

The classification of weighted projective spaces

by

Anthony Bahri (Lawrenceville, NJ), **Matthias Franz** (London, ON),
Dietrich Notbohm (Amsterdam) and **Nigel Ray** (Manchester)

Abstract. We obtain two classifications of weighted projective spaces: up to homeomorphism and up to homotopy equivalence. We show that the former coincides with Al Amrani's classification up to isomorphism of algebraic varieties, and deduce the latter by proving that the Mislin genus of any weighted projective space is rigid.

1. Introduction. Weighted projective spaces are the simplest projective toric varieties that exhibit orbifold singularities. They have been extensively investigated by algebraic geometers, but have attracted only fleeting attention from algebraic topologists since Kawasaki's pioneering work [10], in which he computed their integral cohomology rings. Subsequently, their K -theory was determined by Al Amrani [2], and the study of their KO -theory was initiated by Nishimura–Yosimura [13].

In toric geometry, weighted projective spaces are classified by their fans. Here, we give two classifications that are fundamental to algebraic topology: up to homeomorphism and up to homotopy equivalence. We obtain the latter as a consequence of the fact that the Mislin genus of a weighted projective space is rigid. Our results are stated below, following summaries of the definitions and notation.

A *weight vector* $\chi = (\chi_0, \dots, \chi_n)$ is a finite sequence of positive integers. It gives rise to a weighted action of S^1 on $S^{2n+1} \subset \mathbb{C}^{n+1}$,

$$(1.1) \quad g \cdot z = (g^{\chi_0} z_0, \dots, g^{\chi_n} z_n) \quad \text{for } g \in S^1, z \in S^{2n+1}.$$

The quotient $S^{2n+1}/S^1\langle\chi\rangle$ is the weighted projective space $\mathbb{P}(\chi)$. Alternatively, $\mathbb{P}(\chi)$ may be defined as the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the same weighted action of \mathbb{C}^\times ; this exhibits $\mathbb{P}(\chi)$ as a complex projective variety.

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Scaling the weight vector χ leads to isomorphic weighted projective spaces $\mathbb{P}(\chi)$ and $\mathbb{P}(m\chi)$ for any integer $m \geq 1$. Moreover, if all weights except, say, χ_0 are divisible by some prime p , then the map

$$(1.2) \quad \mathbb{P}(\chi) \rightarrow \mathbb{P}(\chi_0, \chi_1/p, \dots, \chi_n/p), \quad [z_0 : \dots : z_n] \mapsto [z_0^p : z_1 : \dots : z_n],$$

is an isomorphism as well (cf. [8, §5.7]). This leads to the notion of *normalized weights*: a weight vector χ is normalized if for any prime p at least two weights in χ are not divisible by p . Any weight vector can be transformed to a unique normalized vector by repeated application of scaling and (1.2). Consequently, two weighted projective spaces are isomorphic as algebraic varieties and homeomorphic as topological spaces if they have the same normalized weights, up to order. We prove that the converse is also true. In particular, we recover Al Amrani’s classification up to isomorphism of algebraic varieties [1, §8.1].

THEOREM 1.1. *The following are equivalent for any weight vectors χ and χ' :*

- (1) *The normalizations of χ and χ' are the same, up to order.*
- (2) *$\mathbb{P}(\chi)$ and $\mathbb{P}(\chi')$ are isomorphic as algebraic varieties.*
- (3) *$\mathbb{P}(\chi)$ and $\mathbb{P}(\chi')$ are homeomorphic.*

For any prime p , the p -content ${}_p\chi$ of χ is the vector made up of the highest powers of p dividing the individual weights. For example, ${}_2(1, 2, 3, 4) = (1, 2, 1, 4)$. Let χ and χ' be two normalized weight vectors. It follows from Kawasaki’s result that the cohomology rings $H^*(\mathbb{P}(\chi); \mathbb{Z})$ and $H^*(\mathbb{P}(\chi'); \mathbb{Z})$ are isomorphic if and only if, for all primes p , the p -contents ${}_p\chi$ and ${}_p\chi'$ are the same up to order. The same phenomenon can be observed in K -theory and KO -theory. In fact, no cohomology theory can tell such spaces apart:

THEOREM 1.2. *Two weighted projective spaces are homotopy equivalent if and only if for all primes p , the p -contents of their normalized weights are the same, up to order.*

The torus $T = (S^1)^{n+1}/S^1\langle\chi\rangle \cong (S^1)^n$ and its complexification $T_{\mathbb{C}}$ act on $\mathbb{P}(\chi)$ in a canonical way, and the resulting equivariant homotopy type is a finer invariant. As shown in [3, Thm. 5.1], the equivariant cohomology ring $H_T^*(\mathbb{P}(\chi); \mathbb{Z})$ determines the normalized weights up to order.

Let ${}_p\chi^*$ be the vector obtained from ${}_p\chi$ by ordering its coordinates as a non-decreasing sequence, and let χ^* denote the product of the ${}_p\chi^*$, taken coordinatewise. For example, $(1, 2, 3, 4)^* = (1, 1, 2, 12)$. By Theorem 1.2, $\mathbb{P}(\chi)$ is homotopy equivalent to $\mathbb{P}(\chi^*)$. The weights in χ^* form a *divisor chain*, in the sense that each weight divides the next. As a consequence, the space $\mathbb{P}(\chi^*)$ is particularly easy to work with because the differences

$$(1.3) \quad * = \mathbb{P}(\chi_n^*), \mathbb{P}(\chi_{n-1}^*, \chi_n^*) \setminus \mathbb{P}(\chi_n^*), \dots, \mathbb{P}(\chi_0^*, \dots, \chi_n^*) \setminus \mathbb{P}(\chi_1^*, \dots, \chi_n^*)$$

form a cell decomposition of $\mathbb{P}(\chi^*)$ (see Remark 3.2 below), and

$$(1.4) \quad * = \mathbb{P}(\chi_0^*) \subset \mathbb{P}(\chi_0^*, \chi_1^*) \subset \cdots \subset \mathbb{P}(\chi_0^*, \dots, \chi_{n-1}^*) \subset \mathbb{P}(\chi_0^*, \dots, \chi_n^*)$$

displays $\mathbb{P}(\chi^*)$ as an iterated Thom space [4, Cor. 3.8].

The *Mislin genus* of a weighted projective space $\mathbb{P}(\chi)$ is the set of all homotopy classes of simply connected CW complexes Y of finite type such that for all primes p the p -localizations of Y and $\mathbb{P}(\chi)$ are homotopy equivalent. The Mislin genus of a space is *rigid* if it contains only the class of the space itself.

THEOREM 1.3. *The Mislin genus of any weighted projective space is rigid.*

For $\mathbb{C}\mathbb{P}^n$, this has been established by McGibbon [11, Thm. 4.2(ii)].

In Section 2 we review Kawasaki’s results on which our work is based. Theorem 1.1 is proved in Section 3, and Theorems 1.2 and 1.3 in Section 5; necessary conditions for the rigidity of the Mislin genus are established in Section 4.

2. Kawasaki’s results. From now on, $\chi = (\chi_0, \dots, \chi_n)$ always denotes a normalized weight vector, and cohomology is taken with integer coefficients unless otherwise stated. In order to make Kawasaki’s description of $H^*(\mathbb{P}(\chi))$ explicit, it is convenient to recall his notation $(r_0(\chi; p), \dots, r_n(\chi; p))$ for the non-decreasing weight vector ${}_p\chi^*$; given any $0 \leq i \leq n$, we then set

$$(2.1) \quad l_i = l_i(\chi) = \prod_{p \text{ prime}} r_{n-i+1}(\chi; p) \cdots r_n(\chi; p).$$

We also consider the map

$$(2.2) \quad \varphi = \varphi_\chi: \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{P}(\chi), \quad [z_0 : \cdots : z_n] \mapsto [z_0^{\chi_0} : \cdots : z_n^{\chi_n}].$$

THEOREM 2.1 ([10, Thm. 1]). *Additively, $H^*(\mathbb{P}(\chi)) \cong H^*(\mathbb{C}\mathbb{P}^n)$. Furthermore, there exist generators $\xi_i \in H^{2i}(\mathbb{P}(\chi))$ and $\eta \in H^2(\mathbb{C}\mathbb{P}^n)$ such that $\varphi^*(\xi_i) = l_i \eta^i$ for $0 \leq i \leq n$; the multiplicative structure is specified by*

$$\xi_i \xi_j = \frac{l_i l_j}{l_{i+j}} \xi_{i+j}$$

in $H^{2(i+j)}(\mathbb{P}(\chi))$, for $0 \leq i + j \leq n$.

REMARK 2.2. Kawasaki’s proof of Theorem 2.1 shows that the integral homology groups $H_*(\mathbb{P}(\chi))$ are finitely generated and torsion-free, and therefore isomorphic to $\text{Hom}(H^*(\mathbb{P}(\chi)), \mathbb{Z})$ by the Universal Coefficient Theorem.

Moreover, [5, Sec. 3.2] and [9, Cor. 7.2] confirm that $P(\chi)$ is a simply connected finite CW complex for every choice of χ .

Kawasaki also determined the cohomology of the generalized lens space $L(k; \chi) = S^{2n+1}/\mathbb{Z}_k \langle \chi \rangle$, where in this case χ describes the weights of the k th

roots of unity. The answer depends on the augmented weight vector $(\chi, k) = (\chi_0, \dots, \chi_n, k)$.

THEOREM 2.3 ([10, Thm. 2]). *The non-zero cohomology groups of $L = L(k; \chi)$ are $H^0(L) \cong H^{2n+1}(L) \cong \mathbb{Z}$ and $H^{2i}(L) \cong \mathbb{Z}_q$ for $1 \leq i \leq n$, where $q = l_i(\chi, k)/l_i(\chi)$.*

3. Classification up to homeomorphism. Fix a point $z \in \mathbb{P}(\chi)$. Let I and J be the subsets of $\{0, \dots, n\}$ corresponding to the zero and non-zero homogeneous coordinates of z , respectively, and let $q = \gcd\{\chi_i : i \in J\}$. Also, write $\chi_I \in \mathbb{Z}^I$ for the weights indexed by I , and let $U_I = \{[z_0 : \dots : z_n] : z_i \neq 0 \text{ for } i \notin I\}$.

LEMMA 3.1 (cf. [8, §5.15]). *There is an isomorphism of algebraic varieties*

$$U_I \cong (\mathbb{C}^\times)^{|J|-1} \times \mathbb{C}^I / \mathbb{Z}_q \langle \chi_I \rangle,$$

sending z to a point of the form $(\tilde{z}, 0)$.

Observe that $\mathbb{C}^I / \mathbb{Z}_q \langle \chi_I \rangle$ is the unbounded cone over $L(q; \chi_I)$.

Proof of Lemma 3.1. The weight vector χ_J determines a morphism $\mathbb{C}^\times \rightarrow (\mathbb{C}^\times)^J$ with kernel \mathbb{Z}_q . Let T' be its image and $T'' \cong (\mathbb{C}^\times)^{|J|-1}$ a torus complement. Then

$$U_I = ((\mathbb{C}^\times)^J \times \mathbb{C}^I) / \mathbb{C}^\times \langle \chi \rangle = (T'' \times T' \times \mathbb{C}^I) / \mathbb{C}^\times \langle \chi \rangle = T'' \times \mathbb{C}^I / \mathbb{Z}_q \langle \chi_I \rangle. \blacksquare$$

REMARK 3.2. If $\chi_0 = 1$ and $z = [1 : 0 : \dots : 0]$, then $U_I \cong \mathbb{C}^n$. If the weights form a divisor chain, we have $\mathbb{P}(\chi) \setminus U_I = \mathbb{P}(\chi_1, \dots, \chi_n) = \mathbb{P}(1, \chi_2/\chi_1, \dots, \chi_n/\chi_1)$; hence we obtain an inductive decomposition of $\mathbb{P}(\chi)$ into $n + 1$ cells $*$, \mathbb{C} , \mathbb{C}^2 , \dots , \mathbb{C}^n .

LEMMA 3.3. *There is an isomorphism $H^{2n-1}(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}) \cong \mathbb{Z}_q$.*

Proof. Set $X = (\mathbb{C}^\times)^{|J|-1}$, $Y = \mathbb{C}^I / \mathbb{Z}_q \langle \chi_I \rangle$ and $m = |I| - 1$. Note that X is a manifold of dimension $2(n - m - 1)$, so that $H^*(X, X \setminus \{\tilde{z}\})$ is isomorphic to \mathbb{Z} in dimension $2(n - m - 1)$ and zero otherwise. Excision, Lemma 3.1 and the Künneth formula for relative cohomology therefore imply

$$\begin{aligned} H^*(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}) &\cong H^*(U_I, U_I \setminus \{z\}) \\ &\cong H^*(X \times Y, (X \setminus \{\tilde{z}\}) \times Y \cup X \times (Y \setminus \{0\})) \\ &\cong H^*(X, X \setminus \{\tilde{z}\}) \otimes H^*(Y, Y \setminus \{0\}), \end{aligned}$$

because $H^*(X, X \setminus \{\tilde{z}\})$ is free. In particular,

$$H^{2n-1}(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\}) \cong H^{2m+1}(Y, Y \setminus \{0\}) \cong \tilde{H}^{2m}(L(q; \chi_I)).$$

If $m = 0$, then $q = 1$ because χ is normalized, and the claim holds. Otherwise, Theorem 2.3 gives $H^{2m}(L(q; \chi_I)) \cong \mathbb{Z}_q^p$, where the p -content

of q' is given by

$$(3.1) \quad p\text{-content of } \frac{l_m(\chi_I, q)}{l_m(\chi_I)} = \prod_{i=1}^m \frac{r_{m+2-i}(\chi_I, q; p)}{r_{m+1-i}(\chi_I; p)}.$$

We have to show $q' = q$, which means that q' and q have the same p -content for all p . This is clearly true if q is not divisible by p . Otherwise, χ_I inherits from the normalized weight vector χ two weights not divisible by p . (Recall that q is the gcd of the weights appearing in χ , but not in χ_I .) Hence, $r_1(\chi; p) = 1$, and the numerator of (3.1) differs from the denominator by the p -content of q . ■

Proof of Theorem 1.1. By the remarks preceding the theorem, we only have to prove the implication (3) \Rightarrow (1). In order to do so, we show how to read off the normalized weights from topological invariants of a weighted projective space $\mathbb{P}(\chi)$. For $z \in \mathbb{P}(\chi)$, let $q'(z)$ be the order of the finite group $H^{2n-1}(\mathbb{P}(\chi), \mathbb{P}(\chi) \setminus \{z\})$. Lemma 3.3 implies that for all $d \geq 1$ the space

$$X(d) = \{z \in \mathbb{P}(\chi) : d \mid q'(z)\}$$

is again a weighted projective space or empty. In fact,

$$X(d) = \{[z_0 : \cdots : z_n] \in \mathbb{P}(\chi) : z_i = 0 \text{ if } d \nmid \chi_i\}$$

because d divides $q'(z) = q$ if and only if it divides χ_i for all i such that $z_i \neq 0$. For each d , the dimension of $X(d)$ (which equals the degree of the highest non-vanishing cohomology group) therefore tells us the number of weights divisible by d . This determines the normalized weights completely up to order. ■

4. The Mislin genus. This section relies heavily on the theory of localization and homotopy pullbacks. We refer the reader to [7], especially Chapter II, and to [14, Chap. 7], for background information.

Throughout the section, X, Y , and Z denote simply connected CW complexes. A map $f: X \rightarrow Y$ is therefore a homotopy equivalence (written $X \simeq Y$) if and only if it induces an isomorphism $H_*(f)$ of integral homology; in this case, f^{-1} denotes a homotopy inverse for f .

Given any set \mathcal{P} of primes, the algebraic localization of \mathbb{Z} is denoted by $\mathbb{Z}_{\mathcal{P}}$, and the homotopy-theoretic localization of X by $X_{\mathcal{P}}$; the latter is also a CW complex. Every X admits a localization map $l_{\mathcal{P}}: X \rightarrow X_{\mathcal{P}}$, which induces an isomorphism $H_*(l_{\mathcal{P}}; \mathbb{Z}_{\mathcal{P}})$, and every f admits a localization

$f_{\mathcal{P}}: X_{\mathcal{P}} \rightarrow Y_{\mathcal{P}}$, for which the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ l_{\mathcal{P}} \downarrow & & \downarrow l_{\mathcal{P}} \\ X_{\mathcal{P}} & \xrightarrow{f_{\mathcal{P}}} & Y_{\mathcal{P}} \end{array}$$

is homotopy commutative. Any map $g: X_{\mathcal{P}} \rightarrow Y_{\mathcal{P}}$ of localized spaces is a homotopy equivalence if and only if it induces an isomorphism $H_*(g; \mathbb{Z}_{\mathcal{P}})$.

If \mathcal{P} is empty, then the localization map l_{\emptyset} is *rationalization*, and is denoted by $l_0: X \rightarrow X_0$; likewise, f_{\emptyset} is denoted by $f_0: X_0 \rightarrow Y_0$. If \mathcal{P} consists of a single prime p , then the localization map $l_{\{p\}}$ is abbreviated to $l_p: X \rightarrow X_p$, and $f_{\{p\}}$ is abbreviated to $f_p: X_p \rightarrow Y_p$. If \mathcal{P} contains all primes, then $l_{\mathcal{P}}$ is a homotopy equivalence.

The *homotopy pullback* of a diagram $X \rightarrow Z \leftarrow Y$ may be constructed by replacing either map with a fibration, and pulling it back along the other in the standard fashion. The resulting square is unique up to equivalence of diagrams in the homotopy category [14, §7.3]; in particular, the homotopy pullback is well-defined up to homotopy equivalence.

For any set \mathcal{P} of primes, the rationalization of $X_{\mathcal{P}}$ is homotopy equivalent to X_0 , so the rationalization map may be expressed as $l_0: X_{\mathcal{P}} \rightarrow X_0$. If \mathcal{P} and \mathcal{Q} are disjoint, then the homotopy pullback of

$$(4.1) \quad X_{\mathcal{Q}} \xrightarrow{l_0} X_0 \xleftarrow{l_0} X_{\mathcal{P}}$$

is $X_{\mathcal{P} \cup \mathcal{Q}}$ (see [12, Prop. 2.9.3] or [7, proof of Thm. 7.13]).

LEMMA 4.1. *Given two disjoint sets \mathcal{P} and \mathcal{Q} of primes, let $f: Y_{\mathcal{P}} \rightarrow Z_{\mathcal{P}}$ and $g: Y_{\mathcal{Q}} \rightarrow Z_{\mathcal{Q}}$ be homotopy equivalences, and define $h = f_0 g_0^{-1}$; then $Y_{\mathcal{P} \cup \mathcal{Q}}$ is the homotopy pullback of the diagram*

$$(4.2) \quad Z_{\mathcal{Q}} \xrightarrow{hl_0} Z_0 \xleftarrow{l_0} Z_{\mathcal{P}}.$$

If there also exist homotopy equivalences $d: Z_{\mathcal{P}} \rightarrow Z_{\mathcal{P}}$ and $e: Z_{\mathcal{Q}} \rightarrow Z_{\mathcal{Q}}$ such that $h \simeq d_0 e_0^{-1}$, then $Y_{\mathcal{P} \cup \mathcal{Q}}$ and $Z_{\mathcal{P} \cup \mathcal{Q}}$ are homotopy equivalent.

Proof. The vertical maps in the homotopy commutative ladder

$$(4.3) \quad \begin{array}{ccccc} Y_{\mathcal{Q}} & \xrightarrow{l_0} & Y_0 & \xleftarrow{l_0} & Y_{\mathcal{P}} \\ g \downarrow & & \downarrow f_0 & & \downarrow f \\ Z_{\mathcal{Q}} & \xrightarrow{hl_0} & Z_0 & \xleftarrow{l_0} & Z_{\mathcal{P}} \end{array}$$

are homotopy equivalences, and the homotopy pullback of the upper row is $Y_{\mathcal{P} \cup \mathcal{Q}}$, by analogy with (4.1). So the ladder induces a homotopy equivalence of homotopy pullbacks following [14, §7.3], and the first claim follows.

Substituting $Y = Z$, $f = d$ and $g = e$ into (4.3) creates an upper row with homotopy pullback $Z_{\mathcal{P} \cup \mathcal{Q}}$. The second claim is then immediate. ■

The following proposition gives criteria for ensuring that the genus of a finite CW complex is rigid.

PROPOSITION 4.2. *Let Z be a simply connected finite CW complex satisfying*

- (i) *for any space Y in the Mislin genus of Z , there exists a rational homotopy equivalence $k: Y \rightarrow Z$, and*
- (ii) *for any disjoint sets \mathcal{P} and \mathcal{Q} of primes, and any rational homotopy equivalence $h: Z_0 \rightarrow Z_0$, there exist homotopy equivalences $d: Z_{\mathcal{P}} \rightarrow Z_{\mathcal{P}}$ and $e: Z_{\mathcal{Q}} \rightarrow Z_{\mathcal{Q}}$ such that $h \simeq d_0 e_0^{-1}$.*

Then the genus of Z is rigid.

Proof. Let Y belong to the Mislin genus of Z . It follows from [7, p. 105] that there is an isomorphism $H_*(Y; \mathbb{Z}) \cong H_*(Z; \mathbb{Z})$ of graded abelian groups. Since $H_*(k; \mathbb{Q})$ is an isomorphism, there exists a maximal set \mathcal{Q} of primes for which $H_*(k_{\mathcal{Q}})$ is also an isomorphism. Since $H_*(Y; \mathbb{Z})$ and $H_*(Z; \mathbb{Z})$ are finitely generated in each dimension and vanish in large dimensions, the complement $\mathcal{P} = \{p_1, \dots, p_s\}$ of \mathcal{Q} is finite (and possibly empty). Define $\mathcal{Q}_i = \mathcal{Q} \cup \{p_1, \dots, p_i\}$ for $0 \leq i \leq s$. Since \mathcal{Q}_s contains *all* primes, it suffices to show that $Y_{\mathcal{Q}_s} \simeq Z_{\mathcal{Q}_s}$.

In fact we prove that $Y_{\mathcal{Q}_i} \simeq Z_{\mathcal{Q}_i}$ for every $0 \leq i \leq s$, using induction on i . The base case $i = 0$ holds because $H_*(k_{\mathcal{Q}})$ is an isomorphism, so $k_{\mathcal{Q}}$ is a homotopy equivalence. Now assume that $g: Y_{\mathcal{Q}_i} \rightarrow Z_{\mathcal{Q}_i}$ is a homotopy equivalence, and write $p = p_{i+1}$. By choice of Y , there is a homotopy equivalence $f: Y_p \rightarrow Z_p$, so we may apply the first claim of Lemma 4.1. This identifies $Y_{\mathcal{Q}_{i+1}}$ as the homotopy pullback of

$$Z_{\mathcal{Q}_i} \xrightarrow{hg_0} Z_0 \xleftarrow{l_0} Z_p,$$

where h is the homotopy equivalence $f_0 g_0^{-1}: Z_0 \rightarrow Z_0$. By assumption, there exist homotopy equivalences $d: Z_p \rightarrow Z_p$ and $e: Z_{\mathcal{Q}_i} \rightarrow Z_{\mathcal{Q}_i}$ such that $h \simeq d_0 e_0^{-1}$. The second claim of Lemma 4.1 then confirms that $Y_{\mathcal{Q}_{i+1}} \simeq Z_{\mathcal{Q}_{i+1}}$, and completes the inductive step. ■

5. Classification up to homotopy equivalence. Finally, we return to the case of weighted projective space.

In Theorem 2.1 we selected a generator ξ_1 for $H^2(\mathbb{P}(\chi)) \cong \mathbb{Z}$. Given any set \mathcal{P} of primes, its localization in $H^2(\mathbb{P}(\chi)_{\mathcal{P}}; \mathbb{Z}_{\mathcal{P}}) \cong \mathbb{Z}_{\mathcal{P}}$ must also be a generator. We therefore define the *degree* $\text{deg}(h)$ of any self-map h of $\mathbb{P}(\chi)_{\mathcal{P}}$ to be the \mathcal{P} -local integer satisfying $H^*(h; \mathbb{Z}_{\mathcal{P}})(\xi_1) = \text{deg}(h) \xi_1$; this

determines a multiplicative function

$$(5.1) \quad \text{deg}: [\mathbb{P}(\chi)_{\mathcal{P}}, \mathbb{P}(\chi)_{\mathcal{P}}] \rightarrow \mathbb{Z}_{\mathcal{P}}.$$

Remark 2.2 shows that any such h is a homotopy equivalence if and only if $H^*(h; \mathbb{Z}_{\mathcal{P}})$ is an isomorphism.

PROPOSITION 5.1.

- (1) *A self-map of $\mathbb{P}(\chi)_{\mathcal{P}}$ is a homotopy equivalence if and only if its degree is a unit in $\mathbb{Z}_{\mathcal{P}}$.*
- (2) *The degree function (5.1) is surjective.*
- (3) *If \mathcal{P} contains no divisor of any χ_j , then the degree function is a bijection.*

Proof. Since deg is multiplicative and the degree of the identity map is 1, deg maps homotopy equivalences to units. Let h be any self-map of $\mathbb{P}(\chi)_{\mathcal{P}}$, and assume it has degree a . By Theorem 2.1, $H^*(h; \mathbb{Z}_{\mathcal{P}})$ induces multiplication by a^k on $H^{2k}(\mathbb{P}(\chi)_{\mathcal{P}}; \mathbb{Z}_{\mathcal{P}}) \cong \mathbb{Z}_{\mathcal{P}}$ for every $1 \leq k \leq n$. If a is a unit, then $H^*(h; \mathbb{Z}_{\mathcal{P}})$ is an isomorphism, so h is a homotopy equivalence. Thus (1) holds.

Fix a positive integer a , and define the self-map $m_a: \mathbb{P}(\chi) \rightarrow \mathbb{P}(\chi)$ by raising each homogeneous coordinate to the power a ; in particular, write $m'_a: \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{C}\mathbb{P}^n$ for the standard case. Thus m_a and m'_a commute with the map φ of (2.2), leading to the commutative diagram

$$\begin{array}{ccc} H^*(\mathbb{P}(\chi)) & \xrightarrow{H^*(\varphi)} & H^*(\mathbb{C}\mathbb{P}^n) \\ H^*(m_a) \downarrow & & \downarrow H^*(m'_a) \\ H^*(\mathbb{P}(\chi)) & \xrightarrow{H^*(\varphi)} & H^*(\mathbb{C}\mathbb{P}^n) \end{array}$$

Since $H^2(m'_a)$ is multiplication by a , it follows that $\text{deg}(m_a) = a$. But every element $c \in \mathbb{Z}_{\mathcal{P}}$ may be written as a quotient $c = b/a$ of integers, where a is a positive unit in $\mathbb{Z}_{\mathcal{P}}$. Then (2) follows from (1), combined with the observations that complex conjugation on a single coordinate has degree -1 , and constant self-maps have degree 0.

If \mathcal{P} contains no divisor of any weight, then $\varphi_{\mathcal{P}}: \mathbb{C}\mathbb{P}^n_{\mathcal{P}} \rightarrow \mathbb{P}(\chi)_{\mathcal{P}}$ is a homotopy equivalence by Theorem 2.1. To prove (3), it therefore suffices to consider maps $h_1, h_2: \mathbb{C}\mathbb{P}^n_{\mathcal{P}} \rightarrow \mathbb{C}\mathbb{P}^n_{\mathcal{P}}$ of equal degree; in other words, we may restrict attention to the special case $\mathbb{C}\mathbb{P}^n$. Since $\mathbb{C}\mathbb{P}^{\infty}_{\mathcal{P}}$ is an Eilenberg–MacLane space $K(\mathbb{Z}_{\mathcal{P}}, 2)$, the compositions of $i_{\mathcal{P}}: \mathbb{C}\mathbb{P}^n_{\mathcal{P}} \rightarrow \mathbb{C}\mathbb{P}^{\infty}_{\mathcal{P}}$ with h_1 and h_2 are homotopic. Moreover, $\mathbb{C}\mathbb{P}^n_{\mathcal{P}}$ is $2n$ -dimensional and its image is the $(2n + 1)$ -skeleton of $\mathbb{C}\mathbb{P}^{\infty}_{\mathcal{P}}$, so the homotopy corestricts to a homotopy $h_1 \simeq h_2$. Thus deg is injective, and (3) follows. ■

The special case $\mathbb{C}\mathbb{P}^n$ of part (3) is well-known [11, Thm. 2.2], but is stated there without proof.

To complete the proof of Theorem 1.3, it remains only to show that the criteria of Proposition 4.2 apply to $\mathbb{P}(\chi)$.

Proof of Theorem 1.3. Let Y be an element of the Mislin genus of $\mathbb{P}(\chi)$. Since $H_*(Y) \cong H_*(\mathbb{P}(\chi))$ as graded abelian groups, Y is homotopy equivalent to a CW complex of dimension $2n$, by [6, Prop. 4C.1]. Furthermore, $H^*(\mathbb{P}(\chi); \mathbb{Q})$ is multiplicatively generated by a single element of degree 2, so any of the homotopy equivalences $Y_p \simeq \mathbb{P}(\chi)_p$ induces the corresponding structure on $H^*(Y; \mathbb{Q})$. A multiplicative generator γ may be chosen to be integral in $H^2(Y; \mathbb{Q})$, because the Universal Coefficient Theorem confirms that $H^2(Y; \mathbb{Z}) \cong \mathbb{Z}$. Also, γ is represented by a map $j: Y \rightarrow \mathbb{C}\mathbb{P}^\infty \simeq K(\mathbb{Z}, 2)$, for which $H^2(j; \mathbb{Z})$ is an isomorphism. Up to homotopy, j factors through $\mathbb{C}\mathbb{P}^n \subset \mathbb{C}\mathbb{P}^\infty$, so its corestriction $j': Y \rightarrow \mathbb{C}\mathbb{P}^n$ is a rational homotopy equivalence. Since $\varphi: \mathbb{C}\mathbb{P}^n \rightarrow \mathbb{P}(\chi)$ is a rational homotopy equivalence by Theorem 2.1, the same holds for the composition $\varphi j': Y \rightarrow \mathbb{P}(\chi)$. Criterion (i) of Proposition 4.2 is therefore satisfied by $k = \varphi j'$.

Now let $h: \mathbb{P}(\chi)_0 \rightarrow \mathbb{P}(\chi)_0$ be a homotopy equivalence, let $\deg(h) = \pm a/b$ where $a, b \in \mathbb{N}$, and let \mathcal{P} and \mathcal{Q} be two disjoint sets of primes. Write $a = a' a''$ and $b = b' b''$, where a', b' are divisible only by primes not contained in \mathcal{P} , and a'', b'' are divisible only by primes contained in \mathcal{P} . Then $a'/b' \in \mathbb{Z}_{\mathcal{P}}$ and $b''/a'' \in \mathbb{Z}_{\mathcal{Q}}$ are units. So Proposition 5.1(2) guarantees the existence of homotopy equivalences $d: \mathbb{P}(\chi)_{\mathcal{P}} \rightarrow \mathbb{P}(\chi)_{\mathcal{P}}$ and $e: \mathbb{P}(\chi)_{\mathcal{Q}} \rightarrow \mathbb{P}(\chi)_{\mathcal{Q}}$ of degrees $\pm a'/b'$ and b''/a'' respectively, and $h \simeq d_0 e_0^{-1}$ by Proposition 5.1(3). Criterion (ii) of Proposition 4.2 is therefore satisfied, as required. ■

Proof of Theorem 1.2. If χ and χ' have the same p -content up to order, then some permutation of homogeneous coordinates defines a homeomorphism $\mathbb{P}(\chi) \cong \mathbb{P}(\chi')$ for each prime p . This homeomorphism may be localized at p .

Now consider the map

$$g: \mathbb{P}(\chi) \rightarrow \mathbb{P}(\chi), \quad [z_0 : \cdots : z_n] \mapsto [z_0^{\alpha(0)} : \cdots : z_n^{\alpha(n)}],$$

where $\alpha(j) = \chi_j/p\chi_j$ for $0 \leq j \leq n$. Theorem 2.1 implies that $H^*(g; \mathbb{Z}_p)$ is an isomorphism, and Remark 2.2 confirms that g_p is a homotopy equivalence. So g_p^{-1} and g'_p determine a chain of maps

$$\mathbb{P}(\chi)_p \simeq \mathbb{P}(\chi)_p \cong \mathbb{P}(\chi)_p \simeq \mathbb{P}(\chi')_p$$

for any prime p , and the result follows from Theorem 1.3. ■

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Anthony Bahri
 Department of Mathematics
 Rider University
 Lawrenceville, NJ 08648, U.S.A.
 E-mail: bahri@rider.edu

Matthias Franz
 Department of Mathematics
 University of Western Ontario
 London, ON N6A 5B7, Canada
 E-mail: mfranz@uwo.ca

Dietrich Notbohm
 Department of Mathematics
 Vrije Universiteit Amsterdam
 De Boelelaan 1081a
 1081 HV Amsterdam, The Netherlands
 E-mail: notbohm@few.vu.nl

Nigel Ray
 School of Mathematics
 University of Manchester
 Oxford Road
 Manchester M13 9PL, United Kingdom
 E-mail: nigel.ray@manchester.ac.uk

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