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Brunnian links

by

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Abstract. A Brunnian link is a set of n linked loops such that every proper sublink is trivial. Simple Brunnian links have a natural algebraic representation. This is used to determine the form, length and number of minimal simple Brunnian links. Braids are used to investigate when two algebraic words represent equivalent simple Brunnian links that differ only in the arrangement of the component loops.

1. Introduction. A *link* with n components, an *n-link*, is the union of n mutually disjoint smooth embeddings of the circle S^1 in Euclidean 3-space, $\mathbb{R}^3$. An *oriented n-link* is an n-link such that each component has a given orientation. Two links, L_1 and L_2, are *equivalent* if there is an ambient isotopy mapping L_1 onto L_2. Let C_i be the circle $\{\langle x, y, 0\rangle : x^2 + y^2 = 1/(i+1)^2\}$ with anti-clockwise orientation. An oriented n-link is *trivial* if it is equivalent to $\mathcal{C}_n = \bigcup_{i<n} C_i$.

A *Brunnian link* is a non-trivial n-link such that every proper sublink is trivial. The most familiar example is the Borromean rings, a Brunnian 3-link. We will extend the definition of a Brunnian n-link to include the trivial n-link.

If L is any non-trivial oriented Brunnian n-link, then selecting one curve there is an ambient isotopy carrying the remaining curves to $\mathcal{C}_{n-1}$. So L is equivalent to $\mathcal{C}_{n-1} \cup l_d$, where l_d is a simple closed curve looped around all the curves in $\mathcal{C}_{n-1}$. We will refer to l_d as the *distinguished curve*.

Call a Brunnian n-link, L, *simple* if L is equivalent to a link $L' = \mathcal{C}_{n-1} \cup \{l_d\}$ such that the projection of l_d onto the plane $z = 0$ has no self-intersections, meets each ray in the xy-plane emanating from the origin exactly once, and has anti-clockwise orientation. See Figure 1(a) for an example.

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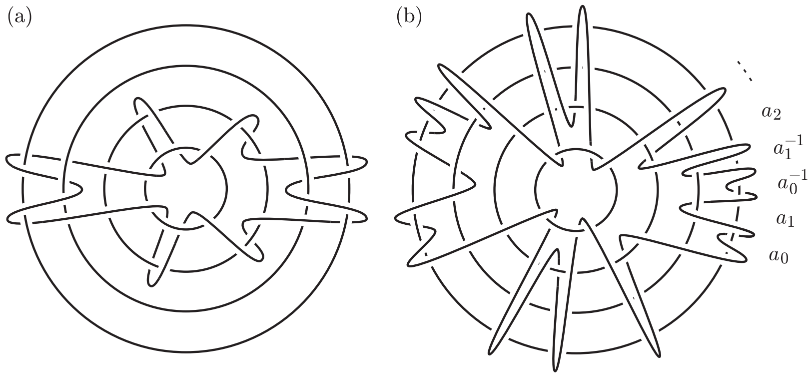

Fig. 1. The simple Brunnian link $a_0a_1a_0^{-1}a_1^{-1}a_2a_3a_2^{-1}a_3^{-1}a_1a_0a_1^{-1}a_0^{-1}a_3a_2a_3^{-1}a_2^{-1}$

Any simple Brunnian link is equivalent to a link L of the following type. Let $O = \{\langle x, y, z\rangle : x^2 + y^2 > 1\}$. Then $L = \mathcal{C}_{n-1} \cup \{l_d\}$ and the projection of l_d onto the plane $z = 0$ can be divided into segments with end points in O. Following each segment in an anti-clockwise direction it passes over the top of $C_0, C_1, \dots, C_{i-1}$, for some $i < n$. It then either passes over the top of C_i and back underneath it, or under C_i and back over the top of it, before returning over the top of $C_{i-1}, \dots, C_0$ to O. Call a Brunnian link of this type *canonical*. See Figure 1(b) for an example.

The geometric representation of a canonical Brunnian link leads to a natural algebraic representation. We can represent such a link by a *word*, $a_{i_0}^{\varepsilon_0} a_{i_1}^{\varepsilon_1} \dots a_{i_m}^{\varepsilon_m}$, $\varepsilon_i \in \mathbb{Z}$. Pick a starting point in $l_d \cap O$. If we follow l_d in an anti-clockwise direction, a_i represents a segment in which l_d passes over then under C_i, and a_i^{-1} represents a segment in which l_d passes under and back over C_i. Figures 1(a) and 1(b) are equivalent links represented by

$$a_0a_1a_0^{-1}a_1^{-1}a_2a_3a_2^{-1}a_3^{-1}a_1a_0a_1^{-1}a_0^{-1}a_3a_2a_3^{-1}a_2^{-1}.$$

Since any simple Brunnian link is equivalent to a canonical Brunnian link, any simple Brunnian link can be represented algebraically by a word. It is immediately clear that such a word is not unique. For example, by choosing a different starting point we will encounter the segments in a different order.

An alternative view of the words introduced above is to recall that the fundamental group of $\mathbb{R}^3 \setminus \mathcal{C}_{n-1}$ is free with letters represented by simple loops about each circle. These letters correspond to the a_i above.

A *minimal* Brunnian link is a simple Brunnian n-link whose associated word is of minimal length (amongst all simple Brunnian n-links). Our key results determine the form and length of minimal Brunnian links. In particular, if $n = 2^m + k$, where $k < 2^m$, then a minimal Brunnian n-link has length $2^m(3k+2^m)$. We also compute the number of distinct words associated with minimal Brunnian n-links. Finally, we investigate when two words represent topologically equivalent Brunnian links.

The plan for the remainder of the paper is as follows. In the next section the basic definitions and notations are introduced. In Section 3 we determine the length of a minimal Brunnian word, and then minimal Brunnian words are classified and counted. In Section 4 we take a brief look at a generalisation of Brunnian links and determine the form and length of minimal words representing these more general links. In Sections 5 and 6 we use braids to determine when two words represent equivalent Brunnian links.

2. Preliminaries. Denote the set of simple Brunnian n-links by $B_{\mathrm{S}}(n)$.

For each n define A_n to be the set $\{a_0, a_1, \ldots, a_{n-1}\}$, and let $A = \bigcup A_n$. Members of A and their inverses will be referred to as *letters*. A *word* is any finite sequence $a_{i_0}^{\varepsilon_0} a_{i_1}^{\varepsilon_1} \ldots a_{i_m}^{\varepsilon_m}$ such that $\varepsilon_i \in \mathbb{Z}$ for all i. Let e denote the empty sequence. We will use the term *string* when we want to refer to a subsequence in a word even though any such sequence is itself a word. When we refer to an arbitrary string we include the possibility that it is an empty string. We denote words by v, w etc, and strings by α, β etc.

If w is a word, denote the set $\{a_i \in A : a_i \text{ occurs in } w\}$ by $A(w)$. A word w is an *n-word* if $A(w) = A_n$ or $w = e$. A word w has the *form of an n-word* if exactly n different letters occur in w, or $w = e$ (hence if w has the form of an n-word then $A(w)$ need not be A_n).

Denote the pth occurrence of a_i in w by $\overset{p}{a}_i$. If w is a word and α is a string, then $w(\alpha/a_i)$ is the word obtained by replacing every occurrence of a_i^ε in w by α^ε. We will abbreviate $w(e/a_i)$ to $w(-a_i)$. If R is a set of occurrences of $a_i^{\pm 1}$ in a word w, then $w(-R)$ is the string obtained by removing from w the occurrences of $a_i^{\pm 1}$ in the set R.

Two words v, w are equivalent if v is convertible into w by a finite sequence of insertions and deletions of subwords $a_i^\varepsilon a_i^{-\varepsilon}$. An n-word is *reduced* if $a_i^\varepsilon a_i^{-\varepsilon}$ does not occur in w for any $i < n$ or $\varepsilon \in \mathbb{Z}$, otherwise w is *reducible*. If w is a word, let $\varrho(w)$ be the reduced word equivalent to w.

We may obtain $\varrho(w)$ from w by successively removing occurrences of $a_i^\varepsilon a_i^{-\varepsilon}$. When $\overset{p}{a}{}_i^\varepsilon \overset{q}{a}{}_i^{-\varepsilon}$ is removed we say that $\overset{p}{a}{}_i^\varepsilon$ *cancels* with $\overset{q}{a}{}_i^{-\varepsilon}$. It will be important to know how cancelling proceeds when obtaining a reduced word for expressions like $w(-a_j)$. In some cases an occurrence of a_i must cancel with a particular occurrence of a_i^{-1}, but (by associativity) if cancelling can proceed in different orders there may be several occurrences of a_i^{-1} with which a_i may cancel.

An n-word w is *Brunnian* if $w(-a_i) = e$ for each $i < n$. Note that a Brunnian n-word represents a Brunnian $(n+1)$-link. Denote the set of Brunnian n-words by $B_{\mathrm{A}}(n)$.

Lemma 1. *For each n, $B_{\mathrm{A}}(n)$ is a normal subgroup of the free group on A_n.*

Proof. Suppose $v, w \in B_{\mathrm{A}}(n)$. Clearly $w^{-1} \in B_{\mathrm{A}}(n)$, $vw \in B_{\mathrm{A}}(n)$ and $vwv^{-1} \in B_{\mathrm{A}}(n)$. If w is a Brunnian n-word and v is equivalent to w, then clearly v is a Brunnian n-word, and hence $B_{\mathrm{A}}(n)$ is normal. ∎

The *length* of a word w, denoted $l(w)$, is zero if $\varrho(w) = e$, otherwise it is the number of letters in $\varrho(w)$ with exponent 1 or -1. A Brunnian n-word, w, is *minimal* if $l(w) \le l(v)$ for all $v \in B_{\mathrm{A}}(n)$.

A word w contains a *copy* of $a_{i_0} a_{i_1} \ldots a_{i_{m-1}}$ if there are strings $\beta_0, \beta_1, \ldots, \beta_m$ such that $w = \beta_0 a_{i_0} \beta_1 a_{i_1} \ldots a_{i_{m-1}} \beta_m$. A Brunnian n-word, w, is *basic* if it does not contain a copy of a word having the form of a Brunnian n-word other than w itself.

The following properties are immediately obvious from the definition of a Brunnian n-word.

1. Let w be a Brunnian n-word, $n > 1$. Then:
   (a) if $w = \alpha \overset{1}{a_i} \beta$ and w is reduced, then β contains a copy of α^{-1};
   (b) a_i^{-1} occurs in w exactly the same number of times as a_i;
   (c) if $w \neq e$ then for each $i < n$ there exist $\varepsilon, \varepsilon' \in \mathbb{Z}$ and $j < n$ such that $a_j^{\varepsilon} a_i^{\varepsilon'} a_j^{-\varepsilon}$ occurs in w;
   (d) if $w \neq e$ then for each $i < n$ there is a non-empty string φ such that $a_i \notin A(\varphi)$ and $a_i \varphi a_i^{-1}$ occurs in w;
   (e) if w is basic then $w \neq a_i \varphi a_i^{-1}$ for any string φ;
   (f) $\varrho((w(-a_i))(-a_j)) = \varrho((w(-a_j))(-a_i)) = e$ for every $i, j < n$ for every $i, j < n$ (by associativity).
2. w is a Brunnian n-word if and only if $a_i w a_i^{-1}$ is a Brunnian n-word for all $i < n$.

Property 2 is equivalent to: $\alpha\beta$ is a Brunnian n-word if and only if $\beta\alpha$ is a Brunnian n-word. Of course $\alpha\beta$ and $\beta\alpha$ represent the same Brunnian link. They simply relate to different starting points.

This relation is important when deducing how words are formed from subwords. Let $\sim_{\mathrm{S}}$ be the conjugacy equivalence relation over $B_{\mathrm{A}}(n)$: $w \sim_{\mathrm{S}} v$ if and only if there exist strings α and β such that $w = \alpha\beta$ and $v = \beta\alpha$.

We introduce another equivalence relation which will help to simplify proofs in Section 3. Many of these proofs involve certain types of strings which occur in a given word. These strings may take a variety of forms depending on whether the occurrence of each letter has index ± 1. To prove these lemmas for each one of the different forms involves repetitious arguments. In order to circumvent such repetition we define $\alpha \equiv \beta$ if and only if there is a sequence $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_m = \beta$ such that for each i we have $\alpha_{i+1} = \alpha_i(a_j^{-1}/a_j)$ for some j. We will prove each lemma for one case (one element of the relevant $\equiv$ equivalence class) whenever it is clear that the other cases follow a similar argument.

The following is a natural extension of the notion of a Brunnian word. An *mn-word*, $m \geq n$, is a word w such that w is an m-word and $w(-a_i) = e$ for all $i < n$. Extending our results on Brunnian words, we will compute the form and length of minimal mn-words.

3. Minimal Brunnian words. In this section we determine the length, form and number of minimal Brunnian n-words. We require several technical lemmas.

LEMMA 2. *Suppose w is a reduced Brunnian n-word, $n > 1$, and suppose $j < n$. Pick integers i, r, s and ε, and strings α and β such that, modulo $\equiv$,*

$$w = \alpha \overset{r}{a_i} a_j^{\varepsilon} (\overset{s}{a_i^{-1}}) \beta.$$

Partition all the occurrences of $a_i^{\pm 1}$ into the maximum number of components such that if $k \neq i$ or j, then any member of any component can only cancel with another member of the same component in the reduction of $w(-a_k)$. Let R be the component containing $\overset{r}{a_i}$, and S the union of all the other components. Let $v_1 = \varrho(w(-R))$ and $v_2 = \varrho(w(-S))$. Let u'_R and u'_S be the words obtained from $w(-a_j)$ by replacing with $(a_j a_i^{-1} a_j^{-1})^{\pm 1}$ each occurrence of $a_i^{\pm 1}$ in R, for u'_R, and each occurrence of $a_i^{\pm 1}$ in S, for u'_S. Let $u_R = \varrho(u'_R)$ and $u_S = \varrho(u'_S)$. Then

(i) $\overset{s}{a_i^{-1}} \notin R$.

If w is minimal then

(ii) $v_l(-a_j) \neq e$ *whenever* $v_l \neq e$, $l = 1, 2$.

If a_j occurs at least as frequently as a_i in w, then:

(iii) *either* $v_1 \neq e$ *or* $v_2 \neq e$;
(iv) $|R| = |S|$, *and S is a single component*;
(v) a_i *occurs the same number of times in w as a_j*;
(vi) $l(u_R) = l(u_S) = l(w)$.

Proof. (i) If $\overset{s}{a_i^{-1}} \in R$ then there is a sequence

$$\overset{r}{a_i} = \overset{p_0}{a_i}, \overset{p_1}{a_i^{-1}}, \ldots, \overset{p_H}{a_i^{-1}} = \overset{s}{a_i^{-1}},$$

such that for all h there is a $k \neq i, j$ such that $\overset{p_h}{a_i^{\varepsilon}}$ cancels with $\overset{p_{h+1}}{a_i^{-\varepsilon}}$ in the reduction of $w(-a_k)$.

Hence a_j occurs the same number of times as a_j^{-1} between each pair $\overset{p_h}{a_i^{\varepsilon}}$ and $\overset{p_{h+1}}{a_i^{-\varepsilon}}$. It follows easily, by induction, that a_j and a_j^{-1} occur the same number of times between $\overset{r}{a_i}$ and any other member of the sequence. This is not the case, however, for $\overset{r}{a_i}$ and $\overset{s}{a_i^{-1}}$. Hence $\overset{s}{a_i^{-1}} \notin R$.

(ii) Suppose $v_1 \neq e$. If $\varrho(v_1(-a_j)) = e$ then v_1 is an n-word since by associativity, for each $k \neq i, j$ members of S cancel with each other in the reduction of $v_1(-a_k)$ so that $\varrho(v_1(-a_k)) = e$. And clearly $\varrho(v_1(-a_i)) = e$. Thus v_1 is a copy of an n-word in w, but $v_1 \neq w$, hence $\varrho(v_1(-a_j)) \neq e$.

(iii) Suppose $v_1 = v_2 = e$. Let w_1 be the word obtained from w by replacing each occurrence of $a_i^{\pm 1}$ in R by $a_n^{\pm 1}$.

If $\varrho(w_1(-a_j)) \neq e$ then $\varrho(w_1(-a_j))$ has the form of a Brunnian n-word since for each $k \neq i, j$, $\varrho(w_1(-a_k)) = \varrho(w(-a_k)) = e$, $\varrho(w_1(-a_n)) = v_1 = e$, and $\varrho(w_1(-a_i)) = e$ because $v_2 = e$. Since $l(\varrho(w_1(-a_j))) < l(w)$ we have a contradiction, and hence $\varrho(w_1(-a_j)) = e$. Thus w_1 is an $(n+1)$-word.

Pick a_{j1} and a_{i1}, and components R_1 and S_1 of w_1 analogously to R and S in w (pick one of the most frequently occurring letters for a_{j1}). If $v_{11} = \varrho(w_1(-R_1)) = e$ and $v_{12} = \varrho(w_1(-S_1)) = e$, obtain u by replacing each $a_{i1}^{\pm 1}$ in R_1 by $a_{n+1}^{\pm 1}$ and reducing. If $\varrho(u(-a_{j1})) \neq e$ then let $w_2 = \varrho(v(-a_{j1}))$, otherwise let $w_2 = u$.

If $v_{11} \neq e$ and $\varrho(v_{11}(-a_j)) = e$, let $w_2 = v_{11}$. It is easy to show that w_2 is an $(n+1)$-word. If $v_{11} \neq e$ and $\varrho(v_{11}(-a_j)) \neq e$, let $w_2 = \varrho(v_{11}(-a_j))$. Continue by induction. For some m we must obtain a word w_m, corresponding to a word in w that has the form of a Brunnian n-word. Since $l(w_m) < l(w)$ we have a contradiction and hence either $v_1 = e$ or $v_2 = e$.

(iv), (v) and (vi). We first prove

Claim. *$u_R \neq e \neq u_S$ and u_R and u_S are Brunnian n-words.*

Proof. We first show that $u_R \neq e \neq u_S$. If $\overset{t}{a_i^{\varepsilon}}$ in u'_R is part of a string that replaced a member of R in w, and $\overset{t'}{a_i^{-\varepsilon}}$ in u'_R was a member of S in w, then $a_j^{\pm 1}$ must occur an odd number of times between $\overset{t}{a_i^{\varepsilon}}$ and $\overset{t'}{a_i^{-\varepsilon}}$, twice for each member from R that occurs between them, and once beside $\overset{t}{a_i^{\varepsilon}}$. Hence when cancelling to obtain u_R, members of R must cancel with each other and members of S must cancel with each other. Members of R cancel to obtain $\varrho(w(-a_j))$ if and only if the corresponding strings $a_j^{\pm\varepsilon} a_i^{\pm\varepsilon} a_j^{\mp\varepsilon}$ cancel when u'_R is reduced. Hence if $u_R = e$, then $\varrho(w(-a_j)(-R)) = e$ and

$\varrho(w(-a_j)(-S)) = e$. But since either $\varrho(v_1(-a_j)) \neq e$ or $\varrho(v_2(-a_j)) \neq e$ by (ii) and (iii), $u_R \neq e$. Similarly, $u_S \neq e$.

We now show that u_R and u_S are Brunnian n-words. We have

$$\varrho(u_R(-a_j)) = \varrho(w(-a_j)) = e, \qquad \varrho(u_R(-a_i)) = \varrho(w(-a_j)(-a_i)) = e.$$

Suppose $k \neq i, j$ and consider $u_R(-a_k)$. Since members of R need only cancel with members of R in the reduction of $w(-a_k)$ and hence in the reduction of $w(-a_j)(-a_k)$, it follows that the strings $a_j^{\pm\varepsilon} a_i^{\pm\varepsilon} a_j^{\mp\varepsilon}$ cancel with each other and

$$\varrho(u_R(-a_k)) = \varrho(w(-a_k)(-a_j)) = e.$$

We may argue similarly for u_S. Thus u_R and u_S are Brunnian n-words, and the claim is proved. ■

The number of $a_j^{\pm 1}$'s occurring in u_R is at most $2|R|$, and in u_S at most $2|S|$. If $a_j^{\pm 1}$ occurs J times in w then $J \geq |R| + |S|$, since a_j occurs at least as often as a_i in w. Since w is minimal, $l(u_R) \geq l(w)$ and $l(u_S) \geq l(w)$, but if $l(u_R) > l(w)$ then $l(u_S) < l(w)$, and if $l(u_S) > l(w)$, then $l(u_R) < l(w)$. Hence a_j occurs $2|S| = 2|R|$ times, $|S| = |R|$, and S is a single component. ■

Lemma 3. *Suppose w is a minimal Brunnian n-word in which a particular collection of letters only appear in strings $\alpha^{\pm 1}$ where α is a Brunnian word, $\alpha a_j^{\varepsilon} \alpha^{-1}$ occurs in w, and a_j does not occur less frequently than α. Then it is possible to replace exactly half the occurrences of $\alpha^{\pm 1}$ in $w(-a_j)$ by $a_j^{\pm 1} \alpha^{\pm 1} a_j^{\mp 1}$, and reduce to obtain an n-word that is also minimal.*

Proof. The proof of Lemma 2 can easily be adjusted to this situation. ■

Lemma 4. *If w is a minimal Brunnian n-word then for each $i < n$ and $\varepsilon = \pm 1$, a_i^{ε} does not occur consecutively in w.*

Proof. Obvious if $n \leq 2$. Suppose $n > 2$ and $w = a_i a_i \alpha$ (it is sufficient to obtain a contradiction for this case). Partition the occurrences of $a_i^{\pm 1}$ in w such that members of each component can only cancel with each other in $\varrho(w(-a_j))$ for each $j \neq i$. If $\overset{1}{a}_i$ and $\overset{2}{a}_i$ are in the same component, then there exists a sequence $\overset{1}{a}_i = \overset{r_0}{a}_i, \overset{r_1}{a}{}_i^{-1}, \overset{r_2}{a}_i, \ldots, \overset{r_m}{a}_i = \overset{2}{a}_i$ such that each $\overset{r_h}{a}{}_i^{\varepsilon}$ cancels with $\overset{r_{h+1}}{a}{}_i^{-\varepsilon}$ in $\varrho(w(-a_j))$ for some j. Then a_i and a_i^{-1} occur an equal number of times between each pair $\overset{r_h}{a}{}_i^{\varepsilon}$ and $\overset{r_{h+1}}{a}{}_i^{-\varepsilon}$.

Let n_h be the total number of occurrences of $a_i^{\pm 1}$ between $\overset{1}{a}_i$ and $\overset{r_h}{a}{}_i^{\varepsilon}$. A simple induction argument shows that n_h is always even if $\varepsilon = -1$ and odd if $\varepsilon = 1$. But $\overset{r_{m-1}}{a}{}_i^{-1}$ cancels with $\overset{2}{a}_i$ so there must be an even number between $\overset{r_{m-1}}{a_i^{-1}}$ and $\overset{2}{a_i}$. However, there are $n_m - 1$ occurrences between $\overset{r_{m-1}}{a_i^{-1}}$ and $\overset{2}{a_i}$, giving a contradiction. ∎

LEMMA 5. *For each $n > 1$, if w is a minimal Brunnian n-word, then there is a Brunnian n-word v such that*:

(i) *all letters occur the same number of times in v as in w*;
(ii) *if n is even, then all letters in v occur in strings $a_i a_j a_i^{-1} a_j^{-1}$ (modulo $\equiv$), and for any two such strings α and β, either $A(\alpha) = A(\beta)$ or $A(\alpha) \cap A(\beta) = \emptyset$; if n is odd, then all letters except one of the least occurring letters, a_k say, occur in such strings, and a_k only occurs in strings $a_k(a_i a_j a_i^{-1} a_j^{-1}) a_k^{-1} (a_i a_j a_i^{-1} a_j^{-1})^{-1}$ (modulo $\equiv$, modulo $\sim_{\mathrm{S}}$).*

Proof. By Lemma 2(v),(vi) and Lemma 4, obtain from w a minimal Brunnian n-word u such that one of the most frequently occurring letters, $a_j^{\pm 1}$, only occurs in strings $a_j^{\varepsilon} a_i^{\varepsilon} a_j^{-\varepsilon}$, $\varepsilon = \pm 1$, and $a_i^{\pm 1}$ and $a_j^{\pm 1}$ occur the same number of times in u as in w. Pick one of the strings $\overset{x}{a_j^{\varepsilon}}\, a_i^{\varepsilon}\, \overset{y}{a_j^{-\varepsilon}}$ occurring in u. Partition the occurrences of a_j in u into a maximal number of components such that members of each component can only cancel with members of the same component in the reduction of $u(-a_k)$ if $k \neq i, j$. Let X be the component containing $\overset{x}{a_j^{\varepsilon}}$ and Y the component containing $\overset{y}{a_j^{-\varepsilon}}$. By arguing as in the proof of Lemma 2(i) we can deduce that for each string $a_j^{\varepsilon} a_i^{\varepsilon} a_j^{-\varepsilon}$ occurring in u, one of $a_j^{\pm 1}$ is in X and the other is in Y.

Let u_1 be the word obtained from u by removing every occurrence of a_i and replacing every member $a_j^{\pm\varepsilon}$ of X by $a_i^{\varepsilon} a_j^{\varepsilon} a_i^{-\varepsilon}$. Then the letters $a_i^{\pm 1}$ and $a_j^{\pm 1}$ only occur in u_1 as part of a string $a_i a_j a_i^{-1} a_j^{-1}$ (modulo $\equiv$), and by Lemma 2(vi), $l(u) = l(u_1)$.

Suppose a_k is one of the most frequently occurring letters other than a_i or a_j. Pick a_l such that $a_l^{\varepsilon} a_k^{\varepsilon} a_l^{-\varepsilon}$ occurs. As above, obtain u_2 such that $a_k^{\pm 1}$ and $a_l^{\pm 1}$ only occur in strings $(a_k a_l a_k^{-1} a_l^{-1})^{\pm 1}$.

Suppose $i = l$. Then all occurrences of $a_i^{\pm 1}$, $a_j^{\pm 1}$ and $a_k^{\pm 1}$ are (modulo $\equiv$) in strings $\alpha^{\pm 1}$ and $\beta^{\pm 1}$, where

$$\alpha = a_i a_j a_i^{-1} a_j^{-1} \quad \text{and} \quad \beta = a_k a_i a_k^{-1} a_i^{-1} a_j a_i a_k a_i^{-1} a_k^{-1} a_j^{-1}.$$

Let $u' = u_2(a_k a_i a_j a_i^{-1} a_j^{-1} a_k^{-1} / \beta)$. Clearly, $\varrho(v) \neq e$, and it is not too difficult to show that v is an n-word. Once again we have a contradiction, since u' is shorter than v_2, and so $l \neq i$. Similarly $l \neq j$.

Continue by induction to obtain a Brunnian n-word u_m in which all the letters, except one if n is odd, only occur in strings $a_i a_j a_i^{-1} a_j^{-1}$ (modulo $\equiv$), and for any two such strings α and β, either $A(\alpha) = A(\beta)$ or $A(\alpha) \cap A(\beta) = \emptyset$.

If n is even we are done. If n is odd, let a_k be the only letter that does not occur in such a string. Pick α such that $\alpha a_k \alpha^{-1}$ occurs in u_m. Then a_k does not occur less often than α in u_m (otherwise swap the occurrences of $\alpha^{\pm 1}$ with those of $a_k^{\pm 1}$ to obtain a Brunnian n-word shorter than w). By Lemma 3 we are done. ∎

3.1. *Length.* We now determine the length of a minimal Brunnian n-word. We first determine an upper bound.

LEMMA 6. *Suppose $n = 2^m + k$, $k < 2^m$ and $n > 0$. Then there is a Brunnian n-word, w_n, such that $l(w_n) = 2^m(3k + 2^m)$, $2^m - k$ letters occur 2^m times and $2k$ letters occur 2^{m+1} times in w_n.*

Proof. Let $w_0 = a_0$, a Brunnian 1-word. Suppose $n = 2^m + k$, $k < 2^m$ and w_n is a Brunnian n-word such that:

- w_n has length $2^m(3k + 2^m)$;
- $2^m - k$ letters occur 2^m times and $2k$ letters occur 2^{m+1} times.

Pick one of the letters a_i that occurs 2^m times and let

$$w_{n+1} = w_n(a_i a_n a_i^{-1} a_n^{-1} / a_i).$$

Then $2^m - (k+1)$ letters occur 2^m times in w_{n+1}, and since both $a_i^{\pm 1}$ and $a_n^{\pm 1}$ occur twice as often in w_{n+1} as a_i occurred in w_n, there are $2k + 2$ letters occurring 2^{m+1} times in w_{n+1}. Also, $l(w_{n+1}) = l(w_n) + 3 \cdot 2^m = 2^m(3(k+1) + 2^m)$. If $k + 1 = 2^m$, then $l(w_{n+1}) = 2^{m+1}2^{m+1}$. ∎

THEOREM 7. *If w is a minimal Brunnian n-word, $0 < n = 2^m + k$ and $k < 2^m$, then $l(w) = 2^m(3k + 2^m)$.*

Proof. Clearly the theorem holds when $n = 1$. Suppose that it holds for each $n' < n$. If n is even, by Lemma 5 pick a minimal Brunnian n-word, w, such that the letters only occur in strings $\alpha_t^{\pm 1}$ where $\alpha_t = a_t a_{n-(t+1)} a_t^{-1} a_{n-(t+1)}^{-1}$, $0 \leq t < n/2$. Let v be the word obtained by replacing each α_t by a_t. Then v is an $n/2$-word. Clearly w is minimal if and only if v is minimal, and $l(w) = 4l(v) = 4 \cdot 2^{m-1}(3k/2 + 2^{m-1}) = 2^m(3k + 2^m)$.

Suppose n is odd. Pick a minimal Brunnian n-word, w, such that all the letters other than a_0 only occur in strings $\alpha_t^{\pm 1}$, where $\alpha_t = a_t a_{n-t} a_t^{-1} a_{n-t}^{-1}$, $0 < t < (n-1)/2$, and a_0 only occurs in strings $\beta = a_0 \alpha_1 a_0^{-1} \alpha_1^{-1}$. Suppose that the number of occurrences of the $\alpha_t^{\pm 1}$'s in w is x, and $\beta^{\pm 1}$ occurs y times. Obtain a Brunnian word u from w by replacing each $\alpha_i^{\pm 1}$ by $a_i^{\pm 1}$, and each $\beta^{\pm 1}$ by $a_0^{\pm 1}$. Then u has the form of a Brunnian $(n-1)/2$-word. Obtain a word v from w with the form of a Brunnian $(n+1)/2$-word, by replacing each $\alpha_i^{\pm 1}$ by $a_i^{\pm 1}$. Then

$$l(w) = 4x + 10y \leq 2^m(3k + 2^m),$$
$$l(u) = x + y \geq 2^{m-1}\left(3\left(\frac{k-1}{2}\right) + 2^{m-1}\right),$$
$$l(v) = x + 4y \geq 2^{m-1}\left(3\left(\frac{k+1}{2}\right) + 2^{m-1}\right),$$

and hence

$$2^m(3k + 2^m) \geq 4x + 10y = 2(l(u) + l(v)) \geq 2^m(3k + 2^m).$$

Hence $l(w) = 2^m(3k + 2^m)$. ■

3.2. *Form.* We can now establish that any minimal Brunnian word must have a certain form.

Let $\mathcal{M}'$ be the collection of words such that:

(i) $\forall a_i \in A$, $a_i, a_i^{-1} \in \mathcal{M}'$;
(ii) $\forall v, w \in \mathcal{M}'$, if $A(v) \cap A(w) = \emptyset$ then $vwv^{-1}w^{-1} \in \mathcal{M}'$;
(iii) $\forall w \in \mathcal{M}'$, if $v \sim_{\mathrm{S}} w$ then $v \in \mathcal{M}'$,

and let

$$\mathcal{M} = \{w \in \mathcal{M}' : (\exists n \in \omega)\ A(w) = A_n\}.$$

Observe that every $w \in \mathcal{M}$ is $\sim_{\mathrm{S}}$ equivalent to a string of the form $\alpha\beta\alpha^{-1}\beta^{-1}$ (provided $l(w) > 1$). Also note that every word in $\mathcal{M}$ is a basic Brunnian word. Finally, we remark that not all Brunnian words have this form (for example, $a_0a_1a_0^{-1}a_2a_0a_1^{-1}a_0^{-1}a_0^{-1}a_1a_0a_2^{-1}a_0^{-1}a_1^{-1}a_0$).

THEOREM 8. *Every minimal Brunnian n-word w is in $\mathcal{M}$.*

Proof. We will assume that all strings in this proof are chosen modulo $\equiv$. We prove this by induction on n. If w is a minimal Brunnian 1-word then $w = a_0^{\pm 1} \in \mathcal{M}$. If w is a minimal Brunnian 2-word then w is a non-trivial arrangement of the letters a_0, a_0^{-1}, a_1 and a_1^{-1} and hence $w \in \mathcal{M}$. If w is a minimal 3-word then one letter occurs twice. Hence w has the form $a_0\alpha a_0^{-1}\alpha^{-1}$ and clearly $\alpha = a_1a_2a_1^{-1}a_2^{-1}$.

Suppose for each $n' < n$ any minimal n'-word is a member of $\mathcal{M}$, and $n > 3$.

Suppose a_j is one of the most frequently occurring letters and $a_ia_ja_i^{-1}$ occurs in w. By Lemma 2(ii),(iii) and Lemma 6, remove half the occurrences of a_i from $w(-a_j)$ and reduce to obtain a minimal $(n-1)$-word v. Then $v = \alpha\beta\alpha^{-1}\beta^{-1}$. We may assume that $a_i^{\pm 1}$ occurs in α and not in β. Pick one of the most frequently occurring letters a_k in β, which only occurs in strings $a_ka_la_k^{-1}a_l^{-1}$ (which is possible since $n > 3$). At least one string $a_la_ka_l^{-1}$ corresponds to such a string in w. If not then $\varrho(w(-a_h)) \neq e$ for $h = i, l$ or k, or a_l must occur consecutively in w, contradicting Lemma 4. Remove half the occurrences of a_l from $w(-a_k)$ and reduce to obtain a minimal $(n-1)$-word v'. Now $v' = \alpha'\beta'(\alpha')^{-1}(\beta')^{-1}$, $a_i^{\pm 1}$ and $a_j^{\pm 1}$ only occur in α' say, and they only occur in strings $a_i a_j a_i^{-1} a_j^{-1}$. Hence it is clear that $w = \alpha'\beta(\alpha')^{-1}\beta^{-1} \in \mathcal{M}$. ∎

3.3. *Number.* In this section we establish the number of different forms of a minimal Brunnian n-word. We consider the words $v, w \in \mathcal{M}$ to have the same *form* if there is a sequence $u_0, u_1, \ldots, u_m$ such that $u_0 = v$, $u_m = w$ and for each $h < m$ either $u_h \equiv u_{h+1}$, u_{h+1} may be obtained from u_h by interchanging all occurrences of $a_i^{\pm 1}$ with $a_j^{\pm 1}$ for some $i, j < n$, or $u_{h+1} = u_h(\beta/\alpha)$ where $\alpha \sim_{\mathrm{S}} \gamma\delta\gamma^{-1}\delta^{-1}$, α occurs in u_h and $\beta \sim_{\mathrm{S}} \alpha$. We associate members of $\mathcal{M}$ with binary trees, so that the number of forms of a word is the number of non-isomorphic trees that are associated with minimal n-words.

For each $w \in \mathcal{M}$ construct a tree T_w such that the root is the word w, if a member of T_w is a string of length 1 then it has no successors, and if it is a string σ and $\sigma \sim_{\mathrm{S}} \alpha\beta\alpha^{-1}\beta^{-1}$, then it has two successors, α and β. Observe that:

- T_w is unique up to isomorphism;
- if $w \in \mathcal{M}$ and $|A(w)| = n$, then T_w has n leaves (members of T with no successors);
- v and w have the same form if and only if T_v and T_w are isomorphic.

Let $f(m,0) = f(m,1) = f(m,2^m) = 1$ for every m. Define recursively, for $m > 1$ and $k < 2^m$,

$$f(m,k) = \begin{cases} \displaystyle\sum_{l=0}^{\lfloor k/2 \rfloor} f(m-1,l) \cdot f(m-1,k-l) & \text{if } k \le 2^{m-1}, \\ \displaystyle\sum_{l=k-2^{m-1}}^{\lfloor k/2 \rfloor} f(m-1,l) \cdot f(m-1,k-l) & \text{if } k > 2^{m-1}. \end{cases}$$

THEOREM 9. *Let $n = 2^m + k$ where $k < 2^m$. The number of forms of a minimal Brunnian n-word is $f(m,k)$.*

Proof. The number of forms is the number of non-isomorphic binary trees with $2^m - k$ leaves at height m, and $2k$ leaves at height $k+1$. Suppose T is a finite binary tree, and l and r are the two immediate successors of the root of T. Let $L(T)$ be the subtree of all successors of l including l, and $R(T)$ the subtree of all successors of r including r. Then two trees S and T are isomorphic if and only if $L(S)$ is isomorphic to $L(T)$ and $R(S)$ is isomorphic to $R(T)$, or $L(S)$ is isomorphic to $R(T)$ and $R(S)$ is isomorphic to $L(T)$. The number of such non-isomorphic trees is $f(m,k)$. ∎

We may also use the binary trees to determine the length of members of $\mathcal{M}$ that have particular forms. Let $\mathcal{B}$ be the isomorphism classes of $\{T_w : w \in \mathcal{M}\}$. We will simply speak of the element T in $\mathcal{B}$ rather than the equivalence classe $[T]$. Now define $l' : \mathcal{B} \to \omega$ as follows. Given $T \in \mathcal{B}$ assign each leaf the number 1. Each vertex is assigned the number equal to twice the sum of the numbers assigned to its immediate successors. Then $l'(T)$ is the number assigned to the root of T. It is not difficult to conclude that $l(w) = l'(T_w)$. Note that l' is well defined but not 1-1.

One can now easily prove:

LEMMA 10. *The maximum length of an n-word in $\mathcal{M}$ is* $2^n + 2^{n-2} + 2^{n-3} + \dots + 2$.

4. mn-words. We now address the minimal length of an mn-word, first establishing the form of a minimal mn-word.

LEMMA 11. *If* $0 < n \le m$ *then any minimal mn-word has the form*

$$a_{i_n}^{\varepsilon_n} a_{i_{n+1}}^{\varepsilon_{n+1}} \dots a_{i_{m-1}}^{\varepsilon_{m-1}} u a_{a_{m-1}}^{-\varepsilon_{m-1}} \dots a_{a_n}^{-\varepsilon_n},$$

where each $\varepsilon_h = \pm 1$, *and u is a minimal Brunnian n-word.*

Proof. Let w be a minimal mn-word which is not an mn'-word for any $n' > n$. When $m = 1$ the claim is obvious. So we argue by induction on m.

Suppose w is a minimal mn-word and any minimal rn-word has the required form if $r < m$. Let $w' = \varrho(w(-a_{m-1}))$. If w is also an mn'-word we are done. It might not be an $m'n'$-word, but if we rename the letters appropriately, we can ensure that it is. Hence, without loss of generality, assume that for some $n \le n' \le m' < m$, w' is an $m'n'$-word but not an $m'(n'-1)$-word. Then $v = a_{m'} a_{m'+1} \dots a_{m-1} w'(-a_{m-1}) a_{m-1}^{-1} \dots a_{m'}^{-1}$ is an mn'-word, and $l(v) \le l(w)$ since the minimum collection of letters removed from w in w' is one occurrence of each of $a_{m'}^{\pm 1}, \dots, a_{m-1}^{\pm 1}$. It follows that w' is a minimal $m'n'$-word

$$a_{i_n'}^{\varepsilon_n} a_{i_{n+1}}^{\varepsilon_{n+1}} \dots a_{i_{m'-1}}^{\varepsilon_{m-1}} u a_{a_{m'-1}}^{-\varepsilon_{m-1}} \dots a_{a_n'}^{-\varepsilon_n},$$

where u is a minimal n'-word, and exactly one occurrence of each of $a_{m'}^{\pm 1}, \dots, a_{m-1}^{\pm 1}$ cancelled in $\varrho(w(a_{m-1}))$.

By Lemma 6 and Theorem 7, if $n' > n$ then the length of u is greater than the length of a minimal n-word by at least $6(n' - n)$, hence $n' = n$ and u is a minimal n-word.

Now suppose w does not have the required form. Then for some $i \ge m'$, a_i occurs within u, and therefore a_i^{-1} also occurs in u. In fact more than one of each must occur in u (contradicting minimality), otherwise there must be an n-word occurring between the single occurrences of a_i and a_i^{-1} in u, giving a contradiction. ∎

COROLLARY 12. *If $n < m$ then the length of any mn-word is strictly greater than the length of a minimal Brunnian n-word.*

From our knowledge of minimal Brunnian words and Lemma 11 we immediately deduce:

THEOREM 13. *A minimal pq-word has length $L = 2(p-q)+2^m(3k+2^m)$ where $q = 2^m + k$ and $k < 2^m$.*

Proof. Follows immediately from Theorem 7 and Lemma 11. ∎

5. Braids. It seems natural to view a simple Brunnian link as a closed braid. In this section we briefly discuss this relationship which we will use in the following section to investigate when two Brunnian words represent the equivalent links that differ only in the order of the component loops.

Call a braid on n threads an *n-braid* and denote the threads in an n-braid by $t_0, t_1, \dots, t_{n-1}$. Let B_n be the group of all braids on n threads, so B_n has letters $\sigma_1, \dots, \sigma_{n-1}$ and defining relations

$$\sigma_i\sigma_k = \sigma_k\sigma_i, \qquad k \neq i-1, i+1,$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}.$$

See [H] for definitions of all standard braid terms used in this section.

The closure of a braid is a link, and any link is equivalent to a closed braid [A]. Two braids define equivalent oriented links if and only if there is a finite sequence of moves involving adding or deleting a thread which shows up as a free factor $\sigma_{n-1}^{\pm 1}$ or by conjugation, taking one braid to the other (this was first stated in [M] and later proved in [B]).

Simple Brunnian links have an obvious braid representation, and Brunnian words correspond naturally to certain braid words. It is easy to find a closed braid equivalent to any given simple Brunnian n-link, L. Take a word in $B_{\mathrm{A}}(n-1)$ representing L. Think of the braid obtained by running $n-1$ straight arcs vertically, and threading t_0 through them in the obvious way. If $w = a_{i_0}^{\varepsilon_0}a_{i_1}^{\varepsilon_1}\dots a_{i_m}^{\varepsilon_m}$, each $\varepsilon_h = \pm 1$, let t_0 run across the top of the threads to t_{i_0}, loop around it by passing over the top and back underneath if $\varepsilon_0 = 1$, or passing underneath and back over the top if $\varepsilon_0 = -1$, and return back across the top. Repeat for a_{i_1} etc. Hence if $w \in B_{\mathrm{A}}(n)$, then a braid whose closure is equivalent to the link represented by w may be obtained by replacing every occurrence of a_i in w by $\sigma_0\sigma_1\dots\sigma_i\sigma_i\sigma_{i-1}^{-1}\dots\sigma_0^{-1}$, and every occurrence of a_i^{-1} by $\sigma_0\dots\sigma_i^{-1}\sigma_i^{-1}\sigma_{i-1}^{-1}\dots\sigma_0^{-1}$. We call a closed braid of this form *canonical*. For example the 2-word $a_0a_1a_0^{-1}a_1^{-1}$ represents a link equivalent to the canonical braid $\sigma_0\sigma_0\sigma_0\sigma_1\sigma_1\sigma_0^{-1}\sigma_0^{-1}\sigma_0^{-1}\sigma_0\sigma_1^{-1}\sigma_1^{-1}\sigma_0^{-1}$. The closure of the braid $\sigma_0\sigma_0\sigma_0\sigma_1\sigma_1\sigma_0^{-1}\sigma_0^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_0^{-1}$ is equivalent to the link represented by $a_0a_1a_0^{-1}a_1^{-1}$.

Let $B_{\mathrm{B}}(n)$ be the set of n-braids whose closures are equivalent to simple Brunnian n-links and call these *Brunnian braids*. To simplify the expression of a Brunnian braid, let $\beta_i = \sigma_0\sigma_1 \ldots \sigma_i\sigma_i\sigma_{i-1}^{-1} \ldots \sigma_0^{-1}$ for each i. Given a canonical Brunnian braid $\beta_{i_0}^{\varepsilon_0}\beta_{i_1}^{\varepsilon_1} \ldots \beta_{i_m}^{\varepsilon_m}$, the corresponding Brunnian word is then $a_{i_0}^{\varepsilon_0}a_{i_1}^{\varepsilon_1} \ldots a_{i_m}^{\varepsilon_m} \in B_{\mathrm{A}}(n)$.

6. Equivalent Brunnian words. The objective of this section is to investigate when two Brunnian words give rise to topologically equivalent simple Brunnian links. Each Brunnian word represents a Brunnian link in canonical form. Hence $n-1$ loops are concentric about the origin and all loops have an anti-clockwise direction. It is clear, for example, that conjugate Brunnian words yield links which are topologically equivalent. The following theorem describes algebraic operations that yield topologically equivalent Brunnian links.

THEOREM 14. *Suppose v and w are two Brunnian n-words and there is a sequence $w_1, \ldots, w_m$ such that $v = w_1$, $w = w_m$ and w_{j+1} can be obtained from w_j by an operation of one of the following types*:

(i) *replace w by $a_i^k w a_i^{-k}$ for some $i < n$ and $k = \pm 1$;*
(ii) *replace all occurrences of $a_i^{\pm 1}$ with $(a_i^{-1}a_{i+1}a_i)^{\pm 1}$ and $a_{i+1}^{\pm 1}$ with $a_i^{\pm 1}$ (or symmetrically in i and $i+1$);*
(iii) *rewrite the word in the form $\delta_0 a_0^{\varepsilon_0}\delta_0^{-1}\delta_1 a_0^{\varepsilon_1}\delta_1^{-1} \ldots \delta_m a_0^{\varepsilon_m}\delta_m^{-1}$ such that for each h, a_0^{-1} does not occur in δ_h and $\delta_h \neq e$, and replace each $\delta_h a_0^{\varepsilon_h}\delta_h^{-1}$ with $\delta_h^{-1}a_0^{\varepsilon_h}\delta_h$.*

Then v and w generate equivalent links.

To prove this theorem we will exploit the connection between braids and links. Each operation corresponds to a straightforward topological operation. Since our motivation is to examine when Brunnian words represent equivalent simple Brunnian links, we are only concerned with canonical Brunnian braids. We describe the topological operation in each case, and then compute the algebraic equivalent to the topological operations.

(i) Conjugation is obvious.

(ii) This relates to swapping two adjacent curves, neither of which is the distinguished curve.

Suppose L is a canonical simple Brunnian n-link and B is the corresponding braid. Let C_i refer to the simple closed curve which is the closure of t_i. Obtain B' from B by swapping t_i and t_{i+1}. We consider two possible cases. Either t_i passes over the top of t_{i+1}, or it passes under it. This relates to C_{i+1} passing through the inside of C_i, or C_i passing through the inside of C_{i+1} (respectively).

Suppose first that t_i passes under t_{i+1}. Define $T : B_{\mathrm{B}}(n) \to B_{\mathrm{B}}(n)$ as follows: If $B \in B_{\mathrm{B}}(n)$, take $\sigma_i B \sigma_i^{-1}$ and pull the threads t_i and t_{i+1} tight to get B'. Let $T(B)$ be the canonical braid equivalent to B'. Then T induces a function, T^*, on the strings β_i occurring in braid words. T^* has no effect on β_j if $j \neq i, i+1$. Since T takes $\sigma_i \sigma_{i+1} \sigma_{i+1} \sigma_i^{-1}$ to $\sigma_i \sigma_i$, it follows that $T^*(\beta_{i+1}) = (\beta_i)$ (see Figure 2).

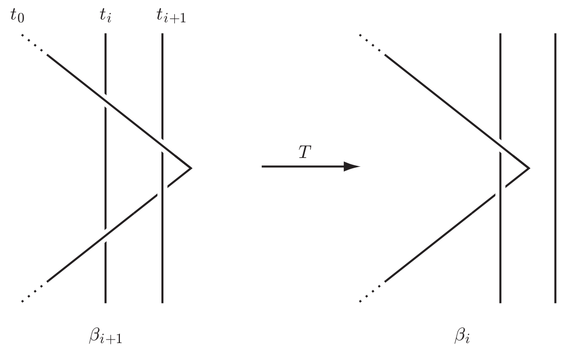

Fig. 2

Moreover, T takes $\sigma_i \sigma_i$ to $\sigma_i^{-1} \sigma_{i+1} \sigma_{i+1} \sigma_i$ and hence $T^*(\beta_i) = \beta_i^{-1} \beta_{i+1} \beta_i$ (see Figure 3).

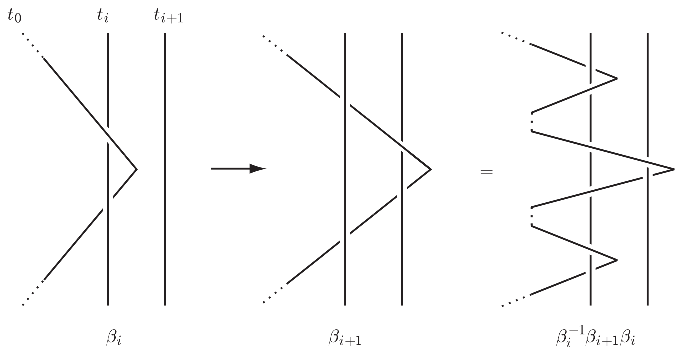

Fig. 3

For example if t_1 passes under t_2 and $B = \beta_0 \beta_1^{-1} \beta_0^{-1} \beta$, then $T(B) = \beta_0^{-1} \beta_1 \beta_0 \beta_1^{-1}$. Thus the words $a_0 a_1 a_0^{-1} a_1^{-1}$ and $a_0^{-1} a_1 a_0 a_1^{-1}$ define equivalent links.

Suppose now that t_i passes over t_{i+1}. Define $T : B_{\mathrm{B}}(n) \to B_{\mathrm{B}}(n)$ similarly to the above, but taking $\sigma_i^{-1} B \sigma_i$ rather than $\sigma B \sigma^{-1}$. Again T^* has no effect on β_j if $j \neq i, i+1$, while $T^*(\beta_i) = \beta_{i+1}$ and $T^*(\beta_{i+1}) = \beta_{i+1} \beta_i \beta_{i+1}^{-1}$.

For example if t_1 passes over t_2 and $B = \beta_0\beta_1\beta_0^{-1}\beta_1^{-1}$, then $T(B) = \beta_1\beta_1\beta_0\beta_1^{-1}\beta_0^{-1}\beta_1^{-1}$.

(iii) This relates to swapping the distinguished curve with C_0.

Suppose the closure of $B \in B_{\mathrm{B}}(n)$ is $L \in B_{\mathrm{S}}(n)$. Since we assume that B is canonical, t_0 corresponds to the distinguished curve in L. Let B'' be a braid with closure equivalent to L, but with t_1 as the distinguished curve. We will take B'' to be the braid derived from B by pulling the threads $t_0, t_2, t_3, \ldots, t_{n-1}$ taut, and letting t_1 loop around them. Then $\sigma_0 B'' \sigma_0^{-1}$ swaps t_0 and t_1. Let B' be the canonical braid equivalent to $\sigma_0 B'' \sigma_0^{-1}$. The process we will now describe will combine these two steps and transform B directly into B'.

Suppose $B = \beta_{i_0}^{\varepsilon_0}\beta_{i_1}^{\varepsilon_1}\ldots\beta_{i_l}^{\varepsilon_l}$ and consider the corresponding word $w = a_{i_0}^{\varepsilon_0}a_{i_1}^{\varepsilon_1}\ldots a_{i_l}^{\varepsilon_l} \in B_{\mathrm{A}}(n)$. We will construct a word w' which reduces to w, with the form $\delta_0 a_0^{\varepsilon_0}\delta_0^{-1}\delta_1 a_0^{\varepsilon_1}\delta_1^{-1}\ldots\delta_m a_0^{\varepsilon_m}\delta_m^{-1}$ such that for each h, a_0^{-1} does not occur in δ_h and $\delta_h \neq e$.

At least one occurrence of a_0^{ε} in w is flanked by some a_i and a_i^{-1}. For the pth occurrence of a_0^{ε} (possibly a string of length greater than 1) in w, let α_p be the maximal string such that $a_0^{\pm 1}$ does not occur in α_p and a_0^{ε} occurs in the string $\gamma_{0p} = \alpha_p a_0^{\pm 1}\alpha_p^{-1}$. If $w = \gamma_{00}\gamma_{01}\ldots\gamma_{0m}$ we are done, otherwise for each possible p let α_{1p} be the maximal string such that $a_0^{\pm 1} \notin \alpha_{1p}$ and $\alpha_{1p}\gamma_{0q}\gamma_{0(q+1)}\cdots\gamma_{0l_p}\alpha_{1p}^{-1}$ occurs in w (at least one such string occurs for cancelling to proceed in $\varrho(w(-a_0))$). Let w_1 be the word obtained from w by replacing each maximal string $\alpha_{1p}\gamma_{0q}\gamma_{0(q+1)}\cdots\gamma_{0l}\alpha_{1p}^{-1}$ by

$$\gamma_{1p} = \alpha_{1p}\gamma_{0q}\alpha_{1p}^{-1}\alpha_{1p}\gamma_{0(q+1)}\alpha_{1p}^{-1}\cdots\alpha_{1p}\gamma_{0l}\alpha_{1p}^{-1}.$$

Note that w_1 reduces to w.

Now change strings $\alpha_{2p}\gamma_{1q}\gamma_{1(q+1)}\cdots\gamma_{1l}\alpha_{2p}^{-1}$ in w_1, where α_{2p} are maximal, to

$$\gamma_{2p} = \alpha_{2p}\gamma_{1q}\alpha_{2p}^{-1}\alpha_{2p}\gamma_{1(q+1)}\alpha_{2p}^{-1}\cdots\alpha_{2p}\gamma_{1l}\alpha_{2p}^{-1},$$

if $w_1 \neq \gamma_{10}\gamma_{11}\ldots\gamma_{1m}$. Continue until $w_m = \gamma_{m0}\gamma_{m1}\ldots\gamma_{ml}$. Note that w_m reduces to w and has the form $\delta_0 a_0^{\varepsilon_0}\delta_0^{-1}\delta_1 a_0^{\varepsilon_1}\delta_1^{-1}\ldots\delta_m a_0^{\varepsilon_m}\delta_m^{-1}$ as required. Then $w' = w_m$.

Now consider the braid corresponding to w'. If α is a string in a Brunnian word w denote the corresponding string in the canonical braid word representing w by $b(\alpha)$. We can unravel each $b(\delta_p)\beta_0 b(\delta_p^{-1})$ to get $b(\delta_p^{-1})\beta_0 b(\delta_p)$. Each $b(\delta_p)\beta_0 b(\delta_p^{-1})$ is a symmetrical bit of the woven thread t_0, which picks up t_1 at the very centre. By unravelling it, t_1 is pulled through following $b(\delta_p)$ (and $b(\delta_p^{-1})$), but in the opposite direction. Turning t_1 into the first thread corresponds to conjugation by σ_0. Thus the final outcome is a replacement of each $b(\delta_p)\beta_0 b(\delta_p^{-1})$ with $b(\delta_p^{-1})\beta_0 b(\delta_p)$ and conjugation by σ_0.

Pulling threads other than t_1 taut and conjugating by σ_0 does not give a canonical braid. It would be necessary to strategically add strings $\sigma_i^{-1}\sigma_{i-1}\ldots\sigma_0^{-1}\sigma_0\sigma_1\ldots\sigma_i$ to the braid. However, this process is incorporated in the algebra.

For example, consider

$$B=\beta_0\beta_1\beta_0^{-1}\beta_1^{-1}\beta_2\beta_1\beta_0\beta_1^{-1}\beta_0^{-1}\beta_2^{-1}.$$

Then

$$\begin{aligned}
w&=a_0a_1a_0^{-1}a_1^{-1}a_2a_1a_0a_1^{-1}a_0^{-1}a_2^{-1},\\
w_1&=(e)a_0(e).(a_1)a_0^{-1}(a_1^{-1}).a_2.(a_1)a_0(a_1^{-1}).(e)a_0^{-1}(e).a_2^{-1},\\
w_2&=(e)a_0(e).(a_1)a_0^{-1}(a_1^{-1}).(a_2)(a_1)a_0(a_1^{-1})(a_2^{-1}a_2)(e)a_0^{-1}(e)(a_2^{-1})\\
&=(e)a_0(e).(a_1)a_0^{-1}(a_1^{-1}).(a_2a_1)a_0(a_1^{-1}a_2^{-1}).(a_2e)a_0^{-1}(ea_2^{-1})\\
&=w',
\end{aligned}$$

and

$$\begin{aligned}
B'&=(e)\beta_0(e).(\beta_1^{-1})\beta_0^{-1}(\beta_1).(\beta_1^{-1}\beta_2^{-1})\beta_0(\beta_2\beta_1)(\beta_2^{-1}e)\beta_0^{-1}(e\beta_2)\\
&=\beta_0\beta_1^{-1}\beta_0^{-1}\beta_1\beta_1^{-1}\beta_2^{-1}\beta_0\beta_2\beta_1\beta_2^{-1}\beta_0^{-1}\beta_2.
\end{aligned}$$

Figure 4(a) is the braid $\beta_0\beta_1\beta_0^{-1}\beta_1^{-1}$. Changing the distinguished loop gives 4(d), the braid $\beta_0\beta_1^{-1}\beta_0^{-1}\beta_1$. Figures 4(b) and 4(c) indicate how t_1 is pulled through as t_0 unravels.

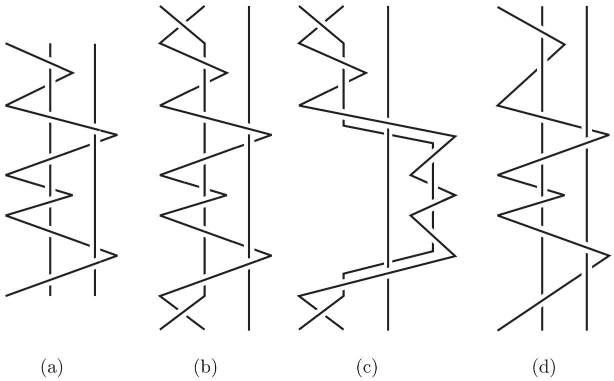

Fig. 2. Changing the distinguished loop

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