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On localizations of torsion abelian groups

by

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Abstract. As is well known, torsion abelian groups are not preserved by localization functors. However, Libman proved that the cardinality of LT is bounded by $|T|^{\aleph_0}$ whenever T is torsion abelian and L is a localization functor. In this paper we study localizations of torsion abelian groups and investigate new examples. In particular we prove that the structure of LT is determined by the structure of the localization of the primary components of T in many cases. Furthermore, we completely characterize the relationship between localizations of abelian p-groups and their basic subgroups.

1. Introduction. Localization functors have a long history and were extensively studied in many fields of mathematics. However, during the last decade, new applications of these functors in homotopy theory due to Bousfield, Casacuberta, Dror Farjoun and others (see e.g. [2], [5], [11]) have pushed several authors to investigate the effect of homotopical localization functors on homotopy or homology groups. For instance, the effect on the fundamental group can often be described by means of group-theoretical localization functors. Motivated by this relationship, important advances have recently been achieved in the study of group localization functors, especially related to their behavior on certain classes of groups, like for example finite or nilpotent groups ([6], [19], [20]), simple groups ([16], [17], [22], [23], [26]) or perfect groups ([1], [24]). The papers by Casacuberta [6] and Libman [19] are good starting points for non-expert readers.

Roughly speaking, a localization functor in the category Grp of all groups is an idempotent functor L : Grp $\to$ Grp together with a natural transformation from the identity functor Id into L. We will recall in Section 2 the basic definitions and properties of such objects.

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One of the most interesting questions about localizations of groups, as motivated in [6], is to determine which algebraic properties are preserved. For instance, every localization of any abelian group is again abelian. Also the fact that a localization of a finite nilpotent group of class three or less is again nilpotent of class three or less has been proved by Aschbacher (preprint in preparation). For higher nilpotency classes the problem is still open. On the other hand, examples showing that finite, perfect, or simple groups are not preserved under localizations have recently been discovered.

Only little is known about localizations of abelian groups, apart from classical localizations or completions at a set of primes (see e.g. [6], [8] and [20]). For example, regarding the size of groups and their localizations it was shown in [25], [8] that any E-ring can appear as a localization of the group of integers, hence localizations of $\mathbb{Z}$ can be arbitrarily large by [13]. Therefore, one cannot expect to control *a priori* all possible localizations of a given torsion-free abelian group. However, recently Dugas [12] has studied localizations of torsion-free abelian groups successfully. On the other hand, Libman proved in [20] that, if $a : T \to LT$ is a localization, where T is torsion abelian, then the cardinality of LT is bounded by $|T|^{\aleph_0}$.

The aim of the present paper is to study in detail the localizations of torsion abelian groups T, find when torsion is preserved, and, if not, what phenomena can occur. If L is a localization functor in the category of abelian groups which preserves torsion, then one may obtain the structure of LT by the structure of $L(T_p)$ where the T_p's are the p-primary components of T (see Proposition 4.6). Studying the problem one prime at a time, we show that LT can be described in a certain way by the effect of the localization functor L on a few test groups, namely the Prüfer groups $\mathbb{Z}(p^\infty)$, the cyclic groups $\mathbb{Z}(p^n)$, and the rationals $\mathbb{Q}$. We separate localization functors into four classes. Let p be a prime.

- $(\mathrm{I})_p$ A localization functor belongs to the class $(\mathrm{I})_p$ if and only if $L\mathbb{Z}(p^\infty) = \mathbb{Z}(p^\infty)$.
- $(\mathrm{II})_p$ A localization functor belongs to the class $(\mathrm{II})_p$ if and only if $L\mathbb{Z}(p^\infty) = 0$ and there exists an integer $n \geq 1$ such that $L\mathbb{Z}(p^n) \neq \mathbb{Z}(p^n)$.
- $(\mathrm{III})_p$ A localization functor belongs to the class $(\mathrm{III})_p$ if and only if $L\mathbb{Z}(p^\infty) = 0$, $L\mathbb{Z}(p^n) = \mathbb{Z}(p^n)$ for any $n \geq 1$, and $L\mathbb{Q} = \mathbb{Q}$.
- $(\mathrm{IV})_p$ A localization functor belongs to the class $(\mathrm{IV})_p$ if and only if $L\mathbb{Q} = 0$ and $L\mathbb{Z}(p^n) = \mathbb{Z}(p^n)$ for any $n \geq 1$.

THEOREM 5.3. *Let L be a localization functor and p a prime. Then*:

(i) *If L belongs to the class* $(\mathrm{I})_p$, *then any abelian p-group is L-local.*
(ii) *If L belongs to the class* $(\mathrm{II})_p$, *then the localization of any abelian p-group is a bounded p-group.*

(iii) *If L belongs to the class* $(\mathrm{III})_p$, *then the localization of any abelian p-group is so as well.*

(iv) *The Ext-completion functor*, *as well as the* $\mathbb{Q}$-*nullification*, *belong to the class* $(\mathrm{IV})_p$.

As a basic subgroup of a reduced p-group is dense and pure inside the group, the localizations coming from an embedding of a basic subgroup into an overgroup should be simpler than arbitrary ones. However, using a construction inside the Ext-completion of a certain torsion group T, we show the following:

Theorem 6.10. *For every prime p there exists a separable non-torsion-complete p-group T and a localization functor L* (*in the class* $(\mathrm{IV})_p$) *such that T is not L-local but any basic subgroup of T is L-local.*

This suggests a way to construct possibly other new localizations. On the other hand we completely characterize the connection between localizations of torsion groups and localizations of their basic subgroups.

Theorem 6.8. *Let T be a reduced p-group for some prime p. Then T is torsion-complete if and only if every embedding of a basic subgroup B of T into T is a localization.*

Our notation is standard and is in accordance with that of [15].

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2. Localizations of abelian groups. Let us recall from [6] or [20] some terminology and basic properties of localization functors. Defined over any reasonable category, they are conceived to formally invert appropriate classes of morphisms. We concentrate here on the category of groups.

A *localization functor* in the category Grp of all groups is a pair (L, a), where $L : \mathrm{Grp} \to \mathrm{Grp}$ is a functor and the *coaugmentation* $a : \mathrm{Id} \to L$ is a natural transformation from the identity functor Id into L such that the homomorphisms a_{LG}, $L(a_G)$ from LG to LLG coincide and are isomorphisms. The groups G for which $a_G : G \cong LG$ are called *L-local*, and the homomorphisms $A \to B$ inducing an isomorphism $LA \cong LB$ are called *L-equivalences*. Hence, for any group G, the coaugmentation homomorphism $a : G \to LG$ is an L-equivalence and furthermore LG is L-local. The classes of L-local groups and of L-equivalences are orthogonal to each other in the following sense: a homomorphism $\varphi : A \to B$ is an L-equivalence if and only if it induces a bijection $\varphi^* : \mathrm{Hom}(B, C) \cong \mathrm{Hom}(A, C)$ for any L-local group C; also, C is L-local if and only if $\varphi^* : \mathrm{Hom}(B, C) \cong \mathrm{Hom}(A, C)$ is a bijection for any L-equivalence $\varphi : A \to B$.

Given a group homomorphism $\varphi : A \to B$ one can always construct a localization functor L_φ having φ as a "generating" equivalence, i.e., the corresponding local groups are those G for which $\varphi^* : \mathrm{Hom}(B, G) \cong \mathrm{Hom}(A, G)$ is bijective (see e.g. [6]). For instance, localization of abelian groups at a prime p given by $A \to A \otimes \mathbb{Z}_{(p)}$ is of the form $A \to L_\varphi A$, where $\varphi = \bigoplus_{q \neq p} \varphi_q$ with $\varphi_q : \mathbb{Z} \to \mathbb{Z}$ multiplication by q on the ring of integers.

Observe that if a homomorphism $f : A \to B$ satisfies $f^* : \mathrm{Hom}(B, B) \cong \mathrm{Hom}(A, B)$ then B is L_f-local. Since f is obviously an L_f-equivalence, it turns out that $f : A \to B$ is indeed $a : A \to L_f A$. For this reason, we say that a homomorphism satisfying this property is a localization, or that B is a localization of A (see [6, Lemma 2.1]).

As mentioned in the introduction, the localization of any abelian group is again an abelian group. Hence, when studying localizations of abelian groups, we may work in the category of abelian groups. We next collect some properties and known results that we shall use in the rest of the paper. Proofs can be found in Libman's paper [20], as well as in the survey [6]. However, except for (4), (5) and (6) all these results were known before. Notice that property (5) gives an estimate for the size of localizations of torsion abelian groups. Recall that an abelian group G is *reduced* if it has no divisible subgroup, and it is *bounded* if there exists a non-zero integer n such that $nG = 0$.

THEOREM 2.1. *Let L be any group localization functor.*

(1) *If G is abelian, then $L(G)$ is abelian.*
(2) *If G is an abelian group which is divisible or bounded, then the coaugmentation map $a : G \to L(G)$ is surjective.*
(3) *$L(\mathbb{Z}(p^\infty)) = 0$ or $L(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)$ and likewise $L\mathbb{Q} = 0$ or $L\mathbb{Q} = \mathbb{Q}$.*
(4) *The direct sum of L-local abelian divisible groups is again L-local.*
(5) *If G is abelian, reduced, and torsion, then $L(G)$ is reduced and $|L(G)| \le |G|^{\aleph_0}$.*
(6) *If $\mathbb{Z}(p^\infty)$ is L-local, then every abelian p-group is L-local.*
(7) *Any retract of an L-local group is L-local (in particular, direct summands of L-local abelian groups are L-local).*
(8) *Any limit of L-local groups is L-local (in particular, the kernel of a homomorphism between L-local groups is L-local as well, and any product of L-local groups is L-local).*
(9) *If $a : G \to L(G)$ is the coaugmentation map and $T \subseteq L(G)$ is L-local such that $\mathrm{Im}(a) \subseteq T$, then $T = L(G)$.*
(10) *If G is L-local and $T \subseteq G$, then the coaugmentation map $a : T \to LT$ is a monomorphism.*

(11) *Any direct limit of L-equivalences is an L-equivalence. In particular, there is an isomorphism $L(\varinjlim A_i) \cong L(\varinjlim LA_i)$ for any direct system $\{A_i\}$ of groups.* ■

3. The Ext-completion. In this section we consider the examples of localizations of torsion abelian groups that motivated our study. Indeed, although localizations of abelian groups are again abelian, a localization of a (p-)torsion group need not be (p-)torsion again. First recall that an abelian group G is called *cotorsion* if $\mathrm{Ext}(\mathbb{Q}, G) = 0$, where $\mathbb{Q}$ is the additive group of rational numbers. The group G is called *adjusted cotorsion* if it is cotorsion, reduced and has no non-zero torsion-free direct summand. It is well known that the so-called *Ext-completion* $T^\bullet = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$ of a torsion abelian group T is always adjusted cotorsion (see [15, Lemma 55.4]). Note that Bousfield and Kan have studied the Ext-completion extensively in a more general setting in [4].

LEMMA 3.1. *The Ext-completion functor* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, -)$ *is a localization functor. In particular, for an abelian group G we have*

(1) $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G) = 0$ *if and only if G is divisible.*
(2) *G embeds into* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ *if G is reduced.*
(3) *G is the torsion part of* $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, G)$ *if G is reduced torsion.*

Proof. Since $\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, -)$ is idempotent (see for instance [4], [15, Chapter IX, Section 54, (H)]) it follows that the Ext-completion functor is a localization functor. Now (3) follows from [15, Lemma 55.1], and (1) is a consequence of (2), which in turn easily follows by applying the functor $\mathrm{Ext}(-, G)$ to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. ■

In fact, Harrison [18] proved that (3) yields a one-to-one correspondence between reduced torsion abelian groups and adjusted cotorsion groups. We obtain two immediate examples from the above lemma, which were noted by Libman in [20] (see also [4, Example 4.2 (ii)]). Let Π denote the set of all primes.

EXAMPLE 3.2. Let $P = \prod_{p\in\Pi} \mathbb{Z}(p)$ and $B = \bigoplus_{p\in\Pi} \mathbb{Z}(p)$. Then it follows from Lemma 3.1 that the natural embedding $f : B \to P$ is a localization since B is reduced torsion. Note that

$$\mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, B) = \prod_{p\in\Pi} \mathrm{Ext}(\mathbb{Z}(p^\infty), B) = \prod_{p\in\Pi} \mathrm{Ext}(\mathbb{Z}(p^\infty), \mathbb{Z}/p\mathbb{Z}) = P.$$

Thus a localization of a reduced torsion group need not be torsion again. Even a localization of a reduced p-group (p a prime) need not be torsion again as is demonstrated by the second example.

EXAMPLE 3.3. Let $B = \bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ for some prime p. Then the natural embedding of B into its Ext-completion $B^\bullet$ is a localization but $B^\bullet$ is not torsion. This follows from Lemma 3.1 and the fact that $\operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, B) = \operatorname{Ext}(\mathbb{Z}(p^\infty), B)$ is not torsion. It actually fits into a short exact sequence

$$\prod J_p \hookrightarrow B^\bullet \to \prod_{n\geq 1} \mathbb{Z}(p^n),$$

where J_p is the group of p-adic integers.

Furthermore, Lemma 3.1 shows that $L(\mathbb{Z}(p^n)) = \mathbb{Z}(p^n)$ for $L = \operatorname{Ext}(\mathbb{Q}/\mathbb{Z}, -)$, hence $\mathbb{Z}(p^n)$ is L-local for all $n \in \omega$ but $B = \bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ is not L-local. Hence this is a particular example of the well known fact that a direct sum of local objects need not be local again. Similarly, Lemma 3.1 demonstrates once more that a direct limit of local objects is not always local. This shows that the structure of a torsion group is not well preserved under localizations.

4. Localizations of torsion abelian groups. In general, if T is a torsion group, then T decomposes as a direct sum of its p-primary components T_p (where p ranges over the set of primes). We shall first show that it is enough to consider the part of T which is reduced and has non-trivial p-components only for those primes p for which $L(\mathbb{Z}(p^\infty)) = 0$. Afterwards we show in Proposition 4.6 that for many localization functors we may deal with p-groups only, which we focus on in the next section.

LEMMA 4.1. *Let L be a localization functor and $Q_L = \{p \in \Pi : L(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)\}$. Then any Q_L-torsion group is L-local.*

Proof. The proof is essentially contained in that of [20, Theorem 2.4] but for the sake of completeness we outline it briefly. Let T be a Q_L-torsion group and E its divisible hull. Then E is a direct sum of copies of groups $\mathbb{Z}(p^\infty)$ with $p \in Q_L$, and so is E/T. Thus E and E/T are L-local by Theorem 2.1(4), and hence T is L-local as well by Theorem 2.1(8). ∎

Thus, given a localization functor L and a torsion group T we may split T into $T = T_1 \oplus T_2$, where $T_1 = \bigoplus_{p\in Q_L} T_p$ and $T_2 = \bigoplus_{p\in P_L} T_p$ with $P_L = \{p \in \Pi : L(\mathbb{Z}(p^\infty)) = 0\}$. We obtain $LT = L(T_1) \oplus L(T_2) = T_1 \oplus L(T_2)$ and therefore it is sufficient to determine $L(T_2)$ in order to find the structure of LT. Moreover T_2 splits as the direct sum of a reduced torsion group R and a divisible one D. Recall that a group H is called (L-)*acyclic* if $L(H) = 0$.

LEMMA 4.2. *If L is a localization functor and $L\mathbb{Z}(p^\infty) = 0$ for a prime p, then $LT = 0$ whenever T is a divisible p-group. In particular D is acyclic.*

Proof. Let T be a divisible p-group. By Theorem 2.1(2) the coaugmentation map $a : T \to LT$ is surjective and hence LT is a divisible p-group. As any direct summand of a local group must be local, LT must be zero. The group D is a direct sum of acyclic Prüfer groups. It must be acyclic by Theorem 2.1(11). ∎

From the above discussion we see that we may restrict ourselves to the study of reduced P_L-torsion groups.

PROPOSITION 4.3. *Let L be a localization functor and T a torsion group. If T_1 denotes the Q_L-torsion part of T and R the reduced P_L-torsion part, then $LT = T_1 \oplus LR$.* ∎

The next lemma tells us furthermore that no new torsion can be created in a localization.

LEMMA 4.4. *Let T be a reduced torsion group and L a localization functor. If T has no p-torsion elements, then neither has LT. Moreover, if $t(LT)$ denotes the torsion part of LT, then $LT/t(LT)$ is divisible. In fact, if $a : T \to LT$ is the coaugmentation map, then $LT/\mathrm{Im}(a)$ is divisible.*

Proof. Since T is a reduced torsion group, if $T_p = 0$ for some prime p, then multiplication by p is an automorphism of T and hence also an automorphism of LT. Thus $t(LT)_p = 0$. Moreover, Libman has shown in [20] that $LT/\mathrm{Im}(a)$ is divisible, hence also $LT/t(LT)$ must be divisible since $\mathrm{Im}(a) \subseteq t(LT)$. ∎

COROLLARY 4.5. *If L is a localization functor and T a p-group, then the torsion part $t(LT)$ of LT is p-torsion.* ∎

If we also know that L preserves torsion groups, i.e. LT is torsion whenever T is torsion, then we may even restrict ourselves to the study of p-groups (for $p \in P_L$).

PROPOSITION 4.6. *If L is a localization functor that preserves torsion groups, then $LT \cong \bigoplus_{p\in\Pi} L(T_p)$ for every torsion group T.*

Proof. Let us fix a prime p. On the one hand we know by assumption that LT is torsion, hence

$$LT = \bigoplus_{q\in\Pi} L(T)_q = L(T)_p \oplus \bigoplus_{q\neq p} L(T)_q.$$

On the other hand $T = T_p \oplus \bigoplus_{q\neq p} T_q$, so that

$$LT = L(T_p) \oplus L\Big(\bigoplus_{q\neq p} T_q\Big).$$

Now $L(T_p)$ is p-torsion by Corollary 4.5. Therefore $L(T_p)$ is contained in

$L(T)_p$, which is L-local. Thus $L(T_p) = L(T)_p$ by Theorem 2.1(9) and hence $LT = \bigoplus_{p\in\Pi} L(T_p)$. ∎

Note that, in general, a localization functor does not commute with the direct decomposition into primary components as Example 3.2 shows.

5. Localizations of p-torsion abelian groups. Let p be a fixed prime. In this section we explain how to divide the class of localization functors into four (disjoint) subclasses, each of which has different behavior on p-torsion groups. The idea is that one has a rather good understanding of a localization functor if one knows its effect on a few "test groups": the Prüfer group $\mathbb{Z}(p^\infty)$, the cyclic groups $\mathbb{Z}(p^n)$, and the rationals $\mathbb{Q}$. Let us define these classes.

(I)$_p$ A localization functor L belongs to the class (I)$_p$ if and only if $L\mathbb{Z}(p^\infty) = \mathbb{Z}(p^\infty)$.

(II)$_p$ A localization functor L belongs to the class (II)$_p$ if and only if $L\mathbb{Z}(p^\infty) = 0$ and there exists an integer $n \geq 1$ such that $L\mathbb{Z}(p^n) \neq \mathbb{Z}(p^n)$.

(III)$_p$ A localization functor L belongs to the class (III)$_p$ if and only if $L\mathbb{Z}(p^\infty) = 0$, $L\mathbb{Z}(p^n) = \mathbb{Z}(p^n)$ for any $n \geq 1$, and $L\mathbb{Q} = \mathbb{Q}$.

(IV)$_p$ A localization functor L belongs to the class (IV)$_p$ if and only if $L\mathbb{Q} = 0$ and $L\mathbb{Z}(p^n) = \mathbb{Z}(p^n)$ for any $n \geq 1$.

In view of Theorem 2.1(6), if the Prüfer group $\mathbb{Z}(p^\infty)$ is local, then so is every abelian p-group. Therefore any localization functor in (I)$_p$ satisfies $LP = P$ for any abelian p-group P. It thus remains to consider localization functors L satisfying $L(\mathbb{Z}(p^\infty)) = 0$.

We now deal with the second class, and show that it consists of localization functors which all preserve p-torsion groups.

PROPOSITION 5.1. *Let L be a localization functor for which $\mathbb{Z}(p^\infty)$ is acyclic. If there exists a natural integer n such that $\mathbb{Z}(p^n)$ is not local, then L preserves p-groups. In fact, if H is a p-group, then LH is a bounded p-group.*

Proof. Suppose $L(\mathbb{Z}(p^n)) \neq \mathbb{Z}(p^n)$ for some integer n. Then $L(\mathbb{Z}(p^n)) = \mathbb{Z}(p^r)$ for some $r < n$ and hence $L(\mathbb{Z}(p^m)) = \mathbb{Z}(p^r)$ for all $m \geq n$ (see [19]). Consider now a p-torsion group H and write H as the direct limit of its finite subgroups H_i. Each H_i is a finite direct sum of cyclic p-groups and by assumption $L(H_i)$ is p^r-bounded for all i. Thus $L(H) = L(\varinjlim L(H_i))$ is p^r-bounded since $\varinjlim L(H_i)$ is p^r-bounded by Theorem 2.1(2). ∎

Our third class preserves torsion groups as well.

Proposition 5.2. *Let L be a localization functor such that $L(\mathbb{Q}) = \mathbb{Q}$. Then L preserves torsion groups.*

Proof. Let T be a torsion group. Without loss of generality we can assume it is reduced (see Proposition 4.3). By Lemma 4.4 we know that $LT/t(LT)$ is divisible and torsion-free, hence a direct sum of copies of $\mathbb{Q}$. Thus we have a short exact sequence

$$0 \to t(LT) \to LT \to LT/t(LT) \to 0,$$

where LT and $LT/t(LT)$ are L-local by 2.1(4) and therefore $t(LT)$ must be L-local by 2.1(8). Thus $t(LT) = LT$ by 2.1(9). ∎

Examples of localization functors satisfying $L(\mathbb{Q}) = \mathbb{Q}$ are for instance localizations at a prime p but there are more interesting and complicated ones as we shall prove in Theorem 6.8.

Our last class has $\mathbb{Q}$ acyclic (by Theorem 2.1(3), $\mathbb{Q}$ must be either local or acyclic). It is easy to see that if $L(\mathbb{Q}) = 0$ then also $L\mathbb{Z}(p^\infty) = 0$ (for all primes p). Therefore our classification of localization functors is complete. The class $(\mathrm{IV})_p$ is the one containing the "wildest" localizations, such as the Ext-completion functor, which does not preserve p-torsion. Let us remark however that the converse of the above proposition is false. In other words there are examples of localization functors in the class $(\mathrm{IV})_p$ which also preserve torsion groups. In fact, reduction by the divisible part of a group (i.e. $\mathbb{Q}$-nullification) is a localization functor sending $\mathbb{Q}$ to zero but preserving torsion groups. Let us sum up the results of this section.

Theorem 5.3. *Let L be a localization functor and p a prime. Then*:

(i) *If L belongs to the class $(\mathrm{I})_p$, then any abelian p-group is local.*

(ii) *If L belongs to the class $(\mathrm{II})_p$, then the localization of any abelian p-group is a bounded p-group.*

(iii) *If L belongs to the class $(\mathrm{III})_p$, then the localization of any abelian p-group is an abelian p-group as well.*

(vi) *The Ext-completion functor and the $\mathbb{Q}$-nullification belong to the class $(\mathrm{IV})_p$.* ∎

We would like to pose an open question at this point which concerns localizations of direct sums of local groups. Note that Bastardas *et al.* proved in [7] that a direct sum of local slender groups is again local. An affirmative answer would help us understand the class $(\mathrm{IV})_p$.

Question 5.4. *If $B = \bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ is local, does it imply that B^κ is local for every cardinal κ?*

6. Localizations of p-groups and their basic subgroups. The first two classes introduced above are perfectly well understood. As for the other two, we need to refine the techniques we used to get a better understanding. A first reduction is to consider only localizations coming from an embedding, i.e. when $T \hookrightarrow LT$ is injective. The general problem is now the following:

PROBLEM 6.1. *Give a characterization of those localization functors which come from an embedding and preserve torsion groups.*

We solve the problem completely in the particular case when T is the basic subgroup of the torsion group LT. More generally we consider the connection between localizations of abelian p-groups and localizations of their basic subgroups. Indeed, the classical theorem of Kulikov asserts that every torsion group has a basic subgroup, hence any torsion group is an extension of a direct sum of cyclic groups by a divisible group. As divisible torsion groups are acyclic in the interesting examples, it is natural to ask what is the relationship between the localizations of torsion groups and their basic subgroups. Recall first the definition of a basic subgroup for p-groups, p a prime.

DEFINITION 6.2. Let H be an abelian group and p a prime. A direct sum $B \subseteq H$ of cyclic groups of order a power of p or of infinite order is a *p-basic subgroup* of H if B is pure and dense in H, i.e. $p^nH \cap B = p^nB$ for all integers n (purity) and the quotient H/B is p-divisible (density). If H is a p-group then we call B a *basic subgroup* of H for short.

Let us first state some properties of basic subgroups (see [15]).

LEMMA 6.3. *Let H be an abelian p-group. Then*:

(1) *All basic subgroups of H are isomorphic.*
(2) *All basic subgroups of H are epimorphic images of H.*
(3) *Every direct sum $B \subseteq H$ of cyclic groups which is pure in H can be extended to a basic subgroup of H.* ∎

Let A and C be abelian groups. A short exact sequence

$$0 \to A \to G \to C \to 0$$

is called *pure exact* if the image of A in G is pure in G. We denote by $\mathrm{Pext}(C, A)$ the class of all equivalence classes of pure exact sequences as above. Recall that

$$\mathrm{Pext}(C, A) = \bigcap_{n\in\omega} n\mathrm{Ext}(C, A)$$

(see [15, Vol. I, p. 228]) and that a reduced p-group T is called *torsion-complete* if $\mathrm{Pext}(P, T) = 0$ for all p-groups P (see [15, p. 18]). This is equivalent to saying that T is a direct summand in every p-group in which

T is contained purely. In fact, torsion-complete groups have to be separable and if B is a basic subgroup of T, then $T = \bar{B}$, the torsion part $t(\widehat{B})$ of the p-adic completion $\widehat{B}$ of B. Recall that a group T is *separable* if it contains no elements of infinite height, i.e. the first Ulm subgroup $p^\omega T = \bigcap_{n\in\mathbb{N}} p^n T$ is trivial.

LEMMA 6.4. *If T is a torsion-complete p-group and B a basic subgroup of T, then any embedding $B \to T$ is a localization* (*belonging to the class* $(\mathrm{III})_p$).

Proof. We consider the short exact sequence

$$0 \to B \to T \to T/B \to 0$$

and apply the Hom-functor. This yields by [15, Theorem 53.7] the short exact sequence

$$0 \to \mathrm{Hom}(T/B, T) \to \mathrm{Hom}(T, T) \to \mathrm{Hom}(B, T) \to \mathrm{Pext}(T/B, T) \\ \to \mathrm{Pext}(T, T) \to \mathrm{Pext}(B, T) \to 0.$$

The first Hom-group is trivial since T is reduced, and $\mathrm{Pext}(T/B, T) = 0$ as T is torsion-complete. It follows that the natural morphism $\mathrm{Hom}(T, T) \to \mathrm{Hom}(B, T)$ is a bijection. Clearly $\mathbb{Q}$ is local because there are morphisms into it neither from B nor from T. ■

EXAMPLE 6.5. Let $B = \bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ for some prime p and consider the corresponding product $B \subseteq P = \prod_{n\in\omega} \mathbb{Z}(p^n)$. The torsion completion $T = t(\widehat{B})$ of B is nothing else but the set of all elements $(c_1, c_2, \ldots)$ in P which are bounded, i.e. for which there is an integer m such that $p^m c_n = 0$ for all $n \geq 1$ (see [15, Vol. II, p. 15]). The above lemma tells us that $B \to T$ is a localization, as is the inclusion of any basic subgroup of T in T.

COROLLARY 6.6. *A basic subgroup of a local p-group need not be local.*

Proof. Let T be a torsion-complete p-group and B a basic subgroup different from T. Then the embedding of B into T is a localization, hence T is local with respect to this localization but B is not. ■

That the embedding of a basic subgroup into a p-group is not always a localization is shown by the next example.

EXAMPLE 6.7. Let $T = \bigoplus_{n\in\omega} \mathbb{Z}(p^n)$ and B be a basic subgroup of T different from T (it exists by [15, Lemma 35.1]; in fact if $T = \bigoplus_{n\in\omega} \langle a_n \rangle$ with $\mathrm{ord}(a_n) = p^n$, then $b_n = a_n - pa_{n+1}$ $(n \in \omega)$ form a basis of such a B). Then the natural embedding $B \to T$ cannot be a localization since B is isomorphic to T.

In fact we can prove that the torsion-complete groups are the only groups such that all embeddings of basic subgroups give rise to a localization.

THEOREM 6.8. *Let T be a reduced p-group. Then T is torsion-complete if and only if every embedding of a basic subgroup B of T into T is a localization.*

Proof. One implication follows from Lemma 6.4. We assume thus that T is a reduced p-group such that every embedding of a basic subgroup B into T is a localization, i.e. for every basic subgroup B of T we have a natural bijection $\mathrm{Hom}(B,T) \cong \mathrm{End}(T)$. Let B_1 and B_2 be two basic subgroups of T and $\psi : B_1 \to B_2$ be an isomorphism. We claim that ψ extends to an automorphism of T. By assumption there exist $\varphi_i \in \mathrm{End}(T)$ for $i = 1, 2$ such that $\varphi_1\restriction_{B_1} = \psi$ and $\varphi_2\restriction_{B_2} = \psi^{-1}$. Thus $\varphi_1\varphi_2$ extends the identity on B_1 and by uniqueness it follows that $\varphi_1\varphi_2 = \mathrm{id}_T$. Similarly we also have $\varphi_2\varphi_1 = \mathrm{id}_T$. Thus ψ extends (even uniquely) to an automorphism φ_1 of T. By [15, Theorem 69.2] it follows that T must be torsion-complete. ∎

On the other hand there exist localization functors L and p-groups T such that T is not L-local but any basic subgroup of T is L-local. In the proof we shall need the definition of *small homomorphisms*, which is the following (see also [15, Chapter VIII, p. 195]). For an abelian torsion group A, the exponent of an element $a \in A$ is denoted by $e(a)$. Note that if the order of a is p^n, then the exponent of a is n.

DEFINITION 6.9. Let A, C be torsion abelian groups and $\varphi : A \to C$ a homomorphism. Then φ is called *small* if for every $k \geq 0$ there exists an integer $n_\varphi(k)$ such that

$$e(a) \geq n_\varphi(k) \quad \text{implies} \quad e(\varphi(a)) \leq e(a) - k$$

for every element $a \in A$.

We denote by $\mathrm{Small}(T)$ the set of all small endomorphisms of T. We will also need the p^n-*socle* $T[p^n]$ of T, which consists of all elements of order dividing p^n.

THEOREM 6.10. *There exists a separable non-torsion-complete p-group T and a localization functor L (in the class $(\mathrm{IV})_p$) such that T is not L-local but any basic subgroup of T is L-local.*

Proof. We shall construct a group G with torsion subgroup T, where T is p-torsion and G/T is divisible, such that the natural embedding $T \to G$ induces a bijection $\mathrm{Hom}(T,G) \cong \mathrm{End}(G)$. Hence G is the localization of T under the natural embedding of T into G. Moreover, we shall prove that the inclusion of every basic subgroup $B \hookrightarrow T$ also induces a bijection $\mathrm{Hom}(T,B) \cong \mathrm{Hom}(G,B)$, i.e. B is local.

Let Q_p^* be the ring of p-adic integers. By [10] we may choose an unbounded separable p-group T of arbitrary cardinality such that $\mathrm{End}(T) = Q_p^* \oplus \mathrm{Small}(T)$. Note that every p-group is obviously also a Q_p^*-module, hence $Q_p^* \subseteq \mathrm{End}(T)$ if T is unbounded. As a torsion-complete group has lots of non-small endomorphisms (e.g. every automorphism of a basic subgroup lifts to an automorphism of T which cannot be small), T is not torsion-complete.

We next consider its Ext-completion $T^\bullet = \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T)$. If $\widehat{T}$ is the p-adic completion of T, notice that

$$p^\omega \mathrm{Ext}(\mathbb{Q}/\mathbb{Z}, T) \cong \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \widehat{T}/T),$$

which contains torsion-free elements (compare also [15, Theorem 56.5]). Choose now such a torsion-free element $x \in T^\bullet$ of p-height ω. We define G to be the purification in $T^\bullet$ of the subgroup generated by x and T, i.e. $G = \bigcup_{n\in\omega} G_n$, where $G_0 = \langle x, T\rangle$ and $G_{n+1} \subseteq T^\bullet$ is obtained by adding solutions of equations of the form $p^k y = g \in G_n$ to G_n (see [15, Vol. 1, Proposition 26.2]). Thus G is a pure subgroup of $T^\bullet$ of torsion-free rank 1, its torsion submodule is T, and G/T is divisible since $T^\bullet/T$ is divisible and $x \in p^\omega T^\bullet$. Moreover, it is easily checked that if $S = \{x_n \in G : n \in \omega\}$ and $p^n x_n = x$ for all $n \in \omega$, then S and T generate G.

We claim that the natural embedding of T into G induces a bijection

$$\mathrm{Hom}(T, G) \cong \mathrm{End}(G).$$

Since G/T is divisible any extension of a homomorphism $\varphi : T \to G$ to an endomorphism of G is unique. Thus it suffices to prove that every such homomorphism extends. Notice also that the image of φ is torsion and must therefore be contained in T. We thus have to check that $\mathrm{End}(T) \cong \mathrm{End}(G)$.

Suppose first that we are given $\beta \in \mathrm{Small}(T)$. We shall prove that β can be extended to $\widetilde{\beta} : G \to G$ with $\widetilde{\beta}(G) = \beta(T)$. Since the Ext-completion is a localization functor, any endomorphism of T extends to an endomorphism of $T^\bullet$. Let $\widehat{\beta}$ be the extension of β and $\widetilde{\beta} = \widehat{\beta}\restriction_G$. We construct elements $x_n \in G$, for $n \geq 1$, such that $p^n x_n = x$ and $\widetilde{\beta}(x_n) = 0$. Since β is small, there exists an increasing sequence k_n of positive integers such that if $t \in T[p^n]$ and the p-height of t is $\geq k_n$, then $\beta(t) = 0$. Since x has p-height ω, we may choose $x_n \in G$ of p-height $\geq k_n$ such that $p^n x_n = x$. Then $x_n - px_{n+1} \in T[p^n]$ is a torsion element of p-height $\geq k_n$ mapping to 0 under β. Thus, $p\widetilde{\beta}(x_{n+1}) = \widetilde{\beta}(x_n)$ for all n. This linear divisibility means that $\widetilde{\beta}(x_n) = 0$ since we are mapping into the reduced group $T^\bullet$. Note that $T^\bullet$ is reduced as T was reduced torsion. So β may be extended to an endomorphism $\widetilde{\beta} : G \to G$ such that $\widetilde{\beta}(G) = \beta(T)$ because G is generated by T and the elements x_n for $n \geq 1$. Hence any small endomorphism of T can be extended to an endomorphism of G.

Since also $Q_p^* \subseteq \mathrm{End}(G)$, we get a natural bijection

$$\mathrm{End}(T) \cong Q_p^* \oplus \mathrm{Small}(T) \cong \mathrm{End}(G).$$

This proves that $T \hookrightarrow G$ is a localization.

Let B be a basic subgroup of T. We have to show that the inclusion $B \hookrightarrow T$ induces a bijection $\mathrm{Hom}(T, B) \cong \mathrm{Hom}(G, B)$. Again it is sufficient to prove that any homomorphism $\alpha : T \to B$ can be extended to G. If the composite $T \xrightarrow{\alpha} B \hookrightarrow T$ is a small endomorphism, we have seen above that it can be extended to a homomorphism $\widetilde{\alpha} : G \to T$ with $\widetilde{\alpha}(G) = \alpha(T) \subseteq B$, i.e. $\widetilde{\alpha} \in \mathrm{Hom}(G, B)$.

The case of non-small endomorphisms remains, but we prove next that there are none. First we claim that any non-small homomorphism $\alpha : T \to B$ has unbounded image. Indeed, suppose that $\alpha \in \mathrm{End}(T)$ is not a small endomorphism. Then $\alpha = p^k u + \beta$, where $k \geq 0$, $u \in Q_p^*$ is a unit, and $\beta \in \mathrm{Small}(T)$. As β is small there exists $m \geq k$ such that if $e(t) \geq m$, then $e(\beta(t)) < e(t) - k$. Thus $e(\alpha(t)) = e(t) - k$. In other words, under a non-small endomorphism, eventually the exponents of torsion elements drop by a fixed amount: the image of α cannot be bounded.

Now let $\alpha \in \mathrm{Hom}(T, B)$ and suppose *ad absurdum* that α is not small. By the above, $\alpha(T)$ cannot be a bounded group. As its image lies in a basic subgroup B, it must be an unbounded direct sum of cyclic groups. Choose next a homomorphism $\gamma : \alpha(T) \to T$ sending some elements of arbitrarily large order to themselves and some other elements of arbitrarily large order to 0. The composition $\gamma\alpha$ is an endomorphism of T which cannot be non-small because of the elements of arbitrarily large exponent going to 0. On the other hand it cannot be small because elements of arbitrarily large exponent have images with exponents reduced by k. This gives a contradiction. ■

Finally, if we pass to separable torsion groups we have the following strengthening of Theorem 6.8.

THEOREM 6.11. *Let T be a separable p-group. Then T is a direct sum of cyclic groups or is torsion-complete if and only if the embedding of some basic subgroup B into T is a localization. In fact, in the last case $T = \overline{B}$.*

Proof. As in the proof of Theorem 6.8 we find that every automorphism of B extends to an automorphism of T and hence [15, Theorem 69.3] implies that $T = B$ or T is torsion-complete. ■

We conclude the paper with an open question.

QUESTION 6.12. *Is there a p-group which is not a direct sum of cyclics such that the embedding of one basic subgroup is a localization but the embedding of another basic subgroup is not?*

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