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On a formula for the asymptotic dimension of free products

by

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Abstract. We prove an exact formula for the asymptotic dimension (asdim) of a free product. Our main theorem states that if A and B are finitely generated groups with $\operatorname{asdim} A = n$ and $\operatorname{asdim} B \le n$, then $\operatorname{asdim}(A * B) = \max\{n, 1\}$.

1. Introduction. The asymptotic dimension of a metric space was defined by Gromov in [9] in his study of asymptotic properties of finitely generated groups. The asymptotic dimension (asdim) of a finitely generated group Γ is defined to be the asymptotic dimension of the metric space $|\Gamma|_S$ associated to a finite generating set S. The metric is the *word metric*, i.e., the maximal left-invariant metric with respect to the property that $\operatorname{dist}(s, e) = 1$ and $\operatorname{dist}(s^{-1}, e) = 1$ for all $s \in S$. Two finite generating sets give rise to Lipschitz equivalent metrics. As asdim is a coarse invariant (so in particular an invariant of Lipschitz equivalence), $\operatorname{asdim} \Gamma$ is well defined without reference to a generating set.

In [13], Yu proved that if Γ has finite asymptotic dimension then the coarse Baum–Connes conjecture and hence the Novikov higher signature conjecture hold for Γ. This result caused a lot of work to be devoted to showing the finiteness of asdim for various groups (see, for instance, [7], [4], and [10]).

The first two authors showed in [1] that the finiteness of asdim is preserved by the construction of the amalgamated free product of groups. The result was concerned with proving finite asdim and so the estimate given there was quite rough. This estimate was improved in [2], and examples were given to show that the upper bound could not be improved.

The purpose of this note is to prove a formula for the asdim of a free product. The proof uses the asymptotic analog of large inductive dimension,

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called asymptotic inductive dimension (asInd), defined by the second author in [5]. In particular we prove that $\operatorname{asdim}(A*B) = \max\{\operatorname{asdim} A, \operatorname{asdim} B, 1\}$. This formula agrees with the conjectured formula for the free product with amalgamation given in [2].

In §2 we recall the definitions of asdim and asInd. In the next section, we recall some technical results which are needed for the proof of the formula. The formula itself appears as the main theorem in the final section.

2. Two notions of asymptotic dimension. In [9] Gromov gave three definitions of asdim (asdim_+ in his notation). We recall them here. A family of subsets of a metric space is called *d-disjoint* if the distance between any two members of the family is at least d.

Definition 1. Let X be a metric space. Then $\operatorname{asdim} X \le n$ if for every (large) number $d > 0$ there exist $n+1$ uniformly bounded families of d-disjoint sets covering X.

Recall that the *Lebesgue number* of a cover $\mathcal{U}$ of a metric space X is $L(\mathcal{U}) = \inf\{\max\{d(x, X \setminus U) \mid U \in \mathcal{U}\} \mid x \in X\}$. Gromov's second definition is the following:

Definition 2. $\operatorname{asdim} X \le n$ if for any large number $L > 0$ there is a uniformly bounded cover $\mathcal{U}$ of X with multiplicity $\le n+1$ and with Lebesgue number $L(\mathcal{U}) > L$.

A map $\psi : X \to K$ from a metric space to a simplicial complex is *uniformly cobounded* if $\operatorname{diam}\{\psi^{-1}(\sigma)\}$ is uniformly bounded, where σ runs over all simplices in K. Any countable simplicial complex K can be realized in $\ell_2 = \ell_2(\mathbb{Z})$ by sending each vertex to an element of an orthonormal basis for ℓ_2. The simplicial complex K is called a *uniform simplicial complex* if it is endowed with the metric inherited from ℓ_2.

Projecting to the nerve of the cover in the second definition, one arrives at the final definition of asdim.

Definition 3. $\operatorname{asdim} X \le n$ if for any (small) $\varepsilon > 0$ there is a uniformly cobounded ε-Lipschitz map $\phi : X \to K$, where K is a uniform simplicial complex of dimension $\le n$.

The equivalence of these three definitions was proved explicitly in [2].

The asymptotic inductive dimension, asInd, of the metric space X was defined by the second author in [5] in order to establish connections between $\operatorname{asdim} X$ and $\operatorname{Ind} \nu X$, where νX is the Higson corona of X.

Let $\varphi : X \to \mathbb{R}$ be a function defined on a metric space X. For every $x \in X$ and every $r > 0$ let $V_r(x) = \sup\{|\varphi(y) - \varphi(x)| \mid y \in N_r(x)\}$, where $N_r(A)$ is the r-neighborhood of the set A. A function φ is called *slowly oscillating* whenever for every $r > 0$ we have $V_r(x) \to 0$ as $x \to \infty$ (the latter means that for every $\varepsilon > 0$ there exists a compact subspace $K \subset X$ such that $|V_r(x)| < \varepsilon$ for all $x \in X \setminus K$). Let $\overline{X}$ be the compactification of X that corresponds to the family of all continuous bounded slowly oscillating functions. The *Higson corona* of X is the remainder $\nu X = \overline{X} \setminus X$ of this compactification.

For any subset A of X we denote by A' its trace on νX, i.e. the intersection of the closure of A in $\overline{X}$ with νX. Obviously, the set A' coincides with the Higson corona νA.

Let X be a proper metric space. A subset $W \subset X$ is called an *asymptotic neighborhood* of a subset $A \subset X$ if $\lim_{r\to\infty} d(X \setminus N_r(x_0), X \setminus W) = \infty$. Two sets A, B in a metric space are called *asymptotically disjoint* if $\lim_{r\to\infty} d(A \setminus N_r(x_0), B \setminus N_r(x_0)) = \infty$. In other words, two sets are asymptotically disjoint if the traces A', B' on νX are disjoint.

A subset C of a metric space X is an *asymptotic separator* between asymptotically disjoint subsets $A_1, A_2 \subset X$ if the trace C' is a separator in νX between A_1' and A_2'. By the definition, $\operatorname{asInd} X = -1$ if and only if X is bounded. Suppose we have defined the class of all proper metric spaces Y with $\operatorname{asInd} Y \le n-1$. Then $\operatorname{asInd} X \le n$ if and only if for any asymptotically disjoint subsets $A_1, A_2 \subset X$ there exists an asymptotic separator C between A_1 and A_2 with $\operatorname{asInd} C \le n-1$. The dimension function asInd is called the *asymptotic inductive dimension*.

3. Previous results. We will use the following result ([6], [8]).

THEOREM 1. *For a proper metric space with bounded geometry X we have the equality*

$$\operatorname{asInd} X = \operatorname{asdim} X$$

provided $\operatorname{asdim} X < \infty$.

We also collect some results from [2] which we shall need.

PROPOSITION 1. *For every k and every $\varepsilon > 0$ there exists a number $\nu = \nu(\varepsilon, k)$ such that for every cover $\mathcal{U}$ of a metric space X of order $\le k+1$ with Lebesgue number $L(\mathcal{U}) > \nu$ the canonical projection to the nerve, $p_{\mathcal{U}} : X \to \operatorname{Nerve}(\mathcal{U})$, is ε-Lipschitz.*

PROPOSITION 2. *For every simplicial map $f : X \to Y$ the mapping cylinder M_f admits a triangulation with the set of vertices equal to the disjoint union of vertices of X and Y.*

PROPOSITION 3. *For every n there is a constant c_n so that for any simplicial map $g : X \to Y$ of an oriented n-dimensional simplicial complex X the quotient map $q : X \times [0,1] \to M_g$ of the product $X \times [0,1]$ to the mapping cylinder M_g is c_n-Lipschitz, where X and M_g are given the uniform metrics and $X \times [0,1]$ has the product metric.*

Proposition 4. *Let $A \subset W \subset X$ be subsets in a geodesic metric space X such that the r-neighborhood $N_r(A)$ is contained in W and let $f : W \to Y$ be a continuous map to a metric space Y. Assume that the restrictions $f|_{N_r(A)}$ and $f|_{W \setminus N_r(A)}$ are ε-Lipschitz. Then f is ε-Lipschitz.*

Lemma 1. *Let A be a closed subset of a geodesic metric space X. Let $r > 8\varepsilon$ and let $\mathcal{V}$ and $\mathcal{U}$ be covers of the r-neighborhood $N_r(A)$ by uniformly bounded open sets such that $\mathcal{V}$ has order $\leq n+1$,* Nerve$(\mathcal{V})$ *is orientable, and $L(\mathcal{U}) > b(\mathcal{V}) > L(\mathcal{V}) \geq \nu(\varepsilon/4c_n, n)$, where c_n is the constant from Proposition* 3. *Then there is an ε-Lipschitz map $f : N_r(A) \to M_g$ to the mapping cylinder, supplied with the uniform metric, of a simplicial map $g :$* Nerve$(\mathcal{V}) \to$ Nerve$(\mathcal{U})$ *between the nerves such that f is uniformly cobounded, $f|_{\partial N_r(A)} = q(p_{\mathcal{V}}|_{\partial N_r(A)}, 0)$, and $f|_A = p_{\mathcal{U}}|_A$, where $p_{\mathcal{U}} : N_r(A) \to$* Nerve$(\mathcal{U})$ *and $p_{\mathcal{V}} : N_r(A) \to$* Nerve$(\mathcal{V})$ *are the canonical projections to the nerves.*

Lemma 2. *Let Γ act by isometries on a tree X with compact quotient such that for all vertices $x \in X$, the stabilizers Γ_x are finitely generated and* $\operatorname{asdim} \Gamma_x \leq n$. *Let $x_0 \in X$, and define $\pi : \Gamma \to X$ by $\pi(\gamma) = \gamma x_0$. Then, for every $R > 0$,* $\operatorname{asdim} \pi^{-1}(B_R(x_0)) \leq n$.

In [1] and [2] the set $\pi^{-1}(B_R(x_0))$ was called the *R-stabilizer* of x_0.

4. Asdim of a free product. A free product is one of the simplest examples of a fundamental group of a graph of groups. According to the Bass–Serre theory, the free product acts isometrically and co-compactly on a tree. The tree and the action are well understood (see [3], [12]).

Proposition 5. *Let $\Gamma = A * B$. Let X denote the tree on which the group Γ acts by isometries, and let $\pi : \Gamma \to X$ be defined by $\pi(\gamma) = \gamma A$. Let W and W' be disjoint bounded subsets of X. Then $\pi^{-1}(W)$ and $\pi^{-1}(W')$ are asymptotically disjoint.*

Proof. First, we observe that if xA and yA are distinct vertices in X, then $x^{-1}y \notin A$, so $\operatorname{dist}(xa, ya') = \|a^{-1}x^{-1}ya'\| \geq \|a\| + \|a'\|$.

Let $\gamma \in \Gamma$. As W and W' are bounded in X, the set $\{xe_A \mid x \in W \cup W'\}$ is also bounded. So, we may take r so large that $B_{r/2}(\gamma) \subset \Gamma$ contains $\{xe_A \mid x \in W \cup W'\}$. Then $\operatorname{dist}(\pi^{-1}(W) \setminus B_r(\gamma), \pi^{-1}(W') \setminus B_r(\gamma)) \geq r/2 + r/2 = r$. Thus, $\lim_{r\to\infty} \operatorname{dist}(\pi^{-1}(W) \setminus B_r(\gamma), \pi^{-1}(W') \setminus B_r(\gamma)) = \infty$, as required. ∎

Theorem 2. *Let A and B be finitely generated groups with* $\operatorname{asdim} A = n$ *and* $\operatorname{asdim} B \leq n$. *Then* $\operatorname{asdim}(A * B) = \max\{n, 1\}$.

Proof. If $n = 0$ then take generators $a \in A$ and $b \in B$ and consider $\{(ab)^k\}_{k\in\mathbb{N}}$. This subset of $A * B$ forms an isometrically embedded copy of $2\mathbb{Z}$, and so $\operatorname{asdim}(A * B) \geq 1$. On the other hand, it was shown in [2] that $\operatorname{asdim}(A * B) \leq 1$ in this case. Hence, we conclude $\operatorname{asdim}(A * B) = 1$.

It remains to consider the case where $n \geq 1$. In any case, $A \subset A * B$, so $\operatorname{asdim}(A * B) \geq n$. Thus, it only remains to show that $\operatorname{asdim}(A * B) \leq n$.

Fix a symmetric generating set $S = S^{-1}$ for Γ. We abuse notation and implicitly identify Γ with the metric space $|\Gamma|_S$. We view Γ as the fundamental group of the segment of groups with vertex groups labeled A and B and with edge labeled $\{1\}$. Let X be the Bass–Serre tree associated to this graph of groups. Let x_0 be a base-point in X and let $\pi : \Gamma \to X$ be given by $\pi(\gamma) = \gamma x_0$, where γx_0 denotes the natural isometric action of Γ on X.

Let $\lambda = \max\{d_X(x_0, sx_0) \mid s \in S\}$. Then π is λ-Lipschitz. Since the metric on Γ is discrete geodesic, it suffices to check the Lipschitz condition on pairs at distance 1 from each other. Any such pair is of the form $(\gamma, \gamma s)$, where $s \in S$. Now,

$$d_X(\pi(\gamma), \pi(\gamma s)) = d_X(\gamma x_0, \gamma s x_0) = d_X(x_0, s x_0) \leq \lambda.$$

Given $\varepsilon > 0$ we will construct an ε-Lipschitz, uniformly cobounded map $\psi : \Gamma \to K$, where K is a uniform simplicial complex of dimension n. Let c_{n-1} denote the constant of uniformization (from Proposition 3) of the product $L^{n-1} \times [0,1]$, where L^{n-1} is a uniform $(n-1)$-dimensional complex. Let $\nu = \nu(\varepsilon/4c_{n-1}, n-1)$ be as in Proposition 1, and let $r > \max\{\nu, 8/\varepsilon\}$.

Let $\mathcal{W}$ be a disjoint cover of Γx_0 such that both the λr-enlargement and the $2\lambda r$-enlargement are covers of multiplicity 2. Denote the enlargements by $N_{\lambda r}(\mathcal{W})$ and $N_{2\lambda r}(\mathcal{W})$, respectively. To see that such a cover (with corresponding enlargements) exists, take the standard cover of the tree by two families of $3d$-bounded, d-disjoint sets (see [11, p. 130], for example) with d much larger than $2\lambda r$.

Since the sets in $\mathcal{W}$ are uniformly bounded there is an $R > 0$ such that for every $W \in \mathcal{W}$ there is a $\gamma_W \in \Gamma$ so that $N_{2\lambda r}(W) \subset B_R(\gamma_W x_0)$. Thus,

$$\pi^{-1}(N_{2\lambda r}((W)) \subset \pi^{-1}(B_R(\gamma_W x_0)).$$

But $\pi^{-1}(B_R(\gamma_W x_0))$ is isometric to $\pi^{-1}(B_R(x_0))$, the R-stabilizer. By Lemma 2, $\operatorname{asdim} \pi^{-1}(B_R(x_0)) \leq n$. Hence, $\operatorname{asdim} \pi^{-1}(N_{2\lambda r}(W)) \leq n$ for all $W \in \mathcal{W}$.

Suppose that W and W' are distinct sets in $\mathcal{W}$ whose λr-enlargements have non-empty intersection, so that W and W' define an edge e in $\operatorname{Nerve}(N_{\lambda r}(\mathcal{W}))$. Since $\operatorname{asdim} \pi^{-1}(N_{2\lambda r}(W)) \leq n$ for all W, and since $\pi^{-1}(N_{2\lambda r}(W))$ is a subset of the finitely generated group Γ, we apply Theorem 1 cited in the previous section to conclude that $\operatorname{asInd} \pi^{-1}(N_{2\lambda r}(W)) \leq n$ for all W. In $\pi^{-1}(N_{\lambda r}(W) \cup N_{\lambda r}(W'))$, the sets $\pi^{-1}(W)$ and $\pi^{-1}(W')$ are asymptotically disjoint, by Proposition 5. Thus, there is an asymptotic separator A_e in $\pi^{-1}(N_{\lambda r}(W) \cup N_{\lambda r}(W'))$ separating $\pi^{-1}(W)$ and $\pi^{-1}(W')$ with $\operatorname{asInd} A_e \leq n-1$. Hence, $\operatorname{asdim} A_e \leq n-1$.

Since $N_r(A_e)$ is coarsely equivalent to A_e, we have $\operatorname{asdim} N_r(A_e) \leq n-1$. Take $\mathcal{V}_e$ to be a uniformly bounded cover of $N_r(A_e)$ with multiplicity $\leq n$ and Lebesgue number $L > r$. Take $\mathcal{U}_W$ to be a cover of $\pi^{-1}(N_{2\lambda r}(W))$ by uniformly bounded sets with multiplicity $\leq n+1$ and Lebesgue number greater than $b(\mathcal{V}_e)$, where $b(\mathcal{V}_e)$ is an upper bound for the diameter of the sets in $\mathcal{V}_e$. Define a similar cover for $\pi^{-1}(N_{2\lambda r}(W'))$.

The conditions on the Lebesgue numbers and the fact that

$$N_r(A_e) \subset \pi^{-1}(N_{2\lambda r}(W)) \cap \pi^{-1}(N_{2\lambda r}(W'))$$

imply that there are simplicial maps $g_W : \mathrm{Nerve}(\mathcal{V}_e) \to \mathrm{Nerve}(\mathcal{U}_W)$ and $g_{W'} : \mathrm{Nerve}(\mathcal{V}_e) \to \mathrm{Nerve}(\mathcal{U}_{W'})$. Take $M_{e,W}$ and $M_{e,W'}$ to be the uniform mapping cylinders of the maps g_W and $g_{W'}$, respectively.

As $r > 8/\varepsilon$, we may apply Lemma 1 to $A_e \subset \Gamma$ and the covers to obtain ε-Lipschitz maps $f_{e,W} : N_r(A_e) \to M_{e,W}$ and $f_{e,W'} : N_r(A_e) \to M_{e,W'}$ to the uniform mapping cylinders.

For each $W \in \mathcal{W}$, construct a uniformly cobounded ε-Lipschitz map $\phi_W : \pi^{-1}(N_{2\lambda r}(W)) \to K_W$ to the uniform n-dimensional simplicial complex K_W by taking the natural projection to the nerve of $\mathcal{V}_W$. Such a mapping exists since $r > \nu$, by Proposition 1.

We note that the $N_r(A_e)$ are disjoint for distinct edges in the nerve. Thus, for each $W \in \mathcal{W}$ define $\psi_W : \pi^{-1}(N_{2\lambda r}(W)) \to K_W \cup \bigcup_{W\in e} M_{e,W} = L_W$ to the uniform complex L_W, with mapping cylinders attached as the union of the map ϕ_W restricted to $\pi^{-1}(N_{2\lambda r}(W)) \setminus \bigcup_{W\in e} N_r(A_e)$ and the restrictions of $f_{e,W}$ to $N_r(A_e) \cap \pi^{-1}(N_{2\lambda r}(W))$, for all edges e in $\mathrm{Nerve}(N_{\lambda r}(\mathcal{W}))$ which contain W as a vertex.

We construct K by gluing together the L_W. Clearly, the dimension of K is at most n. The maps $\psi_W : \Gamma \to K$ agree on the common parts A_e so they define a map $\psi : \Gamma \to K$. The map ψ is ε-Lipschitz by Proposition 4, and uniformly cobounded by Lemma 1. ∎

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