FUNDAMENTA
MATHEMATICAE
172 (2002)

Ordinal indices and Ramsey dichotomies measuring c_0-content and semibounded completeness

by

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Abstract. We study the c_0-content of a seminormalized basic sequence (χ_n) in a Banach space, by the use of ordinal indices (taking values up to ω_1) that determine dichotomies at every ordinal stage, based on the Ramsey-type principle for every countable ordinal, obtained earlier by the author. We introduce two such indices, the c_0-index $\xi_0^{(\chi_n)}$ and the semibounded completeness index $\xi_b^{(\chi_n)}$, and we examine their relationship. The countable ordinal values that these indices can take are always of the form ω^ζ. These results extend, to the countable ordinal level, an earlier result by Odell, which was stated only for the limiting case of the first uncountable ordinal.

Introduction. In this paper we study the precise c_0-content of an arbitrary (seminormalized and basic) sequence (χ_n) in a Banach space, measured by the c_0-*index* $\xi_0^{(\chi_n)}$ defined for any such sequence. As this index is a countable ordinal of the form ω^ζ or equal to the first uncountable ordinal ω_1, on the one hand we give dichotomy conditions, separating the basic classes $\xi_0^{(\chi_n)} = \omega_1$ and $\xi_0^{(\chi_n)} < \omega_1$, and on the other hand, we characterize the spectrum of the states precisely quantified by the countable ordinals.

The main tools, combinatorial in nature, consist of the Ramsey-type principle for every countable ordinal, proved in [F1], and of the Pták-type theorem for every countable ordinal, proved also in [F1]. In the statements of these theorems we make use of the complete thin Schreier system of families $(\mathcal{A}_\xi)_{\xi<\omega_1}$, introduced in [F1]. Closely connected with this system is the generalized Schreier system $(\mathcal{F}_\alpha)_{\alpha<\omega_1}$, defined in [A-A], which is often used in the present paper.

In order to state our main results, we need the following definitions:

(i) (χ_n) has c_0-*spreading model of order* α, for some $1 \le \alpha < \omega_1$, if there exist $A, B > 0$ such that

2000 *Mathematics Subject Classification*: Primary 46B25; Secondary 46B45.

$$A \max_{i \in F} |\lambda_i| \le \Big\| \sum_{i \in F} \lambda_i \chi_i \Big\| \le B \max_{i \in F} |\lambda_i| \quad \text{for all } \mathcal{F} \in \mathcal{F}_\alpha \text{ and } (\lambda_i)_{i \in F} \subseteq \mathbb{R}.$$

(ii) (χ_n) is *null coefficient of order* α, for some $1 \le \alpha < \omega_1$, if every sequence (λ_n) of real numbers with $\sup\{\|\sum_{i \in F} \lambda_i \chi_i\| : F \in \mathcal{F}_\alpha\} < \infty$ converges to zero; ((χ_n) is *null coefficient* if $\mathcal{F}_\alpha$ can be replaced by the family $[\mathbb{N}]^{<\omega}$ of all finite subsets of $\mathbb{N}$).

That these two properties of a sequence (χ_n) are naturally exclusive for every ordinal α, is the content of the following theorem (Theorem 2.15).

Theorem A. *Let* (χ_n) *be a basic bounded sequence in a Banach space, with* $0 < \inf_n \|\chi_n\|$. *Then either*

(1) [Case $\xi_0^{(\chi_n)} = \omega_1$] (χ_n) *has a subsequence equivalent to the* c_0*-basis*; *or*

(2) [Case $\xi_0^{(\chi_n)} < \omega_1$] (χ_n) *is null coefficient.*

In case (2) *there exists a countable ordinal* ζ (*in fact* $\xi_0^{(\chi_n)} = \omega^\zeta$) *such that for each countable ordinal* α, *either*

(2i) [Case $\alpha < \zeta$] (χ_n) *has a subsequence with* c_0*-spreading model of order* α; *or*

(2ii) [Case $\zeta \le \alpha$] (χ_n) *is null coefficient of order* α.

Next (in Section 3) we introduce and study the *semibounded completeness index* $\xi_b^{(\chi_n)}$ of a sequence (χ_n) (Definition 3.1) and its relation to the c_0-index. The index $\xi_b^{(\chi_n)}$ is countable if and only if (χ_n) is semiboundedly complete, i.e., when every sequence (λ_n) of real numbers with $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| < \infty$ converges to zero. In this case $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ (Proposition 3.3); we thus define a sequence (χ_n) to be *semiboundedly complete of order* α, for some $1 \le \alpha < \omega_1$, if $\xi_b^{(\chi_n)} \le \omega^\alpha$ and equivalently if for every $M \in [\mathbb{N}]$ there exists a strictly increasing function $\varphi : \mathbb{N} \to M$ with the property: for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\{\varphi(n) : n \ge n_0 \text{ and } |\lambda_{\varphi(n)}| \ge \varepsilon\} \in (\mathcal{A}_{\xi_{n_0}})^\star \setminus \mathcal{A}_{\xi_{n_0}}$$

for every $(\lambda_n) \subseteq \mathbb{R}$ with $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| \le 1$, where (ξ_n) is a strictly increasing sequence of ordinals with $\sup_n \xi_n = \omega^\alpha$.

The c_0-index is always less than or equal to the semibounded completeness index (Proposition 3.6), but they differ in general. We give an example of a normalized, weakly null, basic sequence (χ_n) with $\xi_0^{(\chi_n)} = \omega$ and $\xi_b^{(\chi_n)} = \omega_1$ (Example 3.14). For normalized c_0-unconditional sequences (Definition 3.8) we prove (in Theorem 3.10) that the c_0-index is indeed equal to the semibounded completeness index. Thus, a normalized c_0-unconditional sequence is semiboundedly complete of order α, for some $1 \le \alpha \le \omega_1$, if and only if it is null coefficient of order α; and equivalently, if it does not contain a subsequence with c_0-spreading model of order α.

Since every normalized, weakly null sequence in a Banach space has a c_0-unconditional subsequence (according to a result of Elton [E]), we have the following dichotomy (Theorem 3.15), which constitutes a countable ordinal analogue of Odell's limiting (for $\alpha = \omega_1$) theorem.

THEOREM B. *Let (χ_n) be a normalized weakly null sequence in a Banach space and α be a countable ordinal. Then either*

(i) *(χ_n) has a subsequence with c_0-spreading model of order α; or*

(ii) *every subsequence of (χ_n) has a subsequence semiboundedly complete of order α.*

NOTATION. We denote by $\mathbb{N} = \{1, 2, \ldots\}$ the set of all natural numbers and by $\mathbb{R}$ the set of real numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\omega}$ the set of all finite subsets of M, by $[M]^k$ for $k \in \mathbb{N}$ the set of all k-element subsets of M and by $[M]$ the set of all infinite subsets of M (considering them as strictly increasing sequences).

If H, F are non-empty finite subsets of $\mathbb{N}$ then we write $H \le F$ if $\max H \le \min F$, while $H < F$ if $\max H < \min F$. By $|H|$ we denote the cardinality of H.

Identifying every subset of $\mathbb{N}$ with its characteristic function, we topologize the set of all subsets of $\mathbb{N}$ by the topology of pointwise convergence.

For a family $\mathcal{F}$ of finite subsets of $\mathbb{N}$ and $M = (m_i) \in [\mathbb{N}]$ we write:

$$\begin{aligned}
\mathcal{F}[M] &= \mathcal{F} \cap [M]^{<\omega},\\
\mathcal{F}(M) &= \{(m_{n_1}, \ldots, m_{n_k}) \in [M]^{<\omega} : (n_1, \ldots, n_k) \in \mathcal{F}\},\\
\mathcal{F}_\star &= \{H \in [\mathbb{N}]^{<\omega} : H \subseteq F \text{ for some } F \in \mathcal{F}\}.\\
\mathcal{F}^\star &= \{H \in [\mathbb{N}]^{<\omega} : H \text{ is an initial segment of some } F \in \mathcal{F}\}.
\end{aligned}$$

$\mathcal{F}$ is *hereditary* if $\mathcal{F}_\star = \mathcal{F}$.

$\mathcal{F}$ is *thin* if there do not exist $H, F \in \mathcal{F}$ such that H is a proper initial segment of F.

1. The basic combinatorial tools. In this section we recall some known combinatorial results which play a major role in our proofs.

DEFINITION 1.1 (The generalized Schreier system; [A-A]). Set

$$\mathcal{F}_0 = \{\{n\} : n \in \mathbb{N}\};$$

if $\mathcal{F}_\alpha$ has been defined then

$$\mathcal{F}_{\alpha+1} = \Big\{ \bigcup_{i=1}^{n} H_i : n \le H_1 < \ldots < H_n \text{ and } H_1, \ldots, H_n \in \mathcal{F}_\alpha \Big\};$$

and if α is a limit ordinal, fix a strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ of ordinal numbers with $\sup_n \alpha_n = \alpha$ and set

$$\mathcal{F}_\alpha = \{H : H \in \mathcal{F}_{\alpha_n} \text{ and } n \leq \min H\}.$$

We finally set $\mathcal{F}_{\omega_1} = \{H : H \text{ is a finite subset of } \mathbb{N}\}$.

REMARK 1.2. (i) If $A \in \mathcal{F}_\alpha$ for some $1 \leq \alpha \leq \omega_1$, and $B \subseteq A$, then $B \in \mathcal{F}_\alpha$. In other words the families $\mathcal{F}_\alpha$ are *hereditary.*

(ii) It is easy to prove by induction that whenever $\{n_1, \ldots, n_k\} \in \mathcal{F}_\alpha$ and $m_i \geq n_i$ for every $i = 1, \ldots, k$, then $(m_1, \ldots, m_k) \in \mathcal{F}_\alpha$.

(iii) For every $\beta < \alpha < \omega_1$ there exists $n_0 = n_0(\beta, \alpha) \in \mathbb{N}$ such that if $F \in \mathcal{F}_\beta$ and $n_0 < F$, then $F \in \mathcal{F}_\alpha$.

Now, we recall the definition of the complete thin Schreier system $(\mathcal{A}_\xi)_{\xi<\omega_1}$, defined in [F1].

DEFINITION 1.3 (The complete thin Schreier system; [F1]). For every non-zero limit ordinal α we fix a strictly increasing sequence (α_n) of successor ordinals smaller than α with $\sup_n \alpha_n = \alpha$. We define the system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ recursively as follows:

(1) [Case $\xi = 1$]

$$\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\};$$

(2) [Case $\xi = \zeta + 1$]

$$\mathcal{A}_\xi = \mathcal{A}_{\zeta+1} = \{s \subseteq \mathbb{N} : s = \{n\} \cup s_1, \text{ where } n \in \mathbb{N},\ \{n\} < s_1 \text{ and } s_1 \in \mathcal{A}_\zeta\};$$

(3) [Case $\xi = \omega^{\beta+1}$, β a countable ordinal]

$$\mathcal{A}_\xi = \mathcal{A}_{\omega^{\beta+1}} = \Big\{s \subseteq \mathbb{N} : s = \bigcup_{i=1}^{n} s_i \text{ with } n = \min s_1,\ s_1 < \ldots < s_n, \\ \text{and } s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\Big\};$$

(4) [Case $\xi = \omega^\alpha$, α a non-zero countable limit ordinal]

$$\mathcal{A}_\xi = \mathcal{A}_{\omega^\alpha} = \{s \subseteq \mathbb{N} : s \in \mathcal{A}_{\omega^{\alpha_n}} \text{ with } n = \min s\}$$

(where (α_n) is the sequence of ordinals converging to α, fixed above);

(5) [Case ξ limit, $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$] Let $\xi = p\omega^\alpha + \sum_{i=1}^m p_i\omega^{\alpha_i}$ be the canonical representation of ξ, where $m \geq 0$, $p, p_1, \ldots, p_m \geq 1$ are natural numbers so that either $p > 1$, or $p = 1$ and $m \geq 1$ and $\alpha > \alpha_1 > \ldots > \alpha_m > 0$ are countable ordinals. Then

$$\mathcal{A}_\xi = \Big\{s \subseteq \mathbb{N} : s = s_0 \cup \bigcup_{i=1}^{m} s_i \text{ with } s_m < \ldots < s_1 < s_0, \\ s_0 = s_1^0 \cup \ldots \cup s_p^0 \text{ with } s_1^0 < \ldots < s_p^0,\ s_j^0 \in \mathcal{A}_{\omega^\alpha},\ 1 \leq j \leq p, \\ s_i = s_1^i \cup \ldots \cup s_{p_i}^i \text{ with } s_1^i < \ldots < s_{p_i}^i,\ s_j^i \in \mathcal{A}_{\omega^{\alpha_i}},\ 1 \leq i \leq m,\ 1 \leq j \leq p_i\Big\}.$$

We set $\mathcal{B}_\alpha = \mathcal{A}_{\omega^\alpha}$ for each $1 \le \alpha < \omega_1$.

REMARK 1.4. (i) Each family $\mathcal{A}_\xi$ for $1 \le \xi < \omega_1$ is *thin* (does not contain proper initial segments of its elements).

(ii) ([F1]) Each finite subset F of $\mathbb{N}$ has a *canonical representation* with respect to the family $\mathcal{A}_\xi$. This means that for every $1 \le \xi < \omega_1$ there exist unique $n \in \mathbb{N}$, sets $s_1, \ldots, s_n \in \mathcal{A}_\xi$ and s_{n+1}, a proper initial segment of some element of $\mathcal{A}_\xi$, with $s_1 < \ldots < s_n < s_{n+1}$, such that $F = \bigcup_{i=1}^{n+1} s_i$. The number n is called the *type* $t_\xi(F)$ of F with respect to $\mathcal{A}_\xi$.

(iii) ([F1]) For every $0 \le \alpha < \omega_1$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha)_\star \subseteq \mathcal{F}_\alpha$.

Now we give the definition of the strong Cantor–Bendixson index of a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$. This index is analogous to the well-known Cantor–Bendixson index ([B], [C]) and has been defined in [A-M-T] and with a different notation in [F1].

DEFINITION 1.5 ([C], [B], [A-M-T]). Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets on $\mathbb{N}$. For $M \in [\mathbb{N}]$ we define the *strong Cantor–Bendixson derivative* $(\mathcal{F})^\xi_M$ of $\mathcal{F}$ on M for every $\xi < \omega_1$ as follows:

$$(\mathcal{F})^1_M = \{F \in \mathcal{F}[M] : F \text{ is a cluster point of } \mathcal{F}[F \cup L] \text{ for each } L \in [M]\}$$

(where $\mathcal{F}[M] = \mathcal{F} \cap [M]^{<\omega}$),

$$(\mathcal{F})^{\xi+1}_M = \big((\mathcal{F})^\xi_M\big)^1_M, \qquad (\mathcal{F})^\xi_M = \bigcap_{\beta<\xi} (\mathcal{F})^\beta_M \quad \text{if } \xi \text{ is a limit ordinal.}$$

The *strong Cantor–Bendixson index* of $\mathcal{F}$ on M is defined to be the smallest countable ordinal ξ such that $(\mathcal{F})^\xi_M = \emptyset$. We denote this index by $s_M(\mathcal{F})$.

REMARK 1.6. (i) $s_M(\mathcal{F})$ is a countable successor ordinal.

(ii) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $s_M(\mathcal{F}_1) \le s_M(\mathcal{F}_2)$ for every $M \in [\mathbb{N}]$.

(iii) If L is almost contained in M (i.e. $L - M$ is finite), then $s_L(\mathcal{F}) \ge s_M(\mathcal{F})$.

(iv) For every $M \in [\mathbb{N}]$ and $F \in [M]^{<\omega}$, according to a remark in [Ju], we have: $F \in (\mathcal{F})^1_M$ if and only if the set $\{m \in M : F \cup \{m\} \notin \mathcal{F}\}$ is finite.

(v) ([A-M-T]) $s_M(\mathcal{F}_\alpha) = \omega^\alpha + 1$ for every $1 \le \alpha < \omega_1$ and $M \in [\mathbb{N}]$.

(vi) ([F1]) $s_M((\mathcal{A}_\xi)_\star) = \xi + 1$ for every $1 \le \xi < \omega_1$ and $M \in [\mathbb{N}]$.

(vii) ([A-M-T], [Ju], [F1]) If $\mathcal{F}$ is a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$ and $M \in [\mathbb{N}]$ is such that $s_M(\mathcal{F}) \ge \omega^\alpha$, then there exists $L \in [M]$ such that $\mathcal{F}_\alpha(L) \subseteq \mathcal{F}$.

We recall the generalization (proved in [F1]) of the classical Ramsey theorem to every countable ordinal.

Theorem 1.7 (ξ-Ramsey type theorem; [F1]). *Let $\mathcal{F}$ be an arbitrary family of finite subsets of $\mathbb{N}$, M an infinite subset of $\mathbb{N}$ and ξ a countable ordinal number. Then there exists an infinite subset L of M such that*

$$\text{either}\quad \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq \mathcal{F} \quad \text{or} \quad \mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

Using the strong Cantor–Bendixson index, we have developed in [F1] a refined form of the above theorem in case $\mathcal{F}$ is in addition hereditary.

Theorem 1.8 (Refined ξ-Ramsey type theorem; [F1]). *Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. We have the following cases*:

Case 1: *If the family $\mathcal{F}\cap[M]^{<\omega}$ is not pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.*

Case 2: *If the family $\mathcal{F} \cap [M]^{<\omega}$ is pointwise closed, then there exists $L \in [M]$ such that $[L]^{<\omega} \subseteq ([\mathbb{N}]^{<\omega} \setminus \mathcal{F})_\star$. Moreover setting*

$$\xi_M^{\mathcal{F}} = \sup\{s_L(\mathcal{F}) : L \in [M]\},$$

which is a countable ordinal, the following hold:

2(i) *For every countable ordinal ξ with $\xi+1 < \xi_M^{\mathcal{F}}$ there exists $L \in [M]$ such that*

$$(\mathcal{A}_\xi)_\star \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

2(ii) *For every countable ordinal ξ with $\xi_M^{\mathcal{F}} < \xi+1$ there exists $L \in [M]$ such that*

$$\mathcal{F} \cap [L]^{<\omega} \subseteq (\mathcal{A}_\xi)^\star \setminus \mathcal{A}_\xi;$$

and equivalently,

$$\mathcal{A}_\xi \cap [L]^{<\omega} \subseteq [\mathbb{N}]^{<\omega} \setminus \mathcal{F}.$$

2(iii) *If $\xi_M^{\mathcal{F}} = \xi + 1$, then both alternatives may materialize.*

Now we recall the ξ-Pták type theorem for some $1 \le \xi < \omega_1$, which has been proved in [F1], using the notion of the weight of a finite subset F of $\mathbb{N}$ with respect to a set of the family $\mathcal{A}_\xi$. The classical Pták theorem is the limiting ω_1-case.

Definition 1.9. For every finite subset F of $\mathbb{N}$, every countable ordinal ξ, and every $s \in \mathcal{A}_\xi$ we define recursively the ξ-*weight* $w_\xi(F;s)$ of F with respect to s to be a real (in fact, rational) number in $[0,1]$, as follows:

(1) [Case $\xi = 1$] Since $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$, for every $n \in \mathbb{N}$ we set

$$w_1(F;\{n\}) = \begin{cases} 1 & \text{if } n \in F, \\ 0 & \text{otherwise.} \end{cases}$$

(2) [Case $\xi = \zeta + 1$] Let $s \in \mathcal{A}_{\zeta+1}$. Then $s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in \mathcal{A}_\zeta$. We set

$$w_{\zeta+1}(F; s) = w_\zeta(F; s_1) \cdot w_1(F; \{n\}).$$

(3) [Case $\xi = \omega^{\beta+1}$ for $0 \leq \beta < \omega_1$] Let $s \in \mathcal{A}_{\omega^{\beta+1}}$. Then $s = s_1 \cup \ldots \cup s_n$, with $n = \min s_1$, $s_1 < \ldots < s_n$ and $s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}$. We set

$$w_{\omega^{\beta+1}}(F; s) = \frac{1}{n} \sum_{i=1}^{n} w_{\omega^\beta}(F; s_i).$$

(4) [Case $\xi = \omega^\alpha$ for α a non-zero countable limit ordinal] Let $s \in \mathcal{A}_{\omega^\alpha}$. Then $s \in \mathcal{A}_{\omega^{\alpha_n}}$ with $n = \min s$, where (α_n) is the fixed sequence of ordinals "converging" to α (Definition 1.3). So,

$$w_{\omega^\alpha}(F; s) = w_{\omega^{\alpha_n}}(F; s), \qquad n = \min s.$$

(5) [Case ξ limit, $\omega^{\alpha_0} < \xi < \omega^{\alpha_0+1}$ for some $0 < \alpha_0 < \omega_1$] In this case, ξ has a unique representation $\xi = p_0\omega^{\alpha_0} + \sum_{i=1}^{m} p_i\omega^{\alpha_i}$, where $m \in \mathbb{N}$, $\alpha_0 > \alpha_1 > \ldots > \alpha_m > 0$ are ordinal numbers and $p_0, p_1, \ldots, p_m \geq 1$ are natural numbers, so that either $p_0 > 1$, or $p_0 = 1$ and $m > 1$.

Let $s \in \mathcal{A}_\xi$. Then $s = s_0 \cup s_1 \cup \ldots \cup s_m$ with $s_m < \ldots < s_1 < s_0$, where $s_i = s_1^i \cup \ldots \cup s_{p_i}^i$ with $s_1^i < \ldots < s_{p_i}^i$ and $s_j^i \in \mathcal{A}_{\omega^{\alpha_i}}$ for every $0 \leq i \leq m$ and $1 \leq j \leq p_i$. We set

$$w_\xi(F; s) = \prod_{i=0}^{m} \prod_{j=1}^{p_i} w_{\omega^{\alpha_i}}(F; s_j^i).$$

Remark 1.10 ([A-O], [F1]). For every countable ordinal α and $s \in \mathcal{A}_{\omega^\alpha} = \mathcal{B}_\alpha$ we define recursively the functions $\varphi_\alpha^s : \mathbb{N} \to [0, \infty)$ as follows:

- $\varphi_{\{k\}}^0(n) = 1$ if $n = k$, and $\varphi_{\{k\}}^0(n) = 0$ otherwise, for every $\{k\} \in \mathcal{B}_0$.
- $\varphi_s^{\beta+1} = k^{-1} \sum_{i=1}^{k} \varphi_{s_i}^\beta$ for every $s = s_1 \cup \ldots \cup s_k \in \mathcal{B}_{\beta+1}$.
- $\varphi_s^\alpha = \varphi_s^{\alpha_k}$, $k = \min s$, for every $s \in \mathcal{B}_\alpha$, where α is a non-zero countable limit ordinal.

It is easy to see that $\sum_{n \in \mathbb{N}} \varphi_s^\alpha(n) = 1$ and that $s = \{n \in \mathbb{N} : \varphi_\alpha^s(n) \neq 0\}$. Moreover $w_{\omega^\alpha}(F; s) = \sum_{n \in F} \varphi_\alpha^s(n)$ for every $F \in [\mathbb{N}]^{<\omega}$.

Theorem 1.11 (ξ-Pták type theorem; [F1]). *Let $\mathcal{F}$ be a hereditary and pointwise closed family of finite subsets of $\mathbb{N}$, $M \in [\mathbb{N}]$, ξ a non-zero countable ordinal and $0 < \varepsilon < 1$. If for every $s \in \mathcal{A}_\xi \cap [M]^{<\omega}$ there exists $F \in \mathcal{F}$ such that $w_\xi(F; s) > \varepsilon$, then*:

(i) *there exists $L \in [M]$ such that $s_L(\mathcal{F}) \geq \xi + 1$;*
(ii) $\xi_M^{\mathcal{F}} \leq \xi + 1$, *and*
(iii) *for every ordinal ζ with $\zeta < \xi$ there exists $L \in [M]$ such that*

$$\mathcal{A}_\zeta \cap [L]^{<\omega} \subseteq \mathcal{F}.$$

THEOREM 1.12 (Pták's theorem; [P]). *Let $\mathcal{F}$ be a hereditary family of finite subsets of $\mathbb{N}$ and $0 < \varepsilon < 1$. If for every non-negative function φ on $\mathbb{N}$ with finite support and $\sum_{n\in\mathbb{N}} \varphi(n) = 1$ there exists $F \in \mathcal{F}$ such that $\sum_{n\in F} \varphi(n) > \varepsilon$, then there exists $L \in [\mathbb{N}]$ such that $[L]^{<\omega} \subseteq \mathcal{F}$.*

2. The c_0-behavior of a sequence. In this section we study the precise "c_0-content" of an arbitrary (seminormalized, basic) sequence in a Banach space, with the help of the c_0-index defined for any such sequence; this is a countable ordinal of the form $\xi_0 = \omega^\zeta$, or the first uncountable ordinal ω_1 (Proposition 2.5). This index is a measure of the c_0-content of the sequence in the following sense:

(i) If $\xi_0 = \omega_1$, then there is a subsequence equivalent to the unit vector basis of c_0 (Remark 2.2).

(ii) If $\xi_0 < \omega_1$, then there exists a countable ordinal ζ such that:

- (iia) on the one hand, for all $\alpha < \zeta$ there is a subsequence with c_0-spreading model of order α (Proposition 2.10), while
- (iib) on the other hand (if $\zeta \le \alpha$) the sequence is far from any higher order c_0-behavior, in the sense that it is a null coefficient sequence of order ζ (Proposition 2.13).

This is the content of the main theorem (Theorem 2.15).

DEFINITION 2.1. Let (χ_n) be a bounded sequence in a Banach space X. For every $\varepsilon > 0$ we set

$$C_\varepsilon^{(\chi_n)} = \Big\{F \in [\mathbb{N}]^{<\omega} : \Big\|\sum_{i\in F} \lambda_i \chi_i\Big\| \le \varepsilon \max_{i\in F} |\lambda_i| \text{ for all } (\lambda_i)_{i\in F} \subseteq \mathbb{R}\Big\}.$$

All the families $C_\varepsilon^{(\chi_n)}$ for $\varepsilon > 0$ are hereditary.

We then define the c_0-*index* $\xi_0^{(\chi_n)}$ of (χ_n) as follows: If the families $C_\varepsilon^{(\chi_n)}$ for all $\varepsilon > 0$ are pointwise closed, we set

$$\xi_0^{(\chi_n)} = \sup\{s_M(C_\varepsilon^{(\chi_n)}) : M \in [\mathbb{N}] \text{ and } \varepsilon > 0\},$$

which is a countable ordinal; otherwise

$$\xi_0^{(\chi_n)} = \omega_1.$$

REMARK 2.2. (i) $\xi_0^{(\chi_n)} = \omega_1$ if and only if there exist $\varepsilon > 0$ and $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq C_\varepsilon^{(\chi_n)}$ (Theorem 1.8).

(ii) For a basic sequence (χ_n) in a Banach space X with $0 < \inf_n \|\chi_n\|$ there exists $A > 0$ such that

$$A \max_{1\le i\le n} |\lambda_i| \le \Big\|\sum_{i=1}^n \lambda_i \chi_i\Big\| \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$

(iii) A basic sequence (χ_n) with $0 < \inf_n \|\chi_n\| \leq \sup_n \|\chi_n\| < \infty$ has a subsequence equivalent to the unit vector basis of c_0 if and only if $\xi_0^{(\chi_n)} = \omega_1$.

DEFINITION 2.3. A sequence (χ_n) in a Banach space X is called

(i) *null coefficient* (of order ω_1) if every sequence (λ_n) of real numbers with $\sup\{\|\sum_{i\in F} \lambda_i \chi_i\| : F \in [\mathbb{N}]^{<\omega}\} < \infty$ converges to zero; and

(ii) *null coefficient of order* α, for some countable ordinal α, if every sequence (λ_n) of real numbers with $\sup\{\|\sum_{i\in F} \lambda_i \chi_i\| : F \in \mathcal{F}_\alpha\} < \infty$ converges to zero.

PROPOSITION 2.4. *Let (χ_n) be a bounded sequence in a Banach space X. The following are equivalent*:

(i) $\xi_0^{(\chi_n)} < \omega_1$;
(ii) (χ_n) *is null coefficient.*

Proof. (i)$\Rightarrow$(ii). Let $\xi_0^{(\chi_n)} < \omega_1$. Assume that (χ_n) is not null coefficient. Then there exist $(\mu_n) \subseteq \mathbb{R}$ and $\varepsilon > 0$ such that $\|\sum_{i\in F} \mu_i \chi_i\| \leq 1$ for every $F \in [\mathbb{N}]^{<\omega}$ and the set $M = \{n \in \mathbb{N} : \mu_n \geq \varepsilon\}$ is infinite.

Let $F \in [M]^{<\omega}$ and $(\lambda_i)_{i\in F} \subseteq \mathbb{R}$. There exists $f \in X^\star$ with $\|f\| \leq 1$ such that $\|\sum_{i\in F} \lambda_i \chi_i\| = f(\sum_{i\in F} \lambda_i \chi_i)$. Since

$$\begin{aligned}\Big\|\sum_{i\in F} \lambda_i \chi_i\Big\| &= \sum_{i\in F} \lambda_i f(\chi_i) \leq \sum_{i\in F} |\lambda_i| \cdot |f(\chi_i)| \\ &= \sum_{i\in F} |\lambda_i| \varepsilon_i f(\chi_i) \quad \text{(for suitable } (\varepsilon_i)_{i\in F} \subseteq \{-1,1\}) \\ &\leq \frac{1}{\varepsilon} \sum_{i\in F} |\lambda_i| \mu_i \varepsilon_i f(\chi_i) \leq \frac{1}{\varepsilon} (\max_{i\in F} |\lambda_i|) \cdot \Big\|\sum_{i\in F} \mu_i \varepsilon_i \chi_i\Big\| \\ &\leq \frac{2}{\varepsilon} \max_{i\in F} |\lambda_i|,\end{aligned}$$

we see that $[M]^{<\omega} \subseteq C_{2/\varepsilon}^{(\chi_n)}$. This is a contradiction (see Remark 2.2(i)); hence, (χ_n) is null coefficient.

(ii)$\Rightarrow$(i). Let (χ_n) be null coefficient. If $\xi_0^{(\chi_n)} = \omega_1$, then there exist $\varepsilon > 0$ and $M \in [\mathbb{N}]$ such that $[M]^{<\omega} \subseteq C_\varepsilon^{(\chi_n)}$. Thus $\|\sum_{i\in F} \chi_i\| \leq \varepsilon$ for every $F \in [M]^{<\omega}$. Setting $\lambda_n = 1$ for every $n \in M$ and $\lambda_n = 0$ for $n \in \mathbb{N} \setminus M$ we have

$$\sup\Big\{\Big\|\sum_{i\in F} \lambda_i \chi_i\Big\| : F \in [\mathbb{N}]^{<\omega}\Big\} < \infty.$$

A contradiction; hence $\xi_0^{(\chi_n)} < \omega_1$.

PROPOSITION 2.5. *Let (χ_n) be a bounded sequence in a Banach space X. Then either $\xi_0^{(\chi_n)} = \omega_1$ or $\xi_0^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ.*

Proof. Let $\xi_0^{(\chi_n)} < \omega_1$. Then there exists a unique countable ordinal ζ such that $\omega^\zeta \le \xi_0^{(\chi_n)} < \omega^{\zeta+1}$. Arguing by contradiction suppose that $\omega^\zeta < \xi_0^{(\chi_n)}$. Then there exist $M \in [\mathbb{N}]$ and $\varepsilon > 0$ such that $\omega^\zeta < s_M(C_\varepsilon^{(\chi_n)})$. According to Remark 1.6(vii) there exists a subsequence (y_n) of (χ_n) such that $\mathcal{F}_\zeta \subseteq C_\varepsilon^{(y_n)}$. This gives $\mathcal{B}_\zeta \subseteq C_\varepsilon^{(y_n)}$, and consequently $\mathcal{A}_{k\omega^\zeta} \subseteq C_\varepsilon^{(y_n)}$ for every $k \in \mathbb{N}$. Hence (see Remark 1.6(vi)), $s_{\mathbb{N}}(C_\varepsilon^{(y_n)}) > k\omega^\zeta$ for every $k \in \mathbb{N}$.

If $y_n = \chi_{m_n}$ for every $n \in \mathbb{N}$ and $M = \{m_n : n \in \mathbb{N}\}$, then

$$s_M(C_\varepsilon^{(\chi_n)}) > k\omega^\zeta \quad \text{for every } k \in \mathbb{N}.$$

Indeed, by induction on ξ it can be proved that if $(n_1, \ldots, n_l) \in (C_\varepsilon^{(\chi_n)})_{\mathbb{N}}^\xi$, then $(m_{n_1}, \ldots, m_{n_l}) \in (C_\varepsilon^{(y_n)})_M^\xi$.

So, we have $\xi_0^{(\chi_n)} > k\omega^\zeta$ for every $k \in \mathbb{N}$. But this is impossible, since $\xi_0^{(\chi_n)} < \omega^{\zeta+1}$; hence $\xi_0^{(\chi_n)} = \omega^\zeta$.

The previous proposition and the refined ξ-Ramsey type theorem (Theorem 1.8) give the following equivalences:

PROPOSITION 2.6. *Let (χ_n) be a bounded sequence in a Banach space X with $\xi_0^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ. For an arbitrary countable ordinal α we have*:

(i) $\alpha < \zeta$ *if and only if there exist* $L \in [\mathbb{N}]$ *and* $\varepsilon > 0$ *with*

$$\mathcal{B}_\alpha \cap [L]^{<\omega} \subseteq C_\varepsilon^{(\chi_n)};$$

(ii) $\zeta \le \alpha$ *if and only if for every* $\varepsilon > 0$ *and* $M \in [\mathbb{N}]$ *there exists* $L \in [M]$ *such that*

$$C_\varepsilon^{(\chi_n)} \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha)^\star \setminus \mathcal{B}_\alpha.$$

Proof. (i) If $\alpha < \zeta$, then $\omega^\alpha + 1 < \xi_0^{(\chi_n)}$, since $\xi_0^{(\chi_n)}$ is a limit ordinal (Proposition 2.5). Hence, there exists $\varepsilon > 0$ such that $\omega^\alpha + 1 < \sup\{s_M(C_\varepsilon^{(\chi_n)}) : M \in [\mathbb{N}]\}$. From Theorem 1.8, there exists $L \in [\mathbb{N}]$ such that $\mathcal{B}_\alpha \cap [L]^{<\omega} \subseteq C_\varepsilon^{(\chi_n)}$.

On the other hand, if $\mathcal{B}_\alpha \cap [L]^{<\omega} \subseteq C_\varepsilon^{(\chi_n)}$, then (Remark 1.6(vi)) $s_L(C_\varepsilon^{(\chi_n)}) \ge \omega^\alpha + 1 > \omega^\alpha$; hence $\xi_0^{(\chi_n)} > \omega^\alpha$.

(ii) Let $\varepsilon > 0$ and $M \in [\mathbb{N}]$. If $\zeta \le \alpha$, then

$$\sup\{s_L(C_\varepsilon^{(\chi_n)}) : L \in [M]\} \le \xi_0^{(\chi_n)} \le \omega^\alpha < \omega^\alpha + 1.$$

From Theorem 1.8, there exists $L \in [M]$ so that $C_\varepsilon^{(\chi_n)} \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha)^\star \setminus \mathcal{B}_\alpha$.

On the other hand, if for every $\varepsilon > 0$ and $M \in [\mathbb{N}]$ there exists $L \in [M]$ such that $C_\varepsilon^{(\chi_n)} \cap [L]^{<\omega} \subseteq (\mathcal{B}_\alpha)^\star \setminus \mathcal{B}_\alpha$, then $s_M(C_\varepsilon^{(\chi_n)}) \le \omega^\alpha + 1$ for every $M \in [\mathbb{N}]$ and $\varepsilon > 0$ (Theorem 1.8). Hence, $\xi_0^{(\chi_n)} \le \omega^\alpha + 1$ and consequently $\zeta \le \alpha$.

So far we have distinguished the cases $\xi_0^{(\chi_n)} = \omega_1$ and $\xi_0^{(\chi_n)} < \omega_1$ (in Remark 2.2(i) and Proposition 2.4) and proved that in case $\xi_0^{(\chi_n)} < \omega_1$ there is $\zeta < \omega_1$ such that $\xi_0^{(\chi_n)} = \omega^\zeta$ (Proposition 2.5). In this last case the set of all countable ordinals is naturally separated by ζ into two classes, those strictly less than ζ, and those greater than or equal to ζ. To examine the behavior resulting from this dichotomy (in Propositions 2.10 and 2.13 below), we need (a) the notion of the c_0-spreading model of order α for some $1 \leq \alpha < \omega_1$ (Definition 2.7 below) and (b) the notion of the null coefficient sequence of order α (Definition 2.3).

Firstly we recall the notion of the c_0-spreading model of order α of a sequence (χ_n) for a countable number α, a notion that extends the usual notion of spreading model equivalent to the unit vector basis of c_0 (case $\alpha = 1$; [B-S]).

Definition 2.7. Let (χ_n) be a basic sequence in a Banach space X and α be a countable ordinal number. We say that (χ_n) has c_0-*spreading model of order* α if there exist $A, B > 0$ such that

$$A \max_{i \in F} |\lambda_i| \leq \Big\| \sum_{i \in F} \lambda_i \chi_i \Big\| \leq B \max_{i \in F} |\lambda_i|$$

for every $F \in \mathcal{F}_\alpha$ and $(\lambda_i)_{i \in F} \subseteq \mathbb{R}$.

Remark 2.8. If a basic sequence (χ_n) has c_0-spreading model of order α for some countable ordinal α, then every subsequence of (χ_n) has c_0-spreading model of order ζ for every ζ with $1 \leq \zeta \leq \alpha$ (see Remark 1.2(ii)).

Proposition 2.9. *Let (χ_n) be a bounded sequence in a Banach space X and α be a countable ordinal number. The following are equivalent*:

(i) *there exists $\varepsilon > 0$ such that*

$$\Big\| \sum_{i \in F} \lambda_i \chi_i \Big\| \leq \varepsilon \max_{i \in F} |\lambda_i| \quad \text{for every } F \in \mathcal{F}_\alpha \text{ and } (\lambda_i)_{i \in F} \subseteq \mathbb{R};$$

(ii) $\sup\{\sum_{i \in F} |f(\chi_i)| : F \in \mathcal{F}_\alpha\} < \infty$ *for every* $f \in X^\star$;

(iii) *there exists $B > 0$ such that*

$$\Big\| \sum_{i \in F} \chi_i \Big\| \leq B \quad \text{for every } F \in \mathcal{F}_\alpha;$$

(iv) *a sequence $(\lambda_n) \subseteq \mathbb{R}$ converges to zero if and only if for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that*

$$\Big\| \sum_{i \in F} \lambda_i \chi_i \Big\| \leq \varepsilon \quad \text{for every } F \in \mathcal{F}_\alpha \text{ with } n_0 \leq F.$$

Proof. (i)$\Rightarrow$(iv). This is easily proved, using the fact that $\{n\} \in \mathcal{F}_\alpha$ for every $n \in \mathbb{N}$.

(iv)⇒(iii). Assume that (iii) does not hold. Then there exists $F_1 \in \mathcal{F}_\alpha$ such that
$$\Big\| \sum_{i\in F_1} \chi_i \Big\| > 1.$$
Set $n_1 = \max F_1$. Then there exists $C_1 > 0$ such that
$$\Big\| \sum_{i=1}^{n_1} \lambda_i \chi_i \Big\| \le C_1 \max_{1\le i\le n_1} |\lambda_i| \quad \text{for every } \lambda_1,\dots,\lambda_{n_1} \in \mathbb{R}.$$
If $\|\sum_{i\in F}\chi_i\| \le 2$ for every $F \in \mathcal{F}_\alpha$ with $n_1 < F$, then for every $F \in \mathcal{F}_\alpha$ we have $\|\sum_{i\in F}\chi_i\| \le C_1 + 2$, contradicting our assumption. Hence, there exists $F_2 \in \mathcal{F}_\alpha$ such that $F_1 < F_2$ and
$$\Big\| \sum_{i\in F_2} \chi_i \Big\| > 2.$$
Inductively, we can define a sequence $(F_k)_{k\in\mathbb{N}}$ in $\mathcal{F}_\alpha$ with $F_k < F_{k+1}$ and
$$\Big\| \sum_{i\in F_k} \chi_i \Big\| > k \quad \text{for every } k \in \mathbb{N}.$$

We define a sequence (λ_n) in $\mathbb{R}$ as follows: $\lambda_n = 1/k$ if $n \in F_k$ for some $k \in \mathbb{N}$ and $\lambda_n = 0$ if $n \in \mathbb{N} \setminus \bigcup_{k\in\mathbb{N}} F_k$. Of course, (λ_n) converges to zero and
$$\Big\| \sum_{i\in F_k} \lambda_i\chi_i \Big\| = \frac{1}{k}\Big\| \sum_{i\in F_k} \chi_i \Big\| > 1 \quad \text{for every } k \in \mathbb{N}.$$
Since $k \le F_k$ for every $k \in \mathbb{N}$, we have a contradiction to (iv).

(iii)⇒(ii). Let $f \in X^\star$ and $F \in \mathcal{F}_\alpha$. Since the family $\mathcal{F}_\alpha$ is hereditary, condition (iii) implies that
$$\sum_{i\in F} |f(\chi_i)| = f\Big(\sum_{i\in F} \varepsilon_i\ \chi_i\Big) \le 2B\|f\|,$$
where $(\varepsilon_i)_{i\in F} \subseteq \{-1,1\}$ with $|f(\chi_i)| = \varepsilon_i f(\chi_i)$ for every $i \in F$.

(ii)⇒(i). If (ii) holds, then from the Baire category theorem we have the existence of some $k \in \mathbb{N}$ such that
$$\sup\Big\{ \sum_{i\in F} |f(\chi_i)| : F \in \mathcal{F}_\alpha \Big\} \le k \quad \text{for every } f \in X^\star \text{ with } \|f\| \le 1.$$
Let $F \in \mathcal{F}_\alpha$ and $(\lambda_i)_{i\in F} \subseteq \mathbb{R}$. Then there exists $f \in X^\star$ with $\|f\| \le 1$ such that
$$\Big\| \sum_{i\in F} \lambda_i\chi_i \Big\| = \sum_{i\in F} \lambda_i f(\chi_i).$$
Thus,
$$\Big\| \sum_{i\in F} \lambda_i\chi_i \Big\| \le (\max_{i\in F} |\lambda_i|) \cdot \sum_{i\in F} |f(\chi_i)| \le k \max_{i\in F} |\lambda_i|.$$
This finishes the proof of the proposition.

Proposition 2.10. *Let (χ_n) be a bounded sequence in a Banach space X and α be a countable ordinal number. The following are equivalent*:

(i) $\omega^\alpha < \xi_0^{(\chi_n)}$;

(ii) *there exist a subsequence (y_n) of (χ_n) and $\varepsilon > 0$ such that*

$$\Big\|\sum_{i\in F}\lambda_i y_i\Big\| \le \varepsilon \max_{i\in F}|\lambda_i| \quad \textit{for every } F\in\mathcal{F}_\alpha \textit{ and } (\lambda_i)_{i\in F}\subseteq\mathbb{R};$$

(iii) *there exist a subsequence (y_n) of (χ_n) and $\varepsilon > 0$ such that $\mathcal{B}_\alpha \subseteq C_\varepsilon^{(y_n)}$*;

(iv) *there exist a subsequence (y_n) of (χ_n), $I\in[\mathbb{N}]$ and $\varepsilon>0$ such that $\|\sum_{i\in H}\varepsilon_i y_i\|\le\varepsilon$ for every $H\in\mathcal{B}_\alpha\cap[I]^{<\omega}$ and $(\varepsilon_i)_{i\in H}\subseteq\{-1,1\}$.*

Proof. (i)$\Rightarrow$(ii). If $\omega^\alpha < \xi_0^{(\chi_n)}$, then there exist $M\in[\mathbb{N}]$ and $\varepsilon>0$ such that $\omega^\alpha < s_M(C_\varepsilon^{(\chi_n)})$. From Remark 1.6(vii) there exists $L\in[M]$ such that $\mathcal{F}_\alpha(L)\subseteq C_\varepsilon^{(\chi_n)}$.

(ii)$\Rightarrow$(iii). Since $\mathcal{B}_\alpha\subseteq\mathcal{F}_\alpha$ (see Remark 1.4(iii)), we have $\mathcal{B}_\alpha\subseteq C_\varepsilon^{(y_n)}$.

(iii)$\Rightarrow$(iv). Set $I=\mathbb{N}$.

(iv)$\Rightarrow$(i). Let $H\in\mathcal{B}_\alpha\cap[I]^{<\omega}$ and $(\lambda_i)_{i\in H}\subseteq\mathbb{R}$. There exists $f\in X^\star$ with $\|f\|\le 1$ such that $\|\sum_{i\in H}\lambda_i y_i\| = \sum_{i\in H}\lambda_i f(y_i)$. Hence,

$$\begin{aligned}\Big\|\sum_{i\in H}\lambda_i y_i\Big\| &\le (\max_{i\in H}|\lambda_i|)\sum_{i\in H}|f(y_i)|\\ &= (\max_{i\in H}|\lambda_i|)\sum_{i\in H}\varepsilon_i f(y_i) \quad (\text{for suitable } (\varepsilon_i)_{i\in H}\subseteq\{-1,1\})\\ &\le (\max_{i\in H}|\lambda_i|)\Big\|\sum_{i\in H}\varepsilon_i y_i\Big\| \le \varepsilon\max_{i\in H}|\lambda_i|.\end{aligned}$$

We have thus proved that $\mathcal{B}_\alpha\cap[I]^{<\omega}\subseteq C_\varepsilon^{(y_n)}$. According to Remark 1.6(ii), (vi), we have

$$s_I\big(C_\varepsilon^{(y_n)}\big)\ge\omega^\alpha+1>\omega^\alpha.$$

This implies that $s_M\big(C_\varepsilon^{(\chi_n)}\big)>\omega^\alpha$, where $M=\{m_n : n\in I\}$, and consequently $\omega^\alpha<\xi_0^{(\chi_n)}$.

This finishes the proof.

Corollary 2.11. *A basic sequence (χ_n) in a Banach space with $0 < \inf_n\|\chi_n\| \le \sup_n\|\chi_n| < \infty$ has a subsequence with c_0-spreading model of order α, for some countable ordinal α, if and only if $\omega^\alpha<\xi_0^{(\chi_n)}$.*

Proof. This follows from Remark 2.2(ii) and the previous proposition.

Remark 2.12. (i) A basic sequence (χ_n) in a Banach space with $0 < \inf_n\|\chi_n\|$ and $\sup_n\|\chi_n\|<\infty$ has a subsequence with c_0-spreading model of the greatest possible order if and only if either $\xi_0^{(\chi_n)} = \omega_1$ or $\xi_0^{(\chi_n)} = \omega^{\alpha+1}$ for some countable ordinal α.

(ii) If a basic sequence has for every countable ordinal α a subsequence with c_0-spreading model of order α, then it has a subsequence equivalent to the unit vector basis of c_0.

Until now we characterized the countable ordinals α with $\omega^\alpha < \xi_0^{(\chi_n)}$ as those for which (χ_n) has a subsequence with c_0-spreading model of order α. Additionally, we know that for each countable ordinal with $\xi_0^{(\chi_n)} \leq \omega^\alpha$, no subsequence has c_0-spreading model of order α. In this last case we prove (in Proposition 2.13 below) that the sequence is null coefficient of order α.

PROPOSITION 2.13. *Let (χ_n) be a bounded sequence in a Banach space X and α be a countable ordinal number. The following are equivalent*:

(i) $\xi_0^{(\chi_n)} \leq \omega^\alpha$;

(ii) *the sequence (χ_n) is null coefficient of order α*;

(iii) *for every subsequence (y_n) of (χ_n) and $M \in [\mathbb{N}]$ there exists $I \in [M]$ such that for each $H \in \mathcal{B}_\alpha \cap [I]^{<\omega}$ there exists $(\varepsilon_i)_{i\in H} \subseteq \{-1,1\}$ such that* $\min H < \|\sum_{i\in H} \varepsilon_i y_i\|$;

(iv) *for every subsequence (y_n) of (χ_n) and $M \in [\mathbb{N}]$ there exist a sequence $(H_m)_{m\in\mathbb{N}}$ in $\mathcal{B}_\alpha \cap [M]^{<\omega}$ with $H_1 < H_2 < \ldots$ and (ε_m) in $\{-1,1\}$ such that*

$$\Big\| \sum_{i\in H_m} \varepsilon_i y_i \Big\| \xrightarrow[m]{} \infty.$$

Proof. (i)⇒(ii). Let $\xi_0^{(\chi_n)} \leq \omega^\alpha$. If (χ_n) is not null coefficient of order α, then there exist $(\lambda_n) \subseteq \mathbb{R}$ and $\varepsilon > 0$ such that $\|\sum_{i\in F} \lambda_i \chi_i\| \leq 1$ for every $F \in \mathcal{F}_\alpha$ and the set $M = \{n \in \mathbb{N} : \lambda_n > \varepsilon\}$ is infinite.

Let $M = (m_n)_{n\in\mathbb{N}}$ and $y_n = \chi_{m_n}$ for every $n \in \mathbb{N}$. For every $F \in \mathcal{F}_\alpha$ and $f \in X^\star$ we have

$$\begin{aligned}\sum_{i\in F} |f(y_i)| &\leq \frac{1}{\varepsilon} \sum_{i\in F} \lambda_{m_i} |f(\chi_{m_i})| \\ &\leq \frac{1}{\varepsilon} \sum_{i\in F_1} \lambda_{m_i} f(\chi_{m_i}) + \frac{1}{\varepsilon} \sum_{i\in F_2} \lambda_{m_i} f(\chi_{m_i}) \leq \frac{2}{\varepsilon}\|f\|,\end{aligned}$$

where

$$\begin{aligned}F_1 &= \{i \in F : |f(\chi_{m_i})| = f(\chi_{m_i})\} \in \mathcal{F}_\alpha, \\ F_2 &= \{i \in F : |f(\chi_{m_i})| = -f(\chi_{m_i}\} \in \mathcal{F}_\alpha.\end{aligned}$$

According to Propositions 2.9 and 2.10 we have $\omega^\alpha < \xi_0^{(\chi_n)}$. A contradiction; hence (χ_n) is null coefficient of order α.

(ii)⇒(i). Let (χ_n) be null coefficient of order α. If $\omega^\alpha < \xi_0^{(\chi_n)}$, then according to Proposition 2.10, there exist a subsequence (y_n) of (χ_n) with

$y_n = \chi_{m_n}$ for every $n \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\Big\| \sum_{i\in F} y_i \Big\| \le \varepsilon \quad \text{for every } F \in \mathcal{F}_\alpha.$$

Let $M = \{m_n : n \in \mathbb{N}\}$. From a result of Androulakis and Odell ([An-O]) there exists $L \in [M]$ such that

$$F \setminus \{\min F\} \in \mathcal{F}_\alpha(L) \quad \text{for every } F \in \mathcal{F}_\alpha \cap [L]^{<\omega}.$$

We consider the sequence (λ_n) in $\mathbb{R}$ with $\lambda_n = 1$ if $n \in L$ and $\lambda_n = 0$ if $n \in \mathbb{N} \setminus L$. Then

$$\begin{aligned}
&\sup\Big\{\Big\| \sum_{i\in F} \lambda_i \chi_i \Big\| : F \in \mathcal{F}_\alpha\Big\}\\
&\quad = \sup\Big\{\Big\| \sum_{i\in F} \chi_i \Big\| : F \in \mathcal{F}_\alpha \cap [L]^{<\omega}\Big\} \quad \text{(since } \mathcal{F}_\alpha \text{ is hereditary)}\\
&\quad \le \sup_n \|\chi_n\| + \sup\Big\{\Big\| \sum_{i\in F} \chi_i \Big\| : F \in \mathcal{F}_\alpha(L)\Big\} \quad \text{(see Remark 1.2(ii))}\\
&\quad \le \sup_n \|\chi_n\| + \sup\Big\{\Big\| \sum_{i\in F} y_i \Big\| : F \in \mathcal{F}_\alpha\Big\}\\
&\quad \le \sup_n \|\chi_n\| + \varepsilon.
\end{aligned}$$

Since (λ_n) does not converge to zero, the sequence (χ_n) is not null coefficient. A contradiction, hence $\xi_0^{(\chi_n)} \le \omega^\alpha$.

(i)⇒(iii). Let $\xi_0^{(\chi_n)} \le \omega^\alpha$. If (y_n) is a subsequence of (χ_n) and $M \in [\mathbb{N}]$, then for every $k \in \mathbb{N}$ we set

$$\mathcal{L}_k = \Big\{ H \in [M]^{<\omega} : k < \Big\| \sum_{i\in H} \varepsilon_i y_i \Big\| \text{ for some } (\varepsilon_i)_{i\in H} \subseteq \{-1,1\}\Big\}.$$

According to Proposition 2.10 we have

$$\mathcal{L}_k \cap \mathcal{B}_\alpha \cap [I]^{<\omega} \ne \emptyset \quad \text{for every } k \in \mathbb{N} \text{ and } I \in [M].$$

Using the ω^α-Ramsey theorem (Theorem 1.7) we can construct a decreasing sequence $(I_k)_{k\in\mathbb{N}}$ in $[M]$ such that

$$\mathcal{B}_\alpha \cap [I_k]^{<\omega} \subseteq \mathcal{L}_k \quad \text{for every } k \in \mathbb{N}.$$

Set $I = (i_k^k)_{k\in\mathbb{N}}$ if $I_k = (i_n^k)_{n\in\mathbb{N}}$ for every $k \in \mathbb{N}$. Every set H in $\mathcal{B}_\alpha \cap [I]^{<\omega}$ belongs to $\mathcal{L}_k$, where $k = \min H$. Hence for each $H \in \mathcal{B}_\alpha \cap [I]^{<\omega}$ there exists $(\varepsilon_i)_{i\in H} \subseteq \{-1,1\}$ such that $\min H < \|\sum_{i\in H} \varepsilon_i y_i\|$.

(iii)⇒(iv). For every $I \in [\mathbb{N}]$ there exists a sequence $(H_m)_{m\in\mathbb{N}}$ in $\mathcal{B}_\alpha \cap [I]^{<\omega}$ such that $H_1 < H_2 < \ldots$ (Remark 1.4(ii)).

(iv)⇒(i). This is obvious, by Proposition 2.10. This finishes the proof.

REMARK 2.14. (i) If a bounded sequence (χ_n) is null coefficient of order α for some countable ordinal α, then every subsequence of (χ_n) is null coefficient of order β for every β with $\alpha \leq \beta \leq \omega_1$.

(ii) If a sequence (χ_n) is null coefficient, then there exists a countable ordinal ζ (in fact $\omega^\zeta = \xi_0^{(\chi_n)}$) such that (χ_n) is null coefficient of order α for every α with $\zeta \leq \alpha \leq \omega_1$.

(iii) If the c_0-index $\xi_0^{(\chi_n)}$ of a sequence (χ_n) is countable, then $\xi_0^{(\chi_n)} = \omega^\zeta$ where ζ is the least ordinal α which makes the sequence (χ_n) null coefficient of order α.

Gathering all the previous results we can finally state the principal theorem of this section.

THEOREM 2.15. *Let (χ_n) be a basic bounded sequence in a Banach space with $0 < \inf_n \|\chi_n\|$. Then either*

(1) [Case $\xi_0^{(\chi_n)} = \omega_1$] (χ_n) *has a subsequence equivalent to the unit vector basis of* c_0; *or*

(2) [Case $\xi_0^{(\chi_n)} < \omega_1$] (χ_n) *is null coefficient.*

In case (2) *there exists a countable ordinal* ζ (*in fact* $\xi_0^{(\chi_n)} = \omega^\zeta$) *such that for each countable ordinal* α, *either*

(2i) [Case $\alpha < \zeta$] (χ_n) *has a subsequence with* c_0*-spreading model of order* α; *or*

(2ii) [Case $\zeta \leq \alpha$] (χ_n) *is null coefficient of order* α.

Proof. This follows from Propositions 2.4, 2.5, 2.10 and 2.13.

3. Semiboundedly complete sequences. An important notion concerning basic sequences is that of semibounded completeness. A sequence (χ_n) is *semiboundedly complete* if every sequence (λ_n) of real numbers with $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| < \infty$ converges to zero. According to a result of Odell ([O]) every normalized weakly null sequence contains a subsequence which is either equivalent to the unit vector basis of c_0 or semiboundedly complete. This happens since every normalized weakly null sequence has a c_0-unconditional subsequence (see Definition 3.8 below; [E]) and since every c_0-unconditional sequence is semiboundedly complete if and only if it does not contain a subsequence equivalent to the unit vector basis of c_0.

In this section we introduce (Definition 3.1) and characterize (Theorem 3.4) the semibounded completeness index $\xi_b^{(\chi_n)}$ of a sequence (χ_n). The index $\xi_b^{(\chi_n)}$ is countable if and only if (χ_n) is semiboundedly complete (Remark 3.2(i)) and in this case $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ (Proposition 3.3). We call a sequence *semiboundedly complete of order* α, for some countable ordinal α, if $\xi_b^{(\chi_n)} \leq \omega^\alpha$.

The c_0-index is always less than or equal to the semibounded completeness index (Proposition 3.6), but these differ in general. In Example 3.14 we give an example of a normalized, weakly null, basic sequence (χ_n) with $\xi_0^{(\chi_n)} = \omega$ and $\xi_b^{(\chi_n)} = \omega_1$.

For normalized c_0-unconditional sequences we prove (Theorem 3.10) that the two indices are equal. Thus a normalized c_0-unconditional sequence is semiboundedly complete of order α, for some $0 \leq \alpha < \omega_1$, if and only if it does not contain a subsequence with c_0-spreading model of order α or, equivalently, if it is null coefficient of order α.

As a corollary, we deduce that for a given countable ordinal α every normalized weakly null sequence has a subsequence either semiboundedly complete of order α or with c_0-spreading model of order α, thus obtaining a countable ordinal analogue of Odell's limiting (for $\alpha = \omega_1$) theorem (Theorem 3.15).

DEFINITION 3.1. Let (χ_n) be a sequence in a Banach space X. For every $\varepsilon > 0$ we set

$$\mathcal{D}_\varepsilon^{(\chi_n)} = \Big\{F \in [\mathbb{N}]^{<\omega} : \text{there exists } (\lambda_n) \subseteq \mathbb{R} \text{ with } \sup_n \Big\| \sum_{i=1}^n \lambda_i \chi_i \Big\| \leq 1 \\ \text{and } |\lambda_i| \geq \varepsilon \text{ for every } i \in F\Big\}.$$

The families $\mathcal{D}_\varepsilon^{(\chi_n)}$, for all $\varepsilon > 0$, are hereditary.

We then define the *semibounded completeness index* $\xi_b^{(\chi_n)}$ of (χ_n) as follows: if there exists $\varepsilon > 0$ such that the family $\mathcal{D}_\varepsilon^{(\chi_n)}$ is not pointwise closed, then we set

$$\xi_b^{(\chi_n)} = \omega_1;$$

otherwise

$$\xi_b^{(\chi_n)} = \sup\{s_M(\mathcal{D}_\varepsilon^{(\chi_n)}) : M \in [\mathbb{N}] \text{ and } \varepsilon > 0\},$$

which is a countable ordinal.

We say that the sequence (χ_n) is:

(1) *semiboundedly complete* (of order ω_1) if all the sequences $(\lambda_n) \subseteq \mathbb{R}$ with $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| \leq 1$ converge to zero;

(2) *semiboundedly complete of order* ζ, for some countable ordinal ζ, if

$$\xi_b^{(\chi_n)} \leq \omega^\zeta.$$

REMARK 3.2. (i) For a sequence (χ_n) with $\inf_n \|\chi_n\| > 0$, using a compactness argument, it is easy to prove that $\xi_b^{(\chi_n)} = \omega_1$ if and only if there exist $M \in [\mathbb{N}]$, $\varepsilon > 0$ and $(\lambda_n) \subseteq \mathbb{R}$ such that $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| \leq 1$ and $|\lambda_n| \geq \varepsilon$ for every $n \in M$. Hence (χ_n) is semiboundedly complete if and only if $\xi_b^{(\chi_n)} < \omega_1$.

(ii) $\omega \leq \xi_b^{(\chi_n)}$ for every bounded sequence (χ_n), since for every $k \in \mathbb{N}$ and $F \in [\mathbb{N}]^k$ setting $\alpha_n = 1/(Ak)$ if $n \in F$ and $\alpha_n = 0$ if $n \in \mathbb{N} \setminus F$, where $A = \sup_n \|\chi_n\|$, we get $\sup_n \|\sum_{i=1}^n \alpha_i \chi_i\| \leq 1$.

(iii) For the summing basis (s_n) of c_0 we have $\xi_b^{(s_n)} = \omega_1$ and $\xi_0^{(s_n)} = \omega$.

(iv) If (y_n) is a subsequence of (χ_n), then $\xi_b^{(y_n)} \leq \xi_b^{(\chi_n)}$. So, if (χ_n) is semiboundedly complete of order ζ and $\zeta \leq \alpha \leq \omega_1$, then (y_n) is semiboundedly complete of order α.

Proposition 3.3. *Let (χ_n) be a sequence in a Banach space X. Then either $\xi_b^{(\chi_n)} = \omega_1$ or $\xi_b^{(\chi_n)} = \omega^\zeta$ for some ordinal ζ with $1 \leq \zeta < \omega_1$.*

Proof. Let $\xi_b^{(\chi_n)} < \omega_1$. Then there exists an ordinal ζ with $1 \leq \zeta < \omega_1$ such that $\omega^\zeta \leq \xi_b^{(\chi_n)} < \omega^{\zeta+1}$. Arguing by contradiction suppose that $\omega^\zeta < \xi_b^{(\chi_n)}$. Then there exist $M \in [\mathbb{N}]$ and $\varepsilon > 0$ such that $\omega^\zeta < s_M(\mathcal{D}_\varepsilon^{(\chi_n)})$.

According to Remark 1.6(vii) there exists $L = (l_n) \in [M]$ such that $\mathcal{F}_\zeta(L) \subseteq \mathcal{D}_\varepsilon^{(\chi_n)}$.

Let $k \in \mathbb{N}$ and $F_1, \ldots, F_k \in \mathcal{F}_\zeta$ with $F_1 < \ldots < F_k$. For each $m \in \{1, \ldots, k\}$ there exist $(\lambda_n^m) \subseteq \mathbb{R}$ such that $\sup_n \|\sum_{i=1}^n \lambda_i^m \chi_i\| \leq 1$ and $|\lambda_{l_i}^m| \geq \varepsilon$ for every $i \in F_m$. For each $m \in \{1, \ldots, k\}$ set $\sigma_m = \max F_m$ and $p_m = \min F_m$; also set $b_n^m = \lambda_n^m$ if $n \in \mathbb{N}$ with $l_{p_m} \leq n \leq l_{\sigma_m}$ and $b_n^m = 0$ if $n \in \mathbb{N}$ with $n < l_{p_m}$ or $n > l_{\sigma_m}$. Then

$$\sup_n \Big\| \sum_{i=1}^n b_i^m \chi_i \Big\| \leq 2 \quad \text{for every } m \in \{1, \ldots, k\}.$$

Set $\lambda_n = (b_n^1 + \ldots + b_n^k)/(2k)$ for every $n \in \mathbb{N}$. Then

$$\sup_n \Big\| \sum_{i=1}^n \lambda_i \chi_i \Big\| \leq 1 \quad \text{and} \quad |\lambda_{l_i}| \geq \frac{\varepsilon}{2k} \ \text{ for every } i \in \bigcup_{m=1}^k F_m.$$

Setting, for every $k \in \mathbb{N}$,

$$\mathcal{F}_\zeta^k = \Big\{ F \in [\mathbb{N}]^{<\omega} : F = \bigcup_{i=1}^k F_i \text{ with } F_i \in \mathcal{F}_\zeta \text{ and } F_1 < \ldots < F_k \Big\},$$

we have thus proved that $\mathcal{F}_\zeta^k(L) \subseteq \mathcal{D}_{\varepsilon/(2k)}^{(\chi_n)}$ for every $k \in \mathbb{N}$.

From a result of Androulakis and Odell ([An-O]) there exists $L_1 \in [L]$ such that $F \setminus \{\min F\} \in \mathcal{F}_\zeta(L)$ for every $F \in \mathcal{F}_\zeta \cap [L_1]^{<\omega}$. This shows that $\mathcal{F}_\zeta \cap [L_1]^{<\omega} \subseteq \mathcal{F}_\zeta^2(L)$. Hence, $\mathcal{F}_\zeta^k \cap [L_1]^{<\omega} \subseteq \mathcal{F}_\zeta^{2k}(L)$ for every $k \in \mathbb{N}$. Thus, we have

$$\mathcal{A}_{k\omega^\zeta} \cap [L_1]^{<\omega} \subseteq \mathcal{F}_\zeta^k \cap [L_1]^{<\omega} \subseteq \mathcal{D}^{(\chi_n)}_{\varepsilon/(4k)} \quad \text{for every } k \in \mathbb{N}.$$

This gives (see Remark 1.6(v))

$$k\omega^\zeta + 1 \le s_{L_1}(\mathcal{D}^{(\chi_n)}_{\varepsilon/(4k)}) \quad \text{for every } k \in \mathbb{N}.$$

But this is impossible, since $\xi_b^{(\chi_n)} < \omega^{\zeta+1}$. Hence, $\xi_b^{(\chi_n)} = \omega^\zeta$.

Recapitulating the previous results, we have already proved that a sequence (χ_n) in a Banach space with $0 < \inf_n \|\chi_n\| \le \sup_n \|\chi_n\| < \infty$ is semiboundedly complete if and only if the index $\xi_b^{(\chi_n)}$ is countable and in this case $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ. The ordinal ζ indicates the least possible order of the semibounded completeness of (χ_n) (see Remark 3.2(iv)).

In the following we will establish a characterization of the sequences semiboundedly complete of order ζ in terms of the complete thin Schreier system (Definition 1.3). The sense of this characterization is that the semibounded completeness of order ζ of the sequence (χ_n) is precisely equivalent to the ω^ζ-uniform convergence to zero of all the sequences (λ_n) of real numbers with $\sup_n \|\sum_{i=1}^n \lambda_i\chi_i\| \le 1$.

THEOREM 3.4. *Let (χ_n) be a basic sequence in a Banach space, ζ a countable ordinal and (ξ_n) a strictly increasing sequence of ordinals with $\sup_n \xi_n = \omega^\zeta$. The following are equivalent*:

(i) *(χ_n) is semiboundedly complete of order ζ*;

(ii) *for every $M \in [\mathbb{N}]$ there exists a strictly increasing function $\varphi : \mathbb{N} \to M$ with the property*: *for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that*

$$\{\varphi(n) : n \ge n_0 \text{ and } |\lambda_{\varphi(n)}| \ge \varepsilon\} \in (\mathcal{A}_{\xi_{n_0}})^\star \setminus \mathcal{A}_{\xi_{n_0}}$$

for every $(\lambda_n) \subseteq \mathbb{R}$ with $\sup_n \|\sum_{i=1}^n \lambda_i\chi_i\| \le 1$.

Proof. (i)⇒(ii). Let $\xi_b^{(\chi_n)} \le \omega^\zeta$. For every $M \in [\mathbb{N}]$ and $\varepsilon > 0$ there exists $I \in [M]$ such that $\sup\{s_N(\mathcal{D}_\varepsilon^{(\chi_n)}) : N \in [I]\} < \omega^\zeta$.

Let $M \in [\mathbb{N}]$. Using Theorem 1.8 we can construct a strictly increasing sequence (k_n) in $[\mathbb{N}]$ and a decreasing sequence (I_n) in $[M]$ such that

$$\mathcal{D}^{(\chi_n)}_{1/n} \cap [I_n]^{<\omega} \subseteq (\mathcal{A}_{\xi_{k_n}})^\star \setminus \mathcal{A}_{\xi_{k_n}} \quad \text{for every } n \in \mathbb{N}.$$

If $I_n = (i_m^n)_{m\in\mathbb{N}}$ for every $n \in \mathbb{N}$, then define $\varphi : \mathbb{N} \to M$ by $\varphi(n) = i_n^n$ for every $n \in \mathbb{N}$. For $\varepsilon > 0$ set $n_0 = n_0(\varepsilon) = k_\lambda$ for some $\lambda \in \mathbb{N}$ with $1/\lambda < \varepsilon$. Then for every sequence (λ_n) in $\mathbb{R}$ with $\sup_n \|\sum_{i=1}^n \lambda_i\chi_i\| \le 1$ we get

$$\{\varphi(n) : n \ge n_0 \text{ and } |\lambda_{\varphi(n)}| \ge \varepsilon\} \in (\mathcal{A}_{\xi_{n_0}})^\star \setminus \mathcal{A}_{\xi_{n_0}}.$$

(ii)⇒(i). Suppose that $\xi_b^{(\chi_n)} > \omega^\zeta$. Then there exist $\varepsilon > 0$ and $M \in [\mathbb{N}]$ such that $s_M(\mathcal{D}_\varepsilon^{(\chi_n)}) > \omega^\zeta > \xi_n + 1$ for every $n \in \mathbb{N}$. By (ii), there exist $L \in [M]$ and $n_0 \in \mathbb{N}$ such that

$$\mathcal{D}_\varepsilon^{(\chi_n)} \cap [L]^{<\omega} \subseteq (\mathcal{A}_{\xi_{n_0}})^\star \setminus \mathcal{A}_{\xi_{n_0}}.$$

Since $s_L(\mathcal{D}_\varepsilon^{(\chi_n)}) > \xi_{n_0} + 1$ (Remark 1.6(iii)), according to Theorem 1.8, there exists $I \in [L]$ such that $\mathcal{A}_{\xi_{n_0}} \cap [I]^{<\omega} \subseteq \mathcal{D}_\varepsilon^{(\chi_n)} \cap [L]^{<\omega}$, which is a contradiction, hence $\xi_b^{(\chi_n)} \leq \omega^\zeta$.

Choosing appropriate sequences (ξ_n) strictly increasing to ω^ζ we can obtain interesting descriptions of being semiboundedly complete of order ζ.

Corollary 3.5. *Let (χ_n) be a sequence in a Banach space, and ζ a countable ordinal.*

(1) $\xi_b^{(\chi_n)} \leq \omega^{\zeta+1}$ *if and only if for every $M \in [\mathbb{N}]$ there exists a strictly increasing function $\varphi : \mathbb{N} \to M$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that the type with respect to $\mathcal{B}_\zeta$* (see Remark 1.4(ii)) *of the set*

$$\{\varphi(n) : n \leq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon\}$$

is at most n_0, for every $(\lambda_n) \subseteq \mathbb{R}$ with $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| \leq 1$.

(2) $\xi_b^{(\chi_n)} \leq \omega^\zeta$ *for some limit ordinal ζ if and only if there exists a sequence (ζ_n) of ordinals strictly increasing to ζ with the following property: for every $M \in [\mathbb{N}]$ there exists a strictly increasing sequence $\varphi : \mathbb{N} \to M$ such that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that the type with respect to $\mathcal{B}_{\zeta_{n_0}}$ of the set*

$$\{\varphi(n) : \ n \leq n_0 \text{ and } |\lambda_{\varphi(n)}| \geq \varepsilon\}$$

is at most n_0, for every $(\lambda_n) \subseteq \mathbb{R}$ with $\sup_n \|\sum_{i=1}^n \lambda_i \chi_i\| \leq 1$.

Proof. This is a consequence of Theorem 3.4: in case (1), set $\xi_n = n\omega^\zeta$ for every $n \in \mathbb{N}$, and in case (2), set $\xi_n = \omega^{\zeta_n+1}$ for every $n \in \mathbb{N}$.

Now, we will study the relation of the semibounded completeness index $\xi_b^{(\chi_n)}$ to the c_0-index $\xi_0^{(\chi_n)}$ of a sequence (χ_n).

Proposition 3.6. *Let (χ_n) be a normalized basic sequence in a Banach space X. Then $\xi_0^{(\chi_n)} \leq \xi_b^{(\chi_n)}$.*

Proof. Let $\xi_b^{(\chi_n)} = \omega^\zeta$ for some $1 \leq \zeta < \omega_1$ (Proposition 3.3). Arguing by contradiction, we assume that $\omega^\zeta < \xi_0^{(\chi_n)}$. Then there exist a subsequence (y_n) of (χ_n) and $\varepsilon > 0$ such that

$$\Big\| \sum_{i\in F} \lambda_i y_i \Big\| \leq \varepsilon \max_{i\in F} |\lambda_i| \quad \text{for every } F \in \mathcal{F}_\zeta \text{ and } (\lambda_i)_{i\in F} \subseteq \mathbb{R}.$$

Let $E = [(y_n)]$ be the closed subspace of X which is generated by the sequence (y_n), and $(y_n^\star) \subseteq E^\star$ the sequence of the biorthogonal functionals of (y_n). Clearly, $\|y_n^\star\| \le 2C$ for every $n \in \mathbb{N}$, where C is the basic constant of (y_n).

Let $s \in \mathcal{B}_\zeta \subseteq \mathcal{F}_\zeta$. We set $\lambda_n = \varphi_s^\zeta(n)$ for every $n \in \mathbb{N}$, according to Remark 1.10. Then

$$\Big\|\sum_{i\in s}\lambda_i y_i^\star\Big\| \ge \frac{1}{\varepsilon}\Big(\sum_{i\in s}\lambda_i y_i^\star\Big)\Big(\sum_{i\in s} y_i\Big) = \frac{1}{\varepsilon}\sum_{i\in s}\lambda_i = \frac{1}{\varepsilon}.$$

Let $f \in E^{\star\star}$ with $\|f\| \le 1$ and $\|\sum_{i\in s}\lambda_i y_i^\star\| = \sum_{i\in s}\lambda_i f(y_i^\star)$. Setting $\mu_n = C^{-1} f(y_n^\star)$ for every $n \in \mathbb{N}$, we get $|\mu_n| \le 2$ for every $n \in \mathbb{N}$ and

$$\begin{aligned}\sup_n\Big\|\sum_{i=1}^n \mu_i y_i\Big\| &= \frac{1}{C}\sup_n\Big\|\sum_{i=1}^n f(y_i^\star) y_i\Big\| = \frac{1}{C}\sup_n\|P_n^{\star\star}(f)\|\\ &\le \frac{1}{C}\sup_n\|P_n^{\star\star}\| = 1\end{aligned}$$

(where the P_n are the natural projections associated to the basis (y_n)).

If $F = \{i \in s : |\mu_i| \ge 1/(2\varepsilon C)\}$, then $F \in \mathcal{D}^{(y_n)}_{1/(2\varepsilon C)}$. We will prove that $w_{\omega^\zeta}(F;s) > 1/(4\varepsilon C)$. Indeed,

$$\begin{aligned}\frac{1}{\varepsilon} \le \Big\|\sum_{i\in s}\lambda_i y_i^\star\Big\| &= C\sum_{i\in s}\lambda_i\mu_i = C\sum_{i\in F}\lambda_i\mu_i + C\sum_{i\in s\setminus F}\lambda_i\mu_i\\ &\le 2Cw_{\omega^\zeta}(F;s) + \frac{C}{2\varepsilon C}.\end{aligned}$$

Hence, $\|w_{\omega^\zeta}(F;s)\| \ge 1/(4\varepsilon C)$.

According to the ω^ζ-Pták type theorem (Theorem 1.11) there exists $L \in [\mathbb{N}]$ such that $s_L(\mathcal{D}_\delta^{(y_n)}) > \omega^\zeta$, where $\delta = 1/(2\varepsilon C)$.

If $y_n = \chi_{m_n}$ for every $n \in \mathbb{N}$, then for $M = \{m_n : n \in L\}$ we have $s_M(\mathcal{D}_\delta^{(\chi_n)}) > \omega^\zeta$, and consequently $\xi_b^{(\chi_n)} > \omega^\zeta$.

This is a contradiction; hence $\xi_0^{(\chi_n)} \le \omega^\zeta = \xi_b^{(\chi_n)}$.

Corollary 3.7. *Let (χ_n) be a normalized basic sequence in a Banach space and ζ an ordinal number with $1 \le \zeta \le \omega_1$. If (χ_n) is semiboundedly complete of order ζ, then (χ_n) is null coefficient of order ζ.*

In Remark 3.2(iii) we gave an example of a normalized basic sequence (s_n) with $\xi_b^{(s_n)} = \omega_1$ and $\xi_0^{(s_n)} = \omega$. According to Proposition 2.13, (s_n) is null coefficient of order α for every countable ordinal α, but it is not semiboundedly complete of order α. As we prove in Theorem 3.10 below these notions are equivalent in the case of a c_0-unconditional sequence.

Definition 3.8. A bounded basic sequence (χ_n) in a Banach space is c_0-*unconditional* if for every $\delta > 0$ there exists a constant $K(\delta) < \infty$ so that for every $n \in \mathbb{N}$, every sequence $(\lambda_i)_{i=1}^n \subseteq \mathbb{R}$ with $|\lambda_i| \leq 1$ for all $i = 1, \ldots, n$ and every $F \subseteq \{1 \leq i \leq n \ : |\lambda_i| \geq \delta\}$ we have

$$\Big\| \sum_{i\in F} \lambda_i \chi_i \Big\| \leq K(\delta) \Big\| \sum_{i=1}^n \lambda_i \chi_i \Big\|.$$

Remark 3.9. Elton [E] proved that every normalized weakly null sequence in a Banach space has a c_0-unconditional subsequence.

Theorem 3.10. *Let (χ_n) be a normalized c_0-unconditional basic sequence in a Banach space X. Then $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)}$.*

Proof. We claim that if $\omega^\zeta < \xi_b^{(\chi_n)}$ for some countable ordinal ζ, then $\omega^\zeta < \xi_0^{(\chi_n)}$. Indeed, let $\omega^\zeta < \xi_b^{(\chi_n)}$. Then there exist $M \in [\mathbb{N}]$ and $\varepsilon > 0$ such that $\omega^\zeta < s_M(\mathcal{D}_\varepsilon^{(\chi_n)})$. According to Remark 1.6(vii) there exists $L \in [M]$ such that

$$\mathcal{F}_\zeta(L) \subseteq \mathcal{D}_\varepsilon^{(\chi_n)}.$$

Set $L = (l_n)$ and $y_n = \chi_{l_n}$ for every $n \in \mathbb{N}$. If $F \in \mathcal{F}_\zeta$, then $(l_i)_{i\in F} \in \mathcal{D}_\varepsilon^{(\chi_n)}$, so there exists a sequence $(\lambda_n) \subseteq \mathbb{R}$ with

$$\sup_n \Big\| \sum_{i=1}^n \lambda_i \chi_i \Big\| \leq 1 \quad \text{and} \quad |\lambda_{l_i}| \geq \varepsilon \quad \text{for all} \ \ i \in F.$$

Thus for every $f \in X^\star$ we have

$$\begin{aligned} \sum_{i\in F} |f(y_i)| &\leq \frac{1}{\varepsilon} \sum_{i\in F} |\lambda_{l_i}| \cdot |f(y_i)| \\ &= \frac{1}{\varepsilon} \sum_{i\in F} \varepsilon_i \lambda_{l_i} f(y_i) \quad \text{(for suitable } (\varepsilon_i)_{i\in F} \subseteq \{-1,1\}) \\ &= \frac{1}{\varepsilon} f\Big(\sum_{i\in F} \varepsilon_i \lambda_{l_i} y_i\Big) \leq \frac{1}{\varepsilon}\|f\| \cdot \Big\| \sum_{i\in F} \varepsilon_i \lambda_{l_i} \chi_{\lambda_i} \Big\| \leq \frac{1}{\varepsilon} \|f\| K\Big(\frac{\varepsilon}{2}\Big); \end{aligned}$$

since $|\lambda_n| \geq 2$ for every $n \in \mathbb{N}$, $|\lambda_{l_i}| \geq \varepsilon$ for every $i \in F$ and the sequence (χ_n) is c_0-unconditional with constraint $K(\delta)$ for $\delta > 0$.

According to Propositions 2.9 and 2.10 we get $\omega^\zeta < \xi_0^{(\chi_n)}$, which finishes the proof of our claim.

In case $\xi_b^{(\chi_n)} = \omega_1$ we have $\omega^\zeta < \xi_b^{(\chi_n)}$ for every countable ordinal ζ. So according to our claim $\omega^\zeta < \xi_0^{(\chi_n)}$ for every $1 \leq \zeta < \omega_1$, which gives $\xi_0^{(\chi_n)} = \omega_1 = \xi_b^{(\chi_n)}$.

In case $\xi_b^{(\chi_n)} < \omega_1$ there exists a countable ordinal ζ such that $\xi_0^{(\chi_n)} = \omega^\zeta \leq \xi_b^{(\chi_n)}$ (Propositions 2.5 and 3.6). If $\omega^\zeta < \xi_b^{(\chi_n)}$, then according to the previous claim we have $\omega^\zeta < \xi_0^{(\chi_n)}$, which is impossible. Hence, $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)}$.

So far we have proved that a normalized c_0-unconditional basic sequence (χ_n) in a Banach space X either has a subsequence equivalent to the unit vector basis of c_0 (in which case $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} = \omega_1$), or it is semiboundedly complete (in case $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} < \omega_1$). In the latter case there exists a countable ordinal ζ such that $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} = \omega^\zeta$. The ordinal ζ separates the set of all the countable ordinals into two classes, the ordinals α with $\alpha < \zeta$ and those with $\alpha \geq \zeta$. We characterized the ordinals α with $\alpha < \zeta$ as those for which the sequence (χ_n) has a subsequence with c_0-spreading model of order α (Proposition 2.10); on the other hand, we characterized the ordinals α with $\zeta \leq \alpha$ as those which make the sequence (χ_n) null coefficient of order α (Proposition 2.13) and moreover semiboundedly complete of order α (Theorem 3.4). In the following two propositions we will give more characterizations of these two classes.

PROPOSITION 3.11. *Let (χ_n) be a normalized c_0-unconditional sequence in a Banach space X with $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ. For each countable ordinal α the following are equivalent*:

(i) $\alpha < \zeta$;

(ii) *there exists a subsequence (y_n) of (χ_n) with c_0-spreading model of order α*;

(iii) *there exist a subsequence (y_n) of (χ_n), $I \in [\mathbb{N}]$ and $\varepsilon > 0$ such that*

$$\Big\| \sum_{i \in H} y_i \Big\| \leq \varepsilon \quad \textit{for every } H \in \mathcal{B}_\alpha \cap [I]^{<\omega}.$$

Proof. (i)$\Leftrightarrow$(ii). This is proved in Proposition 2.10.

(ii)$\Rightarrow$(iii). This is obvious: set $I = \mathbb{N}$.

(iii)$\Rightarrow$(ii). According to Remark 1.4(iii) there exists $L \in [I]$ such that

$$\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha \cap [I]^{<\omega})_\star.$$

Set $L = (l_n)_{n \in \mathbb{N}}$ and $z_n = y_{l_n}$ for every $n \in \mathbb{N}$. The subsequence (z_n) of (χ_n) has c_0-spreading model of order α. Indeed, let $F \in \mathcal{F}_\alpha$. Then there exists $H \in \mathcal{B}_\alpha \cap [I]^{<\omega}$ such that $(l_i)_{i \in F} \subseteq H$. Since the sequence (χ_n) is c_0-unconditional there exists $K = K(1) > 0$ such that

$$\Big\| \sum_{i \in F} z_i \Big\| = \Big\| \sum_{i \in F} y_{l_i} \Big\| \leq K \Big\| \sum_{i \in H} y_i \Big\| \leq K\varepsilon.$$

According to Proposition 2.9 the sequence (z_n) has c_0-spreading model of order α.

PROPOSITION 3.12. *Let (χ_n) be a normalized c_0-unconditional sequence in a Banach space with $\xi_b^{(\chi_n)} = \omega^\zeta$ for some countable ordinal ζ. For each countable ordinal α the following are equivalent*:

(i) $\zeta \leq \alpha$;

(ii) *the sequence* (χ_n) *is semiboundedly complete of order* α;

(iii) *the sequence* (χ_n) *is null coefficient of order* α;

(iv) *whenever a bounded sequence* (λ_n) *of real numbers satisfies* $\sup\{\|\sum_{i\in H}\lambda_i\chi_i\| : H \in \mathcal{B}_\alpha\} < \infty$, *then* (λ_n) *converges to zero*;

(v) *for every subsequence* (y_n) *of* (χ_n) *and* $M \in [\mathbb{N}]$ *there exists* $L \in [M]$ *such that*

$$\min H < \Big\|\sum_{i\in H} y_i\Big\| \quad \textit{for every } H \in \mathcal{B}_\alpha \cap [L]^{<\omega};$$

(vi) *for every subsequence* (y_n) *of* (χ_n) *and* $M \in [\mathbb{N}]$ *there exists a sequence* (H_m) *in* $\mathcal{B}_\alpha \cap [M]^{<\omega}$ *with* $H_1 < H_2 < \ldots$ *and*

$$\Big\|\sum_{i\in H_m} y_i\Big\| \to \infty.$$

Proof. (i)⇔(ii). This follows from Definition 3.1.

(i)⇔(iii). Follows from Theorem 3.10 and Proposition 2.13.

(iii)⇒(iv). Let $(\lambda_n) \subseteq \mathbb{R}$ with

$$\sup\Big\{\Big\|\sum_{i\in H}\lambda_i\chi_i\Big\| : H \in \mathcal{B}_\alpha\Big\} = A < \infty \quad \text{and} \quad \sup_n |\lambda_n| = B < \infty.$$

We assume that (λ_n) does not converge to zero. Then there exists $\varepsilon > 0$ such that the set

$$M = \{n \in \mathbb{N} : |\lambda_n| > \varepsilon\} \quad \text{is infinite.}$$

According to Remark 1.6(iii) and a result of Androulakis and Odell ([An-O]) there exists $L \in [M]$ with $L = (l_n)$ such that

$$\mathcal{F}_\alpha(L) \subseteq (\mathcal{B}_\alpha \cap [M]^{<\omega})_\star, \quad F \setminus \{\min F\} \in \mathcal{F}_\alpha(L) \quad \text{for all } F \in \mathcal{F}_\alpha \cap [L]^{<\omega}.$$

Set $\mu_{l_n} = \lambda_{l_n}$ for every $n \in \mathbb{N}$ and $\mu_n = 0$ for every $n \in \mathbb{N} \setminus L$. If $F \in \mathcal{F}_\alpha$, then

$$\Big\|\sum_{i\in F}\mu_i\chi_i\Big\| = \Big\|\sum_{i\in F_1}\lambda_i\chi_i\Big\| \quad \text{for some } F_1 \in \mathcal{F}_\alpha \cap [L]^{<\omega}.$$

Hence,

$$\Big\|\sum_{i\in F}\mu_i\chi_i\Big\| \leq 2K(\varepsilon/B)A,$$

since the sequence (χ_n) is c_0-unconditional with constraint $K(\delta)$ for $\delta > 0$. This contradicts (iii). Hence (λ_n) converges to zero.

(iv)⇒(iii). This is obvious, since $\mathcal{B}_\alpha \subseteq \mathcal{F}_\alpha$.

(i)⇒(v). Let $\zeta \le \alpha$. If (y_n) is a subsequence of (χ_n) and $M \in [\mathbb{N}]$, then for every $k \in \mathbb{N}$ we set

$$\mathcal{L}_k = \Big\{H \in [M]^{<\omega} : k < \Big\|\sum_{i\in H} y_i\Big\|\Big\}.$$

According to Proposition 3.11 we have

$$\mathcal{L}_k \cap \mathcal{B}_\alpha \cap [I]^{<\omega} \neq \emptyset \quad \text{for every } k \in \mathbb{N} \text{ and } I \in [\mathbb{N}].$$

Using the refined ω^α-Ramsey type theorem (Theorem 1.8), a decreasing sequence (I_k) in $[M]$ can be constructed such that

$$\mathcal{B}_\alpha \cap [I_k]^{<\omega} \subseteq \mathcal{L}_k \quad \text{for every } k \in \mathbb{N}.$$

Set $L = (i_k^k)$ if $I_k = (i_n^k)_{n\in\mathbb{N}}$ for every $k \in \mathbb{N}$. If $H \in \mathcal{B}_\alpha \cap [I]^{<\omega}$, then $H \in \mathcal{L}_k$, where $k = \min H$; hence

$$\min H < \Big\|\sum_{i\in H} y_i\Big\| \quad \text{for every } H \in \mathcal{B}_\alpha \cap [I]^{<\omega}.$$

(v)⇒(vi). For every $L \in [M]$ there exists (Remark 1.4(ii)) a sequence (H_m) in $\mathcal{B}_\alpha \cap [M]^{<\omega}$ with $H_1 < H_2 < \ldots$ and $L = \bigcup_{m\in\mathbb{N}} H_m$.

(vi)⇒(i). This follows from Proposition 3.11.

Gathering the previous results we can state a theorem which completes Theorem 2.15 in the case of a normalized c_0-unconditional sequence.

THEOREM 3.13. *Let (χ_n) be a normalized c_0-unconditional basic sequence in a Banach space. Then either*

(1) [Case $\xi_b^{(\chi_n)} = \omega_1$] (χ_n) *has a subsequence equivalent to the unit vector basis of c_0; or*

(2) [Case $\xi_b^{(\chi_n)} < \omega_1$] (χ_n) *is semiboundedly complete (equivalently, null coefficient).*

In case (2) there exists a countable ordinal ζ such that $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)} = \omega^\zeta$. Then, for each countable ordinal α, either

(2i) [Case $\alpha < \zeta$] (χ_n) *has a subsequence with a c_0-spreading model of order α; or*

(2ii) [Case $\zeta \le \alpha$] (χ_n) *is semiboundedly complete of order α (equivalently, null coefficient of order α).*

Proof. This follows from Theorems 2.15 and 3.10, Remark 3.2(i) and Propositions 3.11 and 3.12.

At this point the following question naturally arises: Is it true that $\xi_0^{(\chi_n)} = \xi_b^{(\chi_n)}$ for every normalized weakly null basic sequence (χ_n)? The answer is negative, as follows from the example below.

Example 3.14 (James' space; [J]). For a sequence (λ_n) of real numbers we set

$$\|(\lambda_n)\| = \sup\{[(\lambda_{p_1}-\lambda_{p_2})^2+\ldots+(\lambda_{p_{m-1}}-\lambda_{p_m})^2+(\lambda_{p_m}-\lambda_{p_1})^2]^{1/2} : m\in\mathbb{N} \text{ and } p_1<\ldots<p_m\}.$$

The vector space

$$X=\{(\lambda_n)\in\mathbb{R}^{\mathbb{N}} : \lim_n \lambda_n = 0 \text{ and } \|(\lambda_n)\|<\infty\}$$

is a Banach space with respect to the norm $\|\ \|$.

For $n\in\mathbb{N}$ let $e_n=(\lambda_m^n)_{m\in\mathbb{N}}$ with $\lambda_m^n=0$ if $n\neq m$ and $\lambda_n^m=1$ if $n=m$. The sequence (e_n) is a normalized, weakly null, basic sequence in X. We will prove that $\xi_0^{(e_n)}=\omega$ and $\xi_b^{(e_n)}=\omega_1$:

(i) $\xi_0^{(e_n)}=\omega$. Indeed, suppose $\xi_0^{(e_n)}>\omega$. Then, according to Proposition 2.10, there exists a subsequence (y_n) of (e_n) and $\varepsilon>0$ such that

$$\Big\|\sum_{i\in F} y_i\Big\|\le\varepsilon \quad \text{for every } F\in\mathcal{F}_1.$$

Set $F_n=(n+2,n+4,\ldots,n+2n)$ for every $n\in\mathbb{N}$. Of course, $F_n\in\mathcal{F}_1$ for every $n\in\mathbb{N}$. Setting $p_i=n+i$ for $1\le i\le 2n$ we have

$$\Big\|\sum_{i\in F_n} y_i\Big\|\ge(2n)^{1/2} \quad \text{for every } n\in\mathbb{N}.$$

This is a contradiction, hence $\xi_0^{(e_n)}=\omega$.

(ii) $\xi_b^{(e_n)}=\omega_1$. Indeed, $\|\sum_{i=1}^n e_i\|\le 1$ for every $n\in\mathbb{N}$. So, according to Remark 3.2(i), we have $\xi_b^{(e_n)}=\omega_1$.

From the previous example it is clear that we cannot hope for a theorem analogous to Theorem 3.13 in the general case of a normalized weakly null sequence (not necessarily c_0-unconditional). However, using Elton's theorem (Remark 3.9), we can prove the following dichotomy, which generalizes (to every countable ordinal) the Odell theorem (case $\alpha=\omega_1$).

Theorem 3.15. *Let (χ_n) be a normalized weakly null sequence in a Banach space and α be a countable ordinal. Then either*

(i) *(χ_n) has a subsequence with c_0-spreading model of order α; or*

(ii) *every subsequence of (χ_n) has a subsequence semiboundedly complete of order α.*

Proof. Let α be a countable ordinal and (y_n) a subsequence of (χ_n). The sequence (y_n) has a subsequence (z_n) which is c_0-unconditional and basic. If $\xi_0^{(z_n)}=\omega_1$, then (z_n) has a subsequence equivalent to the unit vector basis of c_0, hence (χ_n) has a subsequence with c_0-spreading model of order α. If $\xi_0^{(z_n)}<\omega_1$, then $\xi_0^{(z_n)}=\omega^\zeta$ for some countable ordinal ζ. Hence, in case

$\alpha < \zeta$ the sequence (χ_n) has a subsequence with c_0-spreading model of order α, and in case $\zeta \le \alpha$ the sequence (χ_n) has a subsequence semiboundedly complete of order α, according to Theorem 3.13.

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Received 30 April 2001;
in revised form 3 September 2001