\alpha \circ p^{\beta}_\xi(\alpha) \circ p_\beta = \mu^{\xi(\alpha)}_\alpha \circ p^{\beta}_\xi(\alpha) \circ \lambda_\beta \circ \mu_\beta = \pi^{\beta}_\alpha \circ \mu_\beta$. Indeed, for any point $y \in Z_\alpha$, the preimage $(\pi^{\beta}_\alpha)^{-1}(y) = \mu_\beta(\mu_\alpha^{-1}(y))$ is connected, being the image of the connected set $\mu_\alpha^{-1}(y)$.

It is easy to see that $\pi^{\gamma}_\alpha = \pi^{\beta}_\alpha \circ \pi^{\gamma}_\beta$ for any ordinals $\alpha < \beta < \gamma$ in $A$, which means that $S^' = \{Z_\alpha, \pi^{\beta}_\alpha : \alpha, \beta \in A\}$ is an inverse spectrum. Let $Z = \lim S^'$ be the limit of this spectrum. Observe that the monotone maps $\mu_\alpha : X \rightarrow Z_\alpha$, $\alpha \in A$, induce a surjective map $\mu : X \rightarrow Z$ while the light maps $\lambda_\alpha : Z_\alpha \rightarrow X_\alpha$, $\alpha \in A$, induce a surjective map $\lambda : Z \rightarrow X$. Since $\lambda_\alpha \circ \mu_\alpha = p_\alpha$ for all $\alpha \in A$, the composition $\lambda \circ \mu : X \rightarrow X$ is the identity map of $X$. Consequently, both $\lambda$ and $\mu$ are homeomorphisms and thus $X$ can be identified with the limit $Z$ of the spectrum $S^'$ of length $\kappa^+$ consisting of compacta of weight $\leq \kappa$ and monotone bonding maps.

The following particular case of Theorems 3 and 4 answers the remaining part of Problem 1 from [1].

**Corollary 5.** Each non-metrizable Suslinian continuum $X$ has weight $\aleph_1$ and is the limit of an inverse spectrum of length $\aleph_1$ consisting of metrizable Suslinian continua and monotone bonding maps.

In the subsequent proof we shall refer to properties of the hyperspace $\exp(X)$ of a given compact Hausdorff space $X$. The **hyperspace** $\exp(X)$ of $X$ is the space of all non-empty closed subsets of $X$, endowed with the Vietoris topology. It is well known that $\exp(X)$ is a compact Hausdorff space with $w(\exp(X)) = w(X)$. We denote by $\exp_c(X)$ the subspace of $\exp(X)$ consisting of subcontinua of $X$. It is easy to see that $\exp_c(X)$ is a closed subspace in $\exp(X)$. 

Compacta \( X \) with small Suslinian number \( \text{Sln}(X) < c \) share many properties of Suslinian continua.

**Theorem 5.** If \( X \) is a continuum with \( \text{Sln}(X) < c \), then \( \dim X \leq 1 \) and
\[
\text{rim}-w(X) \leq \text{Sln}(X) \leq h\ell(\exp_c(X)) \leq w(X) \leq \text{Sln}(X)^+.
\]

**Proof.** Let \( \kappa = \text{Sln}(X) \). To show that \( \text{rim}-w(X) \leq \text{Sln}(X) \), take any point \( x \in X \) and a neighborhood \( U \subset X \) of \( x \). Let \( f : X \to [0,1] \) be any function with \( f(x_0) = \{0\} \) and \( f^{-1}([0,1)) \subset U \). Since \( \text{Sln}(X) < c \), the set \( \{y \in (0,1) : \dim f^{-1}(y) > 0\} \) has size \( \leq \text{Sln}(X) < c \). Consequently, we can find a point \( y \in (0,1) \) whose preimage \( f^{-1}(y) \subset Z \) is zero-dimensional. By Corollary 2, \( w(f^{-1}(y)) \leq \text{Sln}(f^{-1}(y)) \leq \text{Sln}(X) = \kappa \).

Now consider the neighborhood \( V = f^{-1}([0,y)) \) whose boundary \( \partial V \) lies in \( f^{-1}(y) \) and thus has weight \( w(\partial V) \leq \kappa \) and is zero-dimensional. This proves the inequality \( \text{rim}-w(X) \leq \kappa \), and shows that the small inductive dimension of \( X \) satisfies \( \text{ind}(X) \leq 1 \). By [3 7.2.7], \( \dim X \leq 1 \).

It remains to prove that \( \kappa \leq h\ell(\exp_c(X)) \leq w(X) \leq \text{Sln}(X)^+ \). The third inequality was proved in Theorem [4] while the second inequality follows from \( h\ell(\exp_c(X)) \leq w(\exp_c(X)) \leq w(\exp(X)) = w(X) \). Assuming \( h\ell(\exp_c(X)) < \kappa = \text{Sln}(X) \), let \( \lambda = h\ell(\exp_c(X)) \) and find a disjoint family \( \mathcal{C} \) of size \( |\mathcal{C}| = \lambda^+ \) consisting of non-degenerate subcontinua of \( X \). This family \( \mathcal{C} \) can be considered as a subset of the hyperspace \( \exp_c(X) \) of subcontinua of \( X \). Identify \( X \) with the set of all degenerate subcontinua in \( \exp_c(X) \). Since \( h\ell(\exp_c(X)) = \lambda \), the set \( \mathcal{C} \) contains a subset \( \mathcal{C}' \) of size \( |\mathcal{C}'| = |\mathcal{C}| = \lambda^+ \) whose closure in \( \exp_c(X) \) misses \( X \).

We claim that \( \mathcal{C}' \) is not a scattered subspace of \( \exp_c(X) \). Let us recall that a topological space is scattered if each of its subspaces has an isolated point. It is known (and can be easily shown) that the size of a scattered space is equal to its hereditary Lindelöf number. Since \( |\mathcal{C}'| = \lambda^+ > \lambda = h\ell(\exp_c(X)) \geq h\ell(\mathcal{C}') \), the space \( \mathcal{C}' \) is not scattered and thus contains a subspace \( \mathcal{C}'' \) having no isolated point.

Now we shall construct a subset \( \{C_t\}_{t \in \mathcal{T}} \subset \mathcal{C}'' \) indexed by elements of the binary tree \( \mathcal{T} = \bigcup_{n \in \mathbb{N}} \{0,1\}^n \) as follows. The binary tree \( \mathcal{T} \) consists of finite binary sequences. Given two binary sequences \( t = (t_0, \ldots, t_n) \), \( s = (s_0, \ldots, s_m) \) in \( \mathcal{T} \) we write \( t \leq s \) if \( n \leq m \) and \( t_i = s_i \) for all \( i \leq n \).

Take any distinct elements \( C_0, C_1 \in \mathcal{C}'' \) and observe that the subcontinua \( C_0, C_1 \) are disjoint (because the family \( \mathcal{C} \) is disjoint). Hence, they have open neighborhoods \( U_0, U_1 \subset X \) with disjoint closures.

Assuming that for some binary sequence \( s = (s_0, \ldots, s_n) \) the subcontinuum \( C_s \in \mathcal{C}'' \) and its neighborhood \( U_s \subset X \) are constructed, consider the open subset \( \mathcal{U}_s = \{C \in \mathcal{C}'' : C \subset U_s\} \) of the space \( \mathcal{C}'' \) and take any two distinct (and hence disjoint) subcontinua \( C_{s_{0}}, C_{s_{1}} \in \mathcal{U}_s \). Next, choose two
open neighborhoods $U_{s',0}, U_{s',1} \subset U_s$ of $C_{s',0}, C_{s',1}$ with disjoint closures. This finishes the inductive step.

Now, for any infinite binary sequence $s = (s_i)$ let $C_s$ be a cluster point of the set \{ $C_{(s|n)} : n \in \mathbb{N}$ \} in $\exp(X)$, where $s|n = (s_0, \ldots, s_{n-1})$. It is easy to see that \{ $C_s : s \in \{0,1\}^\omega$ \} is a disjoint family of subcontinua of $X$, lying in the closure of the set $\mathcal{C}''$. Since this closure misses the set $X$, each continuum $C_s$, $s \in \{0,1\}^\omega$, is non-degenerate. Thus, $\kappa = \text{Sln}(X) \geq |\{C_s : s \in \{0,1\}^\omega\}| = \mathfrak{c}$, which is a contradiction. ■

**Problem 1.** Is $\text{rim-w}(X) \leq \text{Sln}(X)$ for any compact Hausdorff space $X$?

Let us remark that all examples of non-metrizable Suslinian continua considered in the introduction or in [1] contain a copy of a Suslin line and hence fail to be hereditarily separable. However (consistent) examples of non-metrizable, hereditarily separable Suslinian continua can be constructed as well. For such a construction we need the following definitions and the lemma.

We recall that a surjective map $f : X \to Y$ is irreducible if $f(Z) \neq Y$ for any proper closed subset $Z$ of $X$. This is equivalent to saying that a set $D \subset X$ is dense in $X$ provided $f(D)$ is dense in $Y$.

Following [4] III.1.15 we call a monotone map $f : X \to Y$ between two continua atomic if for every non-degenerate subcontinuum $Z \subset Y$ the map $f|f^{-1}(Z) : f^{-1}(Z) \to Z$ is irreducible. This is equivalent to saying that $\overline{D} = f^{-1}(f(D))$ for every subset $D \subset X$ whose image $f(D)$ is dense in some non-degenerate subcontinuum of $Y$. An atomic map $f : X \to Y$ will be called $I$-atomic if for every $y \in Y$ the preimage is a singleton or an arc in $X$.

The following lemma will be our basic tool in the subsequent inductive construction.

**Lemma 3.** For any non-degenerate metrizable Suslinian continuum $Y$ and any countable set $Z \subset Y$ there are a metrizable Suslinian continuum $X$ and an $I$-atomic map $f : X \to Y$ whose non-degeneracy set $N(f) = \{y \in Y : |f^{-1}(y)| > 1\}$ equals $Z$.

**Proof.** For every $z \in Z$ fix a decreasing neighborhood base $(O_n(z))_{n \in \omega}$ at $z$ such that $\overline{O_{n+1}(z)} \subset O_n(z)$ for all $n \in \omega$. Let $\{q_n : n \in \omega\}$ be a countable dense set in $I = [0,1]$. Fix a map $h_z : Y \setminus \{z\} \to I$ such that $h_z(\partial O_n(z)) = \{q_n\}$ where $\partial O_n(z)$ stands for the boundary of $O_n(z)$ in $Y$. Such a choice of the map $h_z$ guarantees that $h_z(C \setminus \{z\}) = I$ for any non-degenerate subcontinuum $C \subset Y$ containing $z$.

Now consider the set $X = (Y \setminus Z) \cup (Z \times I)$ and the map $f : X \to Y$ which is the identity on $Y \setminus Z$ and $f(z,t) = z$ for each $(z,t) \in Z \times I \subset X$. For every $z \in Z$ let $r_z : X \to \{z\} \times I$ be a unique map such that
• \( r_z(y) = (z, h_z(y)) \) for every \( y \in Y \setminus Z \subset X \);
• \( r_z(y, t) = (z, h_z(y)) \) for every \( (y, t) \in (Z \setminus \{z\}) \times I \subset X \);
• \( r_z(z, t) = (z, t) \) for every \( t \in I \).

Endow the space \( X \) with the weakest topology making the maps \( f : X \to Y \) and \( r_z : X \to \{z\} \times I, z \in Z, \) continuous. According to [4, III.1.2] the resulting space \( X \) is metrizable and compact. It is easy to check that the map \( f \) is \( I \)-atomic (see also [4, III.1.15]).

Using the atomic property of \( f \) and the Suslinian property of \( Y \) it is easy to check that \( X \) is Suslinian too. \( \blacksquare \)

Now, we are ready for the construction of our example. We note that similar constructions using atomic maps have been done before, for instance in [8], [10] and [11].

**Theorem 6.** Under the negation of the Suslin hypothesis there exists a hereditarily separable non-metrizable Suslinian continuum \( X \). Moreover, each non-degenerate subcontinuum of \( X \) is neither metrizable nor locally connected.

**Proof.** Assuming the negation of the Suslin hypothesis, fix a Suslin tree \((T, \leq)\) such that each node \( t \in T \) has uncountably many successors in \( T \) and infinitely many immediate successors in \( T \). Denote by \( h(t) \) the height of a node \( t \in T \) and for a countable ordinal \( \alpha \) let \( T_\alpha = \{ t \in T : h(t) = \alpha \} \) stand for the \( \alpha \)th level of \( T \). For two countable ordinals \( \alpha < \beta \) let \( pr_\alpha^\beta : T_\beta \to T_\alpha \) denote the map assigning to a node \( t \in T_\beta \) a unique node \( t' \in T_\alpha \) with \( t' < t \). We may additionally assume that the tree \( T \) is continuous in the sense that for any limit countable ordinal \( \alpha \) and distinct nodes \( t, t' \in T_\alpha \) there is \( \beta < \alpha \) such that \( pr_\alpha^\beta(t) \neq pr_\beta^\alpha(t') \).

We shall use transfinite induction to construct a well-ordered continuous spectrum \( \{X_\alpha, \pi^\beta_\alpha : \alpha < \beta < \omega_1\} \) consisting of metrizable Suslinian continua \( X_\alpha \) and atomic bonding maps \( \pi^\beta_\alpha : X_\beta \to X_\alpha \), and a sequence \( (i_\alpha : T_\alpha \to X_\alpha)_{\alpha < \omega_1} \) of injective maps such that

1. for any countable ordinals \( \alpha < \beta \) the diagram

\[
\begin{array}{ccc}
T_\beta & \xrightarrow{i_\beta} & X_\beta \\
pr_\alpha^\beta \downarrow & & \downarrow \pi^\beta_\alpha \\
T_\alpha & \xrightarrow{i_\alpha} & X_\alpha
\end{array}
\]

is commutative;

2. for every \( t \in T_\alpha \) the set \( i_{\alpha+1}(pr_\alpha^{\alpha+1})^{-1}(t) \) is dense in \( (\pi^\alpha_{\alpha+1})^{-1}(i_\alpha(t)) \);

3. the short projections \( \pi^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha \) are \( I \)-atomic maps with non-degeneracy set \( N(\pi^{\alpha+1}_\alpha) = i_\alpha(T_\alpha) \).
We start the induction with a singleton $X_0$ and the injective map $i_0 : T_0 \to X_0$ assigning to the root of $T$ the only point of $X_0$. Assume that for some countable ordinal $\alpha$ the Suslinian continua $X_\beta$, atomic bonding maps $\pi^\beta : X^\beta \to X_\gamma$, and injective maps $i_\beta : T_\beta \to X_\beta$ have been constructed for all $\gamma \leq \beta < \alpha$.

If $\alpha$ is a limit ordinal, let $X_\alpha$ be the inverse limit of the countable spectrum $\{X_\beta, \pi^\beta : \gamma \leq \beta < \alpha\}$ and let $\pi^\alpha : X_\alpha \to X_\beta$ stand for the limit projections of this spectrum. They are atomic as limits of atomic bonding maps. For every $t \in T_\alpha$ let $i_\alpha(t)$ be the unique point of $X_\alpha$ such that $\pi^\alpha_\beta(i_\alpha(t)) = i_\beta(pr^\alpha_\beta(t))$ for every $\beta < \alpha$. The continuity of the tree $T$ implies that the resulting map $i_\alpha : T_\alpha \to X_\alpha$ is injective. The Suslinian property of $X_\alpha$ follows from that property of the continua $X_\beta$, $\beta < \alpha$, and the atomicity of the limit projections $\pi^\alpha_\beta$.

If $\alpha = \beta + 1$ is a successor ordinal, then we can apply Lemma 3 to find a metrizable Suslinian continuum $X_{\alpha+1}$ and an $I$-atomic map $\pi^\alpha_{\alpha+1} : X_{\alpha+1} \to X_\alpha$ whose non-degeneracy set coincides with $i_\alpha(T_\alpha)$. Thus we satisfy the condition (3) of the inductive construction. Since for every $t \in T_\alpha$ the set $(\pi^\alpha_{\alpha+1})^{-1}(i_\alpha(t))$ is an arc in $X_{\alpha+1}$, we can define an injective map $i_{\alpha+1} : T_{\alpha+1} \to X_{\alpha+1}$ so that $\pi^\alpha_{\alpha+1} \circ i_{\alpha+1} = i_\alpha \circ pr^\alpha_{\alpha+1}$ and $i_{\alpha+1}$ satisfies the condition (2) of the inductive construction.

After completing the inductive construction, consider the inverse limit $X$ of the spectrum $S = \{X_\alpha, \pi^\beta_\alpha : \beta < \alpha < \omega_1\}$. Using the atomicity of the bonding projections, one can check that the limit projections $\pi^\alpha : X \to X_\alpha$ are atomic as well.

Now, we establish the desired properties of the continuum $X$. First, we show that each non-degenerate subcontinuum $C$ of $X$ is neither metrizable nor locally connected. Let $\alpha$ be the smallest ordinal such that $|\pi^\alpha_\alpha(C)| > 1$. The continuity of the spectrum $S$ implies that $\alpha = \beta + 1$ for some ordinal $\beta$. Then $\pi^\beta(C)$ is a singleton and hence $\pi^\beta_\beta(C) \subset i_\beta(T_\beta)$ (otherwise $C$ would be a singleton). Let $t \in T_\beta$ be a node of $T$ with $\pi^\beta(C) = \{i_\beta(t)\}$. It follows that $\pi^\alpha_\alpha(C)$ is a non-degenerate subcontinuum of the arc $A_t = (\pi^{\alpha^{-1}}_\beta)^{-1}(i_\beta(t))$. The density of $i_\alpha(T_\alpha)$ in $A_t$ implies the existence of a node $t' \in T_\alpha$ with $i_\alpha(t') \in \pi^\alpha_\alpha(C)$. The atomicity of the projection $pr^\alpha_\alpha$ implies that the continuum $C = \pi^\alpha_\alpha^{-1}(\pi^\alpha_\alpha(C))$ contains the subcontinuum $pr^\alpha_\alpha^{-1}(i_\alpha(t'))$ which is not metrizable (because $t'$ has uncountably many successors in the tree $T$). Consequently, $C$ is not metrizable either.

To show that $C$ is not locally connected, assume the converse and, given any two distinct points $x, x' \in pr^\alpha_\alpha^{-1}(i_\alpha(t'))$, find a closed connected neighborhood $U \subset C$ of $x$ with $x' \notin U$. Since $pr^\alpha_\alpha^{-1}(i_\alpha(t'))$ is nowhere dense in $C$, the set $U$ has non-degenerate projection $pr^\alpha_\alpha(U)$. Then the atomicity of $pr^\alpha_\alpha$ implies that $x' \in pr^\alpha_\alpha^{-1}(pr^\alpha_\alpha(U)) = U$, which is a contradiction.
Next, we shall prove that the continuum $X$ is Suslinian. Take any family $C$ of pairwise disjoint non-degenerate subcontinua in $X$. Repeating the preceding argument, for every $C \in C$ we can find a countable ordinal $\alpha$ and a node $t_C \in T_\alpha$ such that $C \supset \pi^{-1}_\alpha(i_\alpha(t_C))$. It follows that the nodes $t_C, C \in C,$ are pairwise incomparable in $T$ (otherwise the family $C$ would contain two intersecting continua). Since $T$ is a Suslin tree, the antichain $\{t_C : C \in C\}$ is at most countable and so is the family $C$, witnessing the Suslinian property of $X$.

It remains to check that the continuum $X$ is hereditarily separable. By [8, 3.12.9] it suffices to prove that each closed subspace $F$ of $X$ is separable. By Theorem 2, the continuum $X$, being Suslinian, is perfectly normal and hence $F = \pi^{-1}_\alpha(\pi_\alpha(F))$ for some countable ordinal $\alpha$. Let $Z = \text{pr}_\alpha(F)$. Since

$$F = \pi^{-1}_\alpha(Z \setminus i_\alpha(T_\alpha)) \cup \bigcup_{z \in Z \cap i_\alpha(T_\alpha)} \pi^{-1}_\alpha(z)$$

and $\pi^{-1}_\alpha(Z \setminus i_\alpha(T_\alpha))$ is homeomorphic to the metrizable separable space $Z \setminus i_\alpha(T_\alpha)$, it remains to check that for every $z \in i_\alpha(T_\alpha)$ the continuum $\pi^{-1}_\alpha(z)$ is separable. Consider the arc $A = \pi^{\alpha+1}_\alpha(z)$ in $X_{\alpha+1}$ and observe that $D = A \setminus i_{\alpha+1}(T_{\alpha+1})$ is a dense subspace of $A$. It follows from the construction that $\pi^{-1}_{\alpha+1}(D)$ is a topological copy of $D$, dense in $\pi^{-1}_\alpha(A) = \pi^{-1}_\alpha(z)$. Therefore, the continuum $\pi^{-1}_\alpha(z)$ is separable. \(\blacksquare\)

We do not know if the preceding theorem can be generalized to higher cardinals.

**Problem 2.** Does the existence of a $\kappa^+$-Suslin tree imply the existence of a continuum $X$ with $\text{hd}(X) \leq \text{Sln}(X) = \kappa < w(X)$?

**Remark 1.** The existence of a $\kappa^+$-Suslin tree is equivalent to the existence of a linearly ordered continuum $X$ with $\kappa = \text{Sln}(X) = c(X) < d(X) = w(X) = \kappa^+$.

The non-metrizable hereditarily separable Suslinian continuum constructed in Theorem 6 is very far from being locally connected. In [2], it was proved that separable homogeneous Suslinian continua are metrizable. This encourages us to recall the following question of [1].

**Problem 3.** Is each locally connected (hereditarily) separable Suslinian continuum metrizable?

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The Suslinian number

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