Ordinal indices and Ramsey dichotomies measuring $c_0$-content and semibounded completeness

by

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Abstract. We study the $c_0$-content of a seminormalized basic sequence $(\chi_n)$ in a Banach space, by the use of ordinal indices (taking values up to $\omega_1$) that determine dichotomies at every ordinal stage, based on the Ramsey-type principle for every countable ordinal, obtained earlier by the author. We introduce two such indices, the $c_0$-index $\xi_0^{(\chi_n)}$ and the semibounded completeness index $\xi_b^{(\chi_n)}$, and we examine their relationship. The countable ordinal values that these indices can take are always of the form $\omega^\xi$. These results extend, to the countable ordinal level, an earlier result by Odell, which was stated only for the limiting case of the first uncountable ordinal.

Introduction. In this paper we study the precise $c_0$-content of an arbitrary (seminormalized and basic) sequence $(\chi_n)$ in a Banach space, measured by the $c_0$-index $\xi_0^{(\chi_n)}$ defined for any such sequence. As this index is a countable ordinal of the form $\omega^\xi$ or equal to the first uncountable ordinal $\omega_1$, on the one hand we give dichotomy conditions, separating the basic classes $\xi_0^{(\chi_n)} = \omega_1$ and $\xi_0^{(\chi_n)} < \omega_1$, and on the other hand, we characterize the spectrum of the states precisely quantified by the countable ordinals.

The main tools, combinatorial in nature, consist of the Ramsey-type principle for every countable ordinal, proved in [F1], and of the Pták-type theorem for every countable ordinal, proved also in [F1]. In the statements of these theorems we make use of the complete thin Schreier system of families $(A_\xi)_{\xi \in \omega_1}$, introduced in [F1]. Closely connected with this system is the generalized Schreier system $(F_\alpha)_{\alpha < \omega_1}$, defined in [A-A], which is often used in the present paper.

In order to state our main results, we need the following definitions:

(i) $(\chi_n)$ has $c_0$-spreading model of order $\alpha$, for some $1 \leq \alpha < \omega_1$, if there exist $A, B > 0$ such that

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\[ A \max_{i \in F} |\lambda_i| \leq \left\| \sum_{i \in F} \lambda_i x_i \right\| \leq B \max_{i \in F} |\lambda_i| \quad \text{for all } F \in \mathcal{F}_\alpha \text{ and } (\lambda_i)_{i \in F} \subseteq \mathbb{R}. \]

(ii) \((\chi_n)\) is null coefficient of order \(\alpha\), for some \(1 \leq \alpha < \omega_1\), if every sequence \((\lambda_n)\) of real numbers with \(\sup\{\| \sum_{i \in F} \lambda_i x_i \| : F \in \mathcal{F}_\alpha \} < \infty\) converges to zero; \((\chi_n)\) is null coefficient if \(\mathcal{F}_\alpha\) can be replaced by the family \([\mathbb{N}]^{<\omega}\) of all finite subsets of \(\mathbb{N}\).

That these two properties of a sequence \((\chi_n)\) are naturally exclusive for every ordinal \(\alpha\), is the content of the following theorem (Theorem 2.15).

**Theorem A.** Let \((\chi_n)\) be a basic bounded sequence in a Banach space, with \(0 < \inf_n \| \chi_n \|\). Then either

1. [Case \(\xi_0(\chi_n) = \omega_1\)] \((\chi_n)\) has a subsequence equivalent to the \(c_0\)-basis; or
2. [Case \(\xi_0(\chi_n) < \omega_1\)] \((\chi_n)\) is null coefficient.

In case (2) there exists a countable ordinal \(\zeta\) (in fact \(\xi_0(\chi_n) = \omega^\zeta\)) such that for each countable ordinal \(\alpha\), either

1. [Case \(\alpha < \zeta\)] \((\chi_n)\) has a subsequence with \(c_0\)-spreading model of order \(\alpha\); or
2. [Case \(\zeta \leq \alpha\)] \((\chi_n)\) is null coefficient of order \(\alpha\).

Next (in Section 3) we introduce and study the semibounded completeness index \(\xi_b(\chi_n)\) of a sequence \((\chi_n)\) (Definition 3.1) and its relation to the \(c_0\)-index. The index \(\xi_b(\chi_n)\) is countable if and only if \((\chi_n)\) is semiboundedly complete, i.e., when every sequence \((\lambda_n)\) of real numbers with \(\sup_n \| \sum_{i=1}^n \lambda_i x_i \| < \infty\) converges to zero. In this case \(\xi_b(\chi_n) = \omega^\zeta\) for some countable ordinal \(\zeta\) (Proposition 3.3); we thus define a sequence \((\chi_n)\) to be semiboundedly complete of order \(\alpha\), for some \(1 \leq \alpha < \omega_1\), if and equivalently if for every \(M \in [\mathbb{N}]\) there exists a strictly increasing function \(\varphi : \mathbb{N} \to M\) with the property: for every \(\varepsilon > 0\) there exists \(n_0 = n_0(\varepsilon) \in \mathbb{N}\) such that

\[ \{ \varphi(n) : n \geq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon \} \in (A_{\xi_{n_0}})^* \setminus A_{\xi_{n_0}} \]

for every \((\lambda_n) \subseteq \mathbb{R}\) with \(\sup_n \| \sum_{i=1}^n \lambda_i x_i \| \leq 1\), where \((\xi_n)\) is a strictly increasing sequence of ordinals with \(\sup_n \xi_n = \omega^\alpha\).

The \(c_0\)-index is always less than or equal to the semibounded completeness index (Proposition 3.6), but they differ in general. We give an example of a normalized, weakly null, basic sequence \((\chi_n)\) with \(\xi_0(\chi_n) = \omega\) and \(\xi_b(\chi_n) = \omega_1\) (Example 3.14). For normalized \(c_0\)-unconditional sequences (Definition 3.8) we prove (in Theorem 3.10) that the \(c_0\)-index is indeed equal to the semibounded completeness index. Thus, a normalized \(c_0\)-unconditional sequence is semiboundedly complete of order \(\alpha\), for some \(1 \leq \alpha \leq \omega_1\), if and
only if it is null coefficient of order $\alpha$; and equivalently, if it does not contain a subsequence with $c_0$-spreading model of order $\alpha$.

Since every normalized, weakly null sequence in a Banach space has a $c_0$-unconditional subsequence (according to a result of Elton [E]), we have the following dichotomy (Theorem 3.15), which constitutes a countable ordinal analogue of Odell’s limiting (for $\alpha = \omega_1$) theorem.

**Theorem B.** Let $(x_n)$ be a normalized weakly null sequence in a Banach space and $\alpha$ be a countable ordinal. Then either

(i) $(x_n)$ has a subsequence with $c_0$-spreading model of order $\alpha$; or

(ii) every subsequence of $(x_n)$ has a subsequence semiboundedly complete of order $\alpha$.

**Notation.** We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of all natural numbers and by $\mathbb{R}$ the set of real numbers. For an infinite subset $M$ of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of $M$, by $[M]^k$ for $k \in \mathbb{N}$ the set of all $k$-element subsets of $M$ and by $[M]$ the set of all infinite subsets of $M$ (considering them as strictly increasing sequences).

If $H, F$ are non-empty finite subsets of $\mathbb{N}$ then we write $H \leq F$ if $\max H \leq \min F$, while $H < F$ if $\max H < \min F$. By $|H|$ we denote the cardinality of $H$.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence.

For a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and $M = (m_i) \in [\mathbb{N}]$ we write:

$\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}$,

$\mathcal{F}(M) = \{(m_{n_1}, \ldots, m_{n_k}) \in [M]^{<\omega} : (n_1, \ldots, n_k) \in \mathcal{F}\}$,

$\mathcal{F}_\ast = \{H \in [\mathbb{N}]^{<\omega} : H \subseteq F \text{ for some } F \in \mathcal{F}\}$.

$\mathcal{F}^\ast = \{H \in [\mathbb{N}]^{<\omega} : H \text{ is an initial segment of some } F \in \mathcal{F}\}$.

$\mathcal{F}$ is hereditary if $\mathcal{F}_\ast = \mathcal{F}$.

$\mathcal{F}$ is thin if there do not exist $H, F \in \mathcal{F}$ such that $H$ is a proper initial segment of $F$.

1. **The basic combinatorial tools.** In this section we recall some known combinatorial results which play a major role in our proofs.

**Definition 1.1** (The generalized Schreier system; [A-A]). Set

$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\}$;

if $\mathcal{F}_\alpha$ has been defined then

$\mathcal{F}_{\alpha+1} = \left\{ \bigcup_{i=1}^{n} H_i : n \leq H_1 < \ldots < H_n \text{ and } H_1, \ldots, H_n \in \mathcal{F}_\alpha \right\}$;
and if $\alpha$ is a limit ordinal, fix a strictly increasing sequence $(\alpha_n)_{n \in \mathbb{N}}$ of ordinal numbers with $\sup_n \alpha_n = \alpha$ and set

$$\mathcal{F}_\alpha = \{H : H \in \mathcal{F}_{\alpha_n} \text{ and } n \leq \min H\}.$$ 

We finally set $\mathcal{F}_{\omega_1} = \{H : H \text{ is a finite subset of } \mathbb{N}\}$.

**Remark 1.2.** (i) If $A \in \mathcal{F}_\alpha$ for some $1 \leq \alpha \leq \omega_1$, and $B \subseteq A$, then $B \in \mathcal{F}_\alpha$. In other words the families $\mathcal{F}_\alpha$ are hereditary.

(ii) It is easy to prove by induction that whenever $\langle n_1, \ldots, n_k \rangle \in \mathcal{F}_\alpha$ and $m_i \geq n_i$ for every $i = 1, \ldots, k$, then $\langle m_1, \ldots, m_k \rangle \in \mathcal{F}_\alpha$.

(iii) For every $\beta < \alpha < \omega_1$ there exists $n_0 = n_0(\beta, \alpha) \in \mathbb{N}$ such that if $F \in \mathcal{F}_\beta$ and $n_0 \prec F$, then $F \in \mathcal{F}_\alpha$.

Now, we recall the definition of the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi < \omega_1}$, defined in [F1].

**Definition 1.3 (The complete thin Schreier system; [F1]).** For every non-zero limit ordinal $\alpha$ we fix a strictly increasing sequence $\langle n \rangle$ of successor ordinals smaller than $\alpha$ with $\sup_n n = \alpha$. We define the system $(\mathcal{A}_\xi)_{\xi < \omega_1}$ recursively as follows:

1. [Case $\xi = 1$]
   $$\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\};$$

2. [Case $\xi = \zeta + 1$]
   $$\mathcal{A}_\xi = \mathcal{A}_{\xi + 1} = \{s \subseteq \mathbb{N} : s = \{n\} \cup s_1, \text{ where } n \in \mathbb{N}, \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\zeta\};$$

3. [Case $\xi = \omega^{\beta+1}$, $\beta$ a countable ordinal]
   $$\mathcal{A}_\xi = \mathcal{A}_{\omega^{\beta+1}} = \left\{s \subseteq \mathbb{N} : s = \bigcup_{i=1}^{n} s_i \text{ with } n = \min s_1, \ s_1 < \ldots < s_n, \text{ and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\right\};$$

4. [Case $\xi = \omega^\alpha$, $\alpha$ a non-zero countable limit ordinal]
   $$\mathcal{A}_\xi = \mathcal{A}_{\omega^\alpha} = \{s \subseteq \mathbb{N} : s \in \mathcal{A}_{\omega^{\alpha_n}} \text{ with } n = \min s\}$$

(where $(\alpha_n)$ is the sequence of ordinals converging to $\alpha$, fixed above); 

5. [Case $\xi$ limit, $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$] Let $\xi = p_0\omega^\alpha + \sum_{i=1}^{m} p_i\omega^{\alpha_i}$ be the canonical representation of $\xi$, where $m \geq 0$, $p, p_1, \ldots, p_m \geq 1$ are natural numbers so that either $p > 1$, or $p = 1$ and $m \geq 1$ and $\alpha > \alpha_1 > \ldots > \alpha_m > 0$ are countable ordinals. Then

   $$\mathcal{A}_\xi = \left\{s \subseteq \mathbb{N} : s = s_0 \cup \bigcup_{i=1}^{m} s_i \text{ with } s_m < \ldots < s_1 < s_0, \right.$$  
   $$s_0 = s_1^0 \cup \ldots \cup s_p^0 \text{ with } s_1^0 < \ldots < s_p^0, \ s_j^0 \in \mathcal{A}_{\omega^\alpha}, \ 1 \leq j \leq p, \right.$$  
   $$s_i = s_1^i \cup \ldots \cup s_p^i \text{ with } s_1^i < \ldots < s_p^i, \ s_j^i \in \mathcal{A}_{\omega^{\alpha_i}}, \ 1 \leq i \leq m, \ 1 \leq j \leq p_i\right\}. $$
We set $B_\alpha = A_{\omega, \alpha}$ for each $1 \leq \alpha < \omega_1$.

**Remark 1.4.** (i) Each family $A_\xi$ for $1 \leq \xi < \omega_1$ is thin (does not contain proper initial segments of its elements).

(ii) ([F1]) Each finite subset $F$ of $\mathbb{N}$ has a *canonical representation* with respect to the family $A_\xi$. This means that for every $1 \leq \xi < \omega_1$ there exist unique $n \in \mathbb{N}$, sets $s_1, \ldots, s_n \in A_\xi$ and $s_{n+1}$, a proper initial segment of some element of $A_\xi$, with $s_1 < \ldots < s_n < s_{n+1}$, such that $F = \bigcup_{i=1}^{n+1} s_i$. The number $n$ is called the *type* $t_\xi(F)$ of $F$ with respect to $A_\xi$.

(iii) ([F1]) For every $0 \leq s \leq \omega_1$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that $F_\alpha(L) \subseteq (B_\alpha)_s \subseteq F_\alpha$.

Now we give the definition of the strong Cantor–Bendixson index of a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$. This index is analogous to the well-known Cantor–Bendixson index ([B], [C]) and has been defined in [A-M-T] and with a different notation in [F1].

**Definition 1.5 ([C], [B], [A-M-T]).** Let $F$ be a hereditary and pointwise closed family of finite subsets on $\mathbb{N}$. For $M \in [\mathbb{N}]$ we define the *strong Cantor–Bendixson derivative* $(F)_M^\xi$ of $F$ on $M$ for every $\xi < \omega_1$ as follows:

$$(F)_M^1 = \{ F \in [F][M] : F \text{ is a cluster point of } F[F \cup L] \text{ for each } L \in [M] \}$$

(where $[F][M] = F \cap [M]^{<\omega}$),

$$(F)_M^{\xi+1} = ((F)_M^\xi)_M^1, \quad (F)_M^\xi = \bigcap_{\beta < \xi} (F)_M^\beta \quad \text{if } \xi \text{ is a limit ordinal}.$$  

The *strong Cantor–Bendixson index* of $F$ on $M$ is defined to be the smallest countable ordinal $\xi$ such that $(F)_M^\xi = \emptyset$. We denote this index by $s_M(F)$.

**Remark 1.6.** (i) $s_M(F)$ is a countable successor ordinal.

(ii) If $F_1 \subseteq F_2$, then $s_M(F_1) \leq s_M(F_2)$ for every $M \in [\mathbb{N}]$.

(iii) If $L$ is almost contained in $M$ (i.e. $L - M$ is finite), then $s_L(F) \geq s_M(F)$.

(iv) For every $M \in [\mathbb{N}]$ and $F \in [M]^{<\omega}$, according to a remark in [Ju], we have: $F \in (F)_M^1$ if and only if the set $\{ m \in M : F \cup \{ m \} \notin F \}$ is finite.

(v) ([A-M-T]) $s_M((A_\xi)_s) = \omega^\alpha + 1$ for every $1 \leq \alpha < \omega_1$ and $M \in [\mathbb{N}]$.

(vi) ([F1]) $s_M((A_\xi)_s) = \xi + 1$ for every $1 \leq \xi < \omega_1$ and $M \in [\mathbb{N}]$.

(vii) ([A-M-T], [Ju], [F1]) If $F$ is a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$ and $M \in [\mathbb{N}]$ is such that $s_M(F) \geq \omega^\alpha$, then there exists $L \in [M]$ such that $F_\alpha(L) \subseteq F$.

We recall the generalization (proved in [F1]) of the classical Ramsey theorem to every countable ordinal.
Theorem 1.7 (\(\xi\)-Ramsey type theorem; [F1]). Let \(\mathcal{F}\) be an arbitrary family of finite subsets of \(\mathbb{N}\), \(M\) an infinite subset of \(\mathbb{N}\) and \(\xi\) a countable ordinal number. Then there exists an infinite subset \(L\) of \(M\) such that

either \(A_\xi \cap [L]^{<\omega} \subseteq \mathcal{F}\) or \(A_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}\).

Using the strong Cantor–Bendixson index, we have developed in [F1] a refined form of the above theorem in case \(\mathcal{F}\) is in addition hereditary.

Theorem 1.8 (Refined \(\xi\)-Ramsey type theorem; [F1]). Let \(\mathcal{F}\) be a hereditary family of finite subsets of \(\mathbb{N}\) and \(M\) an infinite subset of \(\mathbb{N}\). We have the following cases:

Case 1: If the family \(\mathcal{F} \cap [M]^{<\omega}\) is not pointwise closed, then there exists \(L \in [M]\) such that \([L]^{<\omega} \subseteq \mathcal{F}\).

Case 2: If the family \(\mathcal{F} \cap [M]^{<\omega}\) is pointwise closed, then there exists \(L \in [M]\) such that \([L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_*\). Moreover setting

\[
\xi_M^\mathcal{F} = \sup\{s_L(\mathcal{F}) : L \in [M]\},
\]

which is a countable ordinal, the following hold:

2(i) For every countable ordinal \(\xi\) with \(\xi + 1 < \xi_M^\mathcal{F}\) there exists \(L \in [M]\) such that

\[
(A_\xi)_* \cap [L]^{<\omega} \subseteq \mathcal{F}.
\]

2(ii) For every countable ordinal \(\xi\) with \(\xi_M^\mathcal{F} < \xi + 1\) there exists \(L \in [M]\) such that

\[
\mathcal{F} \cap [L]^{<\omega} \subseteq (A_\xi)^* \setminus A_\xi;
\]

and equivalently,

\[
A_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.
\]

2(iii) If \(\xi_M^\mathcal{F} = \xi + 1\), then both alternatives may materialize.

Now we recall the \(\xi\)-Ptáčk type theorem for some \(1 \leq \xi < \omega_1\), which has been proved in [F1], using the notion of the weight of a finite subset \(F\) of \(\mathbb{N}\) with respect to a set of the family \(A_\xi\). The classical Ptáčk theorem is the limiting \(\omega_1\)-case.

Definition 1.9. For every finite subset \(F\) of \(\mathbb{N}\), every countable ordinal \(\xi\), and every \(s \in A_\xi\) we define recursively the \(\xi\)-weight \(w_\xi(F; s)\) of \(F\) with respect to \(s\) to be a real (in fact, rational) number in \([0, 1]\), as follows:

(1) [Case \(\xi = 1\)] Since \(A_1 = \{\{n\} : n \in \mathbb{N}\}\), for every \(n \in \mathbb{N}\) we set

\[
w_1(F; \{n\}) = \begin{cases} 1 & \text{if } n \in F, \\ 0 & \text{otherwise}. \end{cases}
\]
(2) [Case $\xi = \zeta + 1$] Let $s \in A_{\zeta+1}$. Then $s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in A_{\zeta}$. We set

$$w_{\zeta+1}(F; s) = w_{\zeta}(F; s_1) \cdot w_1(F; \{n\}).$$

(3) [Case $\xi = \omega^\beta + 1$ for $0 \leq \beta < \omega_1$] Let $s \in A_{\omega^\beta+1}$. Then $s = s_1 \cup \ldots \cup s_n$, with $n = \min s_1$, $s_1 < \ldots < s_n$ and $s_1, \ldots, s_n \in A_{\omega^\beta}$. We set

$$w_{\omega^\beta+1}(F; s) = \frac{1}{n} \sum_{i=1}^n w_{\omega^\beta}(F; s_i).$$

(4) [Case $\xi = \omega^\alpha$ for $\alpha$ a non-zero countable limit ordinal] Let $s \in A_{\omega^\alpha}$. Then $s \in A_{\omega^\alpha n}$ with $n = \min s$, where $(\alpha_n)$ is the fixed sequence of ordinals “converging” to $\alpha$ (Definition 1.3). So,

$$w_{\omega^\alpha}(F; s) = w_{\omega^\alpha n}(F; s), \quad n = \min s.$$

(5) [Case $\xi$ limit, $\omega^{\alpha_0} < \xi < \omega^{\alpha_0+1}$ for some $0 < \alpha_0 < \omega_1$] In this case, $\xi$ has a unique representation $\xi = p_0\omega^{\alpha_0} + \sum_{i=1}^m p_i\omega^{\alpha_i}$, where $m \in \mathbb{N}$, $\alpha_0 > \alpha_1 > \ldots > \alpha_m > 0$ are ordinal numbers and $p_0, p_1, \ldots, p_m \geq 1$ are natural numbers, so that either $p_0 > 1$, or $p_0 = 1$ and $m > 1$.

Let $s \in A_{\xi}$. Then $s = s_0 \cup s_1 \cup \ldots \cup s_m$ with $s_m < \ldots < s_1 < s_0$, where $s_i = s_i^1 \cup \ldots \cup s_i^{p_i}$ with $s_i^1 < \ldots < s_i^{p_i}$ and $s_i^j \in A_{\omega^{\alpha_i}}$ for every $0 \leq i \leq m$ and $1 \leq j \leq p_i$. We set

$$w_{\xi}(F; s) = \prod_{i=0}^m \prod_{j=1}^{p_i} w_{\omega^{\alpha_i}}(F; s_i^j).$$

**Remark 1.10** ([A-O], [F1]). For every countable ordinal $\alpha$ and $s \in A_{\omega^\alpha} = B_{\alpha}$ we define recursively the functions $\varphi_\alpha^s : \mathbb{N} \to [0, \infty)$ as follows:

- $\varphi_\alpha^k(n) = 1$ if $n = k$, and $\varphi_\alpha^k(n) = 0$ otherwise, for every $\{k\} \in B_0$.
- $\varphi_{\alpha+1}^s = \varphi_{\alpha+1}^s$, for every $s = s_1 \cup \ldots \cup s_k \in B_{\alpha+1}$.
- $\varphi_\alpha^s = \varphi_\alpha^s$, $k = \min s$, for every $s \in B_\alpha$, where $\alpha$ is a non-zero countable limit ordinal.

It is easy to see that $\sum_{n \in \mathbb{N}} \varphi_\alpha^s(n) = 1$ and that $s = \{n \in \mathbb{N} : \varphi_\alpha^s(n) \neq 0\}$. Moreover $w_{\omega^\alpha}(F; s) = \sum_{n \in F} \varphi_\alpha^s(n)$ for every $F \in [\mathbb{N}]^{<\omega}$.

**Theorem 1.11** ($\xi$-Pták type theorem; [F1]). Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$, $\xi$ a non-zero countable ordinal and $0 < \varepsilon < 1$. If for every $s \in A_{\xi} \cap [M]^{<\omega}$ there exists $F \in \mathcal{F}$ such that $w_{\xi}(F; s) > \varepsilon$, then:

(i) there exists $L \in [M]$ such that $s_L(\mathcal{F}) \geq \xi + 1$;

(ii) $\xi_F^M \leq \xi + 1$, and

(iii) for every ordinal $\zeta$ with $\zeta < \xi$ there exists $L \in [M]$ such that $A_{\zeta} \cap [L]^{<\omega} \subseteq \mathcal{F}$. 


Theorem 1.12 (Pták’s theorem; [P]). Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and $0 < \varepsilon < 1$. If for every non-negative function $\varphi$ on $\mathbb{N}$ with finite support and $\sum_{n \in \mathbb{N}} \varphi(n) = 1$ there exists $F \in \mathcal{F}$ such that $\sum_{n \in F} \varphi(n) > \varepsilon$, then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.

2. The $c_0$-behavior of a sequence. In this section we study the precise “$c_0$-content” of an arbitrary (seminormalized, basic) sequence in a Banach space, with the help of the $c_0$-index defined for any such sequence; this is a countable ordinal of the form $\xi_0 = \omega^\zeta$, or the first uncountable ordinal $\omega_1$ (Proposition 2.5). This index is a measure of the $c_0$-content of the sequence in the following sense:

(i) If $\xi_0 = \omega_1$, then there is a subsequence equivalent to the unit vector basis of $c_0$ (Remark 2.2).

(ii) If $\xi_0 < \omega_1$, then there exists a countable ordinal $\zeta$ such that:

(iia) on the one hand, for all $\alpha < \zeta$ there is a subsequence with $c_0$-spreading model of order $\alpha$ (Proposition 2.10), while

(iib) on the other hand (if $\zeta \leq \alpha$) the sequence is far from any higher order $c_0$-behavior, in the sense that it is a null coefficient sequence of order $\zeta$ (Proposition 2.13).

This is the content of the main theorem (Theorem 2.15).

Definition 2.1. Let $(\chi_n)$ be a bounded sequence in a Banach space $X$. For every $\varepsilon > 0$ we set

$$C_{\varepsilon}(\chi_n) = \left\{ F \in [\mathbb{N}]^{<\omega} : \left\| \sum_{i \in F} \lambda_i \chi_i \right\| \leq \varepsilon \max_{i \in F} |\lambda_i| \quad \text{for all } (\lambda_i)_{i \in F} \subseteq \mathbb{R} \right\}.$$ 

All the families $C_{\varepsilon}(\chi_n)$ for $\varepsilon > 0$ are hereditary.

We then define the $c_0$-index $\xi_0(\chi_n)$ of $(\chi_n)$ as follows: If the families $C_{\varepsilon}(\chi_n)$ for all $\varepsilon > 0$ are pointwise closed, we set

$$\xi_0(\chi_n) = \sup \{ s_M(C_{\varepsilon}(\chi_n)) : M \in [\mathbb{N}] \text{ and } \varepsilon > 0 \},$$

which is a countable ordinal; otherwise

$$\xi_0(\chi_n) = \omega_1.$$ 

Remark 2.2. (i) $\xi_0(\chi_n) = \omega_1$ if and only if there exist $\varepsilon > 0$ and $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq C_{\varepsilon}(\chi_n)$ (Theorem 1.8).

(ii) For a basic sequence $(\chi_n)$ in a Banach space $X$ with $0 < \inf_n \|\chi_n\|$ there exists $A > 0$ such that

$$A \max_{1 \leq i \leq n} |\lambda_i| \leq \left\| \sum_{i=1}^{n} \lambda_i \chi_i \right\| \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$
(iii) A basic sequence \((\chi_n)\) with \(0 < \inf_n \|\chi_n\| \leq \sup_n \|\chi_n\| < \infty\) has a subsequence equivalent to the unit vector basis of \(c_0\) if and only if \(\xi_0^{(\chi_n)} = \omega_1\).

**Definition 2.3.** A sequence \((\chi_n)\) in a Banach space \(X\) is called

(i) null coefficient (of order \(\omega_1\)) if every sequence \((\lambda_n)\) of real numbers with \(\sup \{ \| \sum_{i \in F} \lambda_i \chi_i \| : F \in [N]^{<\omega} \} < \infty\) converges to zero; and

(ii) null coefficient of order \(\alpha\), for some countable ordinal \(\alpha\), if every sequence \((\lambda_n)\) of real numbers with \(\sup \{ \| \sum_{i \in F} \lambda_i \chi_i \| : F \in \mathcal{F}_{\alpha} \} < \infty\) converges to zero.

**Proposition 2.4.** Let \((\chi_n)\) be a bounded sequence in a Banach space \(X\). The following are equivalent:

(i) \(\xi_0^{(\chi_n)} < \omega_1\);

(ii) \((\chi_n)\) is null coefficient.

**Proof.** (i) \(\Rightarrow\) (ii). Let \(\xi_0^{(\chi_n)} < \omega_1\). Assume that \((\chi_n)\) is not null coefficient. Then there exist \((\mu_n) \subseteq \mathbb{R}\) and \(\varepsilon > 0\) such that \(\| \sum_{i \in F} \mu_i \chi_i \| \leq 1\) for every \(F \in [N]^{<\omega}\) and the set \(M = \{ n \in \mathbb{N} : \mu_n \geq \varepsilon \}\) is infinite.

Let \(F \in [M]^{<\omega}\) and \((\lambda_i)_{i \in F} \subseteq \mathbb{R}\). There exists \(f \in X^*\) with \(\|f\| \leq 1\) such that \(\| \sum_{i \in F} \lambda_i \chi_i \| = f(\sum_{i \in F} \lambda_i \chi_i)\). Since

\[
\left\| \sum_{i \in F} \lambda_i \chi_i \right\| = \sum_{i \in F} \lambda_i f(\chi_i) \leq \sum_{i \in F} |\lambda_i| \cdot |f(\chi_i)|
\]

\[
= \sum_{i \in F} |\lambda_i| \varepsilon_i f(\chi_i) \quad \text{(for suitable } (\varepsilon_i)_{i \in F} \subseteq \{-1, 1\})
\]

\[
\leq \frac{1}{\varepsilon} \sum_{i \in F} |\lambda_i| |\mu_i| f(\chi_i) \leq \frac{1}{\varepsilon} (\max_{i \in F} |\lambda_i|) \cdot \left\| \sum_{i \in F} \mu_i \varepsilon_i \chi_i \right\|
\]

\[
\leq \frac{2}{\varepsilon} \max_{i \in F} |\lambda_i|,
\]

we see that \([M]^{<\omega} \subseteq C_{2/\varepsilon}^{(\chi_n)}\). This is a contradiction (see Remark 2.2(i)); hence, \((\chi_n)\) is null coefficient.

(ii) \(\Rightarrow\) (i). Let \((\chi_n)\) be null coefficient. If \(\xi_0^{(\chi_n)} = \omega_1\), then there exist \(\varepsilon > 0\) and \(M \in [N]\) such that \([M]^{<\omega} \subseteq C_{\varepsilon}^{(\chi_n)}\). Thus \(\| \sum_{i \in F} \chi_i \| \leq \varepsilon\) for every \(F \in [M]^{<\omega}\). Setting \(\lambda_n = 1\) for every \(n \in M\) and \(\lambda_n = 0\) for \(n \in \mathbb{N} \setminus M\) we have

\[
\sup \left\{ \left\| \sum_{i \in F} \lambda_i \chi_i \right\| : F \in [N]^{<\omega} \right\} < \infty.
\]

A contradiction; hence \(\xi_0^{(\chi_n)} < \omega_1\).

**Proposition 2.5.** Let \((\chi_n)\) be a bounded sequence in a Banach space \(X\). Then either \(\xi_0^{(\chi_n)} = \omega_1\) or \(\xi_0^{(\chi_n)} = \omega^\zeta\) for some countable ordinal \(\zeta\).
Proof. Let \( \xi_0(\chi_n) < \omega_1 \). Then there exists a unique countable ordinal \( \xi \) such that \( \omega^{\xi} \leq \xi_0(\chi_n) < \omega^{\xi+1} \). Arguing by contradiction suppose that \( \omega^{\xi} < \xi_0(\chi_n) \). Then there exist \( M \in [\mathbb{N}] \) and \( \varepsilon > 0 \) such that \( \omega^{\xi} < s_M(C_\varepsilon(\chi_n)) \).

According to Remark 1.6(vii) there exists a subsequence \((y_n)\) of \((\chi_n)\) such that \( F_\xi \subseteq C_\varepsilon(y_n) \). This gives \( B_\xi \subseteq C_\varepsilon(y_n) \), and consequently \( A_{k\omega^\xi} \subseteq C_\varepsilon(y_n) \) for every \( k \in \mathbb{N} \). Hence (see Remark 1.6(vi)), \( s_M(C_\varepsilon(y_n)) > k\omega^\xi \) for every \( k \in \mathbb{N} \).

If \( y_n = \chi_{m_n} \) for every \( n \in \mathbb{N} \) and \( M = \{ m_n : n \in \mathbb{N} \} \), then

\[
\sup \{ s_M(C_\varepsilon(\chi_n)) : L \in [M] \} \leq \xi_0(\chi_n) \leq \omega^\alpha < \omega^{\alpha+1}.
\]

From Theorem 1.8, there exists \( L \in [M] \) so that \( C_\varepsilon(\chi_n) \cap [L] < \omega \subseteq (B_\alpha)^* \setminus B_\alpha \).

On the other hand, if \( B_\alpha \cap [L] \leq \omega \subseteq C(\varepsilon) \), then (Remark 1.6(vi)) \( s_L(C_\varepsilon(\chi_n)) \geq \omega^{\alpha} + 1 > \omega^\alpha \); hence \( \xi_0(\chi_n) > \omega^\alpha \).

(ii) Let \( \varepsilon > 0 \) and \( M \in [\mathbb{N}] \). If \( \zeta \leq \alpha \), then

\[
\sup \{ s_L(C_\varepsilon(\chi_n)) : L \in [M] \} \leq \xi_0(\chi_n) \leq \omega^{\alpha+1} + 1.
\]

From Theorem 1.8, there exists \( L \in [M] \) so that \( C_\varepsilon(\chi_n) \cap [L] < \omega \subseteq (B_\alpha)^* \setminus B_\alpha \).

On the other hand, if for every \( \varepsilon > 0 \) and \( M \in [\mathbb{N}] \) there exists \( L \in [M] \) such that \( C_\varepsilon(\chi_n) \cap [L] < \omega \subseteq (B_\alpha)^* \setminus B_\alpha \), then \( s_M(C_\varepsilon(\chi_n)) \leq \omega^\alpha + 1 \) for every \( M \in [\mathbb{N}] \) and \( \varepsilon > 0 \) (Theorem 1.8). Hence, \( \xi_0(\chi_n) \leq \omega^{\alpha+1} + 1 \) and consequently \( \zeta \leq \alpha \).
So far we have distinguished the cases $\xi_0^{(\chi_n)} = \omega_1$ and $\xi_0^{(\chi_n)} < \omega_1$ (in Remark 2.2(i) and Proposition 2.4) and proved that in case $\xi_0^{(\chi_n)} < \omega_1$ there is $\zeta < \omega_1$ such that $\xi_0^{(\chi_n)} = \omega_\zeta$ (Proposition 2.5). In this last case the set of all countable ordinals is naturally separated by $\zeta$ into two classes, those strictly less than $\zeta$, and those greater than or equal to $\zeta$. To examine the behavior resulting from this dichotomy (in Propositions 2.10 and 2.13 below), we need (a) the notion of the $c_0$-spreading model of order $\alpha$ for some $1 \leq \alpha < \omega_1$ (Definition 2.7 below) and (b) the notion of the null coefficient sequence of order $\alpha$ (Definition 2.3).

Firstly we recall the notion of the $c_0$-spreading model of order $\alpha$ of a sequence $(\chi_n)$ for a countable number $\alpha$, a notion that extends the usual notion of spreading model equivalent to the unit vector basis of $c_0$ (case $\alpha = 1$; [B-S]).

**Definition 2.7.** Let $(\chi_n)$ be a basic sequence in a Banach space $X$ and $\alpha$ be a countable ordinal number. We say that $(\chi_n)$ has $c_0$-spreading model of order $\alpha$ if there exist $A, B > 0$ such that

$$A \max_{i \in F} |\lambda_i| \leq \left\| \sum_{i \in F} \lambda_i \chi_i \right\| \leq B \max_{i \in F} |\lambda_i|$$

for every $F \in \mathcal{F}_\alpha$ and $(\lambda_i)_{i \in F} \subseteq \mathbb{R}$.

**Remark 2.8.** If a basic sequence $(\chi_n)$ has $c_0$-spreading model of order $\alpha$ for some countable ordinal $\alpha$, then every subsequence of $(\chi_n)$ has $c_0$-spreading model of order $\zeta$ for every $\zeta$ with $1 \leq \zeta \leq \alpha$ (see Remark 1.2(ii)).

**Proposition 2.9.** Let $(\chi_n)$ be a bounded sequence in a Banach space $X$ and $\alpha$ be a countable ordinal number. The following are equivalent:

(i) there exists $\varepsilon > 0$ such that

$$\left\| \sum_{i \in F} \lambda_i \chi_i \right\| \leq \varepsilon \max_{i \in F} |\lambda_i| \quad \text{for every } F \in \mathcal{F}_\alpha \text{ and } (\lambda_i)_{i \in F} \subseteq \mathbb{R};$$

(ii) $\sup \{ \sum_{i \in F} |f(\chi_i)| : F \in \mathcal{F}_\alpha \} < \infty$ for every $f \in X^*$;

(iii) there exists $B > 0$ such that

$$\left\| \sum_{i \in F} \chi_i \right\| \leq B \quad \text{for every } F \in \mathcal{F}_\alpha;$$

(iv) a sequence $(\lambda_n) \subseteq \mathbb{R}$ converges to zero if and only if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\left\| \sum_{i \in F} \lambda_i \chi_i \right\| \leq \varepsilon \quad \text{for every } F \in \mathcal{F}_\alpha \text{ with } n_0 \leq F.$$

**Proof.** (i)⇒(iv). This is easily proved, using the fact that $\{n\} \in \mathcal{F}_\alpha$ for every $n \in \mathbb{N}$. 
(iv)⇒(iii). Assume that (iii) does not hold. Then there exists $F_1 \in \mathcal{F}_\alpha$ such that

$$\left\| \sum_{i \in F_1} \chi_i \right\| > 1.$$ 

Set $n_1 = \max F_1$. Then there exists $C_1 > 0$ such that

$$\left\| \sum_{i=1}^{n_1} \lambda_i \chi_i \right\| \leq C_1 \max_{1 \leq i \leq n_1} |\lambda_i| \quad \text{for every } \lambda_1, \ldots, \lambda_{n_1} \in \mathbb{R}.$$ 

If \( \left\| \sum_{i \in F} \chi_i \right\| \leq 2 \) for every $F \in \mathcal{F}_\alpha$ with $n_1 < F$, then for every $F \in \mathcal{F}_\alpha$ we have \( \left\| \sum_{i \in F} \chi_i \right\| \leq C_1 + 2 \), contradicting our assumption. Hence, there exists $F_2 \in \mathcal{F}_\alpha$ such that $F_1 < F_2$ and

$$\left\| \sum_{i \in F_2} \chi_i \right\| > 2.$$ 

Inductively, we can define a sequence \((F_k)_{k \in \mathbb{N}}\) in $\mathcal{F}_\alpha$ with $F_k < F_{k+1}$ and

$$\left\| \sum_{i \in F_k} \chi_i \right\| > k \quad \text{for every } k \in \mathbb{N}.$$ 

We define a sequence \((\lambda_n)\) in $\mathbb{R}$ as follows: $\lambda_n = 1/k$ if $n \in F_k$ for some $k \in \mathbb{N}$ and $\lambda_n = 0$ if $n \in \mathbb{N} \setminus \bigcup_{k \in \mathbb{N}} F_k$. Of course, \((\lambda_n)\) converges to zero and

$$\left\| \sum_{i \in F_k} \lambda_i \chi_i \right\| = \frac{1}{k} \left\| \sum_{i \in F_k} \chi_i \right\| > 1 \quad \text{for every } k \in \mathbb{N}.$$ 

Since $k \leq F_k$ for every $k \in \mathbb{N}$, we have a contradiction to (iv).

(iii)⇒(ii). Let $f \in X^*$ and $F \in \mathcal{F}_\alpha$. Since the family $\mathcal{F}_\alpha$ is hereditary, condition (iii) implies that

$$\sum_{i \in F} |f(\chi_i)| = f\left( \sum_{i \in F} \varepsilon_i \chi_i \right) \leq 2B\|f\|,$$

where \((\varepsilon_i)_{i \in F} \subseteq \{-1, 1\}\) with $|f(\chi_i)| = \varepsilon_i f(\chi_i)$ for every $i \in F$.

(ii)⇒(i). If (ii) holds, then from the Baire category theorem we have the existence of some $k \in \mathbb{N}$ such that

$$\sup \left\{ \sum_{i \in F} |f(\chi_i)| : F \in \mathcal{F}_\alpha \right\} \leq k \quad \text{for every } f \in X^* \text{ with } \|f\| \leq 1.$$ 

Let $F \in \mathcal{F}_\alpha$ and $(\lambda_i)_{i \in F} \subseteq \mathbb{R}$. Then there exists $f \in X^*$ with $\|f\| \leq 1$ such that

$$\left\| \sum_{i \in F} \lambda_i \chi_i \right\| = \sum_{i \in F} \lambda_i f(\chi_i).$$

Thus,

$$\left\| \sum_{i \in F} \lambda_i \chi_i \right\| \leq (\max_{i \in F} |\lambda_i|) \cdot \sum_{i \in F} |f(\chi_i)| \leq k \max_{i \in F} |\lambda_i|.$$ 

This finishes the proof of the proposition.
Proposition 2.10. Let $(\chi_n)$ be a bounded sequence in a Banach space $X$ and $\alpha$ be a countable ordinal number. The following are equivalent:

(i) $\omega^\alpha < \xi_0(\chi_n)$;

(ii) there exist a subsequence $(y_n)$ of $(\chi_n)$ and $\varepsilon > 0$ such that

$$\left\| \sum_{i \in F} \lambda_i y_i \right\| \leq \varepsilon \max_{i \in F} |\lambda_i| \quad \text{for every } F \in \mathcal{F}_\alpha \text{ and } (\lambda_i)_{i \in F} \subseteq \mathbb{R};$$

(iii) there exist a subsequence $(y_n)$ of $(\chi_n)$ and $\varepsilon > 0$ such that $B_\alpha \subseteq C_\varepsilon(y_n)$;

(iv) there exist a subsequence $(y_n)$ of $(\chi_n)$, $I \in [\mathbb{N}]$ and $\varepsilon > 0$ such that

$$\left\| \sum_{i \in H} \varepsilon_i y_i \right\| \leq \varepsilon \text{ for every } H \in B_\alpha \cap [I]^\omega \text{ and } (\varepsilon_i)_{i \in H} \subseteq \{-1, 1\}.$$

Proof. (i) $\Rightarrow$ (ii). If $\omega^\alpha < \xi_0(\chi_n)$, then there exist $M \in [\mathbb{N}]$ and $\varepsilon > 0$ such that $\omega^\alpha < s_M(C_\varepsilon(\chi_n))$. From Remark 1.6(vii) there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq C_\varepsilon(y_n)$.

(ii) $\Rightarrow$ (iii). Since $B_\alpha \subseteq \mathcal{F}_\alpha$ (see Remark 1.4(iii)), we have $B_\alpha \subseteq C_\varepsilon(y_n)$.

(iii) $\Rightarrow$ (iv). Set $I = \mathbb{N}$.

(iv) $\Rightarrow$ (i). Let $H \in B_\alpha \cap [I]^\omega$ and $(\lambda_i)_{i \in H} \subseteq \mathbb{R}$. There exists $f \in X^*$ with $\|f\| \leq 1$ such that $\left\| \sum_{i \in H} \lambda_i y_i \right\| = \sum_{i \in H} \lambda_i f(y_i)$. Hence,

$$\left\| \sum_{i \in H} \lambda_i y_i \right\| \leq \left( \max_{i \in H} |\lambda_i| \right) \left\| \sum_{i \in H} f(y_i) \right\| = \left( \max_{i \in H} |\lambda_i| \right) \left\| \sum_{i \in H} \varepsilon_i f(y_i) \right\| \quad \text{(for suitable } (\varepsilon_i)_{i \in H} \subseteq \{-1, 1\})$$

$$\leq \left( \max_{i \in H} |\lambda_i| \right) \left\| \sum_{i \in H} \varepsilon_i y_i \right\| \leq \varepsilon \max_{i \in H} |\lambda_i|.$$

We have thus proved that $B_\alpha \cap [I]^\omega \subseteq C_\varepsilon(y_n)$. According to Remark 1.6(ii), (vi), we have

$$s_I(C_\varepsilon(y_n)) \geq \omega^\alpha + 1 > \omega^\alpha.$$

This implies that $s_M(C_\varepsilon(\chi_n)) > \omega^\alpha$, where $M = \{m_n : n \in I\}$, and consequently $\omega^\alpha < \xi_0(\chi_n)$.

This finishes the proof.

Corollary 2.11. A basic sequence $(\chi_n)$ in a Banach space with $0 < \inf_n \|\chi_n\| \leq \sup_n \|\chi_n\| < \infty$ has a subsequence with $c_0$-spreading model of order $\alpha$, for some countable ordinal $\alpha$, if and only if $\omega^\alpha < \xi_0(\chi_n)$.

Proof. This follows from Remark 2.2(ii) and the previous proposition.

Remark 2.12. (i) A basic sequence $(\chi_n)$ in a Banach space with $0 < \inf_n \|\chi_n\| \leq \sup_n \|\chi_n\| < \infty$ has a subsequence with $c_0$-spreading model
of the greatest possible order if and only if either \( \xi_0(\chi_n) = \omega_1 \) or \( \xi_0(\chi_n) = \omega^{\alpha+1} \) for some countable ordinal \( \alpha \).

(ii) If a basic sequence has for every countable ordinal \( \alpha \) a subsequence with \( c_0 \)-spreading model of order \( \alpha \), then it has a subsequence equivalent to the unit vector basis of \( c_0 \).

Until now we characterized the countable ordinals \( \alpha \) with \( \omega^\alpha \) as those for which \( (\chi_n) \) has a subsequence with \( c_0 \)-spreading model of order \( \alpha \). Additionally, we know that for each countable ordinal with \( \xi_0(\chi_n) = \omega^\alpha \), no subsequence has \( c_0 \)-spreading model of order \( \alpha \). In this last case we prove (in Proposition 2.13 below) that the sequence is null coefficient of order \( \alpha \).

**Proposition 2.13.** Let \( (\chi_n) \) be a bounded sequence in a Banach space \( X \) and \( \alpha \) be a countable ordinal number. The following are equivalent:

1. \( \xi_0(\chi_n) \leq \omega^\alpha \);
2. the sequence \( (\chi_n) \) is null coefficient of order \( \alpha \);
3. for every subsequence \( (y_n) \) of \( (\chi_n) \) and \( M \in [\mathbb{N}] \) there exists \( I \in [M] \) such that for each \( H \in \mathcal{B}_\alpha \cap [I]<^\omega \) there exists \( (\varepsilon_i)_{i \in H} \subseteq \{-1, 1\} \) such that \( \min H < \| \sum_{i \in H} \varepsilon_i y_i \| \);
4. for every subsequence \( (y_n) \) of \( (\chi_n) \) and \( M \in [\mathbb{N}] \) there exist a sequence \( (H_m)_{m \in \mathbb{N}} \) in \( \mathcal{B}_\alpha \cap [M]<^\omega \) with \( H_1 < H_2 < \ldots \) and \( (\varepsilon_m) \) in \( \{-1, 1\} \) such that

\[
\left\| \sum_{i \in H_m} \varepsilon_i y_i \right\| \xrightarrow{m} \infty.
\]

**Proof.** (i)\(\Rightarrow\)(ii). Let \( \xi_0(\chi_n) \leq \omega^\alpha \). If \( (\chi_n) \) is not null coefficient of order \( \alpha \), then there exist \( (\lambda_n) \subseteq \mathbb{R} \) and \( \varepsilon > 0 \) such that \( \| \sum_{i \in F} \lambda_i x_i \| \leq 1 \) for every \( F \in \mathcal{F}_\alpha \) and the set \( M = \{ n \in \mathbb{N} : \lambda_n > \varepsilon \} \) is infinite.

Let \( M = (m_n)_{n \in \mathbb{N}} \) and \( y_n = \chi_{m_n} \) for every \( n \in \mathbb{N} \). For every \( F \in \mathcal{F}_\alpha \) and \( f \in X^* \) we have

\[
\sum_{i \in F} |f(y_i)| \leq \frac{1}{\varepsilon} \sum_{i \in F} \lambda_{m_i} |f(\chi_{m_i})| \leq \frac{1}{\varepsilon} \sum_{i \in F_1} \lambda_{m_i} f(\chi_{m_i}) + \frac{1}{\varepsilon} \sum_{i \in F_2} \lambda_{m_i} f(\chi_{m_i}) \leq \frac{2}{\varepsilon} \| f \|,
\]

where

\[
F_1 = \{ i \in F : |f(\chi_{m_i})| = f(\chi_{m_i}) \} \in \mathcal{F}_\alpha, \\
F_2 = \{ i \in F : |f(\chi_{m_i})| = -f(\chi_{m_i}) \} \in \mathcal{F}_\alpha.
\]

According to Propositions 2.9 and 2.10 we have \( \omega^\alpha < \xi_0(\chi_n) \). A contradiction; hence \( (\chi_n) \) is null coefficient of order \( \alpha \).

(ii)\(\Rightarrow\)(i). Let \( (\chi_n) \) be null coefficient of order \( \alpha \). If \( \omega^\alpha < \xi_0(\chi_n) \), then according to Proposition 2.10, there exist a subsequence \( (y_n) \) of \( (\chi_n) \) with
$y_n = \chi_{m_n}$ for every $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\left\| \sum_{i \in F} y_i \right\| \leq \varepsilon \quad \text{for every } F \in \mathcal{F}_\alpha.$$ 

Let $M = \{m_n : n \in \mathbb{N}\}$. From a result of Androulakis and Odell ([An-O]) there exists $L \in [M]$ such that

$$F \setminus \{\min F\} \in \mathcal{F}_\alpha(L) \quad \text{for every } F \in \mathcal{F}_\alpha \cap [L]^{<\omega}.$$ 

We consider the sequence $(\lambda_n)$ in $\mathbb{R}$ with $\lambda_n = 1$ if $n \in L$ and $\lambda_n = 0$ if $n \in \mathbb{N} \setminus L$. Then

$$\sup \left\{ \left\| \sum_{i \in F} \lambda_i \chi_i \right\| : F \in \mathcal{F}_\alpha \right\}$$

$$= \sup \left\{ \left\| \sum_{i \in F} \chi_i \right\| : F \in \mathcal{F}_\alpha \cap [L]^{<\omega} \right\} \quad \text{(since $\mathcal{F}_\alpha$ is hereditary)}$$

$$\leq \sup_n \| \chi_n \| + \sup \left\{ \left\| \sum_{i \in F} \chi_i \right\| : F \in \mathcal{F}_\alpha(L) \right\} \quad \text{(see Remark 1.2(ii))}$$

$$\leq \sup_n \| \chi_n \| + \varepsilon.$$ 

Since $(\lambda_n)$ does not converge to zero, the sequence $(\lambda_n)$ is not null coefficient. A contradiction, hence $\xi_0^{(\chi_n)} \leq \omega^\alpha$.

(i) $\Rightarrow$ (iii). Let $\xi_0^{(\chi_n)} \leq \omega^\alpha$. If $(y_n)$ is a subsequence of $(\chi_n)$ and $M \in [\mathbb{N}]$, then for every $k \in \mathbb{N}$ we set

$$\mathcal{L}_k = \left\{ H \in [M]^{<\omega} : k < \left\| \sum_{i \in H} \varepsilon_i y_i \right\| \text{ for some } (\varepsilon_i)_{i \in H} \subseteq \{-1, 1\} \right\}.$$ 

According to Proposition 2.10 we have

$$\mathcal{L}_k \cap \mathcal{B}_\alpha \cap [I]^{<\omega} \neq \emptyset \quad \text{for every } k \in \mathbb{N} \text{ and } I \in [M].$$ 

Using the $\omega^\alpha$-Ramsey theorem (Theorem 1.7) we can construct a decreasing sequence $(I_k)_{k \in \mathbb{N}}$ in $[M]$ such that

$$\mathcal{B}_\alpha \cap [I_k]^{<\omega} \subseteq \mathcal{L}_k \quad \text{for every } k \in \mathbb{N}.$$ 

Set $I = (i^k_n)_{k \in \mathbb{N}}$ if $I_k = (i^k_n)_{n \in \mathbb{N}}$ for every $k \in \mathbb{N}$. Every set $H$ in $\mathcal{B}_\alpha \cap [I]^{<\omega}$ belongs to $\mathcal{L}_k$, where $k = \min H$. Hence for each $H \in \mathcal{B}_\alpha \cap [I]^{<\omega}$ there exists $(\varepsilon_i)_{i \in H} \subseteq \{-1, 1\}$ such that $\min H < \left\| \sum_{i \in H} \varepsilon_i y_i \right\|$.

(iii) $\Rightarrow$ (iv). For every $I \in [\mathbb{N}]$ there exists a sequence $(H_m)_{m \in \mathbb{N}}$ in $\mathcal{B}_\alpha \cap [I]^{<\omega}$ such that $H_1 < H_2 < \ldots$ (Remark 1.4(ii)).

(iv) $\Rightarrow$ (i). This is obvious, by Proposition 2.10. This finishes the proof.
Remark 2.14. (i) If a bounded sequence \((\chi_n)\) is null coefficient of order \(\alpha\) for some countable ordinal \(\alpha\), then every subsequence of \((\chi_n)\) is null coefficient of order \(\beta\) for every \(\beta \leq \alpha \leq \omega_1\).

(ii) If a sequence \((\chi_n)\) is null coefficient, then there exists a countable ordinal \(\zeta\) (in fact \(\omega^\zeta = \xi_0^{(\chi_n)}\)) such that \((\chi_n)\) is null coefficient of order \(\alpha\) for every \(\alpha\) with \(\zeta \leq \alpha \leq \omega_1\).

(iii) If the \(c_0\)-index \(\xi_0^{(\chi_n)}\) of a sequence \((\chi_n)\) is countable, then \(\xi_0^{(\chi_n)} = \omega^\zeta\) where \(\zeta\) is the least ordinal \(\alpha\) which makes the sequence \((\chi_n)\) null coefficient of order \(\alpha\).

Gathering all the previous results we can finally state the principal theorem of this section.

Theorem 2.15. Let \((\chi_n)\) be a basic bounded sequence in a Banach space with \(0 < \inf_n \|\chi_n\|\). Then either

1. [Case \(\xi_0^{(\chi_n)} = \omega_1\)] \((\chi_n)\) has a subsequence equivalent to the unit vector basis of \(c_0\); or
2. [Case \(\xi_0^{(\chi_n)} < \omega_1\)] \((\chi_n)\) is null coefficient.

In case (2) there exists a countable ordinal \(\zeta\) (in fact \(\xi_0^{(\chi_n)} = \omega^\zeta\)) such that for each countable ordinal \(\alpha\), either

1. [Case \(\alpha < \zeta\)] \((\chi_n)\) has a subsequence with \(c_0\)-spreading model of order \(\alpha\); or
2. [Case \(\zeta \leq \alpha\)] \((\chi_n)\) is null coefficient of order \(\alpha\).

Proof. This follows from Propositions 2.4, 2.5, 2.10 and 2.13.

3. Semiboundedly complete sequences. An important notion concerning basic sequences is that of semibounded completeness. A sequence \((\chi_n)\) is semiboundedly complete if every sequence \((\lambda_n)\) of real numbers with \(\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| < \infty\) converges to zero. According to a result of Odell ([O]) every normalized weakly null sequence contains a subsequence which is either equivalent to the unit vector basis of \(c_0\) or semiboundedly complete. This happens since every normalized weakly null sequence contains a \(c_0\)-unconditional subsequence (see Definition 3.8 below; [E]) and since every \(c_0\)-unconditional sequence is semiboundedly complete if and only if it does not contain a subsequence equivalent to the unit vector basis of \(c_0\).

In this section we introduce (Definition 3.1) and characterize (Theorem 3.4) the semibounded completeness index \(\xi_b^{(\chi_n)}\) of a sequence \((\chi_n)\). The index \(\xi_b^{(\chi_n)}\) is countable if and only if \((\chi_n)\) is semiboundedly complete (Remark 3.2(i)) and in this case \(\xi_b^{(\chi_n)} = \omega^\zeta\) for some countable ordinal \(\zeta\) (Proposi-
We call a sequence **semiboundedly complete of order** \( \alpha \), for some countable ordinal \( \alpha \), if \( \xi_b^{(\chi_n)} \leq \omega^\alpha \).

The \( c_0 \)-index is always less than or equal to the semibounded completeness index (Proposition 3.6), but these differ in general. In Example 3.14 we give an example of a normalized, weakly null, basic sequence \((\chi_n)\) with \( \xi_0^{(\chi_n)} = \omega \) and \( \xi_b^{(\chi_n)} = \omega_1 \).

For normalized \( c_0 \)-unconditional sequences we prove (Theorem 3.10) that the two indices are equal. Thus a normalized \( c_0 \)-unconditional sequence is semiboundedly complete of order \( \alpha \), for some \( 0 \leq \alpha < \omega_1 \), if and only if it does not contain a subsequence with \( c_0 \)-spreading model of order \( \alpha \) or, equivalently, if it is null coefficient of order \( \alpha \).

As a corollary, we deduce that for a given countable ordinal \( \alpha \) every normalized weakly null sequence has a subsequence either semiboundedly complete of order \( \alpha \) or with \( c_0 \)-spreading model of order \( \alpha \), thus obtaining a countable ordinal analogue of Odell’s limiting (for \( \alpha = \omega_1 \) theorem (Theorem 3.15)).

**Definition 3.1.** Let \((\chi_n)\) be a sequence in a Banach space \( X \). For every \( \varepsilon > 0 \) we set

\[
D_\varepsilon^{(\chi_n)} = \left\{ F \in [N]^{< \omega} : \text{there exists } (\lambda_n) \subseteq \mathbb{R} \text{ with } \sup_n \left\| \sum_{i=1}^n \lambda_i \chi_i \right\| \leq 1 \right. \\
\text{and } |\lambda_i| \geq \varepsilon \text{ for every } i \in F \right\}.
\]

The families \( D_\varepsilon^{(\chi_n)} \), for all \( \varepsilon > 0 \), are hereditary.

We then define the **semibounded completeness index** \( \xi_b^{(\chi_n)} \) of \((\chi_n)\) as follows: if there exists \( \varepsilon > 0 \) such that the family \( D_\varepsilon^{(\chi_n)} \) is not pointwise closed, then we set

\[ \xi_b^{(\chi_n)} = \omega_1; \]

otherwise

\[ \xi_b^{(\chi_n)} = \sup\{ s_M(D_\varepsilon^{(\chi_n)}) : M \in [N] \text{ and } \varepsilon > 0 \}, \]

which is a countable ordinal.

We say that the sequence \((\chi_n)\) is:

1. **semiboundedly complete** (of order \( \omega_1 \)) if all the sequences \((\lambda_n) \subseteq \mathbb{R} \) with \( \sup_n \left\| \sum_{i=1}^n \lambda_i \chi_i \right\| \leq 1 \) converge to zero;
2. **semiboundedly complete of order** \( \zeta \), for some countable ordinal \( \zeta \), if \( \xi_b^{(\chi_n)} \leq \omega^\zeta \).

**Remark 3.2.** (i) For a sequence \((\chi_n)\) with \( \inf_n \left\| \chi_n \right\| > 0 \), using a compactness argument, it is easy to prove that \( \xi_b^{(\chi_n)} = \omega_1 \) if and only if there
exist \( M \in [N], \varepsilon > 0 \) and \((\lambda_n) \subseteq \mathbb{R}\) such that \(\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| \leq 1\) and \(|\lambda_n| \geq \varepsilon\) for every \( n \in M \). Hence \((\chi_n)\) is semiboundedly complete if and only if \(\xi_b^{(\chi_n)} < \omega_1\).

(iii) \(\omega \leq \xi_b(\chi_n)\) for every bounded sequence \((\chi_n)\), since for every \( k \in \mathbb{N}\) and \(F \in [N]^k\) setting \(\alpha_n = 1/(Ak)\) if \(n \in F\) and \(\alpha_n = 0\) if \(n \in \mathbb{N} \setminus F\), where \(A = \sup_n \|\chi_n\|\), we get \(\sup_n \|\sum_{i=1}^n \alpha_i \chi_i\| \leq 1\).

(iv) For the summing basis \((s_n)\) of \(c_0\) we have \(\xi_b^{(s_n)} = \omega_1\) and \(\xi_0^{(s_n)} = \omega\).

Proposition 3.3. Let \((\chi_n)\) be a sequence in a Banach space \(X\). Then either \(\xi_b^{(\chi_n)} = \omega_1\) or \(\xi_b^{(\chi_n)} = \omega\) for some ordinal \(\omega\) with \(1 \leq \omega < \omega_1\).

Proof. Let \(\xi_b^{(\chi_n)} < \omega_1\). Then there exists an ordinal \(\omega\) with \(1 \leq \omega < \omega_1\) such that \(\omega \subseteq \xi_b^{(\chi_n)} < \omega^{\omega+1}\). Arguing by contradiction suppose that \(\omega \subseteq \xi_b^{(\chi_n)}\). Then there exist \(M \in [N]\) and \(\varepsilon > 0\) such that \(\omega \subseteq s_M(D_{\varepsilon}^{(\chi_n)})\).

According to Remark 1.6(vii) there exists \(L = (l_n) \in [M]\) such that \(F_{\omega}(L) \subseteq D_{\varepsilon}^{(\chi_n)}\).

Let \(k \in \mathbb{N}\) and \(F_1, \ldots, F_k \in F_{\omega}\) with \(F_1 < \ldots < F_k\). For each \(m \in \{1, \ldots, k\}\) there exist \((\lambda_m) \subseteq \mathbb{R}\) such that \(\sup_n \|\sum_{i=1}^n \lambda_i^m \chi_i\| \leq 1\) and \(|\lambda_i^m| \geq \varepsilon\) for every \(i \in F_m\). For each \(m \in \{1, \ldots, k\}\) set \(\sigma_m = \max F_m\) and \(p_m = \min F_m\); also set \(b_n^m = \lambda_i^m\) if \(n \in \mathbb{N}\) with \(l_{p_m} \leq n \leq l_{\sigma_m}\) and \(b_n^m = 0\) if \(n \in \mathbb{N}\) with \(n < l_{p_m}\) or \(n > l_{\sigma_m}\). Then

\[
\sup_n \left\|\sum_{i=1}^n b_i^m \chi_i\right\| \leq 2 \quad \text{for every } m \in \{1, \ldots, k\}.
\]

Set \(\lambda_n = (b_1^k + \ldots + b_k^k)/(2k)\) for every \(n \in \mathbb{N}\). Then

\[
\sup_n \left\|\sum_{i=1}^n \lambda_i \chi_i\right\| \leq 1 \quad \text{and} \quad |\lambda_i| \geq \frac{\varepsilon}{2k} \quad \text{for every } i \in \bigcup_{m=1}^k F_m.
\]

Setting, for every \(k \in \mathbb{N}\),

\[
F^k_{\omega} = \left\{ F \in [N]^\omega : F = \bigcup_{i=1}^k F_i \text{ with } F_i \in F_{\omega} \text{ and } F_1 < \ldots < F_k \right\},
\]

we have thus proved that \(F^k_{\omega}(L) \subseteq D_{\varepsilon/(2k)}^{(\chi_n)}\) for every \(k \in \mathbb{N}\).

From a result of Androulakis and Odell ([An-O]) there exists \(L_1 \in [L]\) such that \(F \setminus \{\min F\} \in F_{\omega}(L)\) for every \(F \in F_{\omega} \cap [L_1]^\omega\). This shows that \(F_{\omega} \cap [L_1]^\omega \subseteq F^2_{\omega}(L)\). Hence, \(F^k_{\omega} \cap [L_1]^\omega \subseteq F^{2k}_{\omega}(L)\) for every \(k \in \mathbb{N}\). Thus,
we have
\[ A_{k,\omega^\xi} \cap [L_1]^{<\omega} \subseteq \mathcal{F}_\xi^k \cap [L_1]^{<\omega} \subseteq D_\varepsilon^{(\omega_n)}(\mathcal{X}_n) \quad \text{for every } k \in \mathbb{N}. \]

This gives (see Remark 1.6(v))
\[ k\omega^\xi + 1 \leq s_{L_1}(D_\varepsilon^{(\omega_n)}) \quad \text{for every } k \in \mathbb{N}. \]

But this is impossible, since \( \xi_b^{(\omega_n)} < \omega^{\xi+1} \). Hence, \( \xi_b^{(\omega_n)} = \omega^\xi \).

Recapitulating the previous results, we have already proved that a sequence \((\omega_n)\) in a Banach space with \( 0 < \inf_n \|\omega_n\| \leq \sup_n \|\omega_n\| < \infty \) is semiboundedly complete if and only if the index \( \xi_b^{(\omega_n)} \) is countable and in this case \( \xi_b^{(\omega_n)} = \omega^\xi \) for some countable ordinal \( \xi \). The ordinal \( \xi \) indicates the least possible order of the semibounded completeness of \((\omega_n)\) (see Remark 3.2(iv)).

In the following we will establish a characterization of the sequences semiboundedly complete of order \( \xi \) in terms of the complete thin Schreier system (Definition 1.3). The sense of this characterization is that the semibounded completeness of order \( \xi \) of the sequence \((\omega_n)\) is precisely equivalent to the \( \omega^\xi \)-uniform convergence to zero of all the sequences \((\lambda_n)\) of real numbers with \( \sup_n \|\sum_{i=1}^n \lambda_i \omega_i\| \leq 1 \).

**Theorem 3.4.** Let \((\omega_n)\) be a basic sequence in a Banach space, \( \xi \) a countable ordinal and \((\xi_n)\) a strictly increasing sequence of ordinals with \( \sup_n \xi_n = \omega^\xi \). The following are equivalent:

(i) \((\omega_n)\) is semiboundedly complete of order \( \xi \);

(ii) for every \( M \in [N] \) there exists a strictly increasing function \( \varphi : \mathbb{N} \to M \) with the property: for every \( \varepsilon > 0 \) there exists \( n_0 = n_0(\varepsilon) \in \mathbb{N} \) such that
\[ \{ \varphi(n) : n \geq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon \} \in (A_{\xi_n})^* \setminus A_{\xi_{n_0}} \]

for every \( (\lambda_n) \subseteq \mathbb{R} \) with \( \sup_n \|\sum_{i=1}^n \lambda_i \omega_i\| \leq 1 \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( \xi_b^{(\omega_n)} \leq \omega^\xi \). For every \( M \in [N] \) and \( \varepsilon > 0 \) there exists \( I \in [M] \) such that \( \sup\{ s_N(D_\varepsilon^{(\omega_n)}) : N \in [I] \} < \omega^\xi \).

Let \( M \in [N] \). Using Theorem 1.8 we can construct a strictly increasing sequence \((k_n)\) in \([N]\) and a decreasing sequence \((I_n)\) in \([M]\) such that
\[ D_1^{(\omega_n)}(I_n) \subseteq [I_n]^{<\omega} \subseteq (A_{\xi_{k_n}})^* \setminus A_{\xi_{k_n}} \quad \text{for every } n \in \mathbb{N}. \]

If \( I_n = (i_{m,n})_{m \in \mathbb{N}} \) for every \( n \in \mathbb{N} \), then define \( \varphi : \mathbb{N} \to M \) by \( \varphi(n) = i_{n,n}^\varepsilon \) for every \( n \in \mathbb{N} \). For \( \varepsilon > 0 \) set \( n_0 = n_0(\varepsilon) = k_\lambda \) for some \( \lambda \in \mathbb{N} \) with \( 1/\lambda < \varepsilon \). Then for every sequence \((\lambda_n)\) in \( \mathbb{R} \) with \( \sup_n \|\sum_{i=1}^n \lambda_i \omega_i\| \leq 1 \) we get
\[ \{ \varphi(n) : n \geq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon \} \in (A_{\xi_{n_0}})^* \setminus A_{\xi_{n_0}}. \]
(ii)⇒(i). Suppose that \( \xi_b^{(x_n)} > \omega^\varsigma \). Then there exist \( \varepsilon > 0 \) and \( M \in [\mathbb{N}] \) such that \( s_M(D_{\varepsilon}^{(x_n)}) > \omega^\varsigma > \xi_n + 1 \) for every \( n \in \mathbb{N} \). By (ii), there exist \( L \in [M] \) and \( n_0 \in \mathbb{N} \) such that
\[
D_\varepsilon^{(x_n)} \cap [L]^{<\omega} \subseteq (A_{\xi_{n_0}})^* \setminus A_{\xi_{n_0}}.
\]
Since \( s_L(D_{\varepsilon}^{(x_n)}) > \xi_{n_0} + 1 \) (Remark 1.6(iii)), according to Theorem 1.8, there exists \( I \in [L] \) such that \( A_{\xi_{n_0}} \cap [I]^{<\omega} \subseteq D_\varepsilon^{(x_n)} \cap [L]^{<\omega} \), which is a contradiction, hence \( \xi_b^{(x_n)} \leq \omega^\varsigma \).

Choosing appropriate sequences \((x_n)\) strictly increasing to \( \omega^\varsigma \) we can obtain interesting descriptions of being semiboundedly complete of order \( \varsigma \).

**Corollary 3.5.** Let \((x_n)\) be a sequence in a Banach space, and \( \varsigma \) a countable ordinal.

1. \( \xi_b^{(x_n)} \leq \omega^{\varsigma+1} \) if and only if for every \( M \in [\mathbb{N}] \) there exists a strictly increasing function \( \varphi : \mathbb{N} \to M \) such that for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) so that the type with respect to \( B_\varsigma \) (see Remark 1.4(ii)) of the set
\[
\{ \varphi(n) : n \leq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon \}
\]
is at most \( n_0 \), for every \( (\lambda_n) \subseteq \mathbb{R} \) with \( \sup_n \| \sum_{i=1}^{n} \lambda_i x_i \| \leq 1 \).

2. \( \xi_b^{(x_n)} \leq \omega^\varsigma \) for some limit ordinal \( \varsigma \) if and only if there exists a sequence \((\varsigma_n)\) of ordinals strictly increasing to \( \varsigma \) with the following property: for every \( M \in [\mathbb{N}] \) there exists a strictly increasing sequence \( \varphi : \mathbb{N} \to M \) such that for every \( \varepsilon > 0 \) there exists \( n_0 \in \mathbb{N} \) so that the type with respect to \( B_{\varsigma n_0} \) of the set
\[
\{ \varphi(n) : n \leq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon \}
\]
is at most \( n_0 \), for every \( (\lambda_n) \subseteq \mathbb{R} \) with \( \sup_n \| \sum_{i=1}^{n} \lambda_i x_i \| \leq 1 \).

**Proof.** This is a consequence of Theorem 3.4: in case (1), set \( \xi_n = n\omega^\varsigma \) for every \( n \in \mathbb{N} \), and in case (2), set \( \xi_n = \omega^{\varsigma n+1} \) for every \( n \in \mathbb{N} \).

Now, we will study the relation of the semibounded completeness index \( \xi_b^{(x_n)} \) to the co-index \( \xi_0^{(x_n)} \) of a sequence \((x_n)\).

**Proposition 3.6.** Let \((x_n)\) be a normalized basic sequence in a Banach space \( X \). Then \( \xi_0^{(x_n)} \leq \xi_b^{(x_n)} \).

**Proof.** Let \( \xi_b^{(x_n)} = \omega^\varsigma \) for some \( 1 \leq \varsigma < \omega_1 \) (Proposition 3.3). Arguing by contradiction, we assume that \( \omega^\varsigma < \xi_0^{(x_n)} \). Then there exist a subsequence \((y_n)\) of \((x_n)\) and \( \varepsilon > 0 \) such that
\[
\left\| \sum_{i \in F} \lambda_i y_i \right\| \leq \varepsilon \max_{i \in F} |\lambda_i| \quad \text{for every } F \in \mathcal{F}_\varsigma \text{ and } \lambda_i, i \in F \subseteq \mathbb{R}.
\]
Let $E = [(y_n)]$ be the closed subspace of $X$ which is generated by the sequence $(y_n)$, and $(y_n^*) \subseteq E^*$ the sequence of the biorthogonal functionals of $(y_n)$. Clearly, $\|y_n^*\| \leq 2C$ for every $n \in \mathbb{N}$, where $C$ is the basic constant of $(y_n)$.

Let $s \in B_\zeta \subseteq \mathcal{F}_\zeta$. We set $\lambda_n = \varphi_\zeta(n)$ for every $n \in \mathbb{N}$, according to Remark 1.10. Then

$$\left\| \sum_{i \in s} \lambda_i y_i^* \right\| \geq \frac{1}{\epsilon} \left( \sum_{i \in s} \lambda_i y_i^* \right) \left( \sum_{i \in s} y_i \right) = \frac{1}{\epsilon} \sum_{i \in s} \lambda_i = \frac{1}{\epsilon}.$$

Let $f \in E^{**}$ with $\|f\| \leq 1$ and $\left\| \sum_{i \in s} \lambda_i y_i^* \right\| = \sum_{i \in s} \lambda_i f(y_i^*)$. Setting $\mu_n = C^{-1} f(y_n^*)$ for every $n \in \mathbb{N}$, we get $|\mu_n| \leq 2$ for every $n \in \mathbb{N}$ and

$$\sup_n \left\| \sum_{i = 1}^n \mu_i y_i \right\| = \frac{1}{C} \sup_n \left\| \sum_{i = 1}^n f(y_i^*) y_i \right\| = \frac{1}{C} \sup_n \|P_n^{**}(f)\| \leq \frac{1}{C} \sup_n \|P_n^{**}\| = 1$$

(where the $P_n$ are the natural projections associated to the basis $(y_n)$).

If $F = \{i \in s : |\mu_i| \geq 1/(2\epsilon C)\}$, then $F \in D_{1/(2\epsilon C)}^{(y_n)}$. We will prove that $w_{\omega^\zeta}(F; s) > 1/(4\epsilon C)$. Indeed,

$$\frac{1}{\epsilon} \leq \left\| \sum_{i \in s} \lambda_i y_i^* \right\| = C \sum_{i \in s} \lambda_i \mu_i = C \sum_{i \in F} \lambda_i \mu_i + C \sum_{i \notin F} \lambda_i \mu_i \leq 2C w_{\omega^\zeta}(F; s) + \frac{C}{2\epsilon C}.$$

Hence, $\|w_{\omega^\zeta}(F; s)\| \geq 1/(4\epsilon C)$.

According to the $\omega^\zeta$-Ptáčk type theorem (Theorem 1.11) there exists $L \in [\mathbb{N}]$ such that $s_L (D_\delta^{(y_n)}) > \omega^\zeta$, where $\delta = 1/(2\epsilon C)$.

If $y_n = \chi_{m_n}$ for every $n \in \mathbb{N}$, then for $M = \{m_n : n \in L\}$ we have $s_M (D_\delta^{(\chi_n)}) > \omega^\zeta$, and consequently $\xi^{(\chi_n)}_b > \omega^\zeta$.

This is a contradiction; hence $\xi^{(\chi_n)}_0 \leq \omega^\zeta = \xi^{(\chi_n)}_b$.

**Corollary 3.7.** Let $(\chi_n)$ be a normalized basic sequence in a Banach space and $\zeta$ an ordinal number with $1 \leq \zeta \leq \omega_1$. If $(\chi_n)$ is semiboundedly complete of order $\zeta$, then $(\chi_n)$ is null coefficient of order $\zeta$.

In Remark 3.2(iii) we gave an example of a normalized basic sequence $(s_n)$ with $\xi^{(s_n)}_b = \omega_1$ and $\xi^{(s_n)}_0 = \omega$. According to Proposition 2.13, $(s_n)$ is null coefficient of order $\alpha$ for every countable ordinal $\alpha$, but it is not semiboundedly complete of order $\alpha$. As we prove in Theorem 3.10 below these notions are equivalent in the case of a $c_0$-unconditional sequence.

**Definition 3.8.** A bounded basic sequence $(\chi_n)$ in a Banach space is $c_0$-unconditional if for every $\delta > 0$ there exists a constant $K(\delta) < \infty$ so that
for every $n \in \mathbb{N}$, every sequence $(\lambda_i)_{i=1}^n \subseteq \mathbb{R}$ with $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$ and every $F \subseteq \{1 \leq i \leq n : |\lambda_i| \geq \delta\}$ we have
\[
\left\| \sum_{i \in F} \lambda_i \chi_i \right\| \leq K(\delta) \left\| \sum_{i=1}^n \lambda_i \chi_i \right\|.
\]

**Remark 3.9.** Elton [E] proved that every normalized weakly null sequence in a Banach space has a $c_0$-unconditional subsequence.

**Theorem 3.10.** Let $(\chi_n)$ be a normalized $c_0$-unconditional basic sequence in a Banach space $X$. Then $\xi_0(\chi_n) = \xi_b(\chi_n)$.

**Proof.** We claim that if $\omega^\zeta < \xi_b(\chi_n)$ for some countable ordinal $\zeta$, then $\omega^\zeta < \xi_0(\chi_n)$. Indeed, let $\omega^\zeta < \xi_b(\chi_n)$. Then there exist $M \in [\mathbb{N}]$ and $\varepsilon > 0$ such that $\omega^\zeta < s_M(D_\varepsilon(\chi_n))$. According to Remark 1.6(vii) there exists $L \in [M]$ such that
\[
\mathcal{F}_\zeta(L) \subseteq D_\varepsilon(\chi_n).
\]
Set $L = (l_n)$ and $y_n = \chi_{i_n}$ for every $n \in \mathbb{N}$. If $F \in \mathcal{F}_\zeta$, then $(l_i)_{i \in F} \in D_\varepsilon(\chi_n)$, so there exists a sequence $(\lambda_n) \subseteq \mathbb{R}$ with
\[
\sup_n \left\| \sum_{i=1}^n \lambda_i \chi_i \right\| \leq 1 \quad \text{and} \quad |\lambda_n| \geq \varepsilon \quad \text{for all} \quad i \in F.
\]
Thus for every $f \in X^*$ we have
\[
\sum_{i \in F} |f(y_i)| \leq \frac{1}{\varepsilon} \sum_{i \in F} |\lambda_i| \cdot |f(y_i)|
\]
\[
= \frac{1}{\varepsilon} \sum_{i \in F} \varepsilon_i \lambda_i f(y_i) \quad \text{(for suitable $(\varepsilon_i)_{i \in F} \subseteq \{-1, 1\}$)}
\]
\[
= \frac{1}{\varepsilon} f \left( \sum_{i \in F} \varepsilon_i \lambda_i y_i \right) \leq \frac{1}{\varepsilon} \|f\| \cdot \left\| \sum_{i \in F} \varepsilon_i \lambda_i \chi_i \right\| \leq \frac{1}{\varepsilon} \|f\| K \left( \frac{\varepsilon}{2} \right);
\]

since $|\lambda_n| \geq 2$ for every $n \in \mathbb{N}$, $|\lambda_i| \geq \varepsilon$ for every $i \in F$ and the sequence $(\chi_n)$ is $c_0$-unconditional with constraint $K(\delta)$ for $\delta > 0$.

According to Propositions 2.9 and 2.10 we get $\omega^\zeta < \xi_0(\chi_n)$, which finishes the proof of our claim.

In case $\xi_b(\chi_n) = \omega_1$ we have $\omega^\zeta < \xi_b(\chi_n)$ for every countable ordinal $\zeta$. So according to our claim $\omega^\zeta < \xi_0(\chi_n)$ for every $1 \leq \zeta < \omega_1$, which gives $\xi_0(\chi_n) = \omega_1 = \xi_b(\chi_n)$.

In case $\xi_b(\chi_n) < \omega_1$ there exists a countable ordinal $\zeta$ such that $\xi_0(\chi_n) = \omega^\zeta \leq \xi_b(\chi_n)$ (Propositions 2.5 and 3.6). If $\omega^\zeta < \xi_b(\chi_n)$, then according to the previous claim we have $\omega^\zeta < \xi_0(\chi_n)$, which is impossible. Hence, $\xi_0(\chi_n) = \xi_b(\chi_n)$. 
So far we have proved that a normalized $c_0$-unconditional basic sequence $(\chi_n)$ in a Banach space $X$ either has a subsequence equivalent to the unit vector basis of $c_0$ (in which case $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} = \omega_1$), or it is semiboundedly complete (in case $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} < \omega_1$). In the latter case there exists a countable ordinal $\zeta$ such that $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} = \omega^\zeta$. The ordinal $\zeta$ separates the set of all the countable ordinals into two classes, the ordinals $\alpha$ with $\alpha < \zeta$ and those with $\alpha \geq \zeta$. We characterized the ordinals $\alpha$ with $\alpha < \zeta$ as those for which the sequence $(\chi_n)$ has a subsequence with $c_0$-spreading model of order (Proposition 2.10); on the other hand, we characterized the ordinals $\alpha$ with $\zeta \leq \alpha$ as those which make the sequence $(\chi_n)$ null coefficient of order $\alpha$ (Proposition 2.13) and moreover semiboundedly complete of order $\alpha$ (Theorem 3.4). In the following two propositions we will give more characterizations of these two classes.

**Proposition 3.11.** Let $(\chi_n)$ be a normalized $c_0$-unconditional sequence in a Banach space $X$ with $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal $\zeta$. For each countable ordinal $\alpha$ the following are equivalent:

(i) $\alpha < \zeta$;
(ii) there exists a subsequence $(y_n)$ of $(\chi_n)$ with $c_0$-spreading model of order $\alpha$;
(iii) there exist a subsequence $(y_n)$ of $(\chi_n)$, $I \subseteq [\mathbb{N}]$ and $\varepsilon > 0$ such that

$$\left\| \sum_{i \in H} y_i \right\| \leq \varepsilon \quad \text{for every } H \subseteq \mathcal{B}_\alpha \cap [I]^{<\omega}.$$  

**Proof.** (i) $\Rightarrow$ (ii). This is proved in Proposition 2.10.
(ii) $\Rightarrow$ (iii). This is obvious: set $I = \mathbb{N}$.
(iii) $\Rightarrow$ (ii). According to Remark 1.4(iii) there exists $L \subseteq [I]$ such that

$$\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha \cap [I]^{<\omega})^*_\times.$$ 

Set $L = (l_n)_{n \in \mathbb{N}}$ and $z_n = y_{l_n}$ for every $n \in \mathbb{N}$. The subsequence $(z_n)$ of $(\chi_n)$ has $c_0$-spreading model of order $\alpha$. Indeed, let $F \subseteq \mathcal{F}_\alpha$. Then there exists $H \subseteq \mathcal{B}_\alpha \cap [I]^{<\omega}$ such that $(l_i)_{i \in F} \subseteq H$. Since the sequence $(\chi_n)$ is $c_0$-unconditional there exists $K = K(1) > 0$ such that

$$\left\| \sum_{i \in F} z_i \right\| = \left\| \sum_{i \in F} y_i \right\| \leq K \left\| \sum_{i \in H} y_i \right\| \leq K \varepsilon.$$ 

According to Proposition 2.9 the sequence $(z_n)$ has $c_0$-spreading model of order $\alpha$.

**Proposition 3.12.** Let $(\chi_n)$ be a normalized $c_0$-unconditional sequence in a Banach space with $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal $\zeta$. For each countable ordinal $\alpha$ the following are equivalent:
(i) \( \zeta \leq \alpha \);
(ii) the sequence \((\chi_n)\) is semiboundedly complete of order \(\alpha\);
(iii) the sequence \((\chi_n)\) is null coefficient of order \(\alpha\);
(iv) whenever a bounded sequence \((\lambda_n)\) of real numbers satisfies
\[ \sup \left\{ \| \sum_{i \in H} \lambda_i \chi_i \| : H \in \mathcal{B}_\alpha \right\} < \infty, \text{ then } (\lambda_n) \text{ converges to zero}; \]
(v) for every subsequence \((y_n)\) of \((\chi_n)\) and \(M \in [\mathbb{N}]\) there exists \(L \in [M]\) such that
\[ \min H < \left\| \sum_{i \in H} y_i \right\| \quad \text{for every } H \in \mathcal{B}_\alpha \cap [L]^{<\omega}; \]
(vi) for every subsequence \((y_n)\) of \((\chi_n)\) and \(M \in [\mathbb{N}]\) there exists a sequence \((H_m)\) in \(\mathcal{B}_\alpha \cap [M]^{<\omega}\) with \(H_1 < H_2 < \ldots\) and
\[ \left\| \sum_{i \in H_m} y_i \right\| \rightarrow \infty. \]

Proof. (i)\(\Leftrightarrow\) (ii). This follows from Definition 3.1.
(i)\(\Leftrightarrow\) (iii). Follows from Theorem 3.10 and Proposition 2.13.
(iii)\(\Rightarrow\) (iv). Let \((\lambda_n) \subseteq \mathbb{R}\) with
\[ \sup \left\{ \left\| \sum_{i \in H} \lambda_i \chi_i \right\| : H \in \mathcal{B}_\alpha \right\} = A < \infty \quad \text{and} \quad \sup_n |\lambda_n| = B < \infty. \]
We assume that \((\lambda_n)\) does not converge to zero. Then there exists \(\varepsilon > 0\) such that the set
\[ M = \{ n \in \mathbb{N} : |\lambda_n| > \varepsilon \} \quad \text{is infinite.} \]
According to Remark 1.6(iii) and a result of Androulakis and Odell ([An-O]) there exists \(L \in [M]\) with \(L = (l_n)\) such that
\[ \mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha \cap [M]^{<\omega})^*, \quad F \setminus \{ \min F \} \in \mathcal{F}_\alpha(L) \quad \text{for all } F \in \mathcal{F}_\alpha \cap [L]^{<\omega}. \]
Set \(\mu_{i_n} = \lambda_{i_n}\) for every \(n \in \mathbb{N}\) and \(\mu_n = 0\) for every \(n \in \mathbb{N} \setminus L\). If \(F \in \mathcal{F}_\alpha\), then
\[ \left\| \sum_{i \in F} \mu_i \chi_i \right\| = \left\| \sum_{i \in F_1} \lambda_i \chi_i \right\| \quad \text{for some } F_1 \in \mathcal{F}_\alpha \cap [L]^{<\omega}. \]
Hence,
\[ \left\| \sum_{i \in F} \mu_i \chi_i \right\| \leq 2K(\varepsilon/B)A, \]
since the sequence \((\chi_n)\) is \(c_0\)-unconditional with constraint \(K(\delta)\) for \(\delta > 0\). This contradicts (iii). Hence \((\lambda_n)\) converges to zero.
(iv)\(\Rightarrow\) (iii). This is obvious, since \(\mathcal{B}_\alpha \subseteq \mathcal{F}_\alpha\).
(i)⇒(v). Let \( \zeta \leq \alpha \). If \( (y_n) \) is a subsequence of \( (\chi_n) \) and \( M \in [N] \), then for every \( k \in \mathbb{N} \) we set

\[
L_k = \left\{ H \in [M]^{<\omega} : k < \left\| \sum_{i \in H} y_i \right\| \right\}.
\]

According to Proposition 3.11 we have

\[
L_k \cap B_\alpha \cap [I]^{<\omega} \neq \emptyset \quad \text{for every } k \in \mathbb{N} \text{ and } I \in [N].
\]

Using the refined \( \omega^\alpha \)-Ramsey type theorem (Theorem 1.8), a decreasing sequence \( (I_k) \) in \( [M] \) can be constructed such that

\[
B_\alpha \cap [I_k]^{<\omega} \subseteq L_k \quad \text{for every } k \in \mathbb{N}.
\]

Set \( L = (i_k) \) if \( I_k = (i_n^k)_{n \in \mathbb{N}} \) for every \( k \in \mathbb{N} \). If \( H \in B_\alpha \cap [I]^{<\omega} \), then \( H \in L_k \), where \( k = \min H \); hence

\[
\min H < \left\| \sum_{i \in H} y_i \right\| \quad \text{for every } H \in B_\alpha \cap [I]^{<\omega}.
\]

(v)⇒(vi). For every \( L \in [M] \) there exists (Remark 1.4(ii)) a sequence \( (H_m) \) in \( B_\alpha \cap [M]^{<\omega} \) with \( H_1 < H_2 < \ldots \) and \( L = \bigcup_{m \in \mathbb{N}} H_m \).

(vi)⇒(i). This follows from Proposition 3.11.

Gathering the previous results we can state a theorem which completes Theorem 2.15 in the case of a normalized \( c_0 \)-unconditional sequence.

**Theorem 3.13.** Let \( (\chi_n) \) be a normalized \( c_0 \)-unconditional basic sequence in a Banach space. Then either

1. [Case \( \xi^{(\chi_n)}_b = \omega_1 \)] \( (\chi_n) \) has a subsequence equivalent to the unit vector basis of \( c_0 \); or
2. [Case \( \xi^{(\chi_n)}_b < \omega_1 \)] \( (\chi_n) \) is semiboundedly complete (equivalently, null coefficient).

In case (2) there exists a countable ordinal \( \zeta \) such that \( \xi^{(\chi_n)}_0 = \xi^{(\chi_n)}_b = \omega^\zeta \). Then, for each countable ordinal \( \alpha \), either

1. [Case \( \alpha < \zeta \)] \( (\chi_n) \) has a subsequence with a \( c_0 \)-spreading model of order \( \alpha \); or
2. [Case \( \zeta \leq \alpha \)] \( (\chi_n) \) is semiboundedly complete of order \( \alpha \) (equivalently, null coefficient of order \( \alpha \)).

**Proof.** This follows from Theorems 2.15 and 3.10, Remark 3.2(i) and Propositions 3.11 and 3.12.

At this point the following question naturally arises: Is it true that \( \xi^{(\chi_n)}_0 = \xi^{(\chi_n)}_b \) for every normalized weakly null basic sequence \( (\chi_n) \)? The answer is negative, as follows from the example below.
Example 3.14 (James’ space; [J]). For a sequence \((\lambda_n)\) of real numbers we set
\[
\| (\lambda_n) \| = \sup \{ (\lambda_{p_1} - \lambda_{p_2})^2 + \cdots + (\lambda_{p_{m-1}} - \lambda_{p_m})^2 + (\lambda_{p_m} - \lambda_{p_1})^2)^{1/2} : m \in \mathbb{N} \text{ and } p_1 < \cdots < p_m \}.
\]
The vector space
\[
X = \{ (\lambda_n) \in \mathbb{R}^\mathbb{N} : \lim_n \lambda_n = 0 \text{ and } \| (\lambda_n) \| < \infty \}
\]
is a Banach space with respect to the norm \(\| \cdot \|\).

For \(n \in \mathbb{N}\) let \(e_n = (\lambda^n_m)_{m \in \mathbb{N}}\) with \(\lambda^n_m = 0\) if \(n \neq m\) and \(\lambda^n_m = 1\) if \(n = m\). The sequence \((e_n)\) is a normalized, weakly null, basic sequence in \(X\). We will prove that \(\xi_0^{(e_n)} = \omega\) and \(\xi_0^{(e_n)} = \omega_1\):
(i) \(\xi_0^{(e_n)} = \omega\). Indeed, suppose \(\xi_0^{(e_n)} > \omega\). Then, according to Proposition 2.10, there exists a subsequence \((y_n)\) of \((e_n)\) and \(\varepsilon > 0\) such that
\[
\left\| \sum_{i \in F} y_i \right\| \leq \varepsilon \quad \text{for every } F \in \mathcal{F}_1.
\]
Set \(F_n = (n + 2, n + 4, \ldots, n + 2n)\) for every \(n \in \mathbb{N}\). Of course, \(F_n \in \mathcal{F}_1\) for every \(n \in \mathbb{N}\). Setting \(p_i = n + i\) for \(1 \leq i \leq 2n\) we have
\[
\left\| \sum_{i \in F_n} y_i \right\| \geq (2n)^{1/2} \quad \text{for every } n \in \mathbb{N}.
\]
This is a contradiction, hence \(\xi_0^{(e_n)} = \omega\).

(ii) \(\xi_0^{(e_n)} = \omega_1\). Indeed, \(\| \sum_{i=1}^n e_i \| \leq 1\) for every \(n \in \mathbb{N}\). So, according to Remark 3.2(i), we have \(\xi_0^{(e_n)} = \omega_1\).

From the previous example it is clear that we cannot hope for a theorem analogous to Theorem 3.13 in the general case of a normalized weakly null sequence (not necessarily \(c_0\)-unconditional). However, using Elton’s theorem (Remark 3.9), we can prove the following dichotomy, which generalizes (to every countable ordinal) the Odell theorem (case \(\alpha = \omega_1\)).

Theorem 3.15. Let \((\chi_n)\) be a normalized weakly null sequence in a Banach space and \(\alpha\) be a countable ordinal. Then either
(i) \((\chi_n)\) has a subsequence with \(c_0\)-spreading model of order \(\alpha\); or
(ii) every subsequence of \((\chi_n)\) has a subsequence semiboundedly complete of order \(\alpha\).

Proof. Let \(\alpha\) be a countable ordinal and \((y_n)\) a subsequence of \((\chi_n)\). The sequence \((y_n)\) has a subsequence \((z_n)\) which is \(c_0\)-unconditional and basic. If \(\xi_0^{(z_n)} = \omega_1\), then \((z_n)\) has a subsequence equivalent to the unit vector basis of \(c_0\), hence \((\chi_n)\) has a subsequence with \(c_0\)-spreading model of order \(\alpha\). If \(\xi_0^{(z_n)} < \omega_1\), then \(\xi_0^{(z_n)} = \omega^{\zeta}\) for some countable ordinal \(\zeta\). Hence, in case
α < ζ the sequence (χₙ) has a subsequence with c₀-spreading model of order α, and in case ζ ≤ α the sequence (χₙ) has a subsequence semiboundedly complete of order α, according to Theorem 3.13.

References


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