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Products of Baire spaces revisited

by

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Abstract. Generalizing a theorem of Oxtoby, it is shown that an arbitrary product of Baire spaces which are almost locally universally Kuratowski–Ulam (in particular, have countable-in-itself π-bases) is a Baire space. Also, partially answering a question of Fleissner, it is proved that a countable box product of almost locally universally Kuratowski–Ulam Baire spaces is a Baire space.

A topological space is a *Baire space* provided countable collections of dense open subsets have a dense intersection (equivalently, nonempty open subsets are of 2nd category). Products of Baire spaces are not always Baire. Indeed, Oxtoby constructed, under CH, the first example of Baire spaces with a non-Baire product ([Ox]); various absolute examples followed (see [Co], [FK], [Po], [PvM], [Va]). As a result, some restrictions on the coordinate spaces are needed in order to get Baireness of the product space. One possibility is to strengthen the completeness properties of the factor spaces, e.g. the product of Čech-complete or (strongly) α-favorable spaces, respectively, is a Baire space (see [HMC] and [AL] for more completeness type properties). Another option is to add a *countable-in-itself* π*-base* ([1]) (i.e. a π-base each member of which contains only countably many members of the π-base) or a countable π-base to Baireness of the coordinate spaces, as classical results of Oxtoby show:

THEOREM 1 ([Ox]). (i) *Finite products of Baire spaces with countable-in-itself* π*-bases are Baire spaces.*

(ii) *Any product of Baire spaces with countable* π*-bases is a Baire space.*

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([1]) We will use this terminology instead of Oxtoby's original *locally countable pseudo-base* [Ox], since a more established meaning of the latter is a pseudo-base such that each point has a neighborhood meeting only countably many members of the pseudo-base.

The proof of Theorem 1(ii) is based on the fact that a space X having a countable π-base is *universally Kuratowski–Ulam* (for short, *uK-U space* [FNR]), i.e. for any topological space Y and a meager $E \subseteq X \times Y$, the set

$$Y \setminus \{y \in Y : \{x \in X : (x,y) \in E\} \text{ is meager in } X\}$$

is meager in Y (a property first considered in [KU]), and that the product of spaces with countable π-bases has ccc. In fact, using the same technique we can prove:

THEOREM 2. *Any product of Baire uK-U spaces is a Baire space.*

Proof. For countable products we can use an identical argument to that of Theorem 1(ii) (see [Ox, Theorem 3] or [HMC, Lemma 5.6]), if we notice that a countable product of uK-U spaces is a uK-U space ([FNR, Property 2]). Also, a Baire uK-U space has ccc ([FNR, Corollary 4]), so finite products of Baire uK-U spaces have ccc; thus, by the Noble–Ulmer Theorem ([NU]), any product of Baire uK-U spaces has ccc. The rest follows from [HMC, Lemma 5.7]. ∎

Besides meager spaces and spaces with countable π-bases, dyadic and regular quasi-dyadic spaces are also uK-U (see [FNR]), so Theorem 2 is a generalization of Theorem 1(ii); however, ω_1 with the discrete topology is a Baire space with a countable-in-itself π-base which is not a uK-U space (since it is not ccc), so the above argument cannot be modified to extend Theorem 1(i) to infinite products.

Our main theorem will imply that this extension is nevertheless possible (Corollary 6); moreover, we will partially answer a question of Fleissner ([Fl, Question 2]) about Baireness of box products (cf. Theorem 7 and its corollaries). Also note that the technique applied to prove these results can be adjusted to prove Baireness of other product topologies, which in turn can help establish Baireness of hyperspaces (see [Zs2] or [MC], [HMC], [Zs1] for earlier applications).

We will say that X is an *almost locally uK-U space* provided the set of points having an open uK-U neighborhood is dense in X (equivalently, if X has a π-base each member of which is uK-U). Since the uK-U property is open-hereditary and spaces with countable π-bases are uK-U (see [FNR]), it follows that if X is a uK-U space or has a countable-in-itself π-base, then X is almost locally uK-U. Observe that being almost locally uK-U is a genuine generalization of both being a uK-U space (ω_1 with the discrete topology is not uK-U, but it has a countable-in-itself π-base) and having a countable-in-itself π-base ($X = 2^{\omega_1}$ is a uK-U space with no countable π-base—cf. [FNR, Corollary 2]—and since all basic open sets in X are homeomorphic images of X, X has no countable-in-itself π-base either); however, we have the following:

PROPOSITION 3. *Let X be a metrizable Baire space. Then the following are equivalent*:

(i) *X is almost locally uK-U*,
(ii) *X has a countable-in-itself π-base.*

Proof. Only (i)⇒(ii) needs some explanation: let U be an open uK-U subspace of X and Y be a nowhere locally separable space. Let $A \subset X$ and $B \subset Y$ be such that $A \times B$ is meager in $X \times Y$ and B nonmeager in Y. Now, $U \times Y \cap A \times B$ is a meager subset of $U \times Y$, so, since U is uK-U and B is nonmeager, there is $y \in B$ such that $\{x \in U : (x, y) \in A \times B\} = U \cap A$ is meager in U and hence in X; thus, A is meager in X ([HMC, Theorem 1.7]). By a theorem of Pol ([Po, Theorem]), the points in X without a separable neighborhood form a closed meager subset so, since X is a Baire space, X has a dense open locally separable subspace. Finally, locally separable metrizable spaces can be partitioned into clopen separable subspaces, so if we unite the countable bases of this partition's members, we get a countable-in-itself π-base for X. ■

Let (X_i, τ_i) be a topological space for each $i \in I$. We will use bold symbols to denote notions related to the product space $\mathbf{X} = \prod_{i\in I} X_i$. Denote by $\boldsymbol{\tau}$ the product topology on $\mathbf{X}$ and by $\boldsymbol{\tau}_0$ the collection of Tikhonov cubes in $\mathbf{X}$; further, $\boldsymbol{\tau}_\Box$ will stand for the box-product topology on $\mathbf{X}$. If $\Pi_J = \prod_{j\in J}(X_j, \tau_j)$, then the projection maps $\pi_J : \mathbf{X} \to \Pi_J$ are continuous and open for each finite (possibly empty) $J \subseteq I$, if $\mathbf{X}$ is endowed with $\boldsymbol{\tau}$ or $\boldsymbol{\tau}_\Box$, respectively. Denote by $\operatorname{supp}(\mathbf{B})$ the support of $\mathbf{B} \in \boldsymbol{\tau}_0$, which is a subset of I such that $\pi_j^{\rightarrow}(\mathbf{B})$ is a proper nonempty τ_j-open set for all $j \in \operatorname{supp}(\mathbf{B})$ and $\pi_j^{\rightarrow}(\mathbf{B}) = X_j$ for all $j \in I \setminus \operatorname{supp}(\mathbf{B})$.

If $\mathbf{C} \subseteq \mathbf{X}$, $I_0, \ldots, I_t$ are pairwise disjoint finite subsets of I and $x_s \in \Pi_{I_s}$ $(s \le t)$, put $\mathbf{C}[x_0, \ldots, x_t] = \mathbf{C} \cap \bigcap_{s\le t} \pi_{I_s}^{\leftarrow}(x_s)$; further, if $\boldsymbol{\mathcal{C}}$ is a collection of subsets of $\mathbf{X}$, put $\boldsymbol{\mathcal{C}}[x_0, \ldots, x_t] = \{\mathbf{C} \in \boldsymbol{\mathcal{C}} : \mathbf{C}[x_0, \ldots, x_t] \ne \emptyset\}$.

The proof of Theorem 1(i) (see [Ox, Theorem 2] or [HMC, Theorem 5.1(vii)]) works for almost locally uK-U Baire spaces as well:

PROPOSITION 4. *Finite products of almost locally uK-U Baire spaces are Baire spaces.*

The main theorem of the paper reads as follows:

THEOREM 5. *If (X_i, τ_i) is an almost locally uK-U Baire space for each $i \in I$, then $(\mathbf{X}, \boldsymbol{\tau})$ is a Baire space.*

Proof. Since X_i is an almost locally uK-U space, it has a π-base $\mathcal{P}_i$ each member of which is uK-U. Define

$$\boldsymbol{\mathcal{P}} = \Big\{B \times \prod_{i\in I\setminus J} X_i : \emptyset \ne J \subseteq I \text{ finite}, \ B \in \prod_{i\in J} \mathcal{P}_i\Big\}.$$

Let $\{\mathbf{G}_n\}_n$ be a decreasing sequence of dense open subsets of $(\mathbf{X}, \boldsymbol{\tau})$. Fix a nonempty $\boldsymbol{\tau}$-open $\mathbf{V}$ and choose some $\mathbf{V}_0 \in \boldsymbol{\mathcal{P}}$ so that $\mathbf{V}_0 \subseteq \mathbf{V} \cap \mathbf{G}_0$. Put $J_0 = \emptyset$, $J_1 = \operatorname{supp}(\mathbf{V}_0)$ and $\boldsymbol{\mathcal{B}}_0 = \{\mathbf{V}_0\}$. By induction, we can define $\boldsymbol{\mathcal{B}}_i \subseteq \boldsymbol{\mathcal{P}}$ for each $i \geq 1$ so that $\boldsymbol{\mathcal{B}}_i = \bigcup_{\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}} \boldsymbol{\mathcal{B}}_i(\mathbf{B})$, where for all $\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}$, $\boldsymbol{\mathcal{B}}_i(\mathbf{B})$ is a maximal collection such that

(1) $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{G}_i$ for each $\mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})$,
(2) $\operatorname{supp}(\mathbf{A}) \supsetneq \operatorname{supp}(\mathbf{B})$ for each $\mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})$,
(3) $\{\pi^{\rightarrow}_{\operatorname{supp}(\mathbf{B})}(\mathbf{A}) : \mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})\}$ is pairwise disjoint.

Finite products of Baire uK-U spaces are Baire uK-U spaces (see [FNR, Property 2 and the subsequent Applications]) and Baire uK-U spaces have ccc ([FNR, Corollary 4]), so $\pi^{\rightarrow}_{\operatorname{supp}(\mathbf{B})}(\mathbf{B})$ has ccc for each $i \geq 1$ and $\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}$; thus, $\boldsymbol{\mathcal{B}}_i(\mathbf{B})$ is countable for each $i \geq 1$ and $\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}$ and so is the set $\boldsymbol{\mathcal{B}} = \bigcup_{i \in \omega} \boldsymbol{\mathcal{B}}_i$. Define

$$\boldsymbol{\mathcal{P}}' = \{\mathbf{P} \in \boldsymbol{\tau}_0 : \exists \mathbf{B} \in \boldsymbol{\mathcal{B}} \text{ with } \mathbf{P} \subseteq \mathbf{B} \text{ and } \operatorname{supp}(\mathbf{P}) = \operatorname{supp}(\mathbf{B})\}$$

and put $x_0 = \emptyset$ and $W_1 = \pi^{\rightarrow}_{J_1}(\bigcup \boldsymbol{\mathcal{B}}_1(\mathbf{V}_0)[x_0])$. For each $\mathbf{B} \in \boldsymbol{\mathcal{B}}$ and $n \geq 1$ put

$$Y_{\mathbf{B},n,1} = \{x \in W_1 : \exists \mathbf{P} \in \boldsymbol{\mathcal{P}}' \text{ with } \mathbf{P} \subseteq \mathbf{B},\ \mathbf{P}[x_0, x] \neq \emptyset$$
$$\text{such that } \forall \mathbf{P}' \in \boldsymbol{\mathcal{P}}',\ \mathbf{P}' \subseteq \mathbf{G}_n \cap \mathbf{P} \Rightarrow \mathbf{P}'[x_0, x] = \emptyset\}.$$

CLAIM 1. *$Y_{\mathbf{B},n,1}$ is nowhere dense in W_1 for each $\mathbf{B} \in \boldsymbol{\mathcal{B}}$ and $n \geq 1$.*

Indeed, if $Y_{\mathbf{B},n,1}$ is dense in a nonempty open $U \subseteq W_1$, then $\mathbf{P} \cap \pi^{\leftarrow}_{J_1}(U) \neq \emptyset$. Let $\mathbf{B} = \mathbf{U}_{i_0} \in \boldsymbol{\mathcal{B}}_{i_0}$ and assume that $\mathbf{U}_i \in \boldsymbol{\mathcal{B}}_i$ with $\mathbf{P}_i = \mathbf{U}_i \cap \pi^{\leftarrow}_{J_1}(U) \neq \emptyset$ has been defined for $i \geq i_0$. Then there exists a $\mathbf{U}_{i+1} \in \boldsymbol{\mathcal{B}}_{i+1}(\mathbf{U}_i)$ such that $\mathbf{P}_{i+1} = \mathbf{P}_i \cap \mathbf{U}_{i+1} \neq \emptyset$; otherwise, $\mathbf{P}_i \cap \mathbf{A} = \emptyset$ for each $\mathbf{A} \in \boldsymbol{\mathcal{B}}_{i+1}(\mathbf{U}_i)$, so, since $\operatorname{supp}(\mathbf{P}_i) = \operatorname{supp}(\mathbf{U}_i) \subsetneq \operatorname{supp}(\mathbf{A})$, $\pi^{\rightarrow}_{\operatorname{supp}(\mathbf{U}_i)}(\mathbf{P}_i)$ would be disjoint from $\pi^{\rightarrow}_{\operatorname{supp}(\mathbf{U}_i)}(\mathbf{A})$ for each $\mathbf{A} \in \boldsymbol{\mathcal{B}}_{i+1}(\mathbf{U}_i)$. Then choosing $\mathbf{A}' \in \boldsymbol{\mathcal{P}}$ with $\mathbf{A}' \subseteq \mathbf{P}_i \cap \mathbf{G}_{i+1}$ and $\operatorname{supp}(\mathbf{A}') \supsetneq \operatorname{supp}(\mathbf{U}_i)$, we would violate maximality of $\boldsymbol{\mathcal{B}}_{i+1}(\mathbf{U}_i)$. It follows, by (1), that $\mathbf{P}_n \subseteq \mathbf{U}_n \subseteq \mathbf{G}_n$, hence $\mathbf{P}' = \mathbf{P}_n \cap \pi^{\leftarrow}_{J_1}(U) \in \boldsymbol{\mathcal{P}}'$ and $\mathbf{P}' \subseteq \mathbf{P} \cap \mathbf{G}_n \cap \pi^{\leftarrow}_{J_1}(U)$. Then $\pi^{\rightarrow}_{J_1}(\mathbf{P}')$ is a nonempty open subset of U, so it intersects $Y_{\mathbf{B},n,1}$, say, in x. Now $x \in \pi^{\rightarrow}_{J_1}(\mathbf{P}')$ means $\mathbf{P}'[x_0, x] \neq \emptyset$; on the other hand, $x \in Y_{\mathbf{B},n,1}$ implies $\mathbf{P}'[x_0, x] = \emptyset$, since $\mathbf{P}' \in \boldsymbol{\mathcal{P}}'$ and $\mathbf{P}' \subseteq \mathbf{G}_n \cap \mathbf{P}$, a contradiction.

Since Π_{J_1} is a Baire space by Proposition 4, there exists some

$$x_1 \in W_1 \setminus \bigcup_{\mathbf{B} \in \boldsymbol{\mathcal{B}}} \bigcup_{n \geq 1} Y_{\mathbf{B},n,1}.$$

Assume that $\mathbf{V}_{j-1} \in \boldsymbol{\mathcal{B}}_{j-1}$ with $J_j = \operatorname{supp}(\mathbf{V}_{j-1}) \supsetneq J_{j-1}$ and $x_j \in W_j = \pi^{\rightarrow}_{J_j \setminus J_{j-1}}(\bigcup \boldsymbol{\mathcal{B}}_j(\mathbf{V}_{j-1})[x_0, \ldots, x_{j-1}])$ have been defined for $j \geq 1$ so that

(4) $\mathbf{V}_i \in \boldsymbol{\mathcal{B}}_i(\mathbf{V}_{i-1})[x_0, \ldots, x_i]$ for each $1 \leq i < j$,

(5) $\forall \mathbf{B} \in \boldsymbol{\mathcal{B}}\ \forall \mathbf{P} \in \boldsymbol{\mathcal{P}}'$ with $\mathbf{P} \subseteq \mathbf{B}$, $\mathbf{P}[x_0, \ldots, x_j] \neq \emptyset$, $\forall n \geq j\ \exists \mathbf{P}'_n \in \boldsymbol{\mathcal{P}}'$ such that $\mathbf{P}'_n \subseteq \mathbf{G}_n \cap \mathbf{P}$ and $\mathbf{P}'_n[x_0, \ldots, x_j] \neq \emptyset$.

Since $x_j \in W_j$ and $\{\pi^{\rightarrow}_{J_j \setminus J_{j-1}}(\mathbf{A}) : \mathbf{A} \in \boldsymbol{\mathcal{B}}_j(\mathbf{V}_{j-1})[x_0, \ldots, x_{j-1}]\}$ is pairwise disjoint (otherwise, if $x \in \pi^{\rightarrow}_{J_j \setminus J_{j-1}}(\mathbf{A}) \cap \pi^{\rightarrow}_{J_j \setminus J_{j-1}}(\mathbf{A}')$ for distinct $\mathbf{A}, \mathbf{A}' \in \boldsymbol{\mathcal{B}}_j(\mathbf{V}_{j-1})[x_0, \ldots, x_{j-1}]$, then $[x_0, \ldots, x_{j-1}, x] \in \pi^{\rightarrow}_{J_j}(\mathbf{A}) \cap \pi^{\rightarrow}_{J_j}(\mathbf{A}')$, which would violate (3)), there is a unique $\mathbf{V}_j \in \boldsymbol{\mathcal{B}}_j(\mathbf{V}_{j-1})[x_0, \ldots, x_{j-1}]$ with $x_j \in \pi^{\rightarrow}_{J_j \setminus J_{j-1}}(\mathbf{V}_j)$, which means that $\mathbf{V}_j \in \boldsymbol{\mathcal{B}}_j(\mathbf{V}_{j-1})[x_0, \ldots, x_j]$; thus, (4) is satisfied for $i = j$. Then $\mathbf{V}_j \subseteq \mathbf{V}_{j-1} \cap \mathbf{G}_j$ by (1) and, by (2), $J_{j+1} = \operatorname{supp}(\mathbf{V}_j) \supsetneq J_j$.

Since $\mathbf{V}_j[x_0, \ldots, x_j] \neq \emptyset$, it follows from (5) that for all $n \geq j$, there is some $\mathbf{P}'_n \in \boldsymbol{\mathcal{P}}'$ with $\mathbf{P}'_n \subseteq \mathbf{G}_n \cap \mathbf{V}_j$ and $\mathbf{P}'_n[x_0, \ldots, x_j] \neq \emptyset$; we can even assume that $\operatorname{supp}(\mathbf{P}'_n) \supsetneq J_{j+1}$ for some $n \geq j$ (otherwise, $\operatorname{supp}(\mathbf{P}'_n) = J_{j+1}$ for all $n \geq j$ and $\bigcap_{n \geq j} \mathbf{P}'_n[x_0, \ldots, x_j] \neq \emptyset$, whence $\mathbf{V}_j \cap \bigcap_n \mathbf{G}_n \neq \emptyset$ and we are done). It follows that $[x_0, \ldots, x_j] \in \pi^{\rightarrow}_{J_j}(\bigcup \boldsymbol{\mathcal{B}}_{j+1}(\mathbf{V}_j))$, so

$$W_{j+1} = \pi^{\rightarrow}_{J_{j+1} \setminus J_j}\Big(\bigcup \boldsymbol{\mathcal{B}}_{j+1}(\mathbf{V}_j)[x_0, \ldots, x_j]\Big)$$

is a nonempty $\Pi_{J_{j+1} \setminus J_j}$-open set. For each $\mathbf{B} \in \boldsymbol{\mathcal{B}}$ and $n \geq j+1$ define

$$Y_{\mathbf{B},n,j+1} = \{x \in W_{j+1} :\ \exists \mathbf{P} \in \boldsymbol{\mathcal{P}}' \text{ with } \mathbf{P} \subseteq \mathbf{B},\ \mathbf{P}[x_0, \ldots, x_j, x] \neq \emptyset \text{ and } \forall \mathbf{P}' \in \boldsymbol{\mathcal{P}}',\ \mathbf{P}' \subseteq \mathbf{G}_n \cap \mathbf{P} \Rightarrow \mathbf{P}'[x_0, \ldots, x_j, x] = \emptyset\}.$$

CLAIM 2. *$Y_{\mathbf{B},n,j+1}$ is nowhere dense in W_j for each $\mathbf{B} \in \boldsymbol{\mathcal{B}}$ and $n \geq j+1$.*

Indeed, assume, that some $Y_{\mathbf{B},n,j+1}$ is dense in an open $U \subseteq W_{j+1}$. Then $\mathbf{S} = \mathbf{V}_j \cap \mathbf{P} \cap \pi^{\leftarrow}_{J_{j+1} \setminus J_j}(U) \in \boldsymbol{\mathcal{P}}'$ is nonempty and $\mathbf{S}[x_0, \ldots, x_j] \neq \emptyset$, since $[x_0, \ldots, x_j] \in \pi^{\rightarrow}_{J_j}(\mathbf{P} \cap \mathbf{V}_j) = \pi^{\rightarrow}_{J_j}(\mathbf{S})$; thus, by (5), there is some $\mathbf{S}'_n \in \boldsymbol{\mathcal{P}}'$ with $\mathbf{S}'_n \subseteq \mathbf{G}_n \cap \mathbf{S}$ and $\mathbf{S}'_n[x_0, \ldots, x_j] \neq \emptyset$ for each $n \geq j$. Consequently, $\pi^{\rightarrow}_{J_{j+1} \setminus J_j}(\mathbf{S}'_n)$ is a nonempty open subset of U and hence it intersects $Y_{\mathbf{B},n,j+1}$, say, in x. Now, $x \in \pi^{\rightarrow}_{J_{j+1} \setminus J_j}(\mathbf{S}'_n)$ implies $\mathbf{S}'_n[x_0, \ldots, x_j, x] \neq \emptyset$; on the other hand, $x \in Y_{\mathbf{B},n,j+1}$ implies $\mathbf{S}'_n[x_0, \ldots, x_j, x] = \emptyset$, since $\mathbf{S}'_n \in \boldsymbol{\mathcal{P}}'$ and $\mathbf{S}'_n \subseteq \mathbf{G}_n \cap \mathbf{P}$, a contradiction.

Since $\Pi_{J_{j+1} \setminus J_j}$ is a Baire space by Proposition 3, we can find some

$$x_{j+1} \in W_{j+1} \setminus \bigcup_{\mathbf{B} \in \boldsymbol{\mathcal{B}}} \bigcup_{n \geq j+1} Y_{\mathbf{B},n,j+1}.$$

Then (4) and (5) is satisfied for $j+1$ as well; thus, by induction, we have constructed sequences $\{x_j \in \Pi_{J_j \setminus J_{j-1}} : j \geq 1\}$ and $\{\mathbf{V}_j \in \boldsymbol{\mathcal{B}} : j \in \omega\}$ such that $\mathbf{V}_{j+1} \in \boldsymbol{\mathcal{B}}_{j+1}(\mathbf{V}_j)[x_0, \ldots, x_j]$ for all $j \in \omega$.

Define the element $\mathbf{x} \in \mathbf{X}$ as follows: let $z \in \prod_{i \in I \setminus \bigcup_{j \geq 1} J_j} X_i$ be fixed, put $\pi^{\rightarrow}_{J_j \setminus J_{j-1}}(\mathbf{x}) = x_j$ for each $j \geq 1$ and $\pi^{\rightarrow}_{I \setminus \bigcup_{j \geq 1} J_j}(\mathbf{x}) = z$. Then $\mathbf{x} \in \mathbf{V}_n \subseteq \mathbf{V} \cap \mathbf{G}_n$ for each $n \in \omega$, so $\mathbf{V} \cap \bigcap_{n \in \omega} \mathbf{G}_n \neq \emptyset$; thus, $(\mathbf{X}, \boldsymbol{\tau})$ is a Baire space. ■

Clearly, Theorem 2 is a corollary of Theorem 5 and so is:

COROLLARY 6. *If* (X_i, τ_i) *is a Baire space with a countable-in-itself* π-*base for each* $i \in I$, *then* $(\mathbf{X}, \boldsymbol{\tau})$ *is a Baire space.*

A slight modification of the proof of Theorem 5 yields a theorem about Baireness of the *countable* box product; we will sketch the proof for completeness:

THEOREM 7. *If* (X_i, τ_i) *is an almost locally uK-U Baire space for each* $i \in \omega$, *then* $(\mathbf{X}, \boldsymbol{\tau}_\square)$ *is a Baire space.*

Proof. We will adopt the notation from the proof of Theorem 5 whenever applicable. Natural numbers will be viewed as sets of predecessors. By induction, for each $i \geq 1$, define $\boldsymbol{\mathcal{B}}_i = \bigcup_{\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}} \boldsymbol{\mathcal{B}}_i(\mathbf{B}) \subseteq \boldsymbol{\mathcal{P}}$, where $\boldsymbol{\mathcal{B}}_i(\mathbf{B})$ is maximal with respect to property (1) and

(3′) $\{\pi_i^{\rightarrow}(\mathbf{A}) : \mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})\}$ is pairwise disjoint.

Define the countable set $\boldsymbol{\mathcal{B}} = \bigcup_{i \in \omega} \boldsymbol{\mathcal{B}}_i$ and put

$$\boldsymbol{\mathcal{P}}' = \{\mathbf{P} \in \boldsymbol{\mathcal{P}} : \exists \mathbf{B} \in \boldsymbol{\mathcal{B}} \text{ with } \mathbf{P} \subseteq \mathbf{B} \text{ and } \pi_{\omega \setminus i+1}^{\rightarrow}(\mathbf{P}) = \pi_{\omega \setminus i+1}^{\rightarrow}(\mathbf{B})\}.$$

The rest of the proof can be adopted from that of Theorem 5, if we use $J_j = j$ for each $j \in \omega$ and instead of basing the induction on supports, we follow the natural order of ω. ■

COROLLARY 8. *If* (X_i, τ_i) *is a uK-U Baire space for each* $i \in \omega$, *then* $(\mathbf{X}, \boldsymbol{\tau}_\square)$ *is a Baire space.*

COROLLARY 9. *If* (X_i, τ_i) *is a Baire space with a countable-in-itself* π-*base for each* $i \in \omega$, *then* $(\mathbf{X}, \boldsymbol{\tau}_\square)$ *is a Baire space.*

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