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Covering the plane with sprays

by

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Abstract. For any three noncollinear points $c_0, c_1, c_2 \in \mathbb{R}^2$, there are sprays S_0, S_1, S_2 centered at c_0, c_1, c_2 that cover $\mathbb{R}^2$. This improves the result of de la Vega in which c_0, c_1, c_2 were required to be the vertices of an equilateral triangle.

Given a point c in the plane $\mathbb{R}^2$ and a subset $S \subseteq \mathbb{R}^2$, we say (following [2]) that S is a *spray centered at* c if whenever $C \subseteq \mathbb{R}^2$ is a circle centered at c, then $S \cap C$ is finite. It was noted in [2] that if $2^{\aleph_0} \le \aleph_{n-2}$, then the plane can be covered by n sprays. In fact, if $c_0, c_1, \ldots, c_{n-1} \in \mathbb{R}^2$ are n distinct points and $2^{\aleph_0} \le \aleph_{n-2}$, then the plane can be covered by n sprays centered at $c_0, c_1, \ldots, c_{n-1}$. De la Vega [3] proved that if $c_0, c_1, c_2 \in \mathbb{R}^2$ are distinct, collinear points, then the Continuum Hypothesis (CH) is equivalent to the existence of a covering of the plane by three sprays centered at c_0, c_1, c_2. (See Theorem 7 for an extension of this result to n collinear points.) On the other hand, without using any additional set-theoretic hypothesis, he proved the very interesting result that the plane can be covered by three sprays whose centers are the vertices of an equilateral triangle. In the proof of this he used a computer algebra system (such as Maple) to show that a certain fourth degree polynomial in two variables is irreducible over $\mathbb{C}$. This proved to be an obstacle to extending his proof to arbitrary triangles. To quote from [3]:

> We suspect that the same remains true for any triangle (as long as the c_i's don't lie on the same line) and we have checked a couple of examples, but we have not found a reasonable way to prove it simultaneously because in one step of the proof we require a computer algebra system to check for the irreducibility of a certain polynomial.

The main purpose of this note is to confirm this suspicion by presenting a proof of the following theorem. This proof is self-contained without reliance on a computer algebra system.

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Theorem 1. *If $c_0, c_1, c_2 \in \mathbb{R}^2$ are any three noncollinear points, then there are three sprays $S_0, S_1, S_2 \subseteq \mathbb{R}^2$ centered at c_0, c_1, c_2, respectively, that cover $\mathbb{R}^2$.*

We make use of some terminology from [3]. Let X be an arbitrary set, and let E_0, E_1, E_2 be equivalence relations on X. If $x \in X$ and $i < 3$, then $[x]_i$ is the equivalence class of E_i to which x belongs. A coloring $\chi : X \to 3 = \{0, 1, 2\}$ is *acceptable* for $\langle E_0, E_1, E_2\rangle$ if whenever $x \in X$ and $i < 3$, then $[x]_i \cap \chi^{-1}(i)$ is finite. De la Vega [3] gave a characterization of those $\langle E_0, E_1, E_2\rangle$ for which there is an acceptable coloring.

Definition 2 (de la Vega [3, Def. 3.1]). Let E_0, E_1, E_2 be equivalence relations on a set X. Then $\langle E_0, E_1, E_2\rangle$ is *twisted* if, whenever M, N are elementary substructures of the universe, $\{i, j, k\} = \{0, 1, 2\}$, $a \in X$, $E_0, E_1, E_2 \in M \cap N$ and $N \in M$, then the set

$$\{x \in [a]_i : [x]_j \in M \setminus N,\ [x]_k \in N \setminus M\}$$

is finite.

Since the phrase *elementary substructure of the universe* is not formalizable, we must reinterpret it in some formalizable way. One way would be to consider a sufficiently large yet specific $n < \omega$, and then agree that *elementary* really means *Σ_n-elementary*. Alternatively, and the way that it is done in [3], is to agree that *the universe* is the set $H(\theta)$ of sets of hereditary cardinality less than θ, where $\theta = (2^{|X|+\aleph_0})^+$ (1).

Theorem 3 (de la Vega [3, Th. 3.8]). *Suppose E_0, E_1, E_2 are equivalence relations on X. Then the following are equivalent*:

(1) *$\langle E_0, E_1, E_2\rangle$ is twisted.*
(2) *There is an acceptable coloring for $\langle E_0, E_1, E_2\rangle$.*

It should be remarked that the statement of Theorem 3 in [3] required that $X = \mathbb{R}^2$, but the proof there applies equally to any X.

Consider points $c_0, c_1, c_2 \in \mathbb{R}^2$. Let $X = \mathbb{R}^2$, and then let E_i be the equivalence relation on $\mathbb{R}^2$ such that $[x]_i$ is a circle centered at c_i. That is, $y \in [x]_i$ iff $\|y - c_i\|^2 = \|x - c_i\|^2$. Clearly, there is a coloring of $\mathbb{R}^2$ acceptable for $\langle E_0, E_1, E_2\rangle$ iff there are sprays centered at c_0, c_1, c_2 covering $\mathbb{R}^2$. Thus, to prove Theorem 1, it would suffice to prove that if c_0, c_1, c_2 are noncollinear, then $\langle E_0, E_1, E_2\rangle$ is twisted. But, instead, we will consider a slightly different, but very closely related triple of equivalence relations.

(1) The most common definition in the literature of the hereditary cardinality of a set x seems to be the cardinality of its transitive closure $\mathrm{TC}(x)$. Sometimes, although less frequently, the phrase "x has hereditary cardinality less than θ" is rendered as "$|x| < \theta$ and, for all $y \in \mathrm{TC}(x)$, $|y| < \theta$". For us, where θ is regular, these two definitions are equivalent.

Now suppose $c_0, c_1, c_2 \in \mathbb{R}^2$ are distinct points. It is obvious that if $r_0, r_1, r_2 \in \mathbb{R}$ are such that there is a point $x \in \mathbb{R}^2$ with r_i being the square of the radius of the circle $[x]_i$, then $r_0, r_1, r_2 \geq 0$ and $\sqrt{r_i} + \sqrt{r_j} \geq \|c_i - c_j\|$ whenever $i < j < 3$. Conversely, we can get a quadratic polynomial $q(u, v, w) \in \mathbb{R}[u, v, w]$ such that if $r_0, r_1, r_2 \in \mathbb{R}$ satisfy these obvious conditions, then $q(r_0, r_1, r_2) = 0$ iff there is a point $x \in \mathbb{R}^2$ such that r_i is the square of the radius of the circle $[x]_i$. If we let

$$d_0 = \|c_1 - c_2\|^2, \quad d_1 = \|c_2 - c_0\|^2, \quad d_2 = \|c_0 - c_1\|^2, \tag{1}$$

then

$$\begin{aligned} q(u, v, w) = {} & d_0 u^2 + d_1 v^2 + d_2 w^2 + (d_0 - d_1 - d_2) vw \\ & + (d_1 - d_2 - d_0) uw + (d_2 - d_0 - d_1) uv \\ & + (d_0 - d_1 - d_2) d_0 u + (d_1 - d_2 - d_0) d_1 v \\ & + (d_2 - d_0 - d_1) d_2 w + d_0 d_1 d_2 \end{aligned} \tag{2}$$

is such a polynomial ([2]). For example, in [3], c_0, c_1, c_2 were chosen to be the vertices of a carefully selected equilateral triangle with sides of length 2, and then the polynomial

$$u^2 + v^2 + w^2 - vw - uw - uv - 4u - 4v - 4w + 16$$

was obtained.

Now, suppose that $\mathbb{F}$ is any field and that $p(u, v, w) \in \mathbb{F}[u, v, w]$. Let $X \subseteq \mathbb{F}^3$ be its zero-set, and for each $i < 3$, define the equivalence relation E_i on X by: if $\langle a_0, a_1, a_2 \rangle, \langle b_0, b_1, b_2 \rangle \in X$, then

$$\langle a_0, a_1, a_2 \rangle E_i \langle b_0, b_1, b_2 \rangle \ \Leftrightarrow \ a_i = b_i. \tag{3}$$

Then we say that the coloring $\chi : X \to 3$ is *acceptable* for $p(u, v, w)$ if it is acceptable for $\langle E_0, E_1, E_2 \rangle$, and we say that $p(u, v, w)$ is *twisted* if $\langle E_0, E_1, E_2 \rangle$ is twisted.

Clearly, if $c_0, c_1, c_2 \in \mathbb{R}^2$, $q(u, v, w) \in \mathbb{R}[u, v, w]$ is as in (1) and (2), and there is an acceptable coloring for $q(u, v, w)$, then $\mathbb{R}^2$ can be covered by three sprays centered at c_0, c_1, c_2. Furthermore, $\mathbb{R}^2$ can be covered by three sprays centered at c_0, c_1, c_2 iff $q(u, v, w)$ is twisted. Thus, the following lemma suffices to prove Theorem 1.

LEMMA 4. *Suppose $c_0, c_1, c_2 \in \mathbb{R}^2$ are noncollinear and $q(u, v, w) \in \mathbb{R}[u, v, w]$ is as in* (1) *and* (2). *Then $q(u, v, w)$ is twisted.*

([2]) Suppose we consider c_0, c_1, c_2 to be in $\mathbb{R}^3$ rather than $\mathbb{R}^2$, but still define d_0, d_1, d_2 as in (1). Then let $c_3 \in \mathbb{R}^3$ and let $u = \|c_3 - c_0\|^2$, $v = \|c_3 - c_1\|^2$ and $w = \|c_3 - c_2\|^2$. Let V be the volume of the tetrahedron whose vertices are c_0, c_1, c_2, c_3. Then $q(u, v, w) = -144V^2$. This can be verified using the 5×5 Cayley–Menger determinant $D = 288V^2$, which is Tartaglia's 3-dimensional generalization of Heron's formula for the area of a triangle.

It will be convenient to work in the complex field $\mathbb{C}$ rather than $\mathbb{R}$. We will consider the polynomial $q(u,v,w)$ to be over $\mathbb{C}$, and then prove that even then it is twisted. Clearly, this implies that it is twisted as a polynomial over $\mathbb{R}$ (or use Theorem 3). Thus, our goal is to prove the following lemma.

LEMMA 5. *Suppose $c_0, c_1, c_2 \in \mathbb{R}^2$ are noncollinear and $q(u,v,w) \in \mathbb{C}[u,v,w]$ is as in* (1) *and* (2). *Then $q(u,v,w)$ is twisted.*

Most of the rest of this paper is devoted to proving Lemma 5.

LEMMA 5.1. *Let $c_0, c_1, c_2 \in \mathbb{C}^2$ and assume that* (1) *and* (2) *hold. The following are equivalent*:

(a) *c_0, c_1, c_2 are collinear.*
(b) *$q(u,v,w)$ is reducible* (*or is the zero polynomial*).
(c) *$q(u,v,w)$ is the square of a polynomial.*
(d) *$d_0^2 + d_1^2 + d_2^2 = 2d_1d_2 + 2d_0d_1 + 2d_0d_1$.*

Proof. (a)$\Leftrightarrow$(c): Let $s_0, s_1, s_2 \in \mathbb{C}$ be such that $s_i^2 = d_i$ for $i < 3$. Suppose that c_0, c_1, c_2 are collinear, so we can assume that $s_0 + s_1 + s_2 = 0$, in which case $q(u,v,w) = (s_0u + s_1v + s_2w + s_0s_1s_2)^2$. Conversely, suppose that $q(u,v,w)$ is a square, so we can assume that $q(u,v,w) = (s_0u + s_1v + s_2w \pm s_0s_1s_2)^2$. Looking at the coefficients of vw, uw, uv, we get

$$\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} d_0 \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2s_1s_2 \\ 2s_0s_2 \\ 2s_0s_1 \end{pmatrix}.$$

Thus, in terms of s_0, s_1, s_2, there are unique d_0, d_1, d_2, and we have seen that $d_0 = s_0^2, d_1 = s_1^2, d_2 = s_2^2$ works. Thus, $s_0^2 - s_1^2 - s_2^2 = 2s_1s_2$, so $s_0^2 = (s_1 + s_2)^2$, and then either $s_0 - s_1 - s_2 = 0$ or $s_0 + s_1 + s_2 = 0$, implying that c_0, c_1, c_2 are collinear.

(c)$\Leftrightarrow$(b): The zero-set X can be parametrized by $X = \{(\|c_0 - x\|^2, \|c_1 - x\|^2, \|c_2 - x\|^2) \in \mathbb{C}^3 : x \in \mathbb{C}^2\}$, so it is a variety. Then, since the polynomial $q(u,v,w)$ is at most a quadratic, it is either irreducible or the square of a polynomial.

(a)$\Leftrightarrow$(d): This is straightforward by algebraic calculations. ∎

From now on until the end of the proof of Lemma 5, **we fix noncollinear $c_0, c_1, c_2 \in \mathbb{R}^2$ and assume that** (1) **and** (2) **hold.** However, now we will consider $q(u,v,w) \in \mathbb{C}[u,v,w]$. Notice that $d_0d_1d_2 \neq 0$. Actually, the following proof will work for all noncollinear $c_0, c_1, c_2 \in \mathbb{C}^2$ with $d_0d_1d_2 \neq 0$. If we are willing to handle some additional details, we could extend the proof to all noncollinear $c_0, c_1, c_2 \in \mathbb{C}^2$ allowing the possibility that $d_0d_1d_2 = 0$.

By Lemma 5.1, $q(u,v,w)$ is irreducible, but the next lemma shows that even more is true.

LEMMA 5.2. *Let* $w_0 \in \mathbb{C}$. *Then* $q(u, v, w_0) \in \mathbb{C}[u, v]$ *is irreducible iff* $w_0 \neq 0$.

To make the statement a little simpler, we have stated only one-third of the truth in the previous lemma, since either of the other two variables could instead be fixed. By symmetry, the other two-thirds easily follows. It should be understood, here and elsewhere, that the variables may be permuted.

Proof. Without loss of generality, assume that $c_2 = (0, 0)$. Since $d_0 \neq 0$, $q(u, v, w_0)$ is a nonlinear quadratic polynomial. Let $X \subseteq \mathbb{C}^2$ be the zero-set of $q(u, v, w_0)$. Then X can be parametrized by $X = \{(\|c_0 - x\|^2, \|c_1 - x\|^2) : \|x\|^2 = w_0\}$.

Suppose $w_0 = 0$. Then $\{x \in \mathbb{C}^2 : \|x\|^2 = 0\}$ is the union of the two lines $\ell_0 = \{(x, ix) \in \mathbb{C}^2 : x \in \mathbb{C}\}$ and $\ell_1 = \{(x, -ix) \in \mathbb{C}^2 : x \in \mathbb{C}\}$. Then $X = X_0 \cup X_1$, where $X_e = \{(\|c_0 - x\|^2, \|c_1 - x\|^2) \in \mathbb{C}^2 : x \in \ell_e\}$ for $e = 0, 1$, so that $q(u, v, w_0)$ is reducible.

Suppose $w_0 \neq 0$. Then $\{x \in \mathbb{C}^2 : \|x\|^2 = w_0\}$ is a variety, so also X is. Thus, either $q(u, v, w_0)$ is irreducible or is the square of a linear polynomial. Suppose that $q(u, v, w_0)$ is not irreducible and so $q(u, v, w_0) = (au + bv + c)^2$. Then $a^2 = d_0$ and $b^2 = d_1$. Also, $2ab$ is the coefficient of uv in $q(u, v, w_0)$, and a calculation shows that this implies that the equality of Lemma 5.1(d) holds, contradicting that c_0, c_1, c_2 are not collinear. ∎

COROLLARY 5.3. *For every* $w_0 \in \mathbb{C}$, *the equation*

$$q(u, v, w_0) = \frac{\partial q}{\partial u}(u, v, w_0) = 0$$

has at least one and at most finitely many solutions (u, v).

Proof. Since $d_0 \neq 0$, it follows that $\frac{\partial q}{\partial u}(u, v, w_0)$ is not a constant polynomial, so u is a linear function of v. Plugging that into $q(u, v, w_0)$ results in a quadratic polynomial in which the coefficient of v^2 is nonzero (because of Lemma 5.1(d)). Thus, there is at least one solution (u, v) and there are at most two. ∎

LEMMA 5.4. *If* $u_0, v_0 \in \mathbb{C}$, *then* $q(u_0, v_0, w) = 0$ *has at most finitely many solutions.*

Proof. This is immediate since $d_2 \neq 0$. ∎

LEMMA 5.5. *If* $0 \neq w_0 \in \mathbb{C}$, *then the equation*

$$q(u, v, w_0) = \frac{\partial q}{\partial u}(u, v, w_0) = \frac{\partial q}{\partial v}(u, v, w_0) = 0$$

has no solution (u, v).

Proof. If $p(u, v) \in \mathbb{C}[u, v]$ is a nonlinear quadratic polynomial such that

$$p(u, v) = \frac{\partial p}{\partial u}(u, v) = \frac{\partial p}{\partial v}(u, v) = 0$$

has a solution, then $p(u,v)$ is reducible. For, without loss of generality, assume that $(0,0)$ is a solution so that $p(u,v) = au^2 + buv + cv^2$, which is reducible. For any $w_0 \in \mathbb{C}$, it is clear that $q(u,v,w_0)$ is a nonlinear quadratic polynomial. Thus, if the equation in the lemma has a solution, then $q(u,v,w_0)$ is reducible. By Lemma 5.2, this happens only if $w_0 = 0$. ∎

The next lemma is the main algebraic fact used in the proof of Lemma 5.

LEMMA 5.6. *Suppose that $w_0, w_1, w_2, w_3 \in \mathbb{C}$ are such that any three of them are algebraically independent over the coefficients of $q(u,v,w)$. Then the equation*

$$q(u,v,w_0) = q(u,v',w_1) = q(u',v',w_2) = q(u',v,w_3) = 0$$

has at most finitely many solutions $(u,u',v,v') \in \mathbb{C}^4$.

Proof. It suffices to assume that all coefficients of $q(u,v,w)$ are algebraic, so we will do so. [Alternatively, we could reinterpret each use of the term *algebraic* in this proof to mean *algebraic over the coefficients of* $q(u,v,w)$.] Suppose there are infinitely many solutions. It follows from Lemma 5.4 that there are infinitely many $u_0 \in \mathbb{C}$ for which there are $u_0', v_0, v_0' \in \mathbb{C}$ such that (u_0, u_0', v_0, v_0') is a solution. Then, for all but possibly finitely many $u_0 \in \mathbb{C}$ there are u_0', v_0, v_0' such that (u_0, u_0', v_0, v_0') is a solution. Since $d_0 \neq 0 \neq d_1$, it follows that for every bounded $A \subseteq \mathbb{C}$, there is a bounded $B \subseteq \mathbb{C}$ such that whenever (u_0, u_0', v_0, v_0') is a solution and $u_0 \in A$, then $u_0, u_0', v_0, v_0' \in B$. Thus, the set of all u_0 that extend to a solution is a closed subset of $\mathbb{C}$, and therefore all of $\mathbb{C}$. Moreover, since $q(u,v,w_0)$ is irreducible, whenever $q(u_0,v_0,w_0) = 0$, then there are u_0', v_0' such that (u_0, u_0', v_0, v_0') is a solution.

By Corollary 5.3, we can let u_0, v_0 be such that

$$q(u_0,v_0,w_0) = \frac{\partial q}{\partial u}(u_0,v_0,w_0) = 0, \tag{4}$$

and then obtain u_0', v_0' so that

$$q(u_0,v_0',w_1) = q(u_0',v_0',w_2) = q(u_0',v_0,w_3) = 0. \tag{5}$$

Corollary 5.3 and (4) imply that w_0 is algebraic over v_0, and then since w_0 is not algebraic, neither is v_0. Again by Corollary 5.3 and (4), u_0 and v_0 are algebraic over w_0.

From Lemma 5.4 and (5), w_3 is algebraic over $\{u_0', v_0\}$. Since v_0 is algebraic over w_0 and w_3 is not algebraic over w_0, it cannot be that u_0' is algebraic. Similarly, v_0' is not algebraic.

Since w_0 is not algebraic, we see from Lemma 5.5 and (4) that

$$\frac{\partial q}{\partial v}(u_0,v_0,w_0) \neq 0. \tag{6}$$

We deduce from Corollary 5.3 and (5) that

$$\frac{\partial q}{\partial u}(u_0', v_0, w_3) \neq 0, \tag{7}$$

as otherwise w_3 (and also u_0') would be algebraic over v_0 and thus algebraic over w_0. Also, v_0' is algebraic over $\{w_0, w_1\}$, and then since w_3 is not algebraic over $\{w_0, w_1\}$, Corollary 5.3 and (5) yield

$$\frac{\partial q}{\partial v}(u_0, v_0', w_1) \neq 0. \tag{8}$$

Similarly, since w_1 is not algebraic, Corollary 5.3 and (5) imply that

$$\frac{\partial q}{\partial u}(u_0, v_0', w_1) \neq 0, \tag{9}$$

and, likewise, since w_2 is not algebraic over $\{w_0, w_1\}$ and neither of u_0', v_0' is algebraic, Corollary 5.3 and (5) yield

$$\frac{\partial q}{\partial u}(u_0', v_0', w_2) \neq 0 \neq \frac{\partial q}{\partial v}(u_0', v_0', w_2). \tag{10}$$

We can now invoke the Implicit Function Theorem, using (4)–(10), to get neighborhoods U, V, V' of u_0, v_0, v_0' and analytic functions $f_0, f_1 : U \to \mathbb{C}$, $f_2 : V' \to \mathbb{C}$ and $f_3 : V \to \mathbb{C}$ such that

(a) $f_0(u_0) = v_0$, $f_1(u_0) = v_0'$, $f_2(v_0') = u_0'$ and $f_3(v_0) = u_0'$,
(b) $q(u, f_0(u), w_0) = q(u, f_1(u), w_1) = 0$ for all $u \in U$,
(c) $q(f_2(v'), v', w_2) = q(f_3(v), v, w_3) = 0$ for all $v' \in V'$ and $v \in V$,
(d) $f_0'(u_0) = 0$ and $f_1'(u_0) \neq 0 \neq f_2'(v_0')$,
(e) $f_3(f_0(u)) = f_2(f_1(u))$ for all $u \in U$.

Then (a), (e) and the Chain Rule imply that

$$f_3'(v_0)f_0'(u_0) = f_2'(v_0')f_1'(u_0),$$

contradicting (d). ∎

Proof of Lemma 5. Let $X \subseteq \mathbb{C}^3$ be the zero-set of $q(u, v, w)$, and let E_0, E_1, E_2 be the equivalence relations on X as defined in (3). Without loss of generality, in Definition 2, we will let $i = 1$, $j = 0$ and $k = 2$. Let $v_0 \in \mathbb{C}$, and then define

$$B = \{\langle u, w\rangle \in \mathbb{C}^2 : q(u, v_0, w) = 0,\ u \in M \setminus N,\ w \in N \setminus M\}.$$

Our goal is to show that B is finite, so, for a contradiction, assume that B is infinite. Let

$$A = \{w \in \mathbb{C} : \langle u, w\rangle \in B \text{ for some } u \in \mathbb{C}\}.$$

Then $A \subseteq N \setminus M$ and, by Lemma 5.4, A is infinite.

At this point, we would like to get $w_0, w_1 \in A$ that are algebraically independent over the coefficients of $q(u, v, w)$ so as then to make use of

Lemma 5.6. But that may not be possible. One way around this problem is as follows. Enumerate all polynomials whose coefficients are in the field generated by the coefficients of $q(u,v,w)$ and then say, for example, that $w_0, w_1 \in \mathbb{C}$ are *n-algebraically independent* if they are not the zeroes of any of the first n of the enumerated polynomials. Then there is some sufficiently large $n < \omega$ such that Lemma 5.6 is still true when we replace "algebraically independent over the coefficients of $q(u,v,w)$" with "n-algebraically independent". Then there is some $m < \omega$ such that it suffices to choose $w_0, w_1 \in A$ to be m-algebraically independent.

Another, and simpler, way is just take an ultrapower of the universe so that everything is $\aleph_0$-saturated. This is the approach we will take, so we assume that M and N are $\aleph_0$-saturated.

Since A is infinite and definable, by $\aleph_0$-saturation we can get $w_0, w_1 \in A$ which are algebraically independent over the coefficients of $q(u,v,w)$. Then let u_0, u_0' be such that $\langle u_0, w_0\rangle, \langle u_0', w_1\rangle \in B$. Notice that $u_0, u_0' \in M \setminus N$. Define $G \subseteq \mathbb{C}^2$ to be the set of all $\langle b_0, b_1\rangle$ such that:

- $\langle b_0, b_1\rangle \in N$ and $b_0 \neq b_1$;
- there is v such that $q(u_0, v, b_0) = q(u_0', v, b_1) = 0$;
- b_0, b_1 are algebraically independent over the coefficients of $q(u,v,w)$.

Since $\langle w_0, w_1\rangle \in G$, we see that $G \neq \emptyset$. By elementarity, there is $\langle b_0, b_1\rangle \in M$ satisfying the first two of the above conditions and also such that b_0, b_1 are n-algebraically independent over the coefficients of $q(u,v,w)$ for any $n < \omega$. Then, by $\aleph_0$-saturation, it follows that $G \cap M \neq \emptyset$, so let $\langle w_2, w_3\rangle \in G \cap M$. Clearly, $w_2 \neq w_3$, and also $\{w_2, w_3\} \cap \{w_0, w_1\} = \emptyset$ since $w_0, w_1 \in N \setminus M$ and $w_2, w_3 \in M$. In fact, each 3-element subset of $\{w_0, w_1, w_2, w_3\}$ is algebraically independent over the coefficients of $q(u,v,w)$. Let v_0' be such that $q(u_0, v_0', b_0) = q(u_0', v_0', b_1) = 0$. Then, by Lemma 5.6, the system of equations

$$q(u,v,w_0) = q(u',v,w_1) = q(u',v',w_2) = q(u,v',w_3) = 0$$

has only finitely many solutions (u, u', v, v'). Since $w_0, w_1, w_2, w_3 \in N$, all the solutions are in N. In particular, $(u_0, u_0', v_0, v_0') \in N$, so that $u_0 \in N$, which is a contradiction. ∎

Theorem 1, which deals with noncollinear points $c_0, c_1, c_2 \in \mathbb{R}^2$, is now proved. We next consider what happens with collinear points. The $n = 3$ case of Theorem 7 was proved by de la Vega [3] using Theorem 3. The proof of Theorem 7 that is given here is more direct.

We make the obvious generalizations of some of the definitions given right after Theorem 1. Suppose that X is an arbitrary set and $E_0, E_1, \dots, E_{n-1}$ are equivalence relations on X. If $x \in X$ and $i < n$, then $[x]_i$ is the equivalence class of E_i to which x belongs. A coloring $\chi : X \to n = \{0, 1, \dots, n-1\}$

is *acceptable* for $\langle E_0, E_1, \ldots, E_{n-1}\rangle$ if whenever $x \in X$ and $i < n$, then $[x]_i \cap \chi^{-1}(i)$ is finite.

LEMMA 6. *Suppose that $n < \omega$. Let $\ell_0, \ell_1, \ldots, \ell_{n+1} \subseteq \mathbb{R}^2$ be distinct lines passing through the origin, and then let E_i be the equivalence relation on $\mathbb{R}^2$ defined as follows: if $x \in \mathbb{R}^2$, then $[x]_i$ is the line containing x that is parallel (or equal) to ℓ_i. If there is an acceptable coloring of $[0,1]^2$ for $\langle E_0 \cap [0,1]^2, E_1 \cap [0,1]^2, \ldots, E_{n+1} \cap [0,1]^2\rangle$, then $2^{\aleph_0} \le \aleph_n$.*

A slightly weaker result was proved by Davies [1]. His result had the same conclusion but had the weaker hypothesis that there is an acceptable coloring of $\mathbb{R}^2$ for $\langle E_0, E_1, \ldots, E_{n+1}\rangle$. However, the proof in [1] can easily be applied to prove Lemma 6.

THEOREM 7. *Suppose that $n < \omega$ and $c_0, c_1, \ldots, c_{n+1} \in \mathbb{R}^2$ are $n+2$ distinct, collinear points. Then $2^{\aleph_0} \le \aleph_n$ iff $\mathbb{R}^2$ can be covered by $n+2$ sprays $S_0, S_1, \ldots, S_{n+1}$ centered at $c_0, c_1, \ldots, c_{n+1}$, respectively.*

Proof. As already noted, it was shown in [2] that if $2^{\aleph_0} \le \aleph_n$, then $\mathbb{R}^2$ can be covered by sprays centered at $c_0, c_1, \ldots, c_{n+1}$. We consider the converse, so let $c_0, c_1, \ldots, c_{n+1} \in \mathbb{R}^2$ be distinct, collinear points, and suppose that $S_0, S_1, \ldots, S_{n+1}$ are sprays centered at $c_0, c_1, \ldots, c_{n+1}$, respectively, covering $\mathbb{R}^2$. We might as well assume that $c_0, c_1, \ldots, c_{n+1}$ are all on the x-axis and that $c_i = (a_i, 0)$, where $0 = a_0 < a_1 < \cdots < a_{n+1}$. For $i, j \le n+1$, let $s_{ij} = |a_i - a_j| = \|c_i - c_j\|$. Let $X \subseteq \mathbb{R}^{n+2}$ be such that

$$X = \{(\|c_0 - x\|^2, \|c_1 - x\|^2, \ldots, \|c_{n+1} - x\|^2) \in \mathbb{R}^{n+2}) : x \in \mathbb{R}^2\}.$$

It easily follows, as in the proof of (a)$\Leftrightarrow$(c) of Lemma 5.1, that $(r_0, r_1, \ldots, r_{n+1}) \in X$ iff $r_i \ge 0$ whenever $i \le n+1$, $\sqrt{r}_i - \sqrt{r}_j \ge s_{ij}$ whenever $i < j \le n+1$, and $s_{ij} r_k + s_{jk} r_i - s_{ik} r_j = s_{ij} s_{jk} s_{ik}$ whenever $i < j < k \le n+1$. For this last equation, it suffices to restrict to $i = 0$ and $j = 1$, so the zero-set Z of these equations is a 2-dimensional plane in $\mathbb{R}^{n+2}$.

Let $\psi : X \to n+2$ be such that if $\psi(r_0, r_1, \ldots, r_{n-1}) = i$, then there is $x \in S_i$ such that

$$(r_0, r_1, \ldots, r_{n+1}) = (\|c_0 - x\|^2, \|c_1 - x\|^2, \ldots, \|c_{n+1} - x\|^2).$$

Then ψ has the property that if $i \le n+1$ and $H \subseteq \mathbb{R}^{n+2}$ is a hyperplane orthogonal to the ith coordinate axis, then $\psi^{-1}(i) \cap H$ is finite.

Let $f : \mathbb{R}^2 \to Z$ be an isometry such that $f([0,1]^2) \subseteq X$. Then there are distinct lines $\ell_0, \ell_1, \ldots, \ell_{n+1} \subseteq \mathbb{R}^2$ such that whenever $i \le n+1$ and $H \subseteq \mathbb{R}^{n+2}$ is a hyperplane orthogonal to the ith coordinate axis, then $f^{-1}(i) \cap H$ is parallel (or equal) to ℓ_i. Let E_i be defined as in Lemma 6. Then ψf is an acceptable coloring of $[0,1]^2$ for $\langle E_0 \cap [0,1]^2, E_1 \cap [0,1]^2, \ldots, E_{n+1} \cap [0,1]^2\rangle$. Lemma 6 then implies that $2^{\aleph_0} \le \aleph_n$. ∎

The proofs of Theorems 1 and 7 suggest a general problem. First, a definition. If $2 \le n < \omega$ and $X \subseteq \mathbb{R}^n$, then we say that a function $\chi : X \to n$ is *acceptable* if whenever $i < n$ and $H \subseteq \mathbb{R}^n$ is a hyperplane orthogonal to the ith coordinate axis, then $\chi^{-1}(i) \cap H$ is finite.

Problem. Assuming CH, determine those polynomials $p(x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}[x_0, x_1, \dots, x_{n-1}]$ whose zero-sets have acceptable colorings. Do the same assuming other restrictions on $2^{\aleph_0}$.

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