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On the closure of Baire classes under transfinite convergences

by

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Abstract. Let X be a Polish space and Y be a separable metric space. For a fixed $\xi < \omega_1$, consider a family $f_\alpha: X \to Y$ $(\alpha < \omega_1)$ of Baire-ξ functions. Answering a question of Tomasz Natkaniec, we show that if for a function $f: X \to Y$, the set $\{\alpha < \omega_1: f_\alpha(x) \neq f(x)\}$ is finite for every $x \in X$, then f itself is necessarily Baire-ξ. The proof is based on a characterization of Σ^0_η sets which can be interesting in its own right.

1. Introduction. It is a fact of life that the class of continuous real functions is not closed under pointwise convergence: instead, we obtain a realization of the Baire-1 functions. On the other hand, it is an easy exercise that the pointwise limit of a sequence of continuous functions with length ω_1 is necessarily continuous.

This problem and other properties of the pointwise convergence of transfinite sequences of real functions have been first considered by W. Sierpiński [7]. In particular, he studied which class of functions is closed under such convergences. Since most of the classes, for example the class of Baire-ξ functions for $\xi \geq 2$, are not, T. Natkaniec [6] introduced a stronger notion of pointwise convergence. We recall the precise setting in the following definition.

Definition 1. Let λ be a cardinal, (X, τ) be a Polish space, (Y, d) be a separable metric space, and consider an ideal $\mathcal{I}$ on λ. We say that a sequence of functions $f_\alpha: X \to Y$ $(\alpha < \lambda)$ *$\mathcal{I}$-converges* to the function $f: X \to Y$, in notation $f_\alpha \to_{\mathcal{I}} f$, if

$$\{\alpha < \lambda: f_\alpha(x) \neq f(x)\} \in \mathcal{I}$$

for every $x \in X$.

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Similarly, we write $f_\alpha \to_{\mathcal{I}}^{d} f$ if for every $\varepsilon > 0$ and $x \in X$ we have

$$\{\alpha < \lambda\colon d(f(x), f_\alpha(x)) > \varepsilon\} \in \mathcal{I}.$$

In the case of the ordinary ω_1 convergence, as used in [3] and [7], we have $\lambda = \omega_1$ and $\mathcal{I} = [\omega_1]^{\leq\omega}$, that is, the ideal of countable subsets of ω_1. However, our motivating theorem, answering [6, Problem 1, p. 490], is related to the particular case when the ideal consists of the finite subsets of ω_1, that is, $\mathcal{I}_< = [\omega_1]^{<\omega}$.

Theorem 2. *Let (X,τ) be a Polish space, (Y,d) be a separable metric space, and for a fixed $\xi < \omega_1$ consider a family $f_\alpha\colon X \to Y$ $(\alpha < \omega_1)$ of Baire-ξ functions. If $f\colon X \to Y$ is such that $f_\alpha \to_{\mathcal{I}_<}^{d} f$, then f is Baire-ξ.*

We note here that the original question asked by T. Natkaniec referred to $\mathcal{I}_<$-convergence. However, it is easy to see that $\mathcal{I}_<$-convergence implies $\frac{d}{\mathcal{I}_<}$-convergence, so the result above is formally stronger than required. The sufficiency of $\frac{d}{\mathcal{I}_<}$-convergence was pointed out to the author by Petr Holický.

As shown by W. Sierpiński ([7, Theorem 1, p. 133 and Theorem 2, p. 137]), for the class of continuous and Baire-1 functions Theorem 2 also holds for $\mathcal{I} = [\omega_1]^{\leq\omega}$ instead of $\mathcal{I}_<$. On the other hand, it is independent for every $2 \leq \xi < \omega_1$ whether there is an $[\omega_1]^{\leq\omega}$-convergent sequence of Baire-ξ functions whose limit function is Borel but not Baire-ξ (observe that $\frac{d}{\mathcal{I}_<}$-convergence implies $[\omega_1]^{\leq\omega}$-convergence). The first part of the following theorem has already been proved by W. Sierpiński ([7, Section 6, pp. 139 and 140]) and further discussed by P. Komjáth ([3, Theorem 3, p. 499]). Its second part, related to Problem 3 in [6, p. 490], is a simple analogue of Theorem 2.

Theorem 3. *Let (X,τ) be a Polish space and (Y,d) be a separable metric space.*

- (i) (W. Sierpiński, P. Komjáth). *Assuming the continuum hypothesis, there exists an $[\omega_1]^{\leq\omega}$-convergent sequence of real Baire-2 functions whose limit function is not Borel.*
- (ii) *Let $\lambda < 2^{\aleph_0}$ be an infinite cardinal with $\mathrm{cf}(\lambda) > \omega$ and set $\mathcal{J} = [\lambda]^{<\lambda}$. For a fixed $\xi < \omega_1$, consider a family $f_\alpha\colon X \to Y$ $(\alpha < \lambda)$ of Baire-ξ functions and a Borel function $f\colon X \to Y$. If $f_\alpha \to_{\mathcal{J}}^{d} f$ and in our model the union of λ meager sets is meager in Polish spaces, then f is necessarily Baire-ξ.*

The assumption on the additivity of meager sets holds under $\mathrm{MA}(\lambda)$ (see e.g. [1, Theorem 1.2, p. 505] or [5, Theorem, p. 170]). The convergence of transfinite sequences of Baire-2 functions of length ω_2 has also been investigated by P. Komjáth (see [3, Theorems 4 and 5, p. 500]). It is consistent (with $2^{\aleph_0} = \omega_2$ and $\mathrm{MA}(\omega_1)$) that every real function can be obtained as such a limit. It is also consistent, under more complicated assumptions, that the limit function is necessarily Baire-2.

It is not surprising, and we will follow this direction, that the proofs of Theorem 2 and Theorem 3(ii) go via the analogous statements for characteristic functions, i.e. for sets of given Borel classes. As usual, $\Pi^0_\xi(\tau)$ ($\Sigma^0_\xi(\tau)$ resp.) stands for the ξth multiplicative (additive resp.) Borel class in (X,τ), starting with $\Pi^0_1(\tau)$ = closed sets, $\Sigma^0_1(\tau)$ = open sets. With this notation our key lemma, which might be considered as the main result of this paper, can be stated as follows.

THEOREM 4. *Let τ_{C_1} denote the product topology on 2^ω; let (X,τ) be a Polish space. For every $2 \le \xi < \omega_1$, there exist a $\Pi^0_\xi(\tau_{C_1})$ set $P_\xi \subseteq 2^\omega$ and a Polish topology τ_ξ on 2^ω which is finer than τ_{C_1} such that P_ξ is nowhere dense and closed in the topology τ_ξ, and if a Borel set $A \subseteq X$ is*

(i) *in $\Sigma^0_\xi(\tau)$, then whenever for a continuous one-to-one mapping $\varphi\colon (2^\omega,\tau_{C_1}) \to (X,\tau)$ the set $\varphi^{-1}(A)\cap P_\xi$ is of second category in P_ξ in the relative topology $\tau_\xi|_{P_\xi}$, then $\varphi^{-1}(A)\subseteq 2^\omega$ is of second category in the topology τ_ξ;*

(ii) *not in $\Sigma^0_\xi(\tau)$, then there is a continuous one-to-one mapping $\varphi\colon (2^\omega,\tau_{C_1}) \to (X,\tau)$ such that $\varphi(P_\xi)\subseteq A$ and $\varphi^{-1}(A)\subseteq 2^\omega$ is of first category in the topology τ_ξ.*

Moreover, if $\lambda < 2^{\aleph_0}$ is a cardinal and in our model the union of λ meager sets is meager in Polish spaces, then the first statement holds for every (not necessarily Borel) set A which can be obtained as a union of λ many $\Sigma^0_\xi(\tau)$ sets.

Informally, this theorem says that a fixed proper $\Pi^0_\xi(\tau)$ set is so far from being a $\Sigma^0_\xi(\tau)$ set that even Baire category can distinguish them in a suitable topology (a similar result was obtained by S. Solecki in [8, Theorem 2.2, p. 526]). This approach explains the appearance of the condition on the additivity of meager sets in Theorem 3(ii). The last statement is necessary to prove Theorem 3(ii), and in other words it states that our assumption on the additivity of meager sets implies that if the union of λ many $\Sigma^0_\xi(\tau_{C_1})$ sets is Borel, then it is $\Sigma^0_\xi(\tau_{C_1})$ (see also [8, Corollary 2.3, p. 526] and [9]).

Theorem 4 can also be regarded as a qualitative analogue of the following result (see e.g. [4, p. 433] for the $\xi \ge 3$ case and [2, Theorem 21.22, p. 161] for the $\xi = 2$ case), that we will use in the proof.

THEOREM 5 (A. Louveau, J. Saint Raymond). *Let $3 \le \xi < \omega_1$ and (X,τ) be a Polish space. If $P_\xi \subseteq 2^\omega$ is $\Pi^0_\xi(\tau_{C_1})$ but not $\Sigma^0_\xi(\tau_{C_1})$ and $A_0, A_1 \subseteq X$ is any pair of disjoint Borel sets, then either A_0 can be separated from A_1 by a $\Sigma^0_\xi(\tau)$ set or there is a continuous one-to-one map $\varphi\colon (2^\omega,\tau_{C_1}) \to X$ with*

$\varphi(P_\xi) \subseteq A_0$ and $\varphi(2^\omega \setminus P_\xi) \subseteq A_1$. The same conclusion holds for $\xi = 2$ if $P_2 \subseteq 2^\omega$ is the complement of a dense countable set.

Our reference for the basic notions of descriptive set theory is [2]. In the next section we prove Theorems 2 and 3(ii), while the proof of Theorem 4 will be given in the last section.

2. $\mathcal{I}$-convergent functions. In order to establish the connection between function classes and sublevel sets we will use the following classical result (see e.g. [2, Chapter II, Theorem 24.3, p. 190]).

THEOREM 6. *Let (X,τ) be a Polish space and (Y,d) be a separable metric space. Then for every $1 \le \xi < \omega_1$, a function $f\colon X \to Y$ is Baire-ξ if and only if $f^{-1}(U) \subseteq X$ is $\Sigma^0_{\xi+1}(\tau)$ for every open set $U \subseteq Y$.*

In the metric space (Y,d), the open ball centered at $x \in Y$ with radius ρ is denoted by $B_d(x,\rho)$. After these preparations, Theorems 2 and 3(ii) are simple corollaries of Theorem 4.

Proof of Theorem 2. By [7, Theorem 1, p. 133 and Theorem 2, p. 137], the statement holds for $\xi \le 1$. So fix $2 \le \xi < \omega_1$ and suppose that $f_\alpha \to_{\frac{d}{\mathcal{I}_<}} f$ for a family $f_\alpha\colon X \to Y$ $(\alpha < \omega_1)$ of Baire-ξ functions.

Suppose that f is not Baire-ξ. Being the pointwise limit of $\{f_\alpha\colon \alpha < \omega\}$, f is clearly Borel, so by Theorem 6, there is an open ball $B_d(x,\rho) \subseteq Y$ such that $f^{-1}(B_d(x,\rho))$ is Borel but not $\Sigma^0_{\xi+1}(\tau)$. Set

$$H(\varepsilon) = f^{-1}(B_d(x,\rho-\varepsilon)), \qquad H_\alpha(\varepsilon) = f_\alpha^{-1}(B_d(x,\rho-\varepsilon))$$

for every $\alpha < \omega_1$ and $0 < \varepsilon < \varrho$. Note that by Theorem 6, $H_\alpha(\varepsilon)$ is in $\Sigma^0_{\xi+1}(\tau)$ for every $\alpha < \omega_1$ and $0 < \varepsilon < \varrho$.

Since $H(0)$ is not $\Sigma^0_{\xi+1}(\tau)$, by Theorem 4(ii) there is a continuous one-to-one map $\varphi\colon (2^\omega, \tau_{C_1}) \to (X,\tau)$ such that

(i) $\varphi(P_{\xi+1}) \subseteq H(0)$,
(ii) $\varphi^{-1}(H(0)) \subseteq 2^\omega$ is of first category in the topology $\tau_{\xi+1}$.

By (i), there is an $\varepsilon_0 > 0$ such that $\varphi^{-1}(H(\varepsilon_0)) \cap P_{\xi+1}$ is of second category in the topology $\tau_{\xi+1}|_{P_{\xi+1}}$. Let $J_1(\varepsilon)$ denote the set of those $\alpha < \omega_1$ for which $\varphi^{-1}(H_\alpha(\varepsilon))$ is of second category in the topology $\tau_{\xi+1}$.

We prove that $\omega_1 \setminus J_1(\varepsilon)$ is finite for every $\varepsilon < \varepsilon_0$. Suppose that this is not true and take a countably infinite set $J'(\varepsilon) \subseteq \omega_1 \setminus J_1(\varepsilon)$. By the definition of $\frac{d}{\mathcal{I}_<}$-convergence, $\varepsilon < \varepsilon_0$ implies that

$$H(\varepsilon_0) \subseteq H'(\varepsilon) := \bigcup_{\alpha \in J'(\varepsilon)} H_\alpha(\varepsilon),$$

so $\varphi^{-1}(H'(\varepsilon)) \cap P_{\xi+1}$ is of second category in $P_{\xi+1}$ in the topology $\tau_{\xi+1}|_{P_{\xi+1}}$;

that is, since $H'(\varepsilon)$ is $\Sigma^0_{\xi+1}(\tau)$, by Theorem 4.1, $\varphi^{-1}(H'(\varepsilon))$ is of second category in $\tau_{\xi+1}$. This is a contradiction, since by the definition of $J_1(\varepsilon)$, $\varphi^{-1}(H'(\varepsilon))$ is $\tau_{\xi+1}$-meager.

So $J_1(\varepsilon)$ is of cardinality ω_1 for every $\varepsilon < \varepsilon_0$. In particular, given that $(2^\omega, \tau_{\xi+1})$ has countable base, there is a $\tau_{\xi+1}$-open set $U \subseteq 2^\omega$ such that for a countably infinite set $J'' \subseteq J_1(\varepsilon_0/2)$ the set $\varphi^{-1}(H_\alpha(\varepsilon_0/2))$ is residual in U in the topology $\tau_{\xi+1}$ whenever $\alpha \in J''$. Hence for

$$H'' = \bigcap_{\alpha \in J''} H_\alpha(\varepsilon_0/2),$$

$\varphi^{-1}(H'')$ is also $\tau_{\xi+1}$-residual in U, so by (ii) we can find a point $x_0 \in H'' \setminus H(0)$. Thus f_α $(\alpha < \omega_1)$ is not $\frac{d}{\mathcal{I}_<}$-convergent since

$$J'' \subseteq \{\alpha < \omega_1 \colon d(f(x_0), f_\alpha(x_0)) > \varepsilon_0/2\}$$

is infinite; a contradiction. The proof is complete. ∎

Proof of Theorem 3(ii). Again, for $\xi \leq 1$ the statement follows from the proofs in [7]; so let $\xi \geq 2$. Now f is Borel by assumption; and the proof is the same as for Theorem 2, until the definition of J_1. Now we show that $\operatorname{card}(\lambda \setminus J_1(\varepsilon)) < \lambda$ for every $\varepsilon < \varepsilon_0$.

Suppose that this is not true and take a set $J'(\varepsilon) \subseteq \lambda \setminus J_1(\varepsilon)$ of cardinality λ. By the definition of $\frac{d}{\mathcal{J}}$-convergence, $\varepsilon < \varepsilon_0$ implies that

$$H(\varepsilon_0) \subseteq H'(\varepsilon) := \bigcup_{\alpha \in J'(\varepsilon)} H_\alpha(\varepsilon),$$

so $\varphi^{-1}(H'(\varepsilon)) \cap P_{\xi+1}$ is of second category in $P_{\xi+1}$ in the topology $\tau_{\xi+1}|_{P_{\xi+1}}$; that is, by the extension of Theorem 4(i), since $H'(\varepsilon)$ is the union of λ many $\Sigma^0_{\xi+1}(\tau)$ sets $H_\alpha(\varepsilon)$ $(\alpha \in J'(\varepsilon))$, $\varphi^{-1}(H'(\varepsilon))$ is of second category in $\tau_{\xi+1}$. Now this contradicts the assumption that the union of λ meager sets is meager in $(2^\omega, \tau_{\xi+1})$, since by the definition of $J_1(\varepsilon)$, $\varphi^{-1}(H_\alpha(\varepsilon))$ $(\alpha \in J'(\varepsilon))$ is $\tau_{\xi+1}$-meager.

We continue as above; $J_1(\varepsilon)$ is of cardinality λ for every $\varepsilon < \varepsilon_0$. In particular, given that $\operatorname{cf}(\lambda) > \omega$ and $(2^\omega, \tau_{\xi+1})$ has countable base, there is a $\tau_{\xi+1}$-open set $U \subseteq 2^\omega$ such that for a set $J'' \subseteq J_1(\varepsilon_0/2)$ of cardinality λ the set $\varphi^{-1}(H_\alpha(\varepsilon_0/2))$ is $\tau_{\xi+1}$-residual in U whenever $\alpha \in J''$. Since in our model the intersection of λ many $\tau_{\xi+1}$-residual sets is again residual, for

$$H'' = \bigcap_{\alpha \in J''} H_\alpha(\varepsilon_0/2),$$

$\varphi^{-1}(H'')$ is also $\tau_{\xi+1}$-residual in U, so by (ii) we can find a point $x_0 \in H'' \setminus H(0)$. Again, this contradicts the $\frac{d}{\mathcal{J}}$-convergence. The proof is complete. ∎

3. Distinguishing Borel classes. We will define recursively a sequence of compact Polish spaces homeomorphic to $(2^\omega, \tau_{C_1})$, Borel sets of increasing complexity and additional Polish topologies which will serve as test sets and topologies. During this construction we will successively refine Polish topologies by turning countably many pairwise disjoint closed sets into open sets. We do this as described in [2], that is, the open sets of the old topology together with their portions on the members of our collection of closed sets serve as a base of a new, finer topology. We will use the fact that the topology obtained this way is also Polish.

We will also need a precise notion of *basic open sets* in our resulting spaces. In what follows, if a basis $\mathcal{G}_i$ is fixed in the space (X_i, σ_i) for every $i \in I$, which are meant to be the basic open sets in X_i, then the basic open sets of $(\prod_{i\in I} X_i, \prod_{i\in I} \sigma_i)$ are the open sets of the form

$$\prod_{i\in J} G_i \times \prod_{i\in I\setminus J} X_i,$$

where $J \subseteq I$ is finite and $G_i \in \mathcal{G}_i$ for every $i \in J$. Similarly, if the basic open sets $\mathcal{G}$ are fixed in (X, σ) and $\mathcal{F}$ is a countable collection of pairwise disjoint closed subsets of X, then the basic open sets of the finer topology obtained as described above are of the form $G \cap F$ or G with $G \in \mathcal{G}$, $F \in \mathcal{F}$. Observe that the basic open sets thus defined form a basis.

We will have to return to the topologies on the coordinate spaces in product spaces. If (X, σ), (Y, τ) are arbitrary topological spaces and $(\mathcal{X}, \mathcal{S}) = (X \times Y, \sigma \times \tau)$, then we define $\mathrm{Pr}_X(\mathcal{S}) = \sigma$. The projection of product sets in product spaces is defined analogously,

Now we can start the construction. We set $C_1 = 2^\omega$ and

$$P_1 = \{x \in C_1\colon \forall m \in \omega\ (x(m) = 1)\}.$$

We denote by τ_{C_1} the product topology on C_1; so (C_1, τ_{C_1}) is a Polish space. Set $\tau_1 = \tau_{C_1}$ on C_1.

For every ordinal $\xi < \omega_1$ we fix once and for all a sequence

$$\xi_1 \le \cdots \le \xi_i \le \ldots < \xi \quad (i < \omega) \tag{1}$$

of ordinals: if ξ is limit, let $\xi = \lim_{i\to\infty} \xi_i$, while for ξ successor, $\xi = \xi_i + 1$ for every $i < \omega$. To avoid complicated notations, we do not indicate the dependence of the sequence on ξ; it will always be clear which pair of an ordinal and a sequence is considered.

Suppose that the sets C_η and P_η and their topologies are defined for every $\eta < \xi$. Then let $C_\xi = \prod_{i=0}^\infty C_{\xi_i}$ and

$$P_\xi = \{x \in C_\xi\colon \forall m \in \omega\ (x(m, \cdot) \in C_{\xi_m} \setminus P_{\xi_m})\}, \tag{2}$$

$$\tau_{C_\xi} = \prod_{i=0}^\infty \tau_{C_{\xi_i}}, \qquad \tau_\xi^< = \prod_{i=0}^\infty \tau_{\xi_i}, \tag{3}$$

and let τ_ξ be the coarsest topology extending $\tau_\xi^<$ such that

$$U_{\xi,N} = \prod_{i=0}^{N-1} (C_{\xi_i} \setminus P_{\xi_i}) \times P_{\xi_N} \times \prod_{i=N+1}^{\infty} C_{\xi_i} \subseteq \prod_{i=0}^{N-1} C_{\xi_i} \times C_{\xi_N} \times \prod_{i=N+1}^{\infty} C_{\xi_i} = C_\xi \tag{4}$$

is open for every $N < \omega$. It is important that the sets $U_{\xi,N}$ $(N < \omega)$ are pairwise disjoint. Note also that this construction fits into the framework presented in the introduction of this section since after having turned $U_{\xi,i}$ into an open set for $i < N$,

$$U_{\xi,N} = \Big(\prod_{i=0}^{N-1} C_{\xi_i} \times P_{\xi_N} \times \prod_{i=N+1}^{\infty} C_{\xi_i}\Big) \setminus \Big(\bigcup_{i=0}^{N-1} U_{\xi,i}\Big)$$

is indeed closed in this intermediate refinement of $\tau_\xi^<$.

In the following six claims we prove some relations between P_ξ and the topologies τ_{C_ξ}, τ_ξ and $\tau_\xi^<$.

Claim 7. *For every* $1 \le \xi < \omega_1$, $P_\xi \in \Pi^0_\xi(\tau_{C_\xi})$.

Proof. We prove the statement by induction on ξ. For $\xi = 1$ the set P_1 is a single point, which is clearly τ_{C_1}-closed.

Let now $\xi \ge 2$ and suppose that $P_\eta \in \Pi^0_\eta(\tau_{C_\eta})$ for every $\eta < \xi$. Then

$$P_\xi = \bigcap_{m<\omega} \{x \in C_\xi \colon x(m,\cdot) \in C_{\xi_m} \setminus P_{\xi_m}\}. \tag{5}$$

Since τ_{C_ξ} is the product of the topologies $\tau_{C_{\xi_m}}$ and P_{ξ_m} is $\Pi^0_{\xi_m}(\tau_{C_{\xi_m}})$ by the induction hypothesis, P_ξ is the intersection of sets of additive class lower than ξ, so the statement follows. ∎

Claim 8.

(i) *For every* $1 \le \xi < \omega_1$, $P_\xi \subseteq C_\xi$ *is nowhere dense closed*, *hence meager in the topology* τ_ξ.
(ii) *For* $\xi \ge 2$, $P_\xi \subseteq C_\xi$ *is a dense* G_δ, *hence residual in the topology* $\tau_\xi^<$.

Proof. We prove the two statements together, by induction on ξ. For $\xi = 1$, P_1 is a single point, which is clearly closed and nowhere dense.

Let now $\xi \ge 2$ and suppose that (i) holds for every $\eta < \xi$. We prove (ii) for ξ.

By (4) and (5), we have

$$P_\xi = C_\xi \setminus \bigcup_{m<\omega} U_{\xi,m}. \tag{6}$$

By the induction hypothesis, P_{ξ_m} is nowhere τ_{ξ_m}-dense and closed for every $m < \omega$, so, since $\tau_\xi^<$ is the product of the topologies τ_{ξ_m}, $U_{\xi,m}$ is nowhere $\tau_{\xi_m}^<$-dense for every $m < \omega$. Also, $U_{\xi,m}$ is a finite intersection of $\tau_\xi^<$-open and $\tau_\xi^<$-closed sets, thus it is an F_σ set in the topology $\tau_\xi^<$ for every $m < \omega$. Hence (6) shows that P_ξ is $\tau_\xi^<$-dense and G_δ.

Consider now statement (i) for ξ. To obtain τ_ξ, we made open every set on the right hand side of (6), so P_ξ is closed. Again using the fact that P_{ξ_m} is nowhere τ_{ξ_m}-dense, we infer that $\bigcup_{m<\omega} U_{\xi,m}$ meets every $\tau_\xi^<$-open set, hence it is also τ_ξ-dense, so P_ξ is nowhere τ_ξ-dense. This finishes the proof. ∎

Claim 9. *For every* $1 \le \xi < \omega_1$, $\tau_\xi|_{P_\xi} = \tau_\xi^<|_{P_\xi}$.

Proof. By definition, $U_{\xi,N} \cap P_\xi = \emptyset$ $(N < \omega)$, so the statement follows. ∎

Claim 10. *For* $2 \le \xi < \omega_1$, *every basic* $\tau_\xi^<$*-open* [*basic* τ_ξ*-open, resp.*] *subset* G *of* C_ξ *is in* $\Sigma_1^0(\tau_{C_\xi}) \cup \Pi_\eta^0(\tau_{C_\xi})$ [$\Pi_\xi^0(\tau_{C_\xi})$, *resp.*] *for some* $\eta < \xi$ *depending on* G.

Proof. We prove the statements by induction on ξ. For $\xi = 2$, $\tau_2^< = \tau_{C_2}$, so the basic $\tau_2^<$-open sets are $\Sigma_1^0(\tau_{C_2})$, as stated. Every new basic open set of τ_2 compared to $\tau_2^<$ is a finite intersection of τ_{C_2}-open and τ_{C_2}-closed sets, so the basic τ_2-open sets are indeed in $\Pi_2^0(\tau_{C_2})$.

Take now $\xi \ge 3$ and suppose that the statements are true for every $\eta < \xi$. By (3) and the induction hypothesis, the $\tau_\xi^<$-open sets can be obtained as finite intersections of sets in $\Sigma_1^0(\tau_{C_\xi}) \cup \Pi_{\xi_i}^0(\tau_{C_\xi})$ with $\xi_i < \xi$, which clearly gives sets in $\Pi_\eta^0(\tau_{C_\xi})$ for $\eta < \xi$.

Consider now the topology τ_ξ; again it is enough to determine the Borel class of the $U_{\xi,N}$'s. By Claim 7, (4) shows that $U_{\xi,N}$ is a finite intersection of $\Sigma_{\xi_i}^0(\tau_{C_\xi})$ and $\Pi_{\xi_N}^0(\tau_{C_\xi})$ sets, which is a $\Pi_\xi^0(\tau_{C_\xi})$ set, as required. ∎

Claim 11. *If* G *is basic* τ_ξ*-open and* $G \cap P_\xi \ne \emptyset$ *then* G *is also basic* $\tau_\xi^<$*-open.*

Proof. Since $U_{\xi,N}$ $(N < \omega)$ is disjoint from P_ξ, the statement follows from the definition of τ_ξ. ∎

Claim 12. *For every* $1 \le \xi < \omega_1$ *and* $N < \omega$,

$$\tau_\xi|_{U_{\xi,N}} = \mathrm{Pr}_{\prod_{i=0}^{N-1} C_{\xi_i}}(\tau_\xi^<|_{\prod_{i=0}^{N-1} C_{\xi_i} \setminus P_{\xi_i}}) \times \tau_{\xi_N}|_{P_{\xi_N}} \times \mathrm{Pr}_{\prod_{i=N+1}^{\infty} C_{\xi_i}}(\tau_\xi^<).$$

Proof. The sets $U_{\xi,N}$ $(N < \omega)$ are pairwise disjoint, so

$$\tau_\xi|_{U_{\xi,N}} = \tau_\xi^<|_{U_{\xi,N}}.$$

By (3), $\mathrm{Pr}_{C_{\xi_N}}(\tau_\xi^<) = \tau_{\xi_N}$ while by (4), $\mathrm{Pr}_{C_{\xi_N}}(U_{\xi,N}) = P_{\xi_N}$ and

$$\mathrm{Pr}_{\prod_{i=0}^{N-1} C_{\xi_i}}(U_{\xi,N}) = \prod_{i=0}^{N-1} C_{\xi_i} \setminus P_{\xi_i}, \qquad \mathrm{Pr}_{\prod_{i=N+1}^{\infty} C_{\xi_i}}(U_{\xi,N}) = \prod_{i=N+1}^{\infty} C_{\xi_i},$$

so we get

$$\tau_\xi^<|_{U_{\xi,N}} = \mathrm{Pr}_{\prod_{i=0}^{N-1} C_{\xi_i}}(\tau_\xi^<|_{U_{\xi,N}}) \times \mathrm{Pr}_{C_{\xi_N}}(\tau_\xi^<|_{U_{\xi,N}}) \times \mathrm{Pr}_{\prod_{i=N+1}^{\infty} C_{\xi_i}}(\tau_\xi^<|_{U_{\xi,N}})$$
$$= \mathrm{Pr}_{\prod_{i=0}^{N-1} C_{\xi_i}}(\tau_\xi^<|_{\prod_{i=0}^{N-1} C_{\xi_i}\setminus P_{\xi_i}}) \times \tau_{\xi_N}|_{P_{\xi_N}} \times \mathrm{Pr}_{\prod_{i=N+1}^{\infty} C_{\xi_i}}(\tau_\xi^<),$$

as required. This finishes the proof. ∎

We will prove Theorem 4 through the following lemma, which states the same result in a more technical way, which fits better into an inductive argument.

LEMMA 13. *Fix* $1 \le \xi < \omega_1$. *Let* (Z, σ) *be an arbitrary Polish space and* $G \subseteq Z \times C_\xi$ *be a basic* $\sigma \times \tau_\xi$*-open set with*

$$G \cap (Z \times P_\xi) \ne \emptyset.$$

(i) *If* $\xi \ge 2$ *and* $Q \subseteq Z \times C_\xi$ *is* $\Pi^0_\vartheta(\sigma \times \tau_{C_\xi})$ *for some* $\vartheta < \xi$, *and* $Q \cap (Z \times P_\xi)$ *is relatively* $\sigma \times (\tau_\xi|_{P_\xi})$*-residual in* $G \cap (Z \times P_\xi)$, *then* Q *is* $\sigma \times \tau_\xi$*-residual in a* $\sigma \times \tau_\xi$*-open set* G' *satisfying*

$$G \cap (Z \times P_\xi) = G' \cap (Z \times P_\xi). \tag{7}$$

(ii) *If for a set* $W \in \Sigma^0_\xi(\sigma \times \tau_{C_\xi})$, $W \cap (Z \times P_\xi)$ *is relatively* $\sigma \times (\tau_\xi|_{P_\xi})$*-residual in* $G \cap (Z \times P_\xi)$, *then* W *is* $\sigma \times \tau_\xi$*-residual in a* $\sigma \times \tau_\xi$*-open set* $H \subseteq Z \times C_\xi$ *such that* $G \cap (Z \times P_\xi)$ *is contained in the* $\sigma \times \tau_\xi$*-closure of* H.

Proof. Once we prove the statement concerning $\Pi^0_\vartheta(\sigma \times \tau_{C_\xi})$ sets for given $\vartheta < \xi < \omega_1$, the statement for $\Sigma^0_\xi(\sigma \times \tau_{C_\xi})$ sets automatically follows. To see this, write

$$W = \bigcup_{\alpha=0}^{\infty} Q_\alpha,$$

where Q_α is in $\Pi^0_{\vartheta_\alpha}(\sigma \times \tau_{C_\xi})$ with $\vartheta_\alpha < \xi$ for every $\alpha < \omega$, and suppose that $W \cap G \cap (Z \times P_\xi)$ is relatively $\sigma \times (\tau_\xi|_{P_\xi})$-residual in $G \cap (Z \times P_\xi)$. For every $\alpha < \omega$, let H_α denote the maximal $\sigma \times \tau_\xi$-open set in which Q_α is $\sigma \times \tau_\xi$-residual. Then by (i), the $\sigma \times \tau_\xi$-open set $H = \bigcup_{\alpha=0}^{\infty} H_\alpha$ meets every open set intersecting $G \cap (Z \times P_\xi)$, which proves the statement.

So we need only prove (i). We do this by induction on ξ, namely we prove (i) for a fixed $\xi < \omega_1$ by assuming that (ii) holds for every $\eta < \xi$. For $\xi = 1$, by the Baire Category Theorem, $H = W$ can be chosen.

Let now $\xi \ge 2$ and suppose that (ii) holds for every $\eta < \xi$ ([1]). Consider a $\Pi^0_\vartheta(\sigma \times \tau_{C_\xi})$ set $Q \subseteq Z \times C_\xi$ for some $\vartheta < \xi$ and suppose that $Q \cap (Z \times P_\xi)$ is $\sigma \times (\tau_\xi|_{P_\xi})$-residual in $G \cap (Z \times P_\xi)$ for a basic $\sigma \times \tau_\xi$-open set G with $G \cap (Z \times P_\xi) \ne \emptyset$. By Claim 11, G is in fact a basic $\sigma \times \tau_\xi^<$-open set.

([1]) To be precise, we assume that the statements hold for every $\eta < \xi$ and Polish space (Z, σ), no matter how we have fixed in (1) the sequence of ordinals η_i $(i < \omega)$ for every $\eta < \xi$.

By Claim 9, the restrictions of the topologies τ_ξ and $\tau_\xi^<$ to P_ξ coincide, so $Q \cap G \cap (Z \times P_\xi)$ is also relatively $\sigma \times \tau_\xi^<$-residual in $G \cap (Z \times P_\xi)$. But by Claim 8(ii), P_ξ is a $\tau_\xi^<$-residual subset of C_ξ, so

$$Q \cap G \text{ is necessarily } \sigma \times \tau_\xi^< \text{-residual in } G. \tag{8}$$

Let $0 < I < \omega$ be minimal so that $\vartheta \leq \xi_I$. We show that

$$G' = G \cap \Big(Z \times \prod_{i=0}^{I-1} (C_{\xi_i} \setminus P_{\xi_i}) \times C_{\xi_I} \times \prod_{i=I+1}^{\infty} C_{\xi_i} \Big) \tag{9}$$

meets the requirements. It is clearly $\sigma \times \tau_\xi$-open and (7) holds, since

$$Z \times P_\xi \subseteq Z \times \prod_{i=0}^{I-1} (C_{\xi_i} \setminus P_{\xi_i}) \times C_{\xi_I} \times \prod_{i=I+1}^{\infty} C_{\xi_i}.$$

Suppose that $Q \cap G'$ is not $\sigma \times \tau_\xi$-residual in G', that is, $Q \cap \widetilde{G}$ is $\sigma \times \tau_\xi$-meager for some basic $\sigma \times \tau_\xi$-open set $\widetilde{G} \subseteq G'$; by passing to a proper basic $\sigma \times \tau_\xi$-open subset we can assume that $\widetilde{G}$ is not $\sigma \times \tau_\xi^<$-open. Let $\widetilde{G}_0$ be the basic $\sigma \times \tau_\xi^<$-open set such that

$$\widetilde{G} = \widetilde{G}_0 \cap (Z \times U_{\xi,J}) = \widetilde{G}_0 \cap \Big(Z \times \prod_{i=0}^{J-1} C_{\xi_i} \times P_{\xi_J} \times \prod_{i=J+1}^{\infty} C_{\xi_i} \Big)$$

for some $J < \omega$. This decomposition exists since $U_{\xi,N}$ $(N < \omega)$ are pairwise disjoint. Note that $I \leq J$ by (9), and we have $\widetilde{G}_0 \subseteq G$. To summarize, we have shown that

$$Q \cap \widetilde{G}_0 \cap (Z \times U_{\xi,J}) \text{ is } \sigma \times \tau_\xi\text{-meager in } \widetilde{G}_0 \cap (Z \times U_{\xi,J}). \tag{10}$$

Set

$$\underline{Z} = Z \times \prod_{i=0}^{J-1} C_{\xi_i} \times \prod_{i=J+1}^{\infty} C_{\xi_i},$$

$$\underline{\sigma} = \sigma \times \mathrm{Pr}_{\prod_{i=0}^{J-1} C_{\xi_i}} (\tau_\xi^<) \times \mathrm{Pr}_{\prod_{i=J+1}^{\infty} C_{\xi_i}} (\tau_\xi^<)$$

$$\underline{Q} = (Z \times C_\xi) \setminus Q \subseteq \underline{Z} \times C_{\xi_J}, \qquad \underline{G} = \widetilde{G}_0 \subseteq \underline{Z} \times C_{\xi_J}.$$

The space $(\underline{Z}, \underline{\sigma})$ is clearly Polish, and $\underline{G}$ is a basic $\underline{\sigma} \times \tau_{\xi_J}$-open subset of $\underline{Z} \times C_{\xi_J}$. From $\vartheta \leq \xi_I \leq \xi_J < \xi$, $\underline{Q}$ is a $\Sigma^0_{\xi_J}(\underline{\sigma} \times \tau_{C_{\xi_J}})$ set.

According to (10) and Claim 12, $\underline{Q}$ is $\underline{\sigma} \times (\tau_{\xi_J}|_{P_{\xi_J}})$-residual in $\underline{G} \cap (\underline{Z} \times P_{\xi_J})$, so by the induction hypothesis, $\underline{Q}$ is $\underline{\sigma} \times \tau_{\xi_J}$-residual in some $\underline{\sigma} \times \tau_{\xi_J}$-open set $\underline{H} \subseteq \underline{Z} \times C_{\xi_J}$ such that the $\underline{\sigma} \times \tau_{\xi_J}$-closure of $\underline{H}$ contains $\underline{G} \cap (\underline{Z} \times P_{\xi_J})$; so in particular, $\underline{H} \cap \underline{G} \neq \emptyset$. Let $H = \underline{H} \subseteq Z \times C_\xi$. Since $\tau_{\xi_J} = \mathrm{Pr}_{C_{\xi_J}}(\tau_\xi^<)$ by definition, we have $H \cap \widetilde{G}_0 \neq \emptyset$ and $Q \cap H \cap \widetilde{G}_0$ is $\sigma \times \tau_\xi^<$-meager in $H \cap \widetilde{G}_0 \subseteq G$. This contradicts (8). The proof is complete. ■

This lemma proves in particular that P_ξ is a proper $\Pi^0_\xi(\tau_{C_\xi})$ set.

Corollary 14. *For every* $1 \le \xi < \omega_1$, P_ξ *is in* $\Pi^0_\xi(\tau_{C_\xi}) \setminus \Sigma^0_\xi(\tau_{C_\xi})$.

Proof. Clearly, $P_\xi \subseteq C_\xi$ is $\tau_\xi|_{P_\xi}$-residual, but by Claim 8(i), P_ξ is τ_ξ-meager. So by Lemma 13, P_ξ cannot be $\Sigma^0_\xi(\tau_{C_\xi})$. ∎

Proof of Theorem 4. First we prove (i) and (ii) in the case $\xi = 2$ since it is exceptional in Theorem 5. Then we show (i) and (ii) for $3 \le \xi < \omega_1$ and finally we treat the extension to all $2 \le \xi < \omega_1$.

So let $\xi = 2$. We set

$$U'_{2,N} = \{x \in 2^\omega \colon \forall n \ge N\ (x(n) = 0)\} \quad (N < \omega),$$
$$P_2 = 2^\omega \setminus \bigcup_{N<\omega} U'_{2,N}.$$

We define the topology τ_2 as the refinement of τ_{C_1} by turning each point of the finite sets $U'_{2,N}$ $(N < \omega)$ into an open set. Clearly, P_2 is the complement of a dense countable subset in $(2^\omega, \tau_{C_1})$, so in particular P_2 is $\Pi^0_2(\tau_{C_1})$ and τ_{C_1}-residual. Being the complement of the dense τ_2-open set $\bigcup_{N<\omega} U'_{2,N}$, it is also τ_2-meager.

Let $A \subseteq X$ be $\Sigma^0_2(\tau)$ and take a continuous one-to-one mapping $\varphi\colon (2^\omega, \tau_{C_1}) \to (X, \tau)$ such that $\varphi^{-1}(A) \cap P_2$ is of second category in P_2 in the relative topology $\tau_2|_{P_2}$. Then $\varphi^{-1}(A) \subseteq (2^\omega, \tau_{C_1})$ is $\Sigma^0_2(\tau_{C_1})$ and $\varphi^{-1}(A) \cap P_2$ is of second category in $\tau_2|_{P_2}$; thus $\varphi^{-1}(A)$ is of second category in τ_{C_1} as well. Since a $\Sigma^0_2(\tau_{C_1})$ set in $(2^\omega, \tau_{C_1})$ is of second category only if its interior is nonempty, $\varphi^{-1}(A)$ contains a nonempty τ_{C_1}-open set so $\varphi^{-1}(A) \cap U'_{2,N} \ne \emptyset$ for some $N < \omega$. Then $\varphi^{-1}(A)$, having nonempty interior, is of second category in τ_2, as required.

If A is not $\Sigma^0_2(\tau)$, we apply Theorem 5 for $A_0 = A$, $A_1 = X \setminus A$. These sets cannot be separated by a $\Sigma^0_\xi(\tau)$ set, so since P_2 is the complement of a countable dense subset of $(2^\omega, \tau_{C_1})$, there is a continuous one-to-one mapping $\varphi\colon 2^\omega \to X$ with $\varphi(P_2) \subseteq A$, $\varphi(2^\omega \setminus P_2) \subseteq X \setminus A$. So as we have seen above, $\varphi^{-1}(A) = P_2$ is indeed τ_2-meager.

We turn to the $\xi \ge 3$ case. The Polish space (C_ξ, τ_{C_ξ}) is obviously homeomorphic to (C_1, τ_{C_1}) (see e.g. [2, Theorem 7.4, p. 35]). We show that (P_ξ, τ_ξ) satisfies the requirements for every $3 \le \xi < \omega_1$.

Let $A \subseteq X$ be $\Sigma^0_\xi(\tau)$ for some $\xi < \omega_1$ and take a continuous one-to-one mapping $\varphi\colon C_\xi \to X$ such that $\varphi^{-1}(A) \cap P_\xi$ is of second category in P_ξ in the relative topology $\tau_\xi|_{P_\xi}$. Then $\varphi^{-1}(A) \subseteq C_\xi$ is $\Sigma^0_\xi(\tau_{C_\xi})$ and $\varphi^{-1}(A) \cap P_\xi$ is $\tau_\xi|_{P_\xi}$-residual in $G \cap P_\xi$ for some basic τ_ξ-open set G. So according to Lemma 13(i), $\varphi^{-1}(A)$ is of second category in τ_ξ, as required.

Suppose now that A is not $\Sigma^0_\xi(\tau)$. We apply Theorem 5 for $A_0 = A$ and $A_1 = X \setminus A$. These sets cannot be separated by a $\Sigma^0_\xi(\tau)$ set, so since

P_ξ is $\Pi^0_\xi(\tau_{C_\xi})$ but not $\Sigma^0_\xi(\tau_{C_\xi})$, there is a continuous one-to-one mapping $\varphi\colon C_\xi \to X$ with $\varphi(P_\xi) \subseteq A$, $\varphi(2^\omega \setminus P_\xi) \subseteq X \setminus A$. So according to Claim 8.1(i), $\varphi^{-1}(A) = P_\xi$ is indeed τ_ξ-meager.

Finally, suppose that for some cardinal $\lambda < 2^{\aleph_0}$, in our model the union of λ meager sets is meager in Polish spaces. Let A_i $(i < \lambda)$ be $\Sigma^0_\xi(\tau)$ for some $2 \le \xi < \omega_1$ and set $A = \bigcup_{i<\lambda} A_i$. Since $\varphi^{-1}(A) \cap P_\xi$ is of second category in P_ξ in the relative topology $\tau_\xi|_{P_\xi}$, by our assumption $\varphi^{-1}(A_i) \cap P_\xi$ is also of second category in $(P_\xi, \tau_\xi|_{P_\xi})$ for some $i < \lambda$. So by the first statement, $\varphi^{-1}(A_i) \subseteq \varphi^{-1}(A)$ is of second category in τ_ξ. This finishes the proof. ∎

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