

Hopf algebras and dendriform structures arising from parking functions

by

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Abstract. We introduce a graded Hopf algebra based on the set of parking functions (hence of dimension $(n + 1)^{n-1}$ in degree n). This algebra can be embedded into a non-commutative polynomial algebra in infinitely many variables. We determine its structure, and show that it admits natural quotients and subalgebras whose graded components have dimensions respectively given by the Schröder numbers (plane trees), the Catalan numbers, and powers of 3. These smaller algebras are always bialgebras and belong to some family of di- or trialgebras occurring in the works of Loday and Ronco.

Moreover, the fundamental notion of parkization allows one to endow the set of parking functions of fixed length with an associative multiplication (different from the one coming from the Shi arrangement), leading to a generalization of the internal product of symmetric functions. Several of the intermediate algebras are stable under this operation. Among them, one finds the Solomon descent algebra but also a new algebra based on a Catalan set, admitting the Solomon algebra as a left ideal.

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1. INTRODUCTION

Many examples of graded Hopf algebras based on combinatorial structures occur in apparently remote contexts. One of them is the theory of operads. It is quite common there that, in a given operad, the free algebra on one generator admits a Hopf structure [15]. This structure often has an elegant combinatorial description, the best known example being the free dendriform algebra on one generator, also known as the Loday–Ronco Hopf algebra of planar binary trees [21, 12], denoted here by \mathbf{PBT} .

On the other hand, such Hopf algebras also occur in the theory of non-commutative symmetric functions [7], for which a central problem is to understand complicated commutative formulas by means of simpler noncommutative analogues. It has been found over the years that such an understanding required the introduction of larger and larger Hopf algebras, based on more and more complex combinatorial objects. For such algebras to be

useful in this context, it is necessary that their elements can be realized as polynomials in some auxiliary infinite set of variables (commutative or not), so as to recover ordinary symmetric functions after a chain of standard manipulations (such as imposing commutation relations among the variables or taking sums to reestablish complete symmetry). The best illustration of this approach is provided by the algebra **FQSym** of free quasi-symmetric functions [4]. This is an algebra of noncommutative polynomials $\mathbf{F}_\sigma(A)$ labelled by permutations. It contains a subalgebra **FSym** spanned by free Schur functions $\mathbf{S}_t(A)$, labelled by standard Young tableaux. This observation essentially amounts to a one-line proof of the Littlewood–Richardson rule. Abstractly, however, **FQSym** and **FSym** are isomorphic to the Hopf algebras previously introduced by Malvenuto and Reutenauer in [24] and by Poirier and Reutenauer in [32], and it is the polynomial realization which allows such a direct application to symmetric functions.

Interestingly, it is the very same realization which allowed a new understanding of **PBT** [12]. It could be put on the same footing as **FSym**, using the Sylvester correspondence instead of Robinson–Schensted’s, so that both algebras appear now as special cases of a general construction.

The aim of the present article is to introduce a new extension of **FQSym**, that is, a larger Hopf algebra built from the same principles, but leaving enough room to accommodate several new combinatorial Hopf algebras.

It turns out that most of the Hopf algebras arising in the process also have an operadic interpretation, in general as some kind of trialgebra or dialgebra [19, 22], thus providing polynomial realizations of those as well.

Our master algebra, denoted by **PQSym**, for *parking quasi-symmetric functions*, is built on the set of parking functions, a special family of words which can in many respects be regarded as natural generalizations of permutations. Geometrically, permutations correspond to chambers of the Coxeter arrangement of type A_{n-1} , while parking functions label those of the Shi arrangement [2]. But this is not the only possible explanation (see, e.g., [20]), and our choice was rather dictated by elementary combinatorial considerations (see Appendix).

Our first task will be to elucidate the structure of **PQSym**. It will be shown that it is free, cofree, and actually self-dual, with a free primitive Lie algebra. This will be done by means of Foissy’s theory of bidendriform bialgebras [6]. Next, we shall determine explicit generators and multiplicative bases of **PQSym** and **PQSym**^{*}. Then come the realizations, given by simple and explicit noncommutative polynomials for the natural basis of **PQSym**^{*}, and in terms of integer matrices, reminiscent of the construction of **MQSym** [4], for the natural basis of **PQSym** itself. After that, we shall start the investigation of smaller Hopf algebras arising from **PQSym** by natural processes.

Recall that the dimension of **PQSym** in degree n is $(n + 1)^{n-1}$. We shall show that it admits natural quotients and subalgebras whose graded components have dimensions respectively given by the Schröder numbers (plane trees), the Catalan numbers, powers of 3 and powers of 2. Most of those turn out to be related to the theory of operads, and to belong to some family of di- or trialgebras occurring in the works of Loday and Ronco. We shall in particular recover the free dendriform trialgebra (also known as the free tridendriform algebra) on one generator (Schröder numbers) and the free cubical trialgebra. Similarly, we obtain a cocommutative Hopf algebra based on a Catalan set, which is isomorphic to the free dendriform dialgebra on one generator as an algebra, but not as a coalgebra.

Moreover, the fundamental notion of parkization of a word, which is needed from the beginning, allows one to endow the set of parking functions of fixed length with an associative multiplication (different from the one coming from their interpretation as chambers of the Shi arrangement), leading to a generalization of the internal product of symmetric functions. Several of the intermediate algebras are stable under this operation. Among them, one finds the Solomon descent algebra and the Solomon–Tits algebra, but also a new algebra based on a Catalan set, admitting the Solomon algebra as a left ideal.

This paper is structured as follows: the preliminaries present some necessary background about parking functions and dendriform structures and give a realization of the free dendriform trialgebra on one generator in terms of noncommutative polynomials. In Section 3, we present our principal algebra **PQSym**, and investigate its most important features, mostly relying upon its bidendriform bialgebra structure. We then move to a subalgebra **SQSym** of **PQSym**, whose Hilbert series is given by the little Schröder numbers, and prove in particular that it is isomorphic to the free dendriform trialgebra on one generator (Section 4). In Section 5, we study another subalgebra **CQSym** of **PQSym** whose Hilbert series is given by the Catalan numbers, show that it is cocommutative, that it is stable under the internal product of **PQSym** and that its dual is a natural generalization of *QSym*. In Section 6, we present **SCQSym**, a quotient of **SQSym** whose Hilbert series is given by powers of 3, and show in particular that it is isomorphic to the free cubical trialgebra on one generator. Finally, the Appendix presents how the construction of **PQSym** arose from considerations about free probability and an exercise proposed by Kerov in 1995. Most of these results were announced in [27].

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2. PRELIMINARIES

2.1. Notations. Our notations for ordinary symmetric functions will be those of [23]. Other undefined notations can be found in [7, 4], although the essential ones will be recalled when needed.

2.1.1. To start with, we shall need the following two operations on words.

For a word w on the alphabet $\{1, 2, \dots\}$, denote by $w[k]$ the word obtained by replacing each letter i by the integer $i + k$. If u and v are two words, with u of length k , one defines the *shifted concatenation*

$$(1) \quad u \bullet v = u \cdot (v[k])$$

and the *shifted shuffle*

$$(2) \quad u \uplus v = u \sqcup (v[k]).$$

where \sqcup is the usual shuffle product on words defined recursively by

$$(3) \quad (au) \sqcup (bv) = a \cdot (u \sqcup (bv)) + b \cdot ((au) \sqcup v),$$

with $u \sqcup \epsilon = \epsilon \sqcup u = u$ if ϵ is the empty word.

It is immediate to see that the set of permutations is closed under both operations. The subalgebra spanned by those elements is isomorphic to the convolution algebra of symmetric groups [24] or to free quasi-symmetric functions [4], whose definition is recalled below.

2.1.2. Let A be a totally ordered alphabet. We denote by \mathbb{K} a field of characteristic 0, and by $\mathbb{K}\langle A \rangle$ the free associative algebra over A when A is finite, and the projective limit $\text{projlim}_B \mathbb{K}\langle B \rangle$, where B runs over finite subsets of A , when A is infinite, which will be generally assumed in the following.

Given a totally ordered alphabet A , the *evaluation vector* $\text{Ev}(w)$ of a word w is the sequence of the numbers of occurrences of all the elements of A in w .

Recall that the standardization $\text{Std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling 1, 2, ... the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma = \text{Std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w\sigma$ is a nondecreasing word. For example, $\text{Std}(bbacab) = 341625$.

This characterizes completely the sequences of transpositions effected by the bubble sort algorithm on w . An elementary observation, which is at the

basis of the constructions of [4], is that the noncommutative polynomials

$$(4) \quad \mathbf{G}_\sigma(A) = \sum_{w \in A^*; \text{Std}(w)=\sigma} w,$$

where σ runs over the set of all permutations, form a basis of a subalgebra of $\mathbb{K}\langle A \rangle$. Moreover, if A is infinite, this subalgebra admits a natural Hopf algebra structure. This is **FQSym**, the algebra of *free quasi-symmetric functions*.

Let $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$. The coproduct is defined by

$$(5) \quad \Delta \mathbf{F}_\sigma = \sum_{u \cdot v = \sigma} \mathbf{F}_{\text{Std}(u)} \otimes \mathbf{F}_{\text{Std}(v)},$$

where $u \cdot v$ means concatenation. The scalar product is defined by

$$(6) \quad \langle \mathbf{F}_\sigma, \mathbf{G}_\tau \rangle = \delta_{\sigma, \tau},$$

where δ is the Kronecker symbol, and then for all $F, G, H \in \mathbf{FQSym}$ one has

$$(7) \quad \langle FG, H \rangle = \langle F \otimes G, \Delta H \rangle.$$

The product formula in the **F** basis is

$$(8) \quad \mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma \in \alpha \uplus \beta} \mathbf{F}_\gamma.$$

The sum of the inverses of the permutations occurring in $\alpha^{-1} \uplus \beta^{-1}$ is called *convolution* and denoted by $\alpha * \beta$ [33, 24].

2.1.3. A general process for constructing interesting subalgebras of **FQSym** is to take sums of the form

$$(9) \quad \mathcal{P}_\mathbf{x}(A) = \sum_{\mathfrak{P}(\sigma)=\mathbf{x}} \mathbf{F}_\sigma,$$

where \mathfrak{P} is the left symbol of some Robinson–Schensted type correspondence. If we take the original Robinson–Schensted map, we obtain **FSym**, the algebra of free symmetric functions [4]. If we take the sylvester congruence [12], we obtain **PBT**, the Loday–Ronco algebra of planar binary trees. Finally, if we take the hypoplactic correspondence [17], we obtain **Sym**, the algebra of noncommutative symmetric functions. The dual Hopf algebras are obtained in each case by imposing the corresponding congruence (plactic, sylvester, hypoplactic) on A^* .

2.2. Parking functions. In the following, we shall see that it is possible to replace permutations by *parking functions* in all these constructions. It will become obvious that the set of parking functions is stable under shifted concatenation and shifted shuffle, and many other classes of words share this

property. The point is that for parking functions, the resulting algebra has a natural Hopf structure, and that it is again possible to find a polynomial realization. Moreover, an interesting internal product can be defined.

2.2.1. A *parking function* is a word $\mathbf{a} = a_1 \cdots a_n$ of length n on the alphabet $[n] = \{1, \dots, n\}$ whose nondecreasing rearrangement $\mathbf{a}^\uparrow = a'_1 \cdots a'_n$ satisfies $a'_i \leq i$ for all i . Let PF_n be the set of such words.

For example, $\text{PF}_1 = \{1\}$, $\text{PF}_2 = \{11, 12, 21\}$, and

$$(10) \quad \text{PF}_3 = \{111, 112, 121, 211, 113, 131, 311, 122, 212, 221, \\ 123, 132, 213, 231, 312, 321\}.$$

2.2.2. It is well known that $|\text{PF}_n| = (n+1)^{n-1}$, and that the permutation representation of \mathfrak{S}_n naturally supported by PF_n has Frobenius characteristic (see [10])

$$(11) \quad (-1)^n \omega(h_n^*)$$

where $f \mapsto f^*$ is the involution on symmetric functions defined on the generators h_n as follows (see [23, Ex. 24, p. 35]). If we set $H(t) := \sum_{n \geq 0} h_n t^n$ and $H^*(u) := \sum_{n \geq 0} h_n^* u^n$, then

$$(12) \quad u = tH(t) \Leftrightarrow t = uH^*(u).$$

Each nondecreasing parking function generates a sub-permutation representation of PF_n . It is easy to see that the number of nondecreasing parking functions of length n is the Catalan number $C_n = \frac{1}{n+1} \binom{2n}{n}$.

2.2.3. *Prime parking functions.* This important notion has been introduced by Gessel in 1997 (see [37]). Given a parking function of length n , one says that $b \in \{0, 1, \dots, n\}$ is a *breakpoint* of \mathbf{a} if $|\{i \mid a_i \leq b\}| = b$. For example, the parking function 112256679 has the five breakpoints $\{0, 4, 5, 8, 9\}$. Then $\mathbf{a} \in \text{PF}_n$ is said to be *prime* if its only breakpoints are the trivial ones: 0 and n . Let $\text{PPF}_n \subset \text{PF}_n$ be the set of prime parking functions on $[n]$. For example,

$$(13) \quad \text{PPF}_1 = \{1\}, \quad \text{PPF}_2 = \{11\}, \quad \text{PPF}_3 = \{111, 112, 121, 211\}.$$

It can easily be shown that $|\text{PPF}_n| = (n-1)^{n-1}$ for $n \geq 2$ (see [37, 16] and Section 5.5). The number of nondecreasing prime parking functions of length n is the shifted Catalan number C_{n-1} : they are obtained by concatenating a 1 to the left of all nondecreasing parking functions of length $n-1$.

As already mentioned, it is immediate to see that the set of all parking functions is closed under shifted concatenation and shifted shuffle. The prime parking functions are exactly those that do not occur in any nontrivial shifted shuffle of parking functions. This observation is at the basis of our definition of the Hopf algebra of parking functions (see Section 3).

2.2.4. The module of prime parking functions. Parking functions can be classified according to the factorization of their nondecreasing reorderings \mathbf{a}^\uparrow with respect to the operation of shifted concatenation. That is, if

$$(14) \quad \mathbf{a}^\uparrow = w_1 \bullet \cdots \bullet w_r$$

is the unique maximal factorization of \mathbf{a}^\uparrow , each w_i is a nondecreasing prime parking function. Define $i_k = |w_k|$ and let $I = (i_1, \dots, i_r)$. We shall say that \mathbf{a} is of *type I* and denote by PPF_I the set of parking functions of type I . For example, the parking function 966142272 is of type $(1, 4, 3, 1)$ and the number of parking functions of length 4 of each type is

(4)	(31)	(13)	(22)	(211)	(121)	(112)	(1111)
27	16	16	6	12	12	12	24

The set PPF_n of prime parking functions of length n is obviously a sub-permutation representation of PF_n . It can be shown that its Frobenius characteristic is

$$(15) \quad f_n = -\omega(e_n^*)$$

(see the Appendix for a direct proof). One can also obtain it as follows.

The set PPF_I of parking functions of type I is a sub-permutation representation of PF_n , and its Frobenius characteristic is

$$(16) \quad \text{ch}(\text{PPF}_I) = f_{i_1} \cdots f_{i_r},$$

since it is induced from the permutation representation of the Young subgroup \mathfrak{S}_I on the Cartesian product $\text{PPF}_{i_1} \times \cdots \times \text{PPF}_{i_r}$. Now, $\text{PF}_n = \bigsqcup_{I \models n} \text{PPF}_I$, so that

$$(17) \quad g_n = \sum_{I \models n} f_{i_1} \cdots f_{i_r},$$

which amounts to

$$(18) \quad g := \sum_{n \geq 0} g_n = (1 - f)^{-1} \quad \text{where} \quad f = \sum_{n \geq 1} f_n.$$

Thus, if we know that g_n is given by (11), we infer that f_n is given by (15), and conversely. A noncommutative version of these results will be established in Section 5.5.

2.3. Dendriform dialgebras. A *dendriform dialgebra*, as defined by Loday [19], is an associative algebra D whose multiplication \odot (it is denoted by $*$ in the original paper) splits into two binary operations

$$(19) \quad x \odot y = x \ll y + x \gg y,$$

called left and right, satisfying the following three compatibility relations for all a, b , and c different from 1 in D :

$$(20) \quad (a \ll b) \ll c = a \ll (b \odot c),$$

$$(21) \quad (a \gg b) \ll c = a \gg (b \ll c),$$

$$(22) \quad (a \odot b) \gg c = a \gg (b \gg c).$$

These relations are satisfied by shuffle algebras with $\odot = \sqcup$ and for $x = ua$ and $y = vb$ ($a, b \in A$),

$$(23) \quad x \gg y = (ua \sqcup vb)b, \quad x \ll y = (u \sqcup vb)a.$$

It turns out that the free associative algebra $\mathbb{K}\langle A \rangle$ is also a dendriform dialgebra. Actually, it is even a dendriform trialgebra, as explained below.

2.4. Dendriform trialgebras. A *dendriform trialgebra* [22] is an associative algebra whose multiplication \odot splits into three pieces

$$(24) \quad x \odot y = x \prec y + x \circ y + x \succ y,$$

where \circ is associative, and

$$(25) \quad (x \prec y) \prec z = x \prec (y \odot z),$$

$$(26) \quad (x \succ y) \prec z = x \succ (y \prec z),$$

$$(27) \quad (x \odot y) \succ z = x \succ (y \succ z),$$

$$(28) \quad (x \succ y) \circ z = x \succ (y \circ z),$$

$$(29) \quad (x \prec y) \circ z = x \circ (y \succ z),$$

$$(30) \quad (x \circ y) \prec z = x \circ (y \prec z).$$

Let $A = \{a_1 < a_2 < \dots\}$ be an infinite linearly ordered alphabet. Recall that $\mathbb{K}\langle A \rangle$ is understood as the projective limit of the $\mathbb{K}\langle A_n \rangle$ where A_n is the interval $[a_1, a_n]$ of A . We denote by $\max(w)$ the greatest letter occurring in the word $w \in A^*$.

DEFINITION 2.1. For two nonempty words $u, v \in A^*$, we set

$$(31) \quad u \prec v = \begin{cases} uv & \text{if } \max(u) > \max(v), \\ 0 & \text{otherwise,} \end{cases}$$

$$(32) \quad u \circ v = \begin{cases} uv & \text{if } \max(u) = \max(v), \\ 0 & \text{otherwise,} \end{cases}$$

$$(33) \quad u \succ v = \begin{cases} uv & \text{if } \max(u) < \max(v), \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 2.2. *The three operations \prec, \circ, \succ , endow the augmentation ideal $\mathbb{K}\langle A \rangle^+$ with the structure of a dendriform trialgebra.*

Proof. A straightforward verification. ■

Setting $\ll = \prec$ and $\gg = \circ + \succ$, we obtain a dendriform dialgebra.

It is known [22] that the free dendriform trialgebra on one generator, denoted here by \mathfrak{D} , is a free associative algebra with Hilbert series

$$(34) \quad \sum_{n \geq 0} s_n t^n = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4t} = 1 + t + 3t^2 + 11t^3 + 45t^4 + 197t^5 + \dots$$

that is, the generating function of the *super-Catalan*, or *little Schröder numbers*, counting plane trees.

The previous considerations allow us to give a simple polynomial realization of \mathfrak{D} . Consider the polynomial

$$(35) \quad \mathbf{M}_1 = \sum_{i \geq 1} a_i$$

(the sum of all letters). We can then state:

THEOREM 2.3. *The sub-trialgebra of $\mathbb{K}\langle A \rangle^+$ generated by \mathbf{M}_1 is free as a dendriform trialgebra.*

We shall need the following construction on words. With any word w of length n , associate a plane tree $\mathcal{T}(w)$ with $n + 1$ leaves, as follows: if $m = \max(w)$ and if w has exactly k occurrences of m , write

$$(36) \quad w = v_0 m v_1 m v_2 \dots v_{k-1} m v_k,$$

where the v_i may be empty. Then $\mathcal{T}(w)$ is the tree obtained by grafting the subtrees $\mathcal{T}(v_0), \mathcal{T}(v_1), \dots, \mathcal{T}(v_k)$ (in this order) on a common root, with the initial condition $\mathcal{T}(\epsilon) = \emptyset$ for the empty word.

For example, the tree associated with 141324431312 is represented in Figure 1.

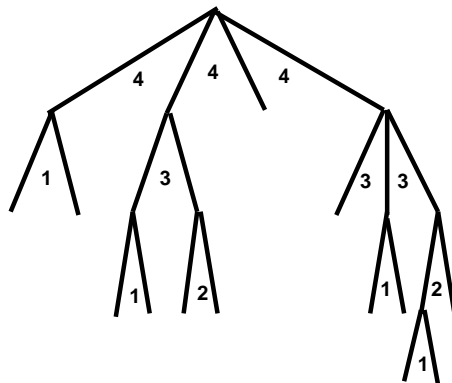


Fig. 1. The tree of 141324431312

Now associate with each plane tree T a polynomial by

$$(37) \quad \mathcal{M}_T := \sum_{\mathcal{T}(w)=T} w.$$

These belong to the subtrialgebra generated by \mathbf{M}_1 , since if T has as subtrees of its root T_1, \dots, T_k , one has

$$(38) \quad \mathcal{M}_T = \mathcal{M}_{T_1} \succ \mathbf{M}_1 \circ (\mathcal{M}_{T_2} \succ \mathbf{M}_1) \circ \dots \circ (\mathcal{M}_{T_{k-1}} \succ \mathbf{M}_1) \prec \mathbf{M}_{T_k}.$$

For example, for the tree T presented in Figure 1, one gets the expression

$$(39) \quad \begin{aligned} \mathbf{M}_1 \succ \mathbf{M}_1 \circ ((\mathbf{M}_1 \succ \mathbf{M}_1 \prec \mathbf{M}_1) \succ \mathbf{M}_1) \circ \mathbf{M}_1 \\ \prec (\mathbf{M}_1 \circ (\mathbf{M}_1 \succ \mathbf{M}_1) \prec (\mathbf{M}_1 \succ \mathbf{M}_1)). \end{aligned}$$

Proof of Theorem 2.3. Since it is already known that the dimension of the free dendriform trialgebra on one generator has dimensions given by the little Schröder numbers, we just need to show that all terms of the Hilbert series of this subalgebra are greater than or equal to the terms of (34). The polynomials \mathcal{M}_T , being sums over disjoint sets of words, are obviously linearly independent, whence the result. ■

COROLLARY 2.4 ([22]). *The free commutative dendriform trialgebra on one generator is $QSym^+$, the augmentation ideal of quasi-symmetric functions.*

Indeed, it is the image of \mathfrak{TD} under the ring homomorphism mapping the letters a_i to commuting variables x_i .

Other applications of this realization of \mathfrak{TD} will be given in Section 4.

2.5. Bidendriform bialgebras. These have been introduced by Foissy in [6]. A *bidendriform bialgebra* is a dendriform dialgebra equipped with a coproduct that splits into two parts, satisfying the *codendriform relations*, obtained by dualizing the dendriform relations, and certain compatibility properties with the two half-products.

A *codendriform coalgebra* is a coalgebra C whose coproduct Δ splits as $\Delta(c) = \bar{\Delta}(c) + c \otimes 1 + 1 \otimes c$ and $\bar{\Delta} = \Delta_{\ll} + \Delta_{\gg}$, such that, for all c in C :

$$(40) \quad (\Delta_{\ll} \otimes \text{Id}) \circ \Delta_{\ll}(a) = (\text{Id} \otimes \bar{\Delta}) \circ \Delta_{\ll}(a),$$

$$(41) \quad (\Delta_{\gg} \otimes \text{Id}) \circ \Delta_{\ll}(a) = (\text{Id} \otimes \Delta_{\ll}) \circ \Delta_{\gg}(a),$$

$$(42) \quad (\bar{\Delta} \otimes \text{Id}) \circ \Delta_{\gg}(a) = (\text{Id} \otimes \Delta_{\gg}) \circ \Delta_{\gg}(a).$$

The Loday–Ronco algebra of planar binary trees introduced in [21] arises as the free dendriform dialgebra on one generator. This is moreover a Hopf algebra, which turns out to be self-dual, so that it is also codendriform.

There is some compatibility between the dendriform and the codendriform structures, leading to what has been called by Foissy [6] a *bidendriform*

bialgebra. A bidendriform bialgebra is both a dendriform dialgebra and a co-dendriform coalgebra satisfying the following four compatibility relations

$$\begin{aligned}
 (43) \quad \Delta_{\gg}(a \gg b) &= a'b'_{\gg} \otimes a'' \gg b''_{\gg} + a' \otimes a'' \gg b \\
 &\quad + b'_{\gg} \otimes a \gg b''_{\gg} + ab'_{\gg} \otimes b''_{\gg} + a \otimes b, \\
 (44) \quad \Delta_{\gg}(a \ll b) &= a'b'_{\gg} \otimes a'' \ll b''_{\gg} + a' \otimes a'' \ll b + b'_{\gg} \otimes a \ll b''_{\gg}, \\
 (45) \quad \Delta_{\ll}(a \gg b) &= a'b'_{\ll} \otimes a'' \gg b''_{\ll} + ab'_{\ll} \otimes b''_{\ll} + b'_{\ll} \otimes a \gg b''_{\ll}, \\
 (46) \quad \Delta_{\ll}(a \ll b) &= a'b'_{\ll} \otimes a'' \ll b''_{\ll} + a'b \otimes a'' + b'_{\ll} \otimes a \ll b''_{\ll} + b \otimes a,
 \end{aligned}$$

where the pairs (x', x'') (resp. (x'_{\ll}, x''_{\ll}) and (x'_{\gg}, x''_{\gg})) correspond to all possible elements occurring in $\overline{\Delta}x$ (resp. $\Delta_{\ll}x$ and $\Delta_{\gg}x$), summation signs being understood (Sweedler's notation).

Foissy has shown [6] that a connected bidendriform bialgebra \mathcal{B} is always free as an associative algebra and self-dual as a Hopf algebra. Moreover, its primitive Lie algebra is free, and as a dendriform dialgebra, \mathcal{B} is also free over the space of totally primitive elements (those annihilated by Δ_{\ll} and Δ_{\gg}).

It is also proved in [6] that **FQSym** is bidendriform, so that it has all these properties.

3. THE HOPF ALGEBRA OF PARKING FUNCTIONS

3.1. The algebra PQSym. Since permutations are special parking functions and parking functions are stable under the shifted shuffle, it is natural to embed the algebra **FQSym** of free quasi-symmetric functions of [4] into an algebra spanned by elements \mathbf{F}_a ($a \in \text{PF}$), with the same multiplication rule:

$$(47) \quad \mathbf{F}_a \mathbf{F}_{a''} := \sum_{a \in a' \cup a''} \mathbf{F}_a.$$

We shall call this algebra **PQSym** (parking quasi-symmetric functions).

For example,

$$(48) \quad \mathbf{F}_1 \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{21}, \quad \mathbf{F}_1 \mathbf{F}_{11} = \mathbf{F}_{122} + \mathbf{F}_{212} + \mathbf{F}_{221},$$

$$(49) \quad \mathbf{F}_1 \mathbf{F}_{12} = \mathbf{F}_{123} + \mathbf{F}_{213} + \mathbf{F}_{231}, \quad \mathbf{F}_1 \mathbf{F}_{21} = \mathbf{F}_{132} + \mathbf{F}_{312} + \mathbf{F}_{321},$$

$$(50) \quad \mathbf{F}_{12} \mathbf{F}_{11} = \mathbf{F}_{1233} + \mathbf{F}_{1323} + \mathbf{F}_{1332} + \mathbf{F}_{3123} + \mathbf{F}_{3132} + \mathbf{F}_{3312},$$

$$\begin{aligned}
 (51) \quad \mathbf{F}_{211} \mathbf{F}_{131} &= \mathbf{F}_{211464} + \mathbf{F}_{214164} + \mathbf{F}_{214614} + \mathbf{F}_{214641} + \mathbf{F}_{241164} \\
 &\quad + \mathbf{F}_{241614} + \mathbf{F}_{241641} + \mathbf{F}_{246114} + \mathbf{F}_{246141} + \mathbf{F}_{246411} \\
 &\quad + \mathbf{F}_{421164} + \mathbf{F}_{421614} + \mathbf{F}_{421641} + \mathbf{F}_{426114} + \mathbf{F}_{426141} \\
 &\quad + \mathbf{F}_{426411} + \mathbf{F}_{462114} + \mathbf{F}_{462141} + \mathbf{F}_{462411} + \mathbf{F}_{464211}.
 \end{aligned}$$

Recall that the prime parking functions are those that do not occur in the decomposition of any nontrivial product $\mathbf{F}_a \mathbf{F}_{a''}$.

3.2. The coalgebra PQSym. There is a coproduct on **PQSym** which appears as a natural extension of the coproduct of **FQSym**. Recall (see [24, 4]) that if σ is a permutation, then

$$(52) \quad \Delta \mathbf{F}_\sigma = \sum_{u \cdot v = \sigma} \mathbf{F}_{\text{Std}(u)} \otimes \mathbf{F}_{\text{Std}(v)},$$

where Std denotes the usual notion of standardization of a word.

Given a word w on $\{1, 2, \dots\}$, it is possible to define a notion of *parkization* $\text{Park}(w)$, a parking function which coincides with $\text{Std}(w)$ when w is a word without repeated letters.

ALGORITHM 3.1. Input: A word w .

Output: A parking function.

Let n be the length of w . Define

$$(53) \quad d(w) := \min\{i \mid |\{w_j \leq i\}| < i\}.$$

- If $d(w) = n + 1$, return w .
- Otherwise, let w' be the word obtained by decreasing by one the values of the elements of w greater than $d(w)$. Then return the parkized word of w' .

The algorithm is correct since $d(w) = n + 1$ iff w is a parking function and since w' is smaller than w in the lexicographic order, it terminates.

For example, the following table displays an execution of the parkization algorithm: on each line, there is a word w and the value of $d(w)$ and the next line contains the element w' as defined in the algorithm.

w									$d(w)$
5	7	3	3	13	1	10	10	4	2
4	6	2	2	12	1	9	9	3	7
4	6	2	2	11	1	8	8	3	7
4	6	2	2	10	1	7	7	3	9
4	6	2	2	9	1	7	7	3	10

We can now define a coproduct on **PQSym** by

$$(54) \quad \Delta \mathbf{F}_a := \sum_{u \cdot v = a} \mathbf{F}_{\text{Park}(u)} \otimes \mathbf{F}_{\text{Park}(v)}.$$

For example,

$$(55) \quad \Delta \mathbf{F}_{121} = 1 \otimes \mathbf{F}_{121} + \mathbf{F}_1 \otimes \mathbf{F}_{21} + \mathbf{F}_{12} \otimes \mathbf{F}_1 + \mathbf{F}_{121} \otimes 1,$$

$$(56) \quad \Delta \mathbf{F}_{131} = 1 \otimes \mathbf{F}_{131} + \mathbf{F}_1 \otimes \mathbf{F}_{21} + \mathbf{F}_{12} \otimes \mathbf{F}_1 + \mathbf{F}_{131} \otimes 1,$$

$$(57) \quad \Delta \mathbf{F}_{3132} = 1 \otimes \mathbf{F}_{3132} + \mathbf{F}_1 \otimes \mathbf{F}_{132} + \mathbf{F}_{21} \otimes \mathbf{F}_{21} \\ + \mathbf{F}_{212} \otimes \mathbf{F}_1 + \mathbf{F}_{3132} \otimes 1,$$

$$(58) \quad \Delta \mathbf{F}_{1643165} = 1 \otimes \mathbf{F}_{1643165} + \mathbf{F}_1 \otimes \mathbf{F}_{532154} + \mathbf{F}_{12} \otimes \mathbf{F}_{32154} \\ + \mathbf{F}_{132} \otimes \mathbf{F}_{2143} + \mathbf{F}_{1432} \otimes \mathbf{F}_{132} + \mathbf{F}_{15431} \otimes \mathbf{F}_{21} \\ + \mathbf{F}_{154315} \otimes \mathbf{F}_1 + \mathbf{F}_{1643165} \otimes 1.$$

PROPOSITION 3.2. *The operation defined by (54) is coassociative and is a morphism for the product. So $(\mathbf{PQSym}, \cdot, \Delta)$ is a bialgebra.*

Proof. The operation is obviously coassociative since the deconcatenation is coassociative. Consider two words w_1 and w_2 and a prefix u_1 (resp. u_2) of w_1 (of w_2). Then the set of the parkized words of all prefixes of $w_1 \cup w_2$ containing only letters of u_1 and u_2 is equal to $u_1 \cup u_2$. So Δ is a morphism for the product, and hence \mathbf{PQSym} is a bialgebra. ■

3.3. The Hopf algebra \mathbf{PQSym} . Since \mathbf{PQSym} is endowed with a bialgebra structure naturally graded by the length of parking functions, one defines the antipode as the inverse of the identity for the convolution product and then endows \mathbf{PQSym} with a Hopf algebra structure.

The standard formula for the antipode, written in the basis (\mathbf{F}_a) , reads

$$(59) \quad \nu(\mathbf{F}_a) = \sum_{r; u_1 \cdots u_r = a; |u_i| \geq 1} (-1)^r \mathbf{F}_{\text{Park}(u_1)} \cdots \mathbf{F}_{\text{Park}(u_r)}.$$

For example,

$$(60) \quad \nu(\mathbf{F}_{122}) = -\mathbf{F}_{122} + \mathbf{F}_1 \mathbf{F}_{11} + \mathbf{F}_{12} \mathbf{F}_1 - \mathbf{F}_1^3 \\ = \mathbf{F}_{212} + \mathbf{F}_{221} - \mathbf{F}_{213} - \mathbf{F}_{231} - \mathbf{F}_{321}.$$

3.4. The graded dual \mathbf{PQSym}^* . Let $\mathbf{G}_a = \mathbf{F}_a^* \in \mathbf{PQSym}^*$ be the dual basis of (\mathbf{F}_a) .

PROPOSITION 3.3. *The product on \mathbf{PQSym}^* is given by*

$$(61) \quad \mathbf{G}_{a'} \mathbf{G}_{a''} = \sum_{a \in a' * p a''} \mathbf{G}_a,$$

where the convolution $a' * p a''$ of two parking functions is defined as

$$(62) \quad a' * p a'' = \sum_{u, v; a = u \cdot v \in \text{PF}, \text{Park}(u) = a', \text{Park}(v) = a''} a.$$

Proof. If $\langle \cdot, \cdot \rangle$ denotes the duality bracket, the product on \mathbf{PQSym}^* is given by

$$(63) \quad \mathbf{G}_{a'} \mathbf{G}_{a''} = \sum_a \langle \mathbf{G}_{a'} \otimes \mathbf{G}_{a''}, \Delta \mathbf{F}_a \rangle \mathbf{G}_a = \sum_{a \in a' * p a''} \mathbf{G}_a. \quad \blacksquare$$

For example,

$$(64) \quad \mathbf{G}_1 \mathbf{G}_1 = \mathbf{G}_{11} + \mathbf{G}_{12} + \mathbf{G}_{21},$$

$$\mathbf{G}_1 \mathbf{G}_{11} = \mathbf{G}_{111} + \mathbf{G}_{122} + \mathbf{G}_{211} + \mathbf{G}_{311},$$

$$(65) \quad \mathbf{G}_1 \mathbf{G}_{12} = \mathbf{G}_{112} + \mathbf{G}_{113} + \mathbf{G}_{123} + \mathbf{G}_{212} + \mathbf{G}_{213} + \mathbf{G}_{312},$$

$$(66) \quad \mathbf{G}_1 \mathbf{G}_{21} = \mathbf{G}_{121} + \mathbf{G}_{131} + \mathbf{G}_{132} + \mathbf{G}_{221} + \mathbf{G}_{231} + \mathbf{G}_{321},$$

$$(67) \quad \mathbf{G}_{12} \mathbf{G}_{11} = \mathbf{G}_{1211} + \mathbf{G}_{1222} + \mathbf{G}_{1233} + \mathbf{G}_{1311} + \mathbf{G}_{1322} \\ + \mathbf{G}_{1411} + \mathbf{G}_{1422} + \mathbf{G}_{2311} + \mathbf{G}_{2411} + \mathbf{G}_{3411},$$

$$(68) \quad \mathbf{G}_{211} \mathbf{G}_{131} = \mathbf{G}_{211131} + \mathbf{G}_{211141} + \mathbf{G}_{211151} + \mathbf{G}_{211161} + \mathbf{G}_{211242} \\ + \mathbf{G}_{211252} + \mathbf{G}_{211262} + \mathbf{G}_{211353} + \mathbf{G}_{211363} + \mathbf{G}_{211464} \\ + \mathbf{G}_{322131} + \mathbf{G}_{322141} + \mathbf{G}_{322151} + \mathbf{G}_{322161} + \mathbf{G}_{433141} \\ + \mathbf{G}_{433151} + \mathbf{G}_{433161} + \mathbf{G}_{433131} + \mathbf{G}_{544131}.$$

When restricted to permutations, the product of \mathbf{G} coincides with the convolution of [33, 24]. Notice also that

$$(69) \quad \mathbf{G}_1^n = \sum_{\mathbf{a} \in \text{PF}_n} \mathbf{G}_{\mathbf{a}}.$$

PROPOSITION 3.4. *The coproduct $\Delta \mathbf{G}_{\mathbf{a}}$ is given by*

$$(70) \quad \Delta \mathbf{G}_{\mathbf{a}} := \sum_{u, v; \mathbf{a} \in u \cup v} \mathbf{G}_u \otimes \mathbf{G}_v.$$

Proof. If $\langle \cdot, \cdot \rangle$ denotes the duality bracket, the coproduct on \mathbf{PQSym}^* is given by

$$(71) \quad \Delta \mathbf{G}_{\mathbf{a}} = \sum_{\mathbf{a}', \mathbf{a}''} \langle \mathbf{G}_{\mathbf{a}}, \mathbf{F}_{\mathbf{a}'} \mathbf{F}_{\mathbf{a}''} \rangle \mathbf{G}_{\mathbf{a}'} \otimes \mathbf{G}_{\mathbf{a}''} = \sum_{\mathbf{a} \in \mathbf{a}' \cup \mathbf{a}''} \mathbf{G}_{\mathbf{a}'} \otimes \mathbf{G}_{\mathbf{a}''}. \blacksquare$$

For example,

$$(72) \quad \Delta \mathbf{G}_{121} = 1 \otimes \mathbf{G}_{121} + \mathbf{G}_{121} \otimes 1,$$

$$(73) \quad \Delta \mathbf{G}_{131} = 1 \otimes \mathbf{G}_{131} + \mathbf{G}_{11} \otimes \mathbf{G}_1 + \mathbf{G}_{131} \otimes 1,$$

$$(74) \quad \Delta \mathbf{G}_{3132} = 1 \otimes \mathbf{G}_{3132} + \mathbf{G}_1 \otimes \mathbf{G}_{221} + \mathbf{G}_{12} \otimes \mathbf{G}_{11} + \mathbf{G}_{3132} \otimes 1$$

$$(75) \quad \Delta \mathbf{G}_{164821657} = 1 \otimes \mathbf{G}_{164821657} + \mathbf{G}_{121} \otimes \mathbf{G}_{315324} \\ + \mathbf{G}_{1421} \otimes \mathbf{G}_{24213} + \mathbf{G}_{14215} \otimes \mathbf{G}_{1312} + \mathbf{G}_{164821657} \otimes 1.$$

There is also a direct way to describe the coproduct of $\mathbf{G}_{\mathbf{a}}$ in terms of breakpoints:

PROPOSITION 3.5. *Let \mathbf{a} be a parking function of length n . For b in $\{0, \dots, n\}$, define $\mathbf{a}'(b)$ and $\mathbf{a}''(b)$ as the restrictions of \mathbf{a} to the respective intervals $[1, b]$ and $[b+1, n]$. Then*

$$(76) \quad \Delta \mathbf{G}_{\mathbf{a}} := \sum_b \mathbf{G}_{\mathbf{a}'(b)} \otimes \mathbf{G}_{\mathbf{a}''(b)},$$

where the sum runs over all breakpoints of \mathbf{a} .

Proof. The term $\mathbf{G}_u \otimes \mathbf{G}_v$ appears in $\Delta \mathbf{G}_a$ iff \mathbf{a} belongs to the shifted shuffle of u and v , so that \mathbf{a} has a breakpoint at $b = |u|$. Then $\mathbf{a}'(b) = u$ and $\mathbf{a}''(b) = v$. ■

For example, the breakpoints of 164821657 are $\{0, 3, 4, 5, 9\}$ so that one recovers the result of (75).

Let us finally mention that the realization provided in Section 3.8.1 allows us to shed an interesting light on the coproduct and the fact that \mathbf{PQSym}^* is a Hopf algebra.

3.5. PQSym as a bidendriform bialgebra. In [6], Foissy has proved that the Hopf algebra \mathbf{FQSym} of free quasi-symmetric functions is bidendriform. A very slight modification of his operations allows us to state:

THEOREM 3.6. *\mathbf{PQSym}^* is a bidendriform bialgebra with the following definitions:*

$$(77) \quad \mathbf{G}_{\mathbf{a}'} \ll \mathbf{G}_{\mathbf{a}''} = \sum_{\mathbf{a}=u \cdot v \in \mathbf{a}' * \mathbf{P} \mathbf{a}'', |u|=|\mathbf{a}'|; \max(v) < \max(u)} \mathbf{G}_{\mathbf{a}},$$

$$(78) \quad \mathbf{G}_{\mathbf{a}'} \gg \mathbf{G}_{\mathbf{a}''} = \sum_{\mathbf{a}=u \cdot v \in \mathbf{a}' * \mathbf{P} \mathbf{a}'', |u|=|\mathbf{a}'|; \max(v) \geq \max(u)} \mathbf{G}_{\mathbf{a}},$$

$$(79) \quad \Delta_{\ll} \mathbf{G}_{\mathbf{a}} = \sum_{\mathbf{a} \in u \uplus v; \text{last}(\mathbf{a}) \leq |u|} \mathbf{G}_u \otimes \mathbf{G}_v,$$

$$(80) \quad \Delta_{\gg} \mathbf{G}_{\mathbf{a}} = \sum_{\mathbf{a} \in u \uplus v; \text{last}(\mathbf{a}) > |u|} \mathbf{G}_u \otimes \mathbf{G}_v,$$

where $|u| \geq 1$ and $|v| \geq 1$, and $\text{last}(\mathbf{a})$ means the last letter of \mathbf{a} .

Proof. First, the three defining relations of a dendriform dialgebra are satisfied. Let us check the first one, for instance. The left side of (20) amounts to considering the elements w in $\mathbf{a} * \mathbf{P} \mathbf{b} * \mathbf{P} \mathbf{c}$ where the last maximum of w belongs to \mathbf{a} . It is the same for the right of (20). The other two relations are proved in the same way, by checking that they build the words in $\mathbf{a} * \mathbf{P} \mathbf{b} * \mathbf{P} \mathbf{c}$ where the last maximum is in \mathbf{b} (equation (21)) or in \mathbf{c} (equation (22)).

The three defining relations of a codendriform coalgebra are also satisfied since they amount to splitting the set of parking functions indexing the elements of $(\bar{\Delta} \otimes \text{Id}) \circ \bar{\Delta}(\mathbf{G}_{\mathbf{a}})$ according to the element of the tensor product containing the last letter of \mathbf{a} .

Since the sum of the four compatibility relations is equivalent to the coassociativity of $\bar{\Delta}$, it is sufficient to check any three of them. We will only prove the first one (more complicated than the second and third one) in detail, the other ones being proved in the same way.

Till the end of this proof we will identify any function $\mathbf{G}_{\mathbf{a}}$ with its index \mathbf{a} . Let \mathbf{a} and \mathbf{b} be two parking functions of length p and q . For any word w

of length $p + q$, let $m_1 = \min(w_1 \dots w_p)$, $m_2 = \min(w_{p+1} \dots w_{p+q})$, $M_1 = \max(w_1 \dots w_p)$, $M_2 = \max(w_{p+1} \dots w_{p+q})$.

Let i be any integer and split into two groups and six subgroups the parking functions indexing the terms in $\mathbf{G}_a \gg \mathbf{G}_b$ having a breakpoint at i according to the criterions:

- $(M_1 \leq i)$, or $(M_1 > i \text{ and } m_1 \leq i)$, or $(m_1 > i)$,
- $(m_2 \leq i)$ or $(m_2 > i)$.

Apply Δ_{\gg} to the first group and let S_1 be the sum of the elements $\mathbf{G}_u \otimes \mathbf{G}_v$ such that u is of length i . Since $M_1 \leq i$ and $m_2 \leq i$, the first p letters of all words are in u , whereas the others are both in u and v (by hypothesis, the last one is in v). Note that the positions of the letters belonging to the right factor of the tensor product are independent of the element of the first group and are the positions of the $p + q - i$ greatest letters of \mathbf{b} . Moreover, there exists a breakpoint of \mathbf{b} separating those letters from the other ones. Since the last letter of any word of the first group goes to the right factor of the tensor product and comes from the last letter of \mathbf{b} , we then deduce that the right factor of S_1 is built with all \mathbf{b}''_{\gg} of length $p + q - i$, using Sweedler's notations. Finally, in the left factor, we find all the elements of the form $\mathbf{a} \uplus \mathbf{b}'_{\gg}$, since they correspond to the restriction of all words of $\mathbf{a} \gg \mathbf{b}$ to letters smaller than i . Finally, summing up over all possible i , we see that the sum of all the elements of all the first groups is $\mathbf{G}_a \mathbf{G}_{\mathbf{b}'_{\gg}} \otimes \mathbf{G}_{\mathbf{b}''_{\gg}}$, that is, the fourth term of the right-hand side of (43).

In the same way, one proves that the second group, corresponding to $M_1 \leq i$, and $m_2 > i$, gives the term $\mathbf{a} \otimes \mathbf{b}$ of (43). The third group, corresponding to $M_1 > i$, $m_1 \leq i$, and $m_2 \leq i$ gives the term $\mathbf{a}'\mathbf{b}'_{\gg} \otimes \mathbf{a}'' \gg \mathbf{b}''_{\gg}$ of (43). The fourth group, corresponding to $M_1 > i$, $m_1 \leq i$, and $m_2 > i$ gives the term $\mathbf{a}' \otimes \mathbf{a}'' \gg \mathbf{b}$ of (43). The fifth group, corresponding to $m_1 > i$ and $m_2 \leq i$, gives the term $\mathbf{b}'_{\gg} \otimes \mathbf{a} \gg \mathbf{b}''_{\gg}$ of (43). The sixth group, corresponding to $m_1 > i$ and $m_2 > i$, gives no term since we would have $|u| = 0$, which is impossible. ■

For example,

$$(81) \quad \mathbf{G}_{12} \ll \mathbf{G}_{212} = \mathbf{G}_{13212} + \mathbf{G}_{14212} + \mathbf{G}_{14313} + \mathbf{G}_{14323} + \mathbf{G}_{15212} \\ + \mathbf{G}_{15313} + \mathbf{G}_{15323} + \mathbf{G}_{24313} + \mathbf{G}_{24212} + \mathbf{G}_{34212} \\ + \mathbf{G}_{23212} + \mathbf{G}_{25212} + \mathbf{G}_{25313} + \mathbf{G}_{35212} + \mathbf{G}_{45212},$$

$$(82) \quad \mathbf{G}_{12} \gg \mathbf{G}_{212} = \mathbf{G}_{12212} + \mathbf{G}_{12313} + \mathbf{G}_{12323} + \mathbf{G}_{12414} + \mathbf{G}_{12424} \\ + \mathbf{G}_{12434} + \mathbf{G}_{13313} + \mathbf{G}_{13323} + \mathbf{G}_{13414} + \mathbf{G}_{13424} \\ + \mathbf{G}_{23313} + \mathbf{G}_{23414},$$

$$(83) \quad \Delta_{\ll} \mathbf{G}_{1252754} = \mathbf{G}_{125254} \otimes \mathbf{G}_1 + \mathbf{G}_{1224} \otimes \mathbf{G}_{131},$$

$$(84) \quad \Delta_{\gg} \mathbf{G}_{1252754} = \mathbf{G}_{122} \otimes \mathbf{G}_{2421} + \mathbf{G}_1 \otimes \mathbf{G}_{141643}.$$

The duality of bidendriform bialgebras implies that the bidendriform relations for **PQSym** are

$$(85) \quad \mathbf{F}_{\mathbf{a}'} \ll \mathbf{F}_{\mathbf{a}''} = \sum_{\mathbf{a} \in \mathbf{a}' \cup \mathbf{a}''; \text{last}(\mathbf{a}) \leq |\mathbf{a}'|} \mathbf{F}_{\mathbf{a}},$$

$$(86) \quad \mathbf{F}_{\mathbf{a}'} \gg \mathbf{F}_{\mathbf{a}''} = \sum_{\mathbf{a} \in \mathbf{a}' \cup \mathbf{a}''; \text{last}(\mathbf{a}) > |\mathbf{a}'|} \mathbf{F}_{\mathbf{a}},$$

$$(87) \quad \Delta_{\ll} \mathbf{F}_{\mathbf{a}} = \sum_{u \cdot v = \mathbf{a}; \max(v) < \max(u)} \mathbf{F}_{\text{Park}(u)} \otimes \mathbf{F}_{\text{Park}(v)},$$

$$(88) \quad \Delta_{\gg} \mathbf{F}_{\mathbf{a}} = \sum_{u \cdot v = \mathbf{a}; \max(v) \geq \max(u)} \mathbf{F}_{\text{Park}(u)} \otimes \mathbf{F}_{\text{Park}(v)},$$

where the sums inside the coproducts range over nontrivial deconcatenations, that is, $|u| \geq 1$ and $|v| \geq 1$.

We then have the following consequences of the results of Foissy [6].

COROLLARY 3.7. ***PQSym** is a self-dual Hopf algebra.*

COROLLARY 3.8. *The Lie algebra of primitive elements of **PQSym** is a free Lie algebra (its generators are not explicitly known).*

Let

$$(89) \quad PF(t) = 1 + \sum_{n \geq 1} (n + 1)^{n-1} t^n.$$

COROLLARY 3.9. ***PQSym** is free as a dendriform dialgebra on its totally primitive elements whose degree generating series is*

$$(90) \quad TP(t) := \frac{PF(t) - 1}{PF(t)^2} = t + t^2 + 7t^3 + 66t^4 + 786t^5 + 11\,278t^6 + 189\,391t^7 + 3\,648\,711t^8 + 79\,447\,316t^9 + O(t^{10}).$$

For example, \mathbf{F}_1 and \mathbf{G}_1 are totally primitive and so are $\mathbf{F}_{12} - \mathbf{F}_{11}$ and \mathbf{G}_{11} . Here are bases of the seven-dimensional space of totally primitive elements of **PQSym** and **PQSym**^{*} in degree 3:

$$(91) \quad \begin{matrix} \mathbf{F}_{123} - \mathbf{F}_{122} - \mathbf{F}_{112} + \mathbf{F}_{111}, & \mathbf{F}_{311} - \mathbf{F}_{211}, & \mathbf{F}_{113} - \mathbf{F}_{112}, \\ \mathbf{F}_{131} - \mathbf{F}_{121}, & \mathbf{F}_{132} - \mathbf{F}_{131}, & \mathbf{F}_{231} - \mathbf{F}_{121}, & \mathbf{F}_{213} - \mathbf{F}_{212}; \end{matrix}$$

$$(92) \quad \begin{matrix} \mathbf{G}_{122} - \mathbf{G}_{212}, & \mathbf{G}_{131} - \mathbf{G}_{311}, & \mathbf{G}_{312} - \mathbf{G}_{132}, \\ \mathbf{G}_{111}, & \mathbf{G}_{112}, & \mathbf{G}_{121}, & \mathbf{G}_{211}. \end{matrix}$$

Thanks to the bidendriform structure of **PQSym**, we know that **PQSym** and **PQSym**^{*} are isomorphic as bidendriform bialgebras and hence

isomorphic as Hopf algebras. We do not know an explicit isomorphism, but restricting to \mathbf{FQSym} , that is, permutations, the linear map φ defined by

$$(93) \quad \varphi(\mathbf{F}_\sigma) := \sum_{\mathbf{a}, \text{Std}(\mathbf{a})=\sigma^{-1}} \mathbf{G}_\mathbf{a},$$

is a bidendriform and hence a Hopf embedding, compatible with the usual realization of \mathbf{FQSym} [4].

3.6. Free generators and multiplicative bases. Let us say that a word w over \mathbb{N}^* is *connected* if it cannot be written as a shifted concatenation $w = u \bullet v$, and *anti-connected* if its mirror image \bar{w} is connected.

PROPOSITION 3.10. \mathbf{PQSym} is free over the set

$$(94) \quad \{\mathbf{F}_\mathbf{c} \mid \mathbf{c} \in \text{PF}, \mathbf{c} \text{ connected}\}$$

and \mathbf{PQSym}^* is free over the set

$$(95) \quad \{\mathbf{G}_\mathbf{d} \mid \mathbf{d} \in \text{PF}, \mathbf{d} \text{ anti-connected}\}.$$

Proof. Clearly, any word w has a unique *maximal factorization* into connected words, $w = w_1 \bullet \cdots \bullet w_k$ where all w_i are connected. Moreover, the lexicographically minimal word in $w_1 \uplus \cdots \uplus w_k$ is w so that the matrix expressing all products of \mathbf{F} indexed by connected words is triangular over the basis $\mathbf{F}_\mathbf{a}$, with ones on the diagonal. The proof is exactly the same for the \mathbf{G} . ■

The ordinary generating function for the numbers c_n of connected parking functions is

$$(96) \quad \begin{aligned} \sum_{n \geq 1} c_n t^n &= 1 - PF(t)^{-1} \\ &= t + 2t^2 + 11t^3 + 92t^4 + 1014t^5 + 13795t^6 + 223061t^7 \\ &\quad + 4180785t^8 + 89191196t^9 + 2135610879t^{10} \\ &\quad + 56749806356t^{11} + 1658094051392t^{12} + O(t^{13}). \end{aligned}$$

Let $\mathbf{a} = \mathbf{a}_1 \bullet \cdots \bullet \mathbf{a}_r$ be the maximal factorization of \mathbf{a} into connected parking functions. We set

$$(97) \quad \mathbf{F}^\mathbf{a} = \mathbf{F}_{\mathbf{a}_1} \cdots \mathbf{F}_{\mathbf{a}_r},$$

$$(98) \quad \mathbf{G}^{\bar{\mathbf{a}}} = \mathbf{G}_{\bar{\mathbf{a}}_r} \cdots \mathbf{G}_{\bar{\mathbf{a}}_1}.$$

PROPOSITION 3.11. *The basis $(\mathbf{F}^\mathbf{a})$ of \mathbf{PQSym} and the basis $(\mathbf{G}^{\bar{\mathbf{a}}})$ of \mathbf{PQSym}^* are both multiplicative.*

Proof. This follows from the proof of Proposition 3.10. ■

Now, if \mathbf{S}_a (resp. \mathbf{T}_a) is the dual basis of \mathbf{F}^a (resp. $\mathbf{G}^{\bar{a}}$) then

$$(99) \quad \{\mathbf{S}_c \mid c \text{ connected}\} \quad \text{and} \quad \{\mathbf{T}_c \mid c \text{ connected}\}$$

are bases of the primitive Lie algebras \mathbf{LPQ}^* (resp. \mathbf{LPQ}) of \mathbf{PQSym}^* (resp. \mathbf{PQSym}).

Thanks again to [6], we know that both Lie algebras are free, on generators whose degree generating series is

$$(100) \quad 1 - \prod_{n \geq 1} (1 - t^n)^{c_n} = 1 - (1 - t)(1 - t^2)^2(1 - t^3)^{11} \dots$$

$$= t + 2t^2 + 9t^3 + 80t^4 + 901t^5 + 12\,564t^6 + 206\,476t^7$$

$$+ 3\,918\,025t^8 + 84\,365\,187t^9 + 2\,034\,559\,143t^{10} + O(t^{11}).$$

3.7. \mathbf{PQSym}^* as a combinatorial Hopf algebra. Since \mathbf{FQSym} can be embedded in \mathbf{PQSym} , we have a canonical Hopf embedding of \mathbf{Sym} in \mathbf{PQSym} given by

$$(101) \quad S_n \mapsto \mathbf{F}_{12 \dots n}.$$

With parking functions, we have other possibilities: for example,

$$(102) \quad j(S_n) := \mathbf{F}_{11 \dots 1}$$

is a Hopf embedding, whose dual j^* maps \mathbf{PQSym}^* to $QSym$ and therefore endows \mathbf{PQSym}^* with a different structure of combinatorial Hopf algebra in the sense of [1].

On the dual side, we have a Hopf embedding

$$(103) \quad S_n \mapsto \sum_{\text{Std}(\mathbf{a})=12 \dots n} \mathbf{G}_a$$

of \mathbf{Sym} into \mathbf{PQSym}^* , given by the restriction of the self-duality isomorphism of (93) to the \mathbf{Sym} subalgebra $S_n = \mathbf{F}_{12 \dots n}$ of \mathbf{PQSym} . Its transpose gives a Hopf epimorphism $\eta : \mathbf{PQSym} \rightarrow QSym$, which maps \mathbf{F}_a to F_I , where I is the descent composition of the word \mathbf{a} .

3.8. Realizations of \mathbf{PQSym}^* and \mathbf{PQSym}

3.8.1. Realization of \mathbf{PQSym}^* . The algebra $\mathbf{PQSym}^*(A)$ admits a simple realization in terms of noncommutative polynomials [28], which is similar to the construction of \mathbf{FQSym} . If A is a totally ordered infinite alphabet, one can set

$$(104) \quad \mathbf{G}_a(A) := \sum_{w \in A^*, \text{Park}(w)=\mathbf{a}} w.$$

THEOREM 3.12 ([28]). *These polynomials satisfy (61) and allow writing the coproduct as $\Delta \mathbf{G}_{\mathbf{a}} = \mathbf{G}_{\mathbf{a}}(A' \hat{+} A'')$ where $A' \hat{+} A''$ denotes the ordered sum of two mutually commuting alphabets isomorphic to A as ordered sets.*

Let us recall the precise way to introduce a coproduct on an algebra realized on words under certain conditions. Start with A' and A'' , two mutually commuting alphabets isomorphic to A as ordered sets. Then build their ordered sum $A' \hat{+} A''$ and compute $\mathbf{G}_{\mathbf{a}}(A' \hat{+} A'')$ separating inside each term what belongs to A' and what belongs to A'' . Assume that one can write, for all parking functions \mathbf{a} ,

$$(105) \quad \mathbf{G}_{\mathbf{a}}(A' \hat{+} A'') = \sum_{\mathbf{a}', \mathbf{a}''} \mathbf{G}_{\mathbf{a}'}(A') \mathbf{G}_{\mathbf{a}''}(A''),$$

where the sum is taken over a set of pairs of parking functions depending on \mathbf{a} . Then the operation

$$(106) \quad \Delta \mathbf{G}_{\mathbf{a}} := \sum_{\mathbf{a}', \mathbf{a}''} \mathbf{G}_{\mathbf{a}'} \otimes \mathbf{G}_{\mathbf{a}''},$$

where the sum is taken over the same set as before, is a coproduct.

For example, $\mathbf{G}_{121} = \sum_i a_i a_{i+1} a_i$, so that

$$(107) \quad \mathbf{G}_{121}(A' \hat{+} A'') = \mathbf{G}_{121}(A') + \mathbf{G}_{121}(A''),$$

since $a_i \in A'$ is equivalent to $a_{i+1} \in A'$ by definition of the ordered sum of alphabets. One then recovers the results of (72).

Now, $\mathbf{G}_{131} = \sum_{i,j; j>i+1} a_i a_j a_i$, so that

$$(108) \quad \mathbf{G}_{131}(A' \hat{+} A'') = \mathbf{G}_{131}(A') + \mathbf{G}_{11}(A') \mathbf{G}_1(A'') + \mathbf{G}_{121}(A''),$$

since a_i and a_j can belong to A (first term), or a_i belongs to A' and a_j belongs to A'' (second term), or a_i and a_j belong to A'' (third term). One then recovers the results of (73).

3.8.2. Realization of PQSym. Although **PQSym** and **PQSym*** are isomorphic as Hopf algebras, no explicit isomorphism is known. We can nevertheless propose a realization of **PQSym** in terms of $(0, 1)$ -matrices instead of words.

This construction is reminiscent of the construction of **MQSym** (see [11, 4]), and coincides with it when restricted to permutation matrices, providing the natural embedding of **FQSym** in **MQSym**.

Let \mathcal{M}_n be the vector space spanned by symbols X_M where M runs over $(0, 1)$ -matrices with n columns and an infinite number of rows, with n nonzero entries, so that at most n rows are nonzero.

Given such a matrix M , we define its *vertical packing* $\text{vp}(M)$ as the finite matrix obtained by removing the null rows of M .

For a vertically packed matrix P , we define

$$(109) \quad \mathbf{M}_P = \sum_{\text{vp}(M)=P} X_M.$$

Now, given a $(0, 1)$ -matrix, we define its *reading* $r(M)$ as the word obtained by reading its entries row-wise, from left to right and from top to bottom and recording the numbers of the columns of the ones. For example, the reading of the matrix

$$(110) \quad \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

is $(2, 3, 1, 2)$.

A matrix M is said to be of *parking type* if $r(M)$ is a parking function. Finally, for a parking function \mathbf{a} , we set

$$(111) \quad \mathbf{F}_{\mathbf{a}} := \sum_{r(P)=\mathbf{a}, P \text{ vertically packed}} \mathbf{M}_P = \sum_{r(M)=\mathbf{a}} X_M.$$

For example,

$$(112) \quad \mathbf{F}_{(1,2,2)} = \mathbf{M}_{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}} + \mathbf{M}_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}}.$$

The multiplication on $\mathcal{M} = \bigoplus_n \mathcal{M}_n$ is defined by columnwise concatenation of matrices:

$$(113) \quad X_M X_N = X_{M \cdot N}.$$

In order to describe explicitly the product of \mathbf{M}_P and \mathbf{M}_Q , we first need a definition. Let P and Q be two vertically packed matrices with respective heights p and q . The *augmented shuffle* of P and Q is defined as follows: let r be an integer in $[\max(p, q), p + q]$. One inserts zero rows in P and Q in all possible ways so that the resulting matrices have $p + q$ rows. Let R be the matrix obtained by concatenation of such pairs of matrices. The augmented shuffle is the set of such matrices R with nonzero rows. We denote this set by $\uplus(P, Q)$.

THEOREM 3.13. *The following formulas hold:*

$$(114) \quad \mathbf{M}_P \mathbf{M}_Q = \sum_{R \in \uplus(P, Q)} \mathbf{M}_R,$$

$$(115) \quad \mathbf{F}_{\mathbf{a}'} \mathbf{F}_{\mathbf{a}''} = \sum_{\mathbf{a} \in \mathbf{a}' \uplus \mathbf{a}''} \mathbf{F}_{\mathbf{a}},$$

This is the same as equation (47).

Proof. Formula (114) comes from the definition of the augmented shuffle of matrices: any matrix in $\uplus(P, Q)$ appears as a product $X_M \cdot X_N$ where $\text{vp}(M) = P$ and $\text{vp}(N) = Q$. Conversely, any element in $\mathbf{M}_P \mathbf{M}_Q$ has as vertical packing a matrix whose number of rows is in the interval $[\max(p, q), p + q]$ and whose left part has M as vertical packing and whose right part has N as vertical packing.

The proof of (115) is almost the same as the previous one if one starts from the definition $\mathbf{F}_a = \sum_{r(M)=a} X_M$. ■

Finally, concerning the coproduct, one has first to define the parkization $\text{Park}(M)$ of a vertically packed matrix M , which consists in iteratively removing column $d(r(M))$ until M becomes a parking matrix.

The coproduct of a matrix \mathbf{M}_P is then defined as

$$(116) \quad \Delta \mathbf{M}_P = \sum_{Q \cdot R = P} \mathbf{M}_{\text{Park}(Q)} \otimes \mathbf{M}_{\text{Park}(R)}.$$

It is then easy to check that

PROPOSITION 3.14. *The following formula holds:*

$$(117) \quad \Delta \mathbf{F}_a = \sum_{u \cdot v = a} \mathbf{F}_{\text{Park}(u)} \otimes \mathbf{F}_{\text{Park}(v)}.$$

This is the same as equation (52).

3.8.3. Realization of \mathbf{FQSym} . A parking matrix M is said to be a *word matrix* if there is exactly one 1 in each column. Then \mathbf{FQSym} is the Hopf subalgebra generated by the parking word matrices.

3.9. \mathbf{PQSym}^* as a dendriform trialgebra. Since we already know that $\mathbb{K}\langle A \rangle^+$ is a dendriform trialgebra (see Definition 2.1 and Lemma 2.2), and since \mathbf{PQSym}^* can be realized on words, it is a natural question to ask whether \mathbf{PQSym}^* is a sub-trialgebra of $\mathbb{K}\langle A \rangle^+$.

THEOREM 3.15. *\mathbf{PQSym}^* is a sub-dendriform trialgebra of $\mathbb{K}\langle A \rangle^+$ with the following product rules:*

$$(118) \quad \mathbf{G}_{a'} \prec \mathbf{G}_{a''} = \sum_{a = u \cdot v \in a' * p a'', |u| = |a'|; \max(v) < \max(u)} \mathbf{G}_a,$$

$$(119) \quad \mathbf{G}_{a'} \circ \mathbf{G}_{a''} = \sum_{a = u \cdot v \in a' * p a'', |u| = |a'|; \max(v) = \max(u)} \mathbf{G}_a,$$

$$(120) \quad \mathbf{G}_{a'} \succ \mathbf{G}_{a''} = \sum_{a = u \cdot v \in a' * p a'', |u| = |a'|; \max(v) > \max(u)} \mathbf{G}_a.$$

Proof. Since \mathbf{PQSym}^* can be realized on words, one only needs to check that \mathbf{PQSym}^* is stable under all three operations, their compatibility coming from the fact that $\mathbb{K}\langle A \rangle^+$ is a dendriform trialgebra. Since all words

having a given parkized word have the same inversions, they have in particular the same relations between the maximum of any prefix and any suffix of given lengths. One then derives the product rules from direct calculation. ■

For example, with the notations of Definition 2.1:

$$(121) \quad \mathbf{G}_{12} \prec \mathbf{G}_{212} = \mathbf{G}_{13212} + \mathbf{G}_{14212} + \mathbf{G}_{14313} + \mathbf{G}_{14323} + \mathbf{G}_{15212} \\ + \mathbf{G}_{15313} + \mathbf{G}_{15323} + \mathbf{G}_{24313} + \mathbf{G}_{24212} + \mathbf{G}_{34212} \\ + \mathbf{G}_{23212} + \mathbf{G}_{25212} + \mathbf{G}_{25313} + \mathbf{G}_{35212} + \mathbf{G}_{45212},$$

$$(122) \quad \mathbf{G}_{12} \circ \mathbf{G}_{212} = \mathbf{G}_{12212} + \mathbf{G}_{13313} + \mathbf{G}_{13323} + \mathbf{G}_{23313},$$

$$(123) \quad \mathbf{G}_{12} \succ \mathbf{G}_{212} = \mathbf{G}_{12313} + \mathbf{G}_{12323} + \mathbf{G}_{12414} + \mathbf{G}_{12424} \\ + \mathbf{G}_{12434} + \mathbf{G}_{13414} + \mathbf{G}_{13424} + \mathbf{G}_{23414}.$$

Based on numerical evidence, we conjecture the following result:

CONJECTURE 3.16. **PQSym*** is a free dendriform trialgebra.

Recall that the generating series $F(t)$ for the dimensions of the free dendriform trialgebra satisfies

$$(124) \quad F(t) - 1 = t(2F(t)^2 - F(t)).$$

Applying the same trick as in [6] for computing the generating series of the totally primitive elements, one gets the generating series of the number g_n of generators in degree n of **PQSym*** as a free dendriform trialgebra:

$$(125) \quad \sum_{n \geq 0} g_n t^n = \frac{PF(t) - 1}{2PF(t)^2 - PF(t)} \\ = t + 5t^3 + 50t^4 + 634t^5 + 9475t^6 + 163843t^7 \\ + 3226213t^8 + 71430404t^9 + O(t^{10}).$$

By self-duality of **PQSym**, one can endow **PQSym** with a structure of dendriform trialgebra.

Note that **FQSym** is not a sub-dendriform trialgebra of $\mathbb{K}\langle A \rangle^+$ since the product is not internal and that, independently of the realization, it cannot be a free dendriform trialgebra since the substitution $F(t) = \sum_n n!t^n$ in (125) does not yield a series with nonnegative integer coefficients.

3.10. The internal product. We shall now recall the definition of the internal product of **PQSym**, introduced in [28]. We first need a few standard notations about biwords. Let $x_{ij} = \binom{i}{j}$ be commuting indeterminates, and $a_{ij} = [j_i]$ be noncommuting ones. We shall denote by $\binom{i_1 \dots i_r}{j_1 \dots j_r}$ the monomial $\binom{i_1}{j_1} \dots \binom{i_r}{j_r}$ and by $[j_1 \dots j_r]_{i_1 \dots i_r}$ the word $[j_1^{i_1}] \dots [j_r^{i_r}]$. Such expressions will be referred to respectively as *bimonomials* and *biwords*.

Recall that Gessel constructed the descent algebra by extending to $QSym$ the coproduct dual to the internal product of symmetric functions. That is, if X and Y are two totally and isomorphically ordered alphabets of commuting variables, we can identify a tensor product $f \otimes g$ of quasi-symmetric functions with $f(X)g(Y)$. Denoting by XY the Cartesian product $X \times Y$ endowed with the lexicographic order, Gessel defined, for $f \in QSym_n$,

$$(126) \quad \delta(f) = f(XY) \in QSym_n \otimes QSym_n.$$

The dual operation on \mathbf{Sym}_n is the internal product $*$, for which it is anti-isomorphic to the descent algebra Σ_n . This construction can be extended to \mathbf{PQSym}^* . Let A' and A'' be two totally and isomorphically ordered alphabets of noncommuting variables, but such that A' and A'' commute with each other. We denote by $A'A''$ the Cartesian product $A' \times A''$ endowed with the lexicographic order. This is a total order in which each element has a successor, so that $G_{\mathbf{a}}(A'A'')$ is a well defined polynomial. Identifying tensor products of words of the same length with words over $A'A''$, we have

$$(127) \quad \mathbf{G}_{\mathbf{a}}(A'A'') = \sum_{\text{Park}(u \otimes v) = \mathbf{a}} u \otimes v.$$

For example, writing tensor products as biwords, one has

$$(128) \quad \mathbf{G}_{4121}(A'A'') = \sum_{a,b,c,d} \begin{bmatrix} b & a & a & a \\ d & c & c+1 & c \end{bmatrix}$$

with $b > a$, or $b = a$ and $d \geq c + 3$.

THEOREM 3.17 ([28]). *The formula $\delta(\mathbf{G}_{\mathbf{a}}) = \mathbf{G}_{\mathbf{a}}(A'A'')$ defines a coassociative coproduct on each homogeneous component \mathbf{PQSym}_n^* . Actually,*

$$(129) \quad \delta(\mathbf{G}_{\mathbf{a}}) = \sum_{\text{Park}(\mathbf{a}' \otimes \mathbf{a}'') = \mathbf{a}} \mathbf{G}_{\mathbf{a}'} \otimes \mathbf{G}_{\mathbf{a}''},$$

where \mathbf{a}' and \mathbf{a}'' are parking functions. By duality, the formula

$$(130) \quad \mathbf{F}_{\mathbf{a}'} * \mathbf{F}_{\mathbf{a}''} = \mathbf{F}_{\text{Park}(\mathbf{a}' \otimes \mathbf{a}'')}$$

defines an associative product on each \mathbf{PQSym}_n .

Since A is infinite, δ is compatible with the product of \mathbf{PQSym}^* .

EXAMPLE 3.18.

$$(131) \quad \begin{aligned} & \delta \mathbf{G}_{4121} \\ &= (\mathbf{G}_{2111} + \mathbf{G}_{3111} + \mathbf{G}_{4111}) \otimes (\mathbf{G}_{1232} + \mathbf{G}_{1121} + \mathbf{G}_{2121} + \mathbf{G}_{3121} + \mathbf{G}_{4121}) \\ & \quad + \mathbf{G}_{1111} \otimes \mathbf{G}_{4121}. \end{aligned}$$

EXAMPLE 3.19.

$$(132) \quad \mathbf{F}_{211} * \mathbf{F}_{211} = \mathbf{F}_{311}, \quad \mathbf{F}_{211} * \mathbf{F}_{112} = \mathbf{F}_{312},$$

$$(133) \quad \mathbf{F}_{211} * \mathbf{F}_{121} = \mathbf{F}_{321}, \quad \mathbf{F}_{112} * \mathbf{F}_{312} = \mathbf{F}_{213},$$

$$(134) \quad \mathbf{F}_{31143231} * \mathbf{F}_{23571713} = \mathbf{F}_{61385451}.$$

Note that although parking functions can be interpreted as chambers of the Shi arrangement, our internal product is not induced by the face semigroup of this arrangement. Indeed, one should obtain in particular an idempotent semigroup, which is clearly not the case.

The main tool for handling internal products of noncommutative symmetric functions is the splitting formula (see [7, Proposition 5.2]). It does not hold in **PQSym**, but one can find subalgebras of **PQSym** larger than **Sym** in which it remains true.

3.11. Subalgebras of PQSym. In the following sections, we present different subalgebras of **PQSym**, summarized in Figure 2 with the names of their natural bases:

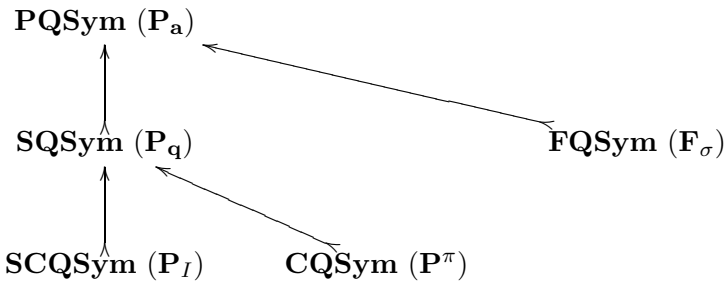


Fig. 2. Some subalgebras of **PQSym**

Here **a** is a parking function, σ is a permutation, **q** is a parking quasi-ribbon, *I* is a composition, and π is a nondecreasing parking function.

4. THE SCHRÖDER QUASI-SYMMETRIC HOPF ALGEBRA SQSym

In Section 2.4, we recalled that the little Schröder numbers build up the Hilbert series of the free dendriform trialgebra on one generator, \mathfrak{SD} . We show in [29] that \mathfrak{SD} realized on words has a natural structure of bidendriform bialgebra. In particular, this proves that there is a natural self-dual Hopf structure on \mathfrak{SD} .

But parking functions provide another way to find little Schröder numbers. Indeed, the number of classes of parking functions of length *n* under the hypoplactic congruence is also equal to *s_n*. This construction leads to a non-self-dual Hopf algebra, denoted by **SQSym**.

4.1. Hypoplactic classes of parking functions. Let \equiv denote the hypoplactic congruence (see [17, 26]). Recall that the equivalence classes of words under this congruence are parametrized by quasi-ribbon tableaux. A *quasi-ribbon tableau* of shape I is a ribbon diagram r of shape I filled by letters in such a way that each row of r is nondecreasing from left to right, and each column of r is strictly increasing from top to bottom. A word is said to be a *quasi-ribbon word* of shape I if it can be obtained by reading from bottom to top and from left to right the columns of a quasi-ribbon diagram of shape I . For example, the word 11425477 is a quasi-ribbon word since it is the reading of the following quasi-ribbon:

$$(135) \quad \begin{array}{cccc} \boxed{1} & \boxed{1} & \boxed{2} & \\ & \boxed{4} & \boxed{4} & \\ & & \boxed{5} & \boxed{7} & \boxed{7} \end{array}$$

The hypoplactic classes of parking functions correspond to *parking quasi-ribbons*, that is, quasi-ribbon words that are parking functions. We denote this set by PQR, and PQR_n is the set of quasi-ribbon parking functions of length n .

We will make use of a simple parametrization of the elements of PQR: define a *segmented word* as a finite sequence of nonempty words, separated by vertical bars, e.g., $232 | 14 | 5 | 746$.

The parking quasi-ribbons can be represented as segmented nondecreasing parking functions where the bars only occur at positions $\cdots a | b \cdots$, with $a < b$. For example, the quasi-ribbon of equation (135) is represented by the word $112 | 44 | 577$.

Clearly, a nondecreasing word containing exactly l different letters admits 2^{l-1} segmentations.

On the other hand, the statistic l (the length of the packed evaluation vector) on nondecreasing parking functions has the same distribution as the number of blocks in noncrossing partitions through the natural bijection. This is given by a classical q -Catalan number, $c_n(q)$ (see, e.g., [25]) and finally, the number of canonical packed words of length n is $c_n(2)$, which is known to be equal to the Schröder number s_n .

For example, $c_1(q) = 1$, $c_2(q) = 1 + q$ and $c_3(q) = 1 + 3q + q^2$, so that $c_1(2) = 1$, $c_2(2) = 3$ and $c_3(2) = 11$ as one can check from (136) and (137). The coefficients of $c_n(q)$ are known as the *Narayana numbers* (sequence A001263 of Sloane's database [34]).

Here is the list of canonical hypoplactic parking functions for $n \geq 3$:

$$(136) \quad \{1\}, \quad \{11, 12, 1 | 2\},$$

$$(137) \quad \{111, 112, 11 | 2, 113, 11 | 3, 122, 1 | 22, 123, 1 | 23, 12 | 3, 1 | 2 | 3\}.$$

In the following, we will identify parking quasi-ribbons and their encodings as segmented words.

4.2. The Schröder quasi-symmetric Hopf algebra \mathbf{SQSym} . Let us denote by $P(w)$ the hypoplactic P -symbol of a word w (its quasi-ribbon). The P -symbols of parking functions are therefore parking quasi-ribbons. With a parking quasi-ribbon \mathbf{q} , we associate the elements

$$(138) \quad \mathbf{P}_{\mathbf{q}} := \sum_{P(\mathbf{a})=\mathbf{q}} \mathbf{F}_{\mathbf{a}} \quad \text{and} \quad \mathbf{Q}_{\mathbf{q}} := \overline{\mathbf{G}}_{\mathbf{a}},$$

where \overline{w} denotes the hypoplactic class of w . For example,

$$(139) \quad \mathbf{P}_{11|3} = \mathbf{F}_{131} + \mathbf{F}_{311}, \quad \mathbf{P}_{113} = \mathbf{F}_{113},$$

$$(140) \quad \mathbf{Q}_{11|3} = \overline{\mathbf{G}}_{131} = \overline{\mathbf{G}}_{311}, \quad \mathbf{Q}_{113} = \overline{\mathbf{G}}_{113},$$

$$(141) \quad \mathbf{Q}_{12|34} = \overline{\mathbf{G}}_{1324} = \overline{\mathbf{G}}_{3124} = \overline{\mathbf{G}}_{1342} = \overline{\mathbf{G}}_{3142} = \overline{\mathbf{G}}_{3412}.$$

THEOREM 4.1. *The $\mathbf{P}_{\mathbf{q}}$ form a basis of a Hopf subalgebra of \mathbf{PQSym} , denoted by \mathbf{SQSym} . Its dual \mathbf{SQSym}^* is the quotient $\mathbf{PQSym}^*/\mathcal{J}$ where \mathcal{J} is the two-sided ideal generated by*

$$(142) \quad \{\mathbf{G}_{\mathbf{a}} - \mathbf{G}_{\mathbf{a}'} \mid \mathbf{a} \equiv \mathbf{a}'\}.$$

Moreover, $\mathbf{G}_{\mathbf{a}} \equiv \mathbf{G}_{\mathbf{a}'}$ iff $\mathbf{a} \equiv \mathbf{a}'$, so that $\mathbf{SQSym}^* \simeq \mathbf{PQSym}^*/\equiv$. The dual basis of $(\mathbf{P}_{\mathbf{q}})$ is then $(\mathbf{Q}_{\mathbf{q}})$.

The dimension of the component of degree n of \mathbf{SQSym} and \mathbf{SQSym}^* is the little Schröder number (or super-Catalan number) s_n .

Proof. Let us begin with the elements $\overline{\mathbf{G}}_{\mathbf{a}}$. Since the hypoplactic equivalence is a congruence:

$$(143) \quad u \equiv u' \text{ and } v \equiv v' \Rightarrow uv \equiv u'v',$$

these elements build up an algebra that we will denote by \mathbf{SQSym}^* . Since the hypoplactic congruence is compatible with the restriction to intervals, one easily checks that the coproduct of $\mathbf{G}_{\mathbf{a}}$ is compatible with the hypoplactic congruence, so that \mathbf{SQSym}^* is a Hopf algebra.

Recall that two words u and v are hypoplactically equivalent iff they have the same evaluation and $\text{Std}(u)$ and $\text{Std}(v)$ are hypoplactically equivalent. Since two words of the same evaluation have parkized words of the same evaluation as well, the same result applies if one replaces standardization by parkization: two words u and v of the same evaluation are hypoplactically equivalent iff their parkized words are. This proves that

$$(144) \quad \mathbf{a} \equiv \mathbf{a}' \Leftrightarrow \overline{\mathbf{G}}_{\mathbf{a}} = \overline{\mathbf{G}}_{\mathbf{a}'}$$

So \mathbf{SQSym}^* is isomorphic to \mathbf{PQSym}^*/\equiv as a Hopf algebra.

Since the dual basis of $\mathbf{G}_{\mathbf{a}}$ in \mathbf{PQSym} is $\mathbf{F}_{\mathbf{a}}$, one can write the duality bracket as

$$(145) \quad \langle \mathbf{G}_{\mathbf{a}}, \mathbf{F}_{\mathbf{a}'} \rangle = \delta_{\mathbf{a},\mathbf{a}'},$$

where δ is the Kronecker symbol. Then the dual basis of $\mathbf{Q}_q = \overline{\mathbf{G}}_{\mathbf{a}}$ inherited from the dual Hopf algebras $\mathbf{PQSym}, \mathbf{PQSym}^*$ is naturally $\sum_{\mathbf{a}' \equiv \mathbf{a}} \mathbf{F}_{\mathbf{a}'}$, that is, \mathbf{P}_q . It then follows without proof that the \mathbf{P}_q form a basis of a Hopf subalgebra of \mathbf{PQSym} we will denote by \mathbf{SQSym} , as it is the dual of \mathbf{SQSym}^* .

The dimensions are given by the little Schröder numbers since these numbers count the hypoplactic classes of parking functions. ■

THEOREM 4.2. *The product and coproduct rules for the \mathbf{Q}_q and \mathbf{P}_q are:*

$$(146) \quad \mathbf{Q}_{q'} \mathbf{Q}_{q''} = \sum_{\mathbf{a} \in \mathbf{a}' * \mathbf{a}''} \mathbf{Q}_{\mathbf{a}},$$

where \mathbf{a}' (resp. \mathbf{a}'') is in the hypoplactic class of q' (resp. q'');

$$(147) \quad \Delta \mathbf{Q}_q = \sum_{\substack{u, v; \mathbf{q} = u|v[|u|] \text{ or } \mathbf{q} = u \cdot v[|u|]}} \mathbf{G}_u \otimes \mathbf{G}_v,$$

$$(148) \quad \mathbf{P}_{q'} \mathbf{P}_{q''} = \mathbf{P}_{q'|r''} + \mathbf{P}_{q'r''},$$

where $r'' = q''[|q'|]$; and

$$(149) \quad \Delta \mathbf{P}_q = \sum_{q', q''} \mathbf{P}_{q'} \otimes \mathbf{P}_{q''},$$

where the sum is taken over the hypoplactic classes q' and q'' such that their canonical elements c' and c'' can be obtained as parkized words of the prefix and the suffix of an element of the hypoplactic class q .

Proof. The formulas for the product and coproduct of the \mathbf{Q} come from the formulas of the \mathbf{G} in \mathbf{PQSym}^* . The formulas for the \mathbf{P} are then easily derived from the previous ones by duality. ■

For example,

$$(150) \quad \mathbf{Q}_{1|2} \mathbf{Q}_1 = \mathbf{Q}_{1|23} + \mathbf{Q}_{1|22} + \mathbf{Q}_{12|3} + \mathbf{Q}_{11|3} + \mathbf{Q}_{11|2} + \mathbf{Q}_{1|2|3},$$

$$(151) \quad \Delta \mathbf{Q}_{11|34|55} = 1 \otimes \mathbf{Q}_{11|34|55} + \mathbf{Q}_{11} \otimes \mathbf{Q}_{12|33} + \mathbf{Q}_{11|3} \otimes \mathbf{Q}_{1|22} \\ + \mathbf{Q}_{11|34} \otimes \mathbf{Q}_{11} + \mathbf{Q}_{11|34|55} \otimes 1,$$

$$(152) \quad \mathbf{P}_{11|335|6} \mathbf{P}_{112} = \mathbf{P}_{11|335|6778} + \mathbf{P}_{11|335|6|778},$$

$$(153) \quad \Delta \mathbf{P}_{11|3} = 1 \otimes \mathbf{P}_{11|3} + \mathbf{P}_1 \otimes (\mathbf{P}_{1|2} + \mathbf{P}_{11}) + (\mathbf{P}_{21} + \mathbf{P}_{12}) \otimes \mathbf{P}_1 + \mathbf{P}_{11|3} \otimes 1.$$

4.3. \mathbf{SQSym} is not self-dual. Some simple computations prove that \mathbf{SQSym} and \mathbf{SQSym}^* are not isomorphic Hopf algebras since the primitive Lie algebra of \mathbf{SQSym}^* is of dimension 6 in degree 3, spanned by

$$(154) \quad \mathbf{Q}_{111}, \mathbf{Q}_{112}, \mathbf{Q}_{11|2}, \mathbf{Q}_{122} - \mathbf{Q}_{1|22}, \mathbf{Q}_{113} - \mathbf{Q}_{11|3}, \\ \mathbf{Q}_{123} - \mathbf{Q}_{1|23} - \mathbf{Q}_{12|3} + \mathbf{Q}_{1|2|3},$$

whereas it is of dimension 7 in \mathbf{SQSym} , spanned by

$$\begin{aligned}
 (155) \quad & \mathbf{P}_{123} - \mathbf{P}_{112} - \mathbf{P}_{122} + \mathbf{P}_{111}, & \mathbf{P}_{1|22} - \mathbf{P}_{11|2} - \mathbf{P}_{112} + \mathbf{P}_{122}, \\
 & \mathbf{P}_{1|23} - \mathbf{P}_{11|2} - \mathbf{P}_{112} + \mathbf{P}_{111}, & \mathbf{P}_{12|3} - \mathbf{P}_{11|2} - \mathbf{P}_{112} + \mathbf{P}_{111}, \\
 & \mathbf{P}_{113} - \mathbf{P}_{112}, & \mathbf{P}_{11|3} - \mathbf{P}_{11|2}, & \mathbf{P}_{1|2|3} - \mathbf{P}_{11|2} + \mathbf{P}_{1|22}.
 \end{aligned}$$

In particular, it is impossible to endow **SQSym** or **SQSym**^{*} with a bidendriform bialgebra structure since both would then be self-dual. We cannot use the machinery of Foissy to investigate the freeness of both algebras and their primitive Lie algebras, but we can do it by hand.

4.4. Algebraic structure of SQSym^{*} and SQSym. Since we know that the primitive Lie algebra of **SQSym** is of dimension seven in degree 3, **SQSym**^{*} cannot be free and, indeed, one finds the relation

$$(156) \quad \mathbf{Q}_1(\mathbf{Q}_{11} + \mathbf{Q}_{12}) = (\mathbf{Q}_{11} + \mathbf{Q}_{12})\mathbf{Q}_1.$$

We now move to **SQSym**. Consider the set PQS of parking quasi-ribbons that cannot be obtained as a nontrivial shifted concatenation of parking quasi-ribbons. They are the parking quasi-ribbons having a bar whenever the underlying nondecreasing parking function has a breakpoint. For example, here are the elements of PQS_n for n ≤ 4.

$$\begin{aligned}
 (157) \quad & \{1\}, \quad \{11, 1|2\}, \quad \{111, 112, 11|2, 11|3, 1|22, 1|2|3\}, \\
 & \{1111, 1112, 111|2, 1113, 111|3, 111|4, 1122, 11|22, \\
 & 1123, 11|23, 112|3, 11|2|3, 112|4, 11|2|4, 11|33, \\
 & 11|3|4, 1|222, 1|223, 1|22|3, 1|22|4, 1|2|33, 1|2|3|4\}.
 \end{aligned}$$

Since the elements of PQS are those that never occur in a nontrivial shifted concatenation of elements of PQR, any element **q** of PQR decomposes uniquely as a shifted product **q**₁ • • • • **q**_k where all the **q**_k are in PQS. Define then

$$(158) \quad \mathbf{P}^{\mathbf{q}} = \mathbf{P}_{\mathbf{q}_1} \cdots \mathbf{P}_{\mathbf{q}_k}.$$

PROPOSITION 4.3. *The **P**^{**q**} form a multiplicative basis of **SQSym**. In particular, **SQSym** is free as an algebra.*

Proof. The **P**^{**q**} generate the same algebra as the **P**_{**q**} since they are triangular over the **P**_{**q**}: each term **P**^{**q**} begins with **P**_{**q**} followed by elements of PQR that are shifted concatenations of strictly lower elements of PQS. ■

Since **SQSym** is free, one can compute the generating series of its generating set. Recall that the generating series of s_n is S(t) := $\frac{1+t-\sqrt{1-6t+t^2}}{4t}$, so that

$$(159) \quad U(t) := 1 - 1/S(t) = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2},$$

that is, the generating series of large Schröder numbers s'_n (A006318 in [34]), obviously equal to 2s_n thanks to the previous formula. So

PROPOSITION 4.4. *The sets PQS_n are enumerated by the large Schröder numbers.*

Proof. Even if the algebraic construction has already proved this result, we provide a bijective proof in order to enlighten the relation between the large and the little Schröder numbers from the point of view of parking functions.

We split PQS_n in two and provide a bijection between both sets and PQR_{n-1} , the set of parking quasi-ribbons of length $n - 1$.

Let PQS'_n be the subset of PQS_n consisting of the elements whose underlying parking function is prime. The bijection between PQS'_n and PQR_{n-1} is trivial: it consists in adding or removing 1 at the beginning of the parking function.

Let PQS''_n be the complementary subset of PQS_n . The bijection is the following: start from an element of PQR_{n-1} . If it belongs to PQS_{n-1} , then add a bar and n to its end. Otherwise, let i be the smallest integer greater than 1 such that $i - 1$ is a breakpoint and there is no bar before the first i . Then insert a bar and an i before the first i . This element satisfies the requirements of PQS_n since it can have breakpoints only to the left of i and, by hypothesis, all those breakpoints are followed by a bar. Moreover, this element has a breakpoint, so belongs to PQS''_n . For example, the image of $11|2|455|669$ is $11|2|4|55|669$, since there is a breakpoint at 4 with no bar before the first 5.

The reverse bijection consists in considering the rightmost breakpoint i of the underlying parking function of an element of PQS''_n and remove $i + 1$ with the bar before it. The result belongs to PQR_{n-1} since we removed the letter just after the *rightmost* breakpoint.

Finally, it is a bijection between PQS''_n and PQR_{n-1} since the operations are inverse to each other and the image of each set is included in the other. ■

The next proposition summarizes the structures of the algebras \mathbf{SQSym} and \mathbf{SQSym}^* .

PROPOSITION 4.5. *The algebra \mathbf{SQSym} is a Hopf algebra of dimension s_n in degree n . It is not self-dual since \mathbf{SQSym} is free as an algebra whereas \mathbf{SQSym}^* is not.*

4.5. \mathbf{SQSym}^* as a combinatorial Hopf algebra. The embedding of (93) induces an embedding

$$(160) \quad \mathcal{QSym} \simeq \mathbf{FQSym}^*/(\mathcal{J} \cap \mathbf{FQSym}^*) \rightarrow \mathbf{PQSym}^*/\mathcal{J} = \mathbf{SQSym}^*.$$

In particular, we see that \mathbf{SQSym}^* contains a large commutative subalgebra.

4.6. Primitive Lie algebras of SQSym and SQSym*. As SQSym* contains QSym as a subalgebra, its primitive Lie algebra cannot be free and one easily finds

$$(161) \quad [\mathbf{Q}_1, \mathbf{Q}_{12} - \mathbf{Q}_{1|2} + \mathbf{Q}_{11}] = 0.$$

The first dimensions for the primitive Lie algebra of SQSym are 1, 2, 7, 25, 102, with no relations between those elements in those degrees so that one can conjecture that it is free as a Lie algebra.

4.7. Schröder ribbons. In the algebra Sym, the products of noncommutative complete functions split into sums of ribbon Schur functions, using a simple order on compositions. To get an analogous construction in our case, we define a partial order on segmented nondecreasing parking functions.

Let π be a segmented nondecreasing parking function and $\text{Ev}(\pi)$ be its segmented evaluation vector, that is, its evaluation vector with separators between the i th and $(i + 1)$ th element if i and $i + 1$ are separated by a bar in π . The successors of π are the segmented nondecreasing parking functions whose evaluations are given by the following algorithm: given two nonzero elements of $\text{Ev}(\pi)$ not separated by a bar with only zeros between them, replace the left one by the sum of both and the right one by 0.

For example, the successors of 11|3346 are 11|3336 and 11|3344.

By transitive closure, the successor map gives rise to a partial order \succeq on segmented nondecreasing parking functions.

Now, define the Schröder ribbons by

$$(162) \quad \mathbf{P}_q =: \sum_{q' \succeq q} \mathbf{R}_{q'},$$

or, by Möbius inversion on the boolean lattice,

$$(163) \quad \mathbf{R}_q := \sum_{q' \succeq q} (-1)^{f(q, q')} \mathbf{P}_{q'},$$

where $f(u, v)$ is the difference between the numbers of different letters in u and in v .

For example,

$$(164) \quad \mathbf{P}_{11|34} = \mathbf{R}_{11|34} + \mathbf{R}_{11|33},$$

$$(165) \quad \mathbf{P}_{11|3346} = \mathbf{R}_{11|3346} + \mathbf{R}_{11|3336} + \mathbf{R}_{11|3344} + \mathbf{R}_{11|3333},$$

$$(166) \quad \mathbf{R}_{11|3346} = \mathbf{P}_{11|3346} - \mathbf{P}_{11|3336} - \mathbf{P}_{11|3344} + \mathbf{P}_{11|3333}.$$

PROPOSITION 4.6. *The product of two ribbons is given by*

$$(167) \quad \mathbf{R}_{q'} \mathbf{R}_{q''} = \mathbf{R}_{q'|r''} + \mathbf{R}_{q'r''} + \mathbf{R}_{q' \triangleright r''}$$

where $r'' = q''[|q'|]$ and $q' \triangleright r''$ is the successor of $q'r''$ obtained by decreasing the smallest letters of r'' down to the value of the greatest letters of q' .

Proof. Let p be the length of \mathbf{q}' . Let us expand $\mathbf{R}_{\mathbf{q}'}\mathbf{R}_{\mathbf{q}''}$ in the \mathbf{P} basis. One gets an alternating sum of \mathbf{P} indexed by the successors of $\mathbf{q}'\mathbf{r}''$ having different p th and $(p+1)$ st letters or indexed by the successors of $\mathbf{q}'|\mathbf{r}'$. The second set obviously sums up to $\mathbf{R}_{\mathbf{q}'|\mathbf{r}''}$. The first set is part of all successors of $\mathbf{q}'\mathbf{r}''$, the missing set being all successors of $\mathbf{q}'\triangleright\mathbf{r}''$.

As the sign of an element depends only on its number of different letters, the result follows. ■

For example,

$$(168) \quad \mathbf{R}_1\mathbf{R}_{1|2} = \mathbf{R}_{1|2|3} + \mathbf{R}_{12|3} + \mathbf{R}_{11|3},$$

$$(169) \quad \mathbf{R}_{11|3}\mathbf{R}_{113} = \mathbf{R}_{11|3|446} + \mathbf{R}_{11|3446} + \mathbf{R}_{11|3336}.$$

4.8. Dendriform structures on \mathbf{SQSym} . Let us now consider the other structures that can be put on \mathbf{SQSym} and \mathbf{SQSym}^* . First note that the product rules of \mathbf{PQSym}^* as a tridendriform algebra are compatible with the hypoplactic congruence, so that \mathbf{SQSym}^* is a tridendriform algebra. But it is not free since

$$(170) \quad \mathbf{Q}_{11|2} = \overline{\mathbf{G}}_{212} = \overline{\mathbf{G}}_{221},$$

which can be rewritten as

$$(171) \quad (\mathbf{Q}_1 \succ \mathbf{Q}_1) \circ \mathbf{Q}_1 = \mathbf{Q}_1 \circ (\mathbf{Q}_1 \succ \mathbf{Q}_1),$$

a relation that is not a consequence of the tridendriform relations.

We already mentioned that \mathbf{SQSym}^* cannot have a bidendriform bialgebra structure since it would imply that \mathbf{SQSym}^* is self-dual. In our realization of the bidendriform bialgebra \mathbf{PQSym} , the explanation comes from the fact that the hypoplactic congruence is not compatible with the codendriform definitions since, for example,

$$(172) \quad \Delta_{\ll}\mathbf{G}_{221} = \mathbf{G}_1 \otimes \mathbf{G}_{11} \quad \text{whereas} \quad \Delta_{\ll}\mathbf{G}_{212} = 0.$$

5. THE CATALAN QUASI-SYMMETRIC HOPF ALGEBRA \mathbf{CQSym}

5.1. The Hopf algebra \mathbf{CQSym}

5.1.1. Nondecreasing parking functions and noncrossing partitions. As already mentioned, nondecreasing parking functions form a Catalan set. There are dozens of possibilities to identify them to other combinatorial objects. However, parking functions are known to be related to noncrossing partitions (see [3, 36, 37]), and there is a simple bijection between nondecreasing parking functions and noncrossing partitions. Starting with a noncrossing partition, e.g.,

$$(173) \quad \pi = 13|2|45,$$

one replaces all the letters of each block by its minimum, and reorders them as a nondecreasing word:

$$(174) \quad 13|2|45 \rightarrow 11244,$$

which is a parking function. In the following, we identify nondecreasing parking functions and noncrossing partitions via this bijection.

5.1.2. The Catalan Hopf algebra \mathbf{CQSym} . For a general $\mathbf{a} \in \text{PF}_n$, let $\text{NC}(\mathbf{a})$ be the noncrossing partition corresponding to \mathbf{a}^\uparrow by the inverse bijection, e.g., $\text{NC}(42141) = \pi$ as above.

Then define \mathbf{P}^π as the sum of all permutations of the nondecreasing word corresponding to the given noncrossing partition:

$$(175) \quad \mathbf{P}^\pi := \sum_{\mathbf{a}; \text{NC}(\mathbf{a})=\pi} \mathbf{F}_{\mathbf{a}}.$$

THEOREM 5.1. *The elements \mathbf{P}^π , where π runs over noncrossing partitions, span a cocommutative Hopf subalgebra of \mathbf{PQSym} with product and coproduct given by*

$$(176) \quad \mathbf{P}^{\pi'} \mathbf{P}^{\pi''} = \mathbf{P}^{\pi' \bullet \pi''},$$

$$(177) \quad \Delta \mathbf{P}^\pi = \sum_{u,v; (u \cdot v)^\uparrow = \pi} \mathbf{P}^{\text{Park}(u)} \otimes \mathbf{P}^{\text{Park}(v)},$$

where u and v run over the set of nondecreasing words.

Moreover, as an algebra, it is isomorphic to the algebra of the free semi-group of noncrossing partitions under the operation of concatenation of diagrams.

Proof. Equation (176) follows from (47): indeed, any permutation of $\pi' \bullet \pi''$ is uniquely obtained as the shifted shuffle of a permutation of π' with a permutation of π'' . The converse is obvious.

Equation (177) comes from (52): consider the relation $P(p, q)$ on words which are pairs (w, w') of words w and w' of length $p + q$ such that the sorted words of the prefix of length p (resp. suffix of length q) of w and w' are equal. By definition of \mathbf{P}^π , it is a sum of such classes, so that $\Delta \mathbf{P}^\pi$ decomposes as a sum of tensor products of the form $\mathbf{P}^{\pi'} \otimes \mathbf{P}^{\pi''}$. The sum on the right-hand side of (177) is exactly over representatives of the equivalence classes, hence the result. Formula (177) proves that the coalgebra \mathbf{CQSym} is cocommutative.

Moreover, since \mathbf{CQSym} is a subalgebra and a sub-coalgebra of \mathbf{PQSym} , the product and the coproduct of \mathbf{CQSym} are compatible, so that \mathbf{CQSym} is endowed with a graded bialgebra structure, and therefore, with a Hopf algebra structure. ■

This algebra will be called the *Catalan subalgebra* of \mathbf{PQSym} and denoted by \mathbf{CQSym} .

For example, one has

$$(178) \quad \mathbf{P}^{11}\mathbf{P}^{1233} = \mathbf{P}^{113455}, \quad \mathbf{P}^{1124}\mathbf{P}^{1223} = \mathbf{P}^{11245667},$$

$$(179) \quad \begin{aligned} \Delta\mathbf{P}^{1124} = & 1 \otimes \mathbf{P}^{1124} + \mathbf{P}^1 \otimes (\mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{123}) \\ & + \mathbf{P}^{11} \otimes \mathbf{P}^{12} + \mathbf{P}^{12} \otimes (\mathbf{P}^{11} + 2\mathbf{P}^{12}) \\ & + (\mathbf{P}^{112} + \mathbf{P}^{113} + \mathbf{P}^{123}) \otimes \mathbf{P}^1 + \mathbf{P}^{1124} \otimes 1. \end{aligned}$$

Since the nondecreasing parking functions that never occur in a nontrivial shifted concatenation of such elements are the connected nondecreasing parking functions, any π decomposes uniquely as a shifted product $\pi_1 \bullet \cdots \bullet \pi_k$ where all the π_k are connected.

PROPOSITION 5.2. *The \mathbf{P} form a multiplicative basis of \mathbf{CQSym} . In particular, \mathbf{CQSym} is free as an algebra.*

Here are the connected nondecreasing parking functions up to length 4:

$$(180) \quad \{1\}, \quad \{11\}, \quad \{111, 112\}, \quad \{1111, 1112, 1113, 1122, 1123\}.$$

Since \mathbf{CQSym} is free, one can compute the generating series of its generating set. Recall that the generating series of C_n is $C(t) := (1 - \sqrt{1 - 4t})/2t$, so that

$$(181) \quad CN(t) := 1 - 1/C(t) = t \frac{1 - \sqrt{1 - 6t + t^2}}{2},$$

that is, the generating series of shifted Catalan numbers C_{n-1} . Indeed, the connected nondecreasing parking functions are obtained by concatenating a 1 to the left of all nondecreasing parking functions.

5.1.3. Algebraic structure of \mathbf{CQSym} . Following Reutenauer [33, p. 58], denote by π_1 the Eulerian idempotent, i.e., the endomorphism of \mathbf{CQSym} defined by $\pi_1 = \log^*(\text{Id})$ where \log^* means that the logarithm is taken in the convolution algebra $\text{End}^{\text{gr}}(\mathbf{CQSym})$ of graded endomorphisms. It is obvious, thanks to the definition of \mathbf{P}^π , that

$$(182) \quad \pi_1(\mathbf{P}^\pi) = \mathbf{P}^\pi + \cdots,$$

where the dots stand for terms \mathbf{P}^γ where γ is not connected. So the family $\pi_1(\mathbf{P}^\alpha)$ where α runs over all connected nondecreasing parking functions is a free set of primitive generators of \mathbf{CQSym} . In particular, they generate a free Lie algebra (see, e.g., [13] for more details) whose Hilbert series is given by

$$(183) \quad t + t^2 + 3t^3 + 8t^4 + 25t^5 + 75t^6 + 245t^7 + 800t^8 + O(t^9).$$

The sequence is referenced in Sloane's database as A022553 [34]. It counts Lyndon words l of even length $2n$ with an equal number of a and b . So

the free Lie algebra of primitive elements of **CQSym** is isomorphic to the Lie subalgebra \mathfrak{L} of the free Lie algebra $\text{Lie}(a, b)$ consisting of the elements with an equal number of a and b . One can then prove that the standard bracketings of the Lyndon words l with the same number of a and b such that $l = l' \cdot b$ with l' also being a Lyndon word generate a free Lie algebra. Since those particular Lyndon words are enumerated by the shifted Catalan numbers, one can conclude that they generate \mathfrak{L} .

5.2. The dual Hopf algebra \mathbf{CQSym}^* . Let us denote by \mathcal{M}_π the dual basis of \mathbf{P}^π in the commutative algebra \mathbf{CQSym}^* . Since **CQSym** is the subalgebra of **PQSym** obtained by summing all permutations of non-decreasing parking functions, \mathbf{CQSym}^* is the quotient of \mathbf{PQSym}^* by the relations $\mathbf{G}_a \equiv \mathbf{G}_b$ if $\mathbf{a}^\uparrow = \mathbf{b}^\uparrow$.

It is then immediate (see (61)) that the multiplication in this basis is given by

$$(184) \quad \mathcal{M}_{\pi'} \mathcal{M}_{\pi''} = \sum_{\mathbf{a} \in \pi' * \pi''} \mathcal{M}_{\mathbf{a}^\uparrow}.$$

For example,

$$(185) \quad \mathcal{M}_1 \mathcal{M}_{12} = \mathcal{M}_{112} + \mathcal{M}_{113} + \mathcal{M}_{122} + 3\mathcal{M}_{123},$$

$$(186) \quad \mathcal{M}_{12} \mathcal{M}_{11} = \mathcal{M}_{1112} + \mathcal{M}_{1113} + \mathcal{M}_{1114} + \mathcal{M}_{1123} + \mathcal{M}_{1124} \\ + \mathcal{M}_{1134} + \mathcal{M}_{1222} + \mathcal{M}_{1223} + \mathcal{M}_{1224} + \mathcal{M}_{1233}.$$

THEOREM 5.3. *The algebra \mathbf{CQSym}^* is embedded in the polynomial algebra $\mathbb{C}[x_1, x_2, \dots]$ by*

$$(187) \quad \mathcal{M}_\pi = \sum_{\text{Park}(w)=\pi} \underline{w},$$

where \underline{w} is the commutative image of w (i.e., $a_i \mapsto x_i$).

Proof. The result follows from (104) and from the fact that quotienting \mathbf{PQSym}^* by the relations $\mathbf{G}_a \equiv \mathbf{G}_b$ if $\mathbf{a}^\uparrow = \mathbf{b}^\uparrow$ amounts to taking the commutative image of words, transforming these into monomials. ■

For example,

$$(188) \quad \mathcal{M}_{111} = \sum_i x_i^3,$$

$$(189) \quad \mathcal{M}_{112} = \sum_i x_i^2 x_{i+1},$$

$$(190) \quad \mathcal{M}_{113} = \sum_{i,j; j \geq i+2} x_i^2 x_j,$$

$$(191) \quad \mathcal{M}_{122} = \sum_{i,j;i < j} x_i x_j^2,$$

$$(192) \quad \mathcal{M}_{123} = \sum_{i,j,k;i < j < k} x_i x_j x_k.$$

The *packed evaluation vector* $t(w)$ of w is obtained from $\text{Ev}(w)$ by removing all its zeros. For example, for $w = 3117291781329$, $\text{Ev}(w) = (4, 2, 2, 0, 0, 0, 2, 1, 2)$ and $t(w) = (4, 2, 2, 2, 1, 2)$.

We can now see that \mathbf{CQSym}^* contains $QSym$ as a subalgebra. The embedding of $QSym$ into \mathbf{CQSym}^* is given by

$$(193) \quad \gamma(M_I) := \sum_{t(\pi)=I} \mathcal{M}_\pi.$$

For example,

$$(194) \quad \begin{aligned} M_3 &= \mathcal{M}_{111}, & M_{21} &= \mathcal{M}_{112} + \mathcal{M}_{113}, \\ M_{12} &= \mathcal{M}_{122}, & M_{111} &= \mathcal{M}_{123}. \end{aligned}$$

5.3. Catalan ribbons. As already done for the Schröder algebras, we define a partial order on nondecreasing parking functions.

Let π be a nondecreasing parking function and $\text{Ev}(\pi)$ be its evaluation vector. The successors of π are the nondecreasing parking functions whose evaluations are given by the following algorithm: given two nonzero elements of $\text{Ev}(\pi)$ with only zeros between them, replace the left one by the sum of both and the right one by 0.

For example, the successors of 113346 are 111146, 113336, and 113344.

By transitive closure, the successor map gives rise to a partial order on nondecreasing parking functions. We will write $\pi' \succeq \pi$ if π' is obtained from π by successive applications of successor maps.

Now, define the Catalan ribbon functions by

$$(195) \quad \mathbf{P}^\pi =: \sum_{\pi' \succeq \pi} \mathbf{R}_{\pi'}.$$

This last equation completely defines the \mathbf{R}_π .

For example,

$$(196) \quad \begin{aligned} \mathbf{P}^{113346} &= \mathbf{R}_{113346} + \mathbf{R}_{113344} + \mathbf{R}_{113336} + \mathbf{R}_{113333} \\ &\quad + \mathbf{R}_{111146} + \mathbf{R}_{111144} + \mathbf{R}_{111116} + \mathbf{R}_{111111}, \end{aligned}$$

$$(197) \quad \begin{aligned} \mathbf{R}_{113346} &= \mathbf{P}^{113346} - \mathbf{P}^{113344} - \mathbf{P}^{113336} + \mathbf{P}^{113333} \\ &\quad - \mathbf{P}^{111146} + \mathbf{P}^{111144} + \mathbf{P}^{111116} - \mathbf{P}^{111111}. \end{aligned}$$

Note that, by Möbius inversion of the boolean lattice, the coefficient of $\mathbf{P}^{\pi'}$

in \mathbf{R}^π is -1 to the number of different letters in π minus the number of different letters in π' .

This definition is compatible with the definition of commutative ribbon Schur functions since if one considers the morphism

$$(198) \quad \phi : \mathbf{CQSym} \rightarrow \mathit{Sym}, \quad \mathbf{P}^\pi \mapsto S^{c(\pi)},$$

then the image $\phi(\mathbf{R}_\pi)$ is equal to $R_{c(\pi)}$.

PROPOSITION 5.4. *The product of two \mathbf{R} functions is*

$$(199) \quad \mathbf{R}_{\pi'}\mathbf{R}_{\pi''} = \mathbf{R}_{\pi' \bullet \pi''} + \mathbf{R}_{\pi' \triangleright \pi''},$$

where $\pi' \triangleright \pi''$ is the successor of $\pi' \bullet \pi''$ obtained by decreasing the smallest letters of π'' down to the greatest letters of π' .

Proof. Let p be the length of π' . Let us expand $\mathbf{R}_{\pi' \bullet \pi''}$ in the \mathbf{P} basis. One gets the alternating sum of \mathbf{P} indexed by successors of $\pi' \bullet \pi''$. Those successors split into two disjoint subsets: the successors having the p th and $(p + 1)$ th letters equal and the others. The first set corresponds to the successors of $\pi' \triangleright \pi''$ whereas the second set corresponds to the $w' \bullet w''$ where $w' \succeq \pi'$ and $w'' \succeq \pi''$.

As the sign of an element depends only on its number of different letters, the alternating sum of the first set gives $-\mathbf{R}_{\pi' \triangleright \pi''}$, whereas the sum of the second set yields $\mathbf{R}_{\pi'}\mathbf{R}_{\pi''}$. ■

For example,

$$(200) \quad \begin{aligned} \mathbf{R}_{11224}\mathbf{R}_{113} &= \mathbf{R}_{11224668} + \mathbf{R}_{11224448}, \\ \mathbf{R}_{113}\mathbf{R}_{11224} &= \mathbf{R}_{11344557} + \mathbf{R}_{11333557}. \end{aligned}$$

5.4. Internal product. Define the parkized word of a bimonomial as the nondecreasing parking function obtained by parkizing its lexicographically sorted biword. Recall that bimonomials can be encoded as matrices, the entry A_{ij} being the number of bi-letters (ij) in the biword, so that it makes sense to speak of the parkized word of a matrix.

THEOREM 5.5 ([28]). *The homogeneous components \mathbf{CQSym}_n of the Catalan algebra are stable under the internal product $*$. More precisely,*

$$(201) \quad \mathbf{P}^{\pi'} * \mathbf{P}^{\pi''} = \sum_{\pi} \mathbf{P}^{\pi}$$

where π runs over the parkized words of all nonnegative integer matrices with row sum $\text{Ev}(\pi')$ and column sum $\text{Ev}(\pi'')$.

EXAMPLE 5.6.

$$(202) \quad \mathbf{P}^{1123} * \mathbf{P}^{1111} = \mathbf{P}^{1134}, \quad \mathbf{P}^{1111} * \mathbf{P}^{1123} = \mathbf{P}^{1123},$$

$$(203) \quad \begin{aligned} \mathbf{P}^{1123} * \mathbf{P}^{1112} &= 2\mathbf{P}^{1134} + \mathbf{P}^{1234}, \\ \mathbf{P}^{1122} * \mathbf{P}^{1224} &= \mathbf{P}^{1134} + \mathbf{P}^{1233} + 2\mathbf{P}^{1234}, \end{aligned}$$

$$(204) \quad \mathbf{P}^{1123} * \mathbf{P}^{1224} = 2\mathbf{P}^{1134} + 5\mathbf{P}^{1234}.$$

The matrices appearing in the last product are

$$(205) \quad \begin{aligned} &\begin{pmatrix} 1 & 1 & . & . \\ . & 1 & . & . \\ . & . & . & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & . & . \\ . & . & . & 1 \\ . & 1 & . & . \end{pmatrix}, \quad \begin{pmatrix} 1 & . & . & 1 \\ . & 1 & . & . \\ . & 1 & . & . \end{pmatrix}, \quad \begin{pmatrix} . & 2 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \end{pmatrix} \\ &\begin{pmatrix} . & 2 & . & . \\ . & . & . & 1 \\ 1 & . & . & . \end{pmatrix}, \quad \begin{pmatrix} . & 1 & . & 1 \\ 1 & . & . & . \\ . & 1 & . & . \end{pmatrix}, \quad \begin{pmatrix} . & 1 & . & 1 \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}, \end{aligned}$$

the fourth and the fifth matrices having 1134 as parkized word whereas the other ones yield 1234.

It is interesting to observe that these algebras are nonunital. Indeed, (201) yields

COROLLARY 5.7. *The element $\mathbf{J}_n = \mathbf{P}^{(1^n)}$ is a left unit for $*$, but not a right unit. ■*

The description of $\mathbf{P}^{\pi'} * \mathbf{P}^{\pi''}$ in terms of integer matrices being essentially identical to that of $S^I * S^J$ in \mathbf{Sym} , the same argument as in [7, proof of Proposition 5.2] shows that the splitting formula remains valid in \mathbf{CQSym}_n :

PROPOSITION 5.8. *Let μ_r denote the r -fold product map from $\mathbf{CQSym}^{\otimes r}$ to \mathbf{CQSym} , Δ^r the r -fold coproduct with values in $\mathbf{CQSym}^{\otimes r}$, and $*_r$ the internal product of the r -fold tensor product of algebras $\mathbf{CQSym}^{\otimes r}$. Then, for $f_1, \dots, f_r, g \in \mathbf{CQSym}$,*

$$(206) \quad (f_1 \cdots f_r) * g = \mu_r[(f_1 \otimes \cdots \otimes f_r) *_r \Delta^r(g)]. \quad \blacksquare$$

This is indeed the same formula as for internal product of \mathbf{Sym} , actually, an extension of it, since we have

COROLLARY 5.9. *The Hopf subalgebra of \mathbf{CQSym} generated by the elements \mathbf{J}_n , which is isomorphic to \mathbf{Sym} by $j : S_n \mapsto \mathbf{J}_n$, is stable under $*$, and thus also $*$ -isomorphic to \mathbf{Sym} . Moreover, the map $f \mapsto f * \mathbf{J}_n$ is a projector onto \mathbf{Sym}_n , which is therefore a left $*$ -ideal of \mathbf{CQSym}_n . ■*

More precisely, if $i < j < \cdots < r$ are the letters occurring in π , so that

as a word $\pi = i^{m_i} j^{m_j} \dots r^{m_r}$, then

$$(207) \quad \mathbf{P}^\pi * \mathbf{J}_n = \mathbf{J}_{m_i} \mathbf{J}_{m_j} \cdots \mathbf{J}_{m_r}.$$

It follows from Theorem 5.5 that the \mathbf{R}_π are the pre-images of the ordinary ribbons under the projection $f \mapsto f * \mathbf{J}_n$:

COROLLARY 5.10. *Let I be the composition obtained by discarding the zeros of the evaluation of a nondecreasing parking function π . Then*

$$(208) \quad \mathbf{R}_\pi * \mathbf{J}_n = j(R_I).$$

More precisely, if $I = (i_1, \dots, i_p)$, this last element is equal to $\mathbf{R}_{1^{i_1} \bullet \dots \bullet 1^{i_p}}$, that is, the Catalan ribbon indexed by the only nondecreasing word of evaluation $d(\pi)$. ■

The internal product of \mathbf{CQSym} is dual to the coproduct $\delta f = f(XY)$ on the commutative algebra $CQSym$, quotient of \mathbf{PQSym}^* . For example, we have

$$(209) \quad \begin{aligned} \mathcal{M}_{113}(XY) &= (\mathcal{M}_{112}(X) + \mathcal{M}_{113}(X))(\mathcal{M}_{111}(Y) + \mathcal{M}_{112}(Y) + \mathcal{M}_{113}(Y) + \mathcal{M}_{122}(Y)) \\ &\quad + \mathcal{M}_{111}(X)\mathcal{M}_{113}(Y), \end{aligned}$$

$$(210) \quad \mathcal{M}_{112}(XY) = \mathcal{M}_{111}(X)\mathcal{M}_{112}(Y).$$

5.4.1. Cauchy kernel. Define the *Cauchy kernel* by

$$(211) \quad K(X; A) = \sum_{\mathbf{a} \in \text{PF}} \mathbf{G}_{\mathbf{a}}(X) \mathbf{F}_{\mathbf{a}}(A) = \sum_{\pi} \mathcal{M}_{\pi}(X) \mathbf{P}^{\pi}(A).$$

PROPOSITION 5.11. *The kernel K has the reproducing property*

$$(212) \quad K(X; A) * K(Y; A) = K(XY; A).$$

Proof.

$$(213) \quad \begin{aligned} \langle K(X) * K(Y), \mathcal{M}_{\pi} \rangle &= \sum_{\pi', \pi''} \mathcal{M}_{\pi'}(X) \mathcal{M}_{\pi''}(Y) \langle \mathbf{P}^{\pi'} * \mathbf{P}^{\pi''}, \mathcal{M}_{\pi} \rangle \\ &= \sum_{\pi', \pi''} \mathcal{M}_{\pi'}(X) \mathcal{M}_{\pi''}(Y) \langle \mathbf{P}^{\pi'} \otimes \mathbf{P}^{\pi''}, \Delta \mathcal{M}_{\pi} \rangle \\ &= \Delta \mathcal{M}_{\pi}(X, Y) = \mathcal{M}_{\pi}(XY). \quad \blacksquare \end{aligned}$$

5.5. Compositions, Lagrange inversion, and $H_n(0)$

5.5.1. Recall that nondecreasing parking functions (or noncrossing partitions) can be classified according to the factorization $\pi = \pi_1 \bullet \dots \bullet \pi_r$ into irreducible nondecreasing parking functions (or noncrossing partitions). Let $i_k := |\pi_k|$ and $c(\pi) := (i_1, \dots, i_k)$, regarded as a composition of n .

We set

$$(214) \quad \mathbf{V}^I := \sum_{c(\pi)=I} \mathbf{P}^\pi$$

considered as an element of \mathbf{PQSym} . If one defines $\mathbf{V}_n = \mathbf{V}^{(n)}$, we have

$$(215) \quad \mathbf{V}_n = \sum_{\mathbf{a} \in \text{PPF}_n} \mathbf{F}_\mathbf{a}$$

and

$$(216) \quad \mathbf{V}^I = \mathbf{V}_{i_1} \cdots \mathbf{V}_{i_r} = \sum_{\mathbf{a} \in \text{PPF}_I} \mathbf{F}_\mathbf{a}.$$

This can be reformulated as

$$(217) \quad \sum_{\mathbf{a} \in \text{PF}} \mathbf{F}_\mathbf{a} = \left(1 - \sum_{\mathbf{b} \in \text{PPF}} \mathbf{F}_\mathbf{b}\right)^{-1},$$

which is the lift to \mathbf{PQSym} of the well known identity

$$(218) \quad \sum_{n \geq 0} (n+1)^{n-1} \frac{t^n}{n!} = \left(1 - \sum_{n \geq 1} (n-1)^{n-1} \frac{t^n}{n!}\right)^{-1}.$$

Indeed, the map $\mathbf{F}_\mathbf{a} \mapsto 1/n!$ ($n = |\mathbf{a}|$) is a character of \mathbf{PQSym} .

At this point, it is useful to observe that if $C(w)$ denotes the descent composition of a word w , the map

$$(219) \quad \eta : \mathbf{F}_\mathbf{a} \mapsto F_{C(\mathbf{a})},$$

which is a Hopf algebra morphism $\mathbf{PQSym} \rightarrow QSym$, maps \mathbf{V}^I to the Frobenius characteristic of the underlying permutation representation of \mathfrak{S}_n on PPF_I :

$$(220) \quad \eta(\mathbf{V}^I) = \sum_{\mathbf{a} \in \text{PPF}_I} \mathbf{F}_{C(\mathbf{a})} = \text{ch}(\text{PPF}_I).$$

Indeed, if $V \subseteq A^n$ is any set of words invariant under the right action of \mathfrak{S}_n , the characteristic of the underlying permutation representation is always equal to $\sum_{w \in V} F_{C(w)}$. This is because V splits as a disjoint union

$$(221) \quad V = \bigsqcup_{\nu} A_\nu, \quad \text{where } A_\nu = \{w \in A^n \mid \text{Ev}(w) = \nu\}.$$

The characteristic of A_ν is clearly h_ν , and it is well known that

$$(222) \quad h_\nu = \sum_{w \in A_\nu} F_{C(w)}.$$

Actually, each $\mathbb{C}A_\nu$ is also a projective $H_n(0)$ -module with noncommutative characteristic $\text{ch}(\mathbb{C}A_\nu) = S^I$, where $I = t(\nu)$.

5.5.2. As a consequence, the number of parking functions of type I with descent composition J is equal to the scalar product of symmetric functions

$$(223) \quad \langle r_J, f^I \rangle$$

where $f^I = f_{i_1} \cdots f_{i_r} = \text{ch}(\text{PPF}_I)$ and r_J is the ribbon Schur function. This extends Proposition 3.2(a) of [36]. Note that in particular, by inversion of

$$(224) \quad \mathbf{F}_{\text{PF}_n} := \sum_{\mathbf{a} \in \text{PF}_n} \mathbf{F}_{\mathbf{a}} = \sum_{I \vDash n} \mathbf{V}^I,$$

one obtains

$$(225) \quad \mathbf{F}_{\text{PPF}_n} = \sum_{I \vDash n} (-1)^{n-l(I)} \mathbf{F}_{\text{PF}_I},$$

where

$$(226) \quad \text{PF}_I := \text{PF}_{i_1} \uplus \cdots \uplus \text{PF}_{i_r}.$$

These identities are easily visualized on the encoding of parking functions with skew Young diagrams as in [30] or in [9].

5.5.3. The transpose γ^* of the map γ defined in (193) is the map

$$(227) \quad \text{ch} : \mathbf{CQSym} \rightarrow \mathbf{Sym}, \quad \mathbf{P}^\pi \mapsto S^{t(\pi)},$$

which sends \mathbf{P}^π to the characteristic noncommutative symmetric function of the natural projective $H_n(0)$ -module with basis $\{\mathbf{a} \in \text{PF}_n \mid \text{NC}(\mathbf{a}) = \pi\}$.

5.5.4. One can show that

$$(228) \quad g := \sum_{n \geq 0} g_n := \sum_{n \geq 0} \text{ch}(\mathbf{F}_{\text{PF}_n}) = \sum_I \text{ch}(\mathbf{V}^I)$$

is the series obtained by applying the noncommutative Lagrange inversion formula of [8, 31] to the generating series of complete functions, i.e., g is the unique solution of the equation

$$(229) \quad g = 1 + S_1 g + S_2 g^2 + \cdots = \sum_{n \geq 0} S_n g^n.$$

Indeed, let g be defined by (228) and set

$$(230) \quad f := \sum_{n \geq 1} f_n = \sum_{n \geq 1} \text{ch}(\mathbf{F}_{\text{PPF}_n}) = \sum_{n \geq 1} \text{ch}(\mathbf{V}_n).$$

Recall that the prime nondecreasing parking functions of length n are obtained by concatenating a 1 to the left of a nondecreasing parking function of length $n - 1$. This gives a recurrence for f_n and g_n . From each nondecreasing parking function with packed evaluation $J \vDash n - 1$, contributing a term S^J to g_{n-1} , we get a prime nondecreasing parking function with packed evaluation $I = (j_1 + 1, j_2, \dots, j_r) \vDash n$, contributing a term $S^I := \Omega S^J$ to f_n ,

where Ω is the linear operator incrementing the first part in the basis S^J of **Sym**. Hence, $f_n = \Omega g_{n-1}$, and we have a system of two equations

$$(231) \quad \begin{cases} f = \Omega g, \\ g = (1 - f)^{-1}, \end{cases}$$

which, with the initial condition $g_0 = 1$, admits a unique solution: $f_1 = \Omega g_0 = S^1$, $g_1 = f_1$, $f_2 = \Omega g_1 = S^2$, $g_2 = f_2 + f^{11} = S^2 + S^{11}$, $f_3 = \Omega g_2 = S^3 + S^{21}$, $g_3 = f_3 + f^{21} + f^{12} + f^{111} = S^3 + 2S^{21} + S^{12} + S^{111}$, and so on.

But the unique solution of (229) satisfies

$$(232) \quad \Omega g = S_1 + S_2 g + S_3 g^2 + \cdots$$

and also

$$(233) \quad 1 = g^{-1} + S_1 + S_2 g + S_3 g^2 + \cdots = g^{-1} + \Omega g$$

so that if we set $f = \Omega g$, we solve (231) as well.

We remark that the commutative images of these equations give the \mathfrak{S}_n -characteristics of PF_n and PPF_n , and that we have derived them from first principles, using only the multiplication rule of **PQSym** and the notion of a prime parking function.

5.5.5. The Hilbert series of **SQSym revisited.** It is also possible to obtain it by a character calculation, derived from the above considerations. If we decompose the noncommutative characteristic of the $H_n(0)$ -module CPF_n into ribbons

$$(234) \quad \mathbf{ch}(\text{CPF}_n) = \sum_{I \models n} m_I R_I,$$

the number of hypoplactic classes of parking functions of length n is

$$(235) \quad \sum_{I \models n} m_I.$$

Indeed, as already mentioned, if $V \subset A^n$ is any set of words which is a disjoint union of evaluation classes A_ν , then $\mathbb{C}V$ is a projective $H_n(0)$ -module since it is the direct sum $\bigoplus \mathbb{C}A_\nu$, where $\mathbf{ch}(\mathbb{C}A_\nu) = S^I$, with $I = t(\nu)$.

Now, each A_ν is itself a disjoint union of hypoplactic classes

$$(236) \quad A_{\nu,I} = \{w \in A_\nu \mid C(\text{Std}(w)^{-1}) = I\},$$

and each such class is the support of an indecomposable projective module

$$(237) \quad \mathbf{ch}(A_{\nu,I}) = R_I.$$

By duality between the bases F_I and R_I ,

$$(238) \quad \sum m_I = \left\langle \sum F_I, \mathbf{ch}(\text{CPF}_n) \right\rangle$$

and taking into account the identity [38]

$$(239) \quad \sum_I F_I = \frac{1}{2} \left[1 + \prod_{i \geq 1} \frac{1+x_i}{1-x_i} \right] = 1 + \frac{1}{2} \sum_{n \geq 1} \sum_{k=0}^n e_k h_{n-k},$$

we obtain

$$(240) \quad \begin{aligned} \dim(\mathbf{SQSym}_n) &= \left\langle \sum_{I \vdash n} F_I, \mathbf{ch}(\mathbf{F}_{\mathbf{PF}_n}) \right\rangle \\ &= \left\langle \frac{1}{2} \sum_{k=0}^n e_k h_{n-k}, \frac{1}{n+1} h_n((n+1)X) \right\rangle \\ &= \frac{1}{2n+2} \sum_{k=0}^n \binom{n+1}{k} \binom{2n-k}{n-k} = s_n. \end{aligned}$$

6. A HOPF ALGEBRA OF SEGMENTED COMPOSITIONS

6.1. Segmented compositions. Define a *segmented composition* as a finite sequence of positive integers, separated by vertical bars or commas, e.g., $(2, 1 | 2 | 1, 2)$.

The number of segmented compositions having the same underlying composition is obviously 2^{l-1} where l is the length of the composition, so that the total number of segmented compositions with sum n is 3^{n-1} since $(1+2)^{n-1} = 3^{n-1}$.

6.2. A Hopf subalgebra of \mathbf{SQSym}^*

6.2.1. Hypoplactic packed words. Let $A = \{a_1 < a_2 < \dots\}$ be an infinite totally ordered alphabet. The *packed word* $u = \text{pack}(w)$ associated with a word $w \in A^*$ is obtained by the following process. If $b_1 < \dots < b_r$ are the letters occurring in w , then u is the image of w under the homomorphism $b_i \mapsto a_i$.

A word u is said to be *packed* if $\text{pack}(u) = u$. We denote by \mathbf{PW} the set of packed words.

Let us consider packed quasi-ribbons, that is, quasi-ribbons that are packed words. For example, the word 11324355 is a packed quasi-ribbon word since it is the reading of the following quasi-ribbon:

$$(241) \quad \begin{array}{cccc} \boxed{1} & \boxed{1} & \boxed{2} & \\ & \boxed{3} & \boxed{3} & \\ & & \boxed{4} & \boxed{5} & \boxed{5} \end{array}$$

These objects are in bijection with segmented compositions. Indeed, start from a packed quasi-ribbon \mathbf{q} and write the evaluation vector \mathbf{I} of \mathbf{q} , putting a separator between \mathbf{I}_i and \mathbf{I}_{i+1} iff i and $i+1$ are not in the same row of \mathbf{q} . For example, the segmented composition corresponding to the quasi-ribbon of (241) is $21|2|12$. This element will be denoted by $\text{ps}(\mathbf{q})$. The reverse

bijection consists in writing the unique nondecreasing word with evaluation \mathbf{I} and put the letters $i + 1$ on the row next to the row of the letters of i iff i and $i + 1$ are separated by $|$ in \mathbf{I} .

EXAMPLE 6.1. For $n = 2$, we have three packed quasi-ribbons

$$(242) \quad 11, \quad 12, \quad 1|2,$$

For $n = 3$, we have nine packed quasi-ribbons

$$(243) \quad 111, \quad 112, \quad 11|2, \quad 122, \quad 1|22, \quad 123, \quad 1|23, \quad 12|3, \quad 1|2|3.$$

respectively encoded as the nine segmented compositions

$$(244) \quad 3, \quad 21, \quad 2|1, \quad 12, \quad 1|2, \quad 111, \quad 1|11, \quad 11|1, \quad 1|1|1.$$

In the following, we will identify packed quasi-ribbons and their encodings as segmented compositions.

6.2.2. A Hopf subalgebra of \mathbf{SQSym}^* . Let us denote by $P(w)$ the hypoplactic P -symbol of a word w (its quasi-ribbon). The P -symbols of packed words are therefore packed quasi-ribbons.

For each packed quasi-ribbon \mathbf{I} , define

$$(245) \quad P_{\mathbf{I}} := \mathbf{P}_{\text{unp}(\mathbf{I})} \quad \text{and} \quad Q_{\mathbf{I}} := \sum_{\text{ps}(\mathbf{q})=\mathbf{I}} \mathbf{Q}_{\mathbf{q}} \in \mathbf{SQSym}^*.$$

where $\text{unp}(\mathbf{q})$ is the maximal parking quasi-ribbon for the lexicographic order of evaluation \mathbf{I} .

For example,

$$(246) \quad P_{11|2} = \mathbf{P}_{12|33}, \quad P_{112} = \mathbf{P}_{1233},$$

$$(247) \quad Q_{12|1} = \mathbf{Q}_{122|3} + \mathbf{Q}_{122|4}, \quad Q_{121} = \mathbf{Q}_{1223} + \mathbf{Q}_{1224},$$

$$(248) \quad Q_{12|21} = \mathbf{Q}_{122|334} + \mathbf{Q}_{122|335} + \mathbf{Q}_{122|336} + \mathbf{Q}_{122|445} + \mathbf{Q}_{122|446}.$$

THEOREM 6.2. *The $P_{\mathbf{I}}$ span a Hopf subalgebra \mathbf{SCQSym} of \mathbf{SQSym} . This subalgebra is also the quotient of \mathbf{SCQSym} by the relations $\mathbf{P}_{\mathbf{q}} = \mathbf{P}_{\mathbf{q}'}$ if w and w' have the same packed word.*

The $Q_{\mathbf{I}}$ span a Hopf subalgebra \mathbf{SCQSym}^ of \mathbf{SQSym}^* , and*

$$(249) \quad \dim \mathbf{SCQSym}_n = 3^{n-1} \quad \text{for } n \geq 1.$$

Proof. The product and coproduct rules of the \mathbf{P} of \mathbf{SQSym} imply that the P of \mathbf{SQSym}^* span a Hopf subalgebra of \mathbf{SQSym} . It is also obvious that both operations are compatible with the relations $\mathbf{P}_{\mathbf{q}} = \mathbf{P}_{\mathbf{q}'}$ if \mathbf{q} and \mathbf{q}' have same packed word, so that \mathbf{SCQSym} is a Hopf quotient of \mathbf{SQSym} . Then, by the standard argument, the Q are a basis of \mathbf{SCQSym}^* .

The dimension of \mathbf{SCQSym}_n is given by the number of segmented compositions, that is, 3^{n-1} . ■

To describe the product and coproduct rules of both bases, we need a new operation on segmented compositions.

Recall that the product of the monomial basis M_I on $QSym$ is defined with the augmented shuffle of two compositions recursively defined as

$$(250) \quad (I_1, I') \uplus (J_1, J') = I_1(I' \uplus (J_1, J')) + J_1((I_1, I') \uplus J') + I_1 + J_1(I' \uplus J'),$$

with the extra condition $I' \uplus \epsilon = \epsilon \uplus I'$ where ϵ is the empty word.

This construction is generalized to **SCQSym** as follows: the *augmented shuffle* $\mathbf{I}' \uplus \mathbf{I}''$ of two segmented compositions is obtained from the usual augmented shuffle $I' \uplus I''$ of their underlying compositions by inserting bars between two blocks I_k and I_{k+1} of a composition I iff

- I_k and I_{k+1} both contain elements coming from \mathbf{I}' and those elements were separated by a bar,
- I_k and I_{k+1} both contain elements coming from \mathbf{I}'' and those elements were separated by a bar,
- I_k contains an element coming from \mathbf{I}'' and I_{k+1} contains an element coming from \mathbf{I}' .

For example,

$$(251) \quad \begin{aligned} 1 \uplus 2|1 &= 12|1 + 3|1 + 2|11 + 2|2 + 2|1|1, \\ 1 \uplus 21 &= 121 + 31 + 2|11 + 2|2 + 21|1. \end{aligned}$$

THEOREM 6.3. *The product and coproduct rules for the $P_{\mathbf{I}}$ and the $Q_{\mathbf{I}}$ are*

$$(252) \quad P_{\mathbf{I}'} P_{\mathbf{I}''} = P_{\mathbf{I}'|\mathbf{I}''} + P_{\mathbf{I}'\mathbf{I}''},$$

$$(253) \quad \Delta P_{\mathbf{I}} = \sum_{\mathbf{I} \in \mathbf{I}' \uplus \mathbf{I}''} \mathbf{P}_{\mathbf{I}'} \otimes \mathbf{P}_{\mathbf{I}''},$$

$$(254) \quad Q_{\mathbf{I}'} Q_{\mathbf{I}''} = \sum_{\mathbf{I} \in \mathbf{I}' \uplus \mathbf{I}''} Q_{\mathbf{I}},$$

$$(255) \quad \Delta Q_{\mathbf{I}} = \sum_{\mathbf{I} = \mathbf{I}' \cdot \mathbf{I}'' \text{ or } \mathbf{I} = \mathbf{I}'|\mathbf{I}''} Q_{\mathbf{I}'} \otimes Q_{\mathbf{I}''}.$$

Proof. The product of two P of **SCQSym** directly comes from the product of two \mathbf{P} of **SQSym**. The coproduct of a Q then follows by duality. The shifted shuffle of two compositions obviously gives all the possible evaluations of the convolution of two parking functions of the given evaluations. Finally, the rules to place the bars correspond to the different cases where there is an i to the right of an $i+1$ in one of the resulting parking functions. ■

For example,

$$(256) \quad P_{12|1} P_{2|11} = P_{12|12|11} + P_{12|1|2|11},$$

$$(257) \quad \begin{aligned} \Delta P_{12|1} &= 1 \otimes P_{12|1} + P_1 \otimes (P_{12} + P_{2|1}) + P_{11} \otimes (P_{1|1} + P_2) \\ &\quad + P_{1|1} \otimes P_2 + (P_{111} + P_{11|1}) \otimes P_1 + P_{12|1} \otimes 1, \end{aligned}$$

$$(258) \quad Q_1 Q_{2|1} = Q_{3|1} + Q_{12|1} + Q_{2|2} + Q_{2|1|1} + Q_{2|11},$$

$$(259) \quad Q_1 Q_{11|1} = Q_{111|1} + Q_{21|1} + Q_{1|11|1} + Q_{1|2|1} + Q_{11|11} + Q_{11|2} + Q_{11|1|1},$$

$$(260) \quad Q_{1|1} Q_{1|1} = 2Q_{1|11|1} + Q_{1|1|11} + Q_{11|11} + Q_{11|1|1} + Q_{1|1|1|1} \\ + Q_{2|11} + Q_{2|1|1} + Q_{2|2} + 2Q_{1|2|1} + Q_{1|1|2} + Q_{11|2},$$

$$(261) \quad \Delta Q_{12|1} = 1 \otimes Q_{12|1} + Q_1 \otimes Q_{2|1} + Q_{12} \otimes Q_1 + Q_{12|1} \otimes 1.$$

6.3. Algebraic structure of \mathbf{SCQSym} and \mathbf{SCQSym}^* . The algebra \mathbf{SCQSym}^* is not free for exactly the same reason \mathbf{SQSym}^* is not: one has the relation

$$(262) \quad Q_1(Q_2 + Q_{11}) = (Q_2 + Q_{11})Q_1.$$

Let us now move to \mathbf{SCQSym} .

Since \mathbf{SCQSym} is the subalgebra of \mathbf{SQSym} spanned by the parking quasi-ribbons that are maximally unpacked, and since \mathbf{SQSym} is free, \mathbf{SCQSym} is automatically free and generated by the maximal elements of PQS. For example, the generators of \mathbf{SCQSym} for $n \leq 4$ are

$$(263) \quad \{1\}, \quad \{11, 1|2\}, \quad \{111, 11|3, 1|22, 1|2|3\}, \\ \{1111, 111|4, 11|33, 11|3|4, 1|222, 1|22|4, 1|2|33, 1|2|3|4\},$$

that can be rewritten on segmented compositions as

$$(264) \quad \{1\}, \quad \{2, 1|1\}, \quad \{3, 2|1, 1|2, 1|1|1\}, \\ \{4, 3|1, 2|2, 2|1|1, 1|3, 1|2|1, 1|1|2, 1|1|1|1\},$$

By the same argument on generating series as in \mathbf{SQSym} , one finds that there are 2^{n-1} generators of \mathbf{SCQSym} of degree n . And indeed, these generators are in natural bijection with compositions of n since they have separators between all elements.

The next proposition summarizes the structures of \mathbf{SCQSym} and \mathbf{SCQSym}^* .

PROPOSITION 6.4. *The algebra \mathbf{SCQSym} is a Hopf algebra of dimension 3^{n-1} . It is not self-dual since \mathbf{SCQSym} is free as an algebra whereas \mathbf{SCQSym}^* is not. Moreover, \mathbf{SCQSym} is free over a graded alphabet labelled by all compositions.*

6.4. Primitive Lie algebras of \mathbf{SCQSym}^* . Since \mathbf{SCQSym}^* contains $QSym$ as a subalgebra, its primitive Lie algebra cannot be free and one easily finds that

$$(265) \quad [Q_1, Q_{12} - Q_{1|2} + Q_{11}] = 0.$$

6.5. A quasi-ribbon basis of SCQSym^* . The elements $Q_{\mathbf{I}}$ are segmented analogs of the basis (M_I) of $QSym$. So we can define analogs of the (F_I) of $QSym$ in the same way as we did in SQSym^* .

Recall that the *refinement order* denoted by \succeq on compositions is such that $I = (i_1, \dots, i_k) \succeq J = (j_1, \dots, j_l)$ iff $\{i_1, i_1+i_2, \dots, i_1+\dots+i_k\}$ contains $\{j_1, j_1+j_2, \dots, j_1+\dots+j_l\}$. In this case, we say that I is finer than J . For example, $(2, 1, 2, 3, 1, 2) \succeq (3, 2, 6)$.

Let $\mathbf{I} = (I_1 \mid \dots \mid I_r)$ and let

$$(266) \quad F_{\mathbf{I}} := \sum_{\mathbf{I}'} Q_{\mathbf{I}'},$$

where the sum is taken over sequences of compositions (I'_1, \dots, I'_r) where I'_k is finer than I_k . For example,

$$(267) \quad F_{2|2} = Q_{11|11} + Q_{2|1} + Q_{11|2} + Q_{2|2}.$$

By a triangularity argument, we have

THEOREM 6.5. *The $F_{\mathbf{I}}$ form a basis of SCQSym^* .*

The basis $F_{\mathbf{I}}$ satisfies a product formula similar to the F_I of $QSym$ (whence the choice of notation). To state it, we need an analogue of the shifted shuffle.

A *segmented permutation* is a permutation with separators. The *descent composition* $C(\alpha)$ of a segmented permutation α is a segmented composition: it is the sequence of descent compositions of the blocks of α separated by bars.

For example, $\alpha = 248|517|3$ is a segmented permutation whose descent composition is $(3|12|1)$.

The *shifted shuffle* $\alpha \uplus \beta$ of two segmented permutations is obtained from the usual shifted shuffle $\sigma \uplus \tau$ of the underlying permutations σ and τ by inserting bars:

- after each descent which was originally followed by a bar in α or in the shift of β ,
- after each descent created by the shuffling process.

For example,

$$(268) \quad 2|1 \uplus 21 = 2|143 + 24|13 + 243|1 + 4|2|13 + 4|23|1 + 43|2|1.$$

THEOREM 6.6. *Let \mathbf{I}' and \mathbf{I}'' be two segmented compositions and let α and β be any two segmented permutations whose descent compositions are respectively \mathbf{I}' and \mathbf{I}'' . Then*

$$(269) \quad F_{\mathbf{I}'} F_{\mathbf{I}''} = \sum_{\mathbf{I}} F_{\mathbf{I}},$$

where the sum runs over the descent compositions of the segmented permutations γ occurring in $\alpha \uplus \beta$.

Proof. The product of the Q can be easily rewritten in terms of segmented permutations as follows: associate with two segmented compositions \mathbf{I}' and \mathbf{I}'' two segmented permutations α and β such that $C(\alpha) = \mathbf{I}'$ and $C(\beta) = \mathbf{I}''$. Consider the elements in the shifted shuffle $\alpha \uplus \beta$ such that two elements in increasing order of α (resp. β) not separated by a bar have no β (resp. α) between them. From all those elements, build the set of all segmented permutations with at least those bars and at most new bars between the elements of α and the elements of β . For all those segmented permutations, compute first their descent compositions and *then* remove the bars added lately. The set of the descent compositions obtained by this process corresponds to the product $Q_\alpha Q_\beta$.

Express both F in the Q basis and group the terms in their product where the letters of α have been inserted at the same place. By construction, the lexicographically minimum element s in each group with the smallest number of bars belongs to $\alpha \uplus \beta$. Now, given the product rule of the Q , we have all elements obtained from s by adding any number of bars, thus F_s . ■

For example,

$$(270) \quad F_1 F_{11|1} = F_{21|1} + F_{1|2|1} + F_{11|2} + F_{11|1|1},$$

$$(271) \quad F_1 F_{2|1} = F_{3|1} + F_{1|2|1} + F_{2|2} + F_{2|1|1}.$$

THEOREM 6.7. *Let \mathbf{I} be a segmented composition. Then*

$$(272) \quad \Delta F_{\mathbf{I}} = \sum_{\mathbf{I}=\mathbf{I}'\cdot\mathbf{I}'' \text{ or } \mathbf{I}=\mathbf{I}'|\mathbf{I}'' \text{ or } \mathbf{I}=\mathbf{I}'\triangleright\mathbf{I}''} F_{\mathbf{I}'} \otimes F_{\mathbf{I}''},$$

where $(I_1, \dots, I_k) \triangleright (I'_1, \dots, I'_l)$ denotes the segmented composition $(I_1, \dots, I_{k-1}, I_k \triangleright' I'_1, I'_2, \dots, I'_l)$ where $(i_1, \dots, i_k) \triangleright' (j_1, \dots, j_l) = (i_1, \dots, i_{k-1}, i_k + j_1, j_2, \dots, j_l)$.

Proof. This result will follow by duality from the considerations in the forthcoming section. ■

6.6. A ribbon basis of SCQSym. Let $(R_{\mathbf{I}})$ be the dual basis of $(F_{\mathbf{I}})$. Then we have

PROPOSITION 6.8. *The $(R_{\mathbf{I}})$ are a basis of SCQSym related to the P by*

$$(273) \quad P_{\mathbf{I}} =: \sum_{\mathbf{I}'} R_{\mathbf{I}'},$$

where the sum is taken over sequences of segmented compositions $(I'_1 | \dots | I'_r)$ where I_k is finer than I'_k .

Since **SCQSym** is a subalgebra of **SQSym** such that the image of $P_{\mathbf{I}}$ is $\mathbf{P}_{\mathbf{q}}$, given both orders on parking quasi-ribbons and on segmented compositions, the image of $R_{\mathbf{I}}$ is $\mathbf{R}_{\mathbf{q}}$. This remark immediately proves the product

rule of the R , whereas its coproduct rule comes from the duality between the R and the F .

THEOREM 6.9. *The product and coproduct rules of R are*

$$(274) \quad R_{\mathbf{I}} \cdot R_{\mathbf{I}''} = R_{\mathbf{I}'\mathbf{I}''} + R_{\mathbf{I}'|\mathbf{I}''} + R_{\mathbf{I}'\triangleright\mathbf{I}''},$$

$$(275) \quad \Delta R_{\mathbf{I}} = \sum_{\mathbf{I} \in \mathbf{I}' \uplus \mathbf{I}''} R_{\mathbf{I}'} \otimes R_{\mathbf{I}''}.$$

From (274), we see that **SCQSym** is the free cubical trialgebra on one generator (see [22]).

7. APPENDIX

7.1. Relations with free probability theory. The free cumulants R_n of a probability measure μ on \mathbb{R} are defined (see, e.g., [35]) by means of the generating series of its moments M_n ,

$$(276) \quad G_{\mu}(z) := \int_{\mathbb{R}} \frac{\mu(dx)}{z-x} = z^{-1} + \sum_{n \geq 1} M_n z^{-n-1}$$

as the coefficients of its compositional inverse

$$(277) \quad K_{\mu}(z) := G_{\mu}(z)^{\langle -1 \rangle} = z^{-1} + \sum_{n \geq 1} R_n z^{n-1}.$$

It is in general instructive to interpret the coefficients of a formal power series as the specializations of the elements of some generating family of the algebra of symmetric functions. In this context, it is the interpretation

$$(278) \quad M_n = \phi(h_n) = h_n(A)$$

that is relevant. Indeed, the process of functional inversion (Lagrange inversion) admits a simple expression within this formalism (see [23, Ex. 24, p. 35]). If the symmetric functions h_n^* are defined by the equations

$$(279) \quad u = tH(t) \Leftrightarrow t = uH^*(u)$$

where $H(t) := \sum_{n \geq 0} h_n t^n$, $H^*(u) := \sum_{n \geq 0} h_n^* u^n$, then, in the λ -ring notation,

$$(280) \quad h_n^*(X) = \frac{1}{n+1} (-1)^n e_n((n+1)X) := \frac{1}{n+1} [t^n] E(-t)^{n+1}$$

where $E(t)$ is defined by $E(t)H(t) = 1$. This defines an involution $f \mapsto f^*$ of the ring of symmetric functions.

Now, if one sets $M_n = h_n(A)$ as above, then

$$(281) \quad G_{\mu}(z) = z^{-1} H(z^{-1}) = u$$

$$\Leftrightarrow z = K_{\mu}(u) = \frac{1}{u} E^*(-u) = u^{-1} + \sum_{n \geq 1} (-1)^n e_n^* u^{n-1}.$$

Hence,

$$(282) \quad R_n = (-1)^n e_n^*(A).$$

7.2. An exercise on permutation representations. It follows immediately from the explicit formula (see [23, p. 35])

$$(283) \quad -e_n^* = \frac{1}{n-1} \sum_{\lambda \vdash n} \binom{n-1}{l(\lambda)} \binom{l(\lambda)}{m_1, m_2, \dots, m_n} e_\lambda$$

(where $\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n}$) that $-e_n^*$ is Schur positive. Clearly, $-e_n^*$ is the Frobenius characteristic of a permutation representation Π_n , twisted by the sign character. Let us set

$$(284) \quad (-1)^{n-1} R_n = -e_n^* =: \omega(f_n)$$

so that

$$(285) \quad f_n := \sum_{\lambda \vdash n} \frac{1}{n-1} \binom{n-1}{l(\lambda)} \binom{l(\lambda)}{m_1, m_2, \dots, m_n} h_\lambda$$

and f_n is the character of Π_n .

The problem of constructing such a representation had been raised by Kerov in 1995. We shall see that Π_n corresponds to prime parking functions. We note that our construction of Π_n is merely a variation on previously known results (see in particular [18, 30]). However, since this is this precise version of the question that led us to the Hopf algebra of parking functions and some of its properties, we decided to include its discussion.

7.3. Solution of the exercise

PROPOSITION 7.1. *The Frobenius characteristic of the permutation representation of PPF_n is f_n .*

Proof. We first show that the number of nondecreasing prime parking functions whose reordered evaluation is a given partition λ is equal to

$$(286) \quad \frac{1}{n-1} \binom{n-1}{l(\lambda)} \binom{l(\lambda)}{m_1, m_2, \dots, m_n}$$

where $\lambda = 1^{m_1} 2^{m_2} \dots n^{m_n}$ (see (285)). Indeed, this number corresponds to the number of ways of putting the λ_i over $n-1$ places in a circle. For such a placement P , number in all possible clockwise ways the places of the circle and consider the $n-1$ nondecreasing words i^{c_i} where c_i is the content of place number i . Then, by [5], exactly one of those words is a prime parking function.

For example, for $\lambda = (3, 2, 2)$, there are ten possible circles. Consider the circle where the 3 is followed by one empty place, a 2, two empty places, and the last 2. Then the six nondecreasing words are:

$$(287) \quad 1113366, 2255666, 1144555, 3344466, 2233355, 1122244.$$

Now, since all permutations of a nondecreasing prime parking function are parking functions, the Frobenius characteristic of the permutation representation of this set of words is h_λ . It then easily follows that

$$(288) \quad \text{ch}(\text{PPF}_n) = f_n,$$

so that Π_n can be identified with PPF_n , as claimed before. ■

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