

Size functions

by

Niel Shell (New York)

Abstract. We introduce the notion of a nonarchimedean size function similar to the notion of a size function introduced by Marcos. We describe a class of ring topologies on fields that are complete, neither first countable nor locally bounded, but have topologically nilpotent elements.

Two types of valuations are commonly considered: real-valued valuations (also called absolute values) and nonarchimedean valuations (also called Krull valuations). In [3] Marcos introduced axioms for a function which we will call a real-valued size function. We introduce the notion of a nonarchimedean size function. As for valuations, the classes of real and nonarchimedean size functions overlap, but neither subsumes the other.

Marcos introduced size functions to construct topologies which yield an affirmative answer to a thirty year old open question in [2]: Do there exist topological fields which are not locally bounded but have topologically nilpotent elements? We describe here another class of ring topologies on fields with these properties.

A nonzero element x in a topological ring is called *topologically nilpotent* if $x^n \rightarrow 0$. A^* denotes the set of nonzero elements of a subset A of an additive group, and $G_{\geq a}$ (resp. $G_{>a}$) denotes the set of elements greater than or equal to (resp. strictly greater than) a in an ordered group G . In an ordered group (in particular, the real numbers) $a \vee b$ (resp. $a \wedge b$) denotes the larger (resp. smaller) of a and b .

DEFINITION 1. A *real-valued size function* on a ring A is a function $N : A \rightarrow [1, \infty)$ such that

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- (N1) $N(0) = 1$;
- (N2) $N(-a) = N(a)$;
- (N3) $N(a + b) \leq N(a) + N(b)$;
- (N4) $N(ab) \leq N(a) + N(b)$;
- (N5) for each $g \geq 1$ there exists $a \in A$ such that $N(a) \geq g$.

A real-valued size function is called *nonarchimedean* if

- (N3') $N(a + b) \leq N(a) \vee N(b)$;
- (N4') $N(ab) \leq N(a) \vee N(b)$.

Condition (N1) is dispensable, in the sense that the topologies induced in [3] by functions satisfying (N1)–(N5) can be induced by functions satisfying only (N2)–(N5). Indeed, in [3] Marcos requires $N(0) = 2$, except for size functions that satisfy conditions (N3') and (N4').

The topology constructed in Section 3 of [3] is described in Theorem 1 below. We have replaced statements in [3] using logarithms with corresponding ones using exponents in order to make the analogy with Theorem 2 more apparent.

THEOREM 1. *Let N be a real-valued size function on a field F . Let $\gamma : (1, \infty) \rightarrow \mathbb{R}$ be a strictly decreasing function such that $\lim_{t \rightarrow 1+} \gamma(t) = \infty$. Let R consist of all formal series $\sum a_n X^n \in F[[X]]$ such that, for all $t > 1$, there exists m such that $N(a_n) \leq t^n$ for $n \geq m$. Then R is a local ring whose maximal ideal consists of all series in R with constant coefficient equal to 0, and $K = \bigcup_{d \geq 0} X^{-d}R$ is the quotient field $R(R^*)^{-1}$. The sets*

$$V_t = \left\{ \sum a_n X^n \in R : a_n = 0 \text{ for } n < \gamma(t) \text{ and } N(a_n) \leq t^n \text{ for } n \geq \gamma(t) \right\},$$

$t > 1$, form a neighborhood base at zero for a complete field topology on K . This topology is not locally bounded, and X is topologically nilpotent in this topology. The topology is independent of the particular choice of γ .

DEFINITION 2. Let G be an additive totally ordered group. An arbitrary element of $G_{>0}$ is labeled as 1. Since no multiplication is defined, the label 1 is not intended to have any significance except to match notation in the real-valued case. A *nonarchimedean G -valued size function* on a ring A is a function $N : A \rightarrow G_{\geq 1}$ which satisfies (N1), (N2), (N3'), (N4') and (N5).

If N is a nonarchimedean G -valued size function and $g \in G_{\geq 1}$, then $A_g = \{a \in A : N(a) \leq g\}$ is a proper subring of A . The rings A_g increase with g and have union A . Conversely, given an increasing indexed set $\{A_g\}_{g \in G_{>0}}$ of proper rings whose union is A and given a cofinal well ordered subset P of $G_{>0}$ whose smallest element we label as 1, then letting $N(a)$ be the least element of $\{p \in P : a \in A_p\}$ defines a nonarchimedean size function.

A nonarchimedean size function N on a field satisfies the condition

$$(N6) \quad N(1) = 1 \text{ and } N(a^{-1}) = N(a) \text{ for all } a \neq 0$$

if and only if $\{a \in A : N(a) \leq g\}$ is a field for each $g \in G_{\geq 1}$.

EXAMPLE. Let F_0 be a field and let P be a well-ordered cofinal subset of positive elements of an ordered group G . Let 1 denote the least element of P , and let $T = \{t_p : p \in P \setminus \{1\}\}$ be a set of elements algebraically independent over F_0 . Let $F = F_0(T)$, $T_p = \{t_q : q \leq p\}$ (so $T_1 = \emptyset$), and define $N(a)$ to be the least element of $\{p \in P : a \in F_0(T_p)\}$. Then N is a nonarchimedean G -valued size function on F satisfying (N6).

A nonarchimedean G -valued size function on a field F may be used to induce a ring topology on a subfield of the field $F((G))$ of formal power series in somewhat the same way as for real-valued size functions. (See [1] or [5, Appendix B] for basic information about power series fields.)

The following result follows immediately from [8] (see also [5, Th. 2.3.2], [6] and [7]):

LEMMA 2.1. *Let R be a ring with identity and with quotient field K , and let \mathcal{T} be a nondiscrete ring topology on R . If the filter $\mathcal{B}(\mathcal{T})$ of \mathcal{T} -neighborhoods of zero satisfies the condition $xU \in \mathcal{B}(\mathcal{T})$ for all $x \in R^*$ and $U \in \mathcal{B}(\mathcal{T})$, then $\mathcal{B}(\mathcal{T})$ is a neighborhood base at zero for a nondiscrete ring topology on K .*

THEOREM 2. *Let F be a field with a nonarchimedean G -valued size function N such that $N(1) = 1$ and $N(a^{-1}) = N(a)$ for $a \neq 0$. The set*

$$R = \left\{ \sum a_g X^g \in F[[G]] : N(a_g) \leq g \vee 1 \text{ for all } g \in G_{\geq 0} \right\}$$

is a local ring whose maximal ideal consists of all series in R with constant coefficient equal to 0; and $K = \bigcup_{d \geq 0} X^{-d}R$ is the quotient field $R(R^)^{-1}$.*

The sets U_h , where, for $h \geq 1$,

$$U_h = \left\{ \sum a_g X^g \in F[[G]] : a_g = 0 \text{ for all } g < h, \right. \\ \left. \text{and } N(a_g) \leq g - h + 1 \text{ for all } g \geq h \right\},$$

are ideals in R , and $\{U_h\}_{h \geq 1}$ is a neighborhood base at zero for a nondiscrete complete locally bounded Hausdorff field topology on K .

Proof. We will verify the following:

- (1) R is an additive group;
- (2) $RR \subset R$;
- (3) the inverse of an element in R which has nonzero constant coefficient is again in R ;
- (4) $R(R^*)^{-1} = K$;

- (5) U_h is an additive group for $h \geq 1$;
- (6) $RU_h \subset U_h$ for $h \geq 1$;
- (7) $X^{-d}U_{h+d} \subset U_h$ for $d \geq 0, h \geq 1$;
- (8) $(1 + U_h)^{-1} = 1 + U_h$ for $h \geq 1$.

From (1)–(3) it will follow that R is a local ring whose maximal ideal consists of the series with constant coefficient equal to zero; (5) and (6) state that the sets U_h are (clearly nonzero) ideals of R . Obviously $\{U_h\}$ is decreasing and $\bigcap\{U_h : h \geq 1\} = \{0\}$. Thus, the sets U_h form a neighborhood base at zero for a nondiscrete Hausdorff ring topology on R . Using (4), (6), (7) and Lemma 2.1, we see that the sets U_h also are a neighborhood base at zero for a ring topology on K : Given $x \in R^*$, write $x^{-1} = X^{-d}y$, where $d \geq 0$ and $y \in R$. Then $X^{-d}yU_{h+d} \subset U_h$ or, equivalently, $U_{h+d} \subset xU_h$. By (8), inversion is continuous. The topology is locally bounded, since, by the definition of U_h , $X^{h-1}U_1 \subset U_h$ for $h \geq 1$ (see [5, Th. 4.1.3]). Completeness will be verified after (1)–(8) are proved.

(1), (5) and (7) are obvious. The remaining statements all require consideration of the coefficient c_g of X^g in the product xy of the series

$$x = \sum a_g X^g, \quad y = \sum b_g X^g,$$

where $a_g = b_g = 0$ for $g < 0$. For some positive integer n , which depends on g , $c_g = a_{g_1}b_{g'_1} + a_{g_2}b_{g'_2} + \dots + a_{g_n}b_{g'_n}$, where, if $c_g \neq 0$, then $a_{g_i}b_{g'_i} \neq 0$ for any i , and $g_i, g'_i \geq 0$ and $g_i + g'_i = g$; thus $g_i, g'_i \leq g$.

(2) If $x, y \in R$, then $N(a_{g_i}) \leq g_i \vee 1 \leq g \vee 1$ and $N(b_{g'_i}) \leq g'_i \vee 1 \leq g \vee 1$, so $N(c_g) \leq g \vee 1$.

(3) For $x \in R$ with constant coefficient not zero, $y = x^{-1} \in R[[G]]$. We verify by induction on the index g that $N(b_g) \leq g \vee 1$. We label indices so that $g'_1 < g'_2 < \dots < g'_n$. If $g = 0$, then $n = 1, g_1 = g'_1 = 0, b_0 = a_0^{-1}$ (since $xy = 1$) and $N(b_0) = N(a_0^{-1}) = N(a_0) \leq 1$. We assume that $N(b_k) \leq k \vee 1$ for $k < g$. For $g > 0, c_g = 0$ (since $xy = 1$). If $b_g = 0$ we are done. Otherwise, we may take $g'_n = g$ and $g_n = 0$, and

$$N(b_g) = N(-a_0^{-1}[a_{g_1}b_{g'_1} + \dots + a_{g_{n-1}}b_{g'_{n-1}}]),$$

and $N(b_g) \leq g \vee 1$ by the inductive hypothesis.

(4) Certainly $K \subset R(R^*)^{-1}$ and K is a ring. Upon verifying that $z^{-1} \in K^*$ for $z \in K^*$, we find that K is a field containing R and, hence, containing the smallest field $R(R^*)^{-1}$ containing R . With $d, h \geq 0, N(a_g) \leq g \vee 1$ and $a_h \neq 0$, we may write

$$z = X^{-d} \sum_{g \geq h} a_g X^g = X^{-d+h} \sum_{g \geq h} a_g X^{g-h} = X^{-d+h} \sum_{g \geq 0} a_{g+h} X^g.$$

If we write $z^{-1} = X^{d-h} \sum_{g \geq 0} b_g X^g$, where $b_g \in F$, it follows, as in (3), from

$N(a_{g+h}) \leq (g+h) \vee 1$ for all g that $N(b_g) \leq (g+h) \vee 1$ for all g . Thus,

$$z^{-1} = X^{d-h} X^{-h} \sum_{g \geq h} b_g X^{g+h} \in X^{d-2h} R.$$

(6) If $x \in R$, $y \in U_h$ and $c_g \neq 0$, then $g \geq h$ and $g'_i \geq h$ for all i . Then $N(a_{g_i}) \leq g_i \vee 1 = (g - g'_i) \vee 1 \leq (g - h) \vee 1$. It follows that $N(c_g) \leq g - h + 1$.

(8) Suppose $x \in 1 + U_h$, where $h \geq 1$, and $y = x^{-1}$. Then $b_0 = 1$ and $b_g = 0$ for $0 < g < h$. For $g \geq h$, $c_g = 0$ (since $xy = 1$) and, as in the proof of (3), $N(b_g) \leq g - h + 1$. Thus, $(1 + U_h)^{-1} \subset 1 + U_h$. Taking the inverse of both sides reverses the containment.

Finally, suppose that $x_\lambda = \sum a_g^\lambda X^g$, $\lambda \in \Lambda$, is a Cauchy net in R . Now, $x_\lambda - x_\mu \in U_h$ implies $a_g^\lambda = a_g^\mu$ for $g < h$. Thus, a_g^λ is eventually constant for any fixed g ; call the eventual value a_g . One readily verifies $x = \sum a_g X^g \in R$ and $x_\lambda \rightarrow x$. If $\{x_\lambda\}_{\lambda \in \Lambda}$ is a Cauchy net in K then, for some $\mu \in \Lambda$, $\{x_\lambda - x_\mu\}_{\lambda \geq \mu}$ is a Cauchy net in R .

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The City College of New York (CUNY)
 Convent Avenue at 138th Street
 New York, NY 10031, U.S.A.
 E-mail: nsxcc@cunyvm.cuny.edu

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