

Stretched shadings and a Banach measure that is not scale-invariant

by

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Abstract. It is shown that if $A \subset \mathbb{R}$ has the same constant shade with respect to all Banach measures, then the same is true of any similarity transformation of A and the shade is not changed by the transformation. On the other hand, if $A \subset \mathbb{R}$ has constant μ -shade with respect to some fixed Banach measure μ , then the same need not be true of a similarity transformation of A with respect to μ . But even if it is, the μ -shade might be changed by the transformation. To prove such a μ exists, a Hamel basis with some weak closure properties with respect to multiplication is used to construct sets with some convenient scaling properties. The notion of *shade-almost invariance* is introduced, aiding in the construction of a variety of Banach measures, in particular, one that is not scale-invariant.

1. Introduction. Let \mathfrak{L} denote the family of Lebesgue measurable subsets of the set \mathbb{R} of real numbers and let λ denote the usual Lebesgue measure on \mathfrak{L} .

A *Banach measure* on \mathbb{R} is an isometry-invariant total extension of λ , i.e., an isometry-invariant extension of λ defined on $2^{\mathbb{R}}$. It is well known that Banach measures exist as a consequence of the axiom of choice and that they are only finitely additive. (In this paper, unless explicitly noted, a *measure* will be assumed to be only finitely additive.) Let \mathcal{B} denote the class of all Banach measures on \mathbb{R} .

It is also known that Banach measures on \mathbb{R} exist that are *scale-invariant* (see [19, Cor. 11.5] and [9]). What seems not to be known up to now, or at least not mentioned, is whether Banach measures exist which are *not* scale-invariant. That such measures do indeed exist is the main result of this paper (Section 10).

Of course the Lebesgue measure λ is scale-invariant; the question is whether or not Banach measures should inherit that property directly from λ

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or perhaps have it as a consequence of isometry-invariance. In [9] it is shown that scale-invariant Banach measures do exist (in \mathbb{R}^2 as well as \mathbb{R} ; see also [13] and [19, Cor. 11.5]) but there is no corresponding proof or suggestion that Banach measures do not come already equipped with this property. By comparison, the Lebesgue measure on \mathbb{R}^n gets its scale-invariance *gratis*—it is the unique *translation-invariant countably additive* measure defined on the Lebesgue measurable sets and normalizing $[0, 1]^n$; its invariance with respect to all other isometries and its respect for similarity are natural consequences. But this is not the case for Banach measures, as they lack countable additivity. This sort of thing is noted following Theorem 11.21 in [19], where it is pointed out that there exist total extensions of λ which are translation-invariant but not reflection-invariant. (A more general result is proved in [18].)

To see how much we can stretch things, we try to be somewhat general where convenient, obtaining examples of Banach measures in which scale-invariance is violated in different, fairly customizable fashions. To do this we exploit properties of sets we call *shade-almost invariant*. Our examples of such sets are created using Hamel bases, their shade-almost invariance being used to extend λ , creating large families of distinct Banach measures. By giving our Hamel bases some additional algebraic properties, we make our resulting Banach measures have some convenient properties with respect to scaling. Along the way, we explore some interesting behavior of *shadings* under similarity transformations.

Most of our set-theoretic notation is fairly standard. The rationals and integers are denoted by \mathbb{Q} and \mathbb{Z} , respectively. The cardinality of a set X is denoted by $|X|$. The cardinality of the continuum is denoted by c , as is the least ordinal with this cardinality. If X and Y are subsets of reals, we let $X + Y$ denote $\{x + y : x \in X, y \in Y\}$. When $x \in \mathbb{R}$, $x + Y$ is synonymous with $\{x\} + Y$. Similarly, $X - Y$ and $x - Y$ have the obvious meanings, $xY = \{xy : y \in Y\}$, and combinations of these notations are freely used. Symmetric differences of sets are indicated by Δ and we emphasize disjoint unions with \uplus .

The term “scale-invariance” has the traditional meaning here. We shall deal exclusively with subsets of the real line, so for us a measure m is scale-invariant if $m(hS) = |h|m(S)$ whenever $h \in \mathbb{R}$ and S is m -measurable.

2. Basic properties of shadings

DEFINITION 2.1. For a fixed $t \in [0, 1]$, a subset A of \mathbb{R} is said to be a *shading* of \mathbb{R} , and to have *shade* (or *shade-density*) equal to t , if

$$(1) \quad \mu(A \cap E) = t\mu(E)$$

for every bounded $E \in \mathfrak{L}$ and every $\mu \in \mathfrak{B}$. We then write $\text{sh}(A) = t$. (This notion was introduced in [10].)

We will show (Theorem 4.1) that if $A \subset \mathbb{R}$ has constant shade and $g : \mathbb{R} \rightarrow \mathbb{R}$ is any similarity transformation, then $\text{sh}(g(A)) = \text{sh}(A)$. Thus, a stretch does not affect constant shade-density. This is easy to prove and has an intuitive appeal (if intuition can be brought to bear on such shady consequences of the axiom of choice). Supposing, for instance, that A has shade $1/3$, we may think of A as taking up $1/3$ of each interval, no matter how small. Stretching A uniformly by any amount should not change that. But we shall see that this view (which is only heuristic) depends also on the “uniformity with respect to μ ” in the definition of shade. We will make the last statement more precise (it is the main point of this article) in Section 4.

DEFINITION 2.2. For a fixed $\mu \in \mathfrak{B}$ we call $A \subset \mathbb{R}$ a μ -*shading* with μ -*shade* equal to t , and write $\text{sh}_\mu(A) = t$, if $\mu(A \cap E) = t\mu(E)$ for each bounded $E \in \mathfrak{L}$. (To be more specific, A has *constant* μ -shade in this case.)

It is clear that $\text{sh}(A) = t$ iff $\text{sh}_\mu(A) = t$ for all $\mu \in \mathfrak{B}$.

In the definitions of shading and μ -shading, it is equivalent to require that (1) hold whenever E is a bounded interval. (This is explicitly shown for shadings in [10]—see, in particular, Definition 3.7 in conjunction with Theorem 3.11 there—but easily extends to μ -shadings.) In this paper we will mainly concern ourselves with sets of constant shade or μ -shade, but it is possible for a set to have, for example, smoothly varying shade. In fact, for any continuous $\varphi : \mathbb{R} \rightarrow [0, 1]$, there exists a set $F \subset \mathbb{R}$ having the property that

$$(2) \quad \lim_{\mu(J) \rightarrow 0} \mu(F \cap J) / \mu(J) = \varphi(x)$$

for every $\mu \in \mathfrak{B}$, where J represents a bounded interval containing x . It then makes sense to write $\text{sh}(F) = \varphi$ (see Thm. 5.7 in [10]). Similarly, for a fixed $\mu \in \mathfrak{B}$, the notation $\text{sh}_\mu(F) = \varphi$ has the obvious meaning. Shade is an extension of the usual Lebesgue density.

For other basic properties of shadings, see [10]. For an introduction to the main ideas and motivations behind various extensions of Lebesgue measure, see [2] or [3]. For an extensive look, see [19] or [7].

3. Shadings versus μ -shadings. There are examples of sets that show the distinction between shadings and μ -shadings is not vacuous.

DEFINITION 3.1. A set $A \subset \mathbb{R}$ is called *Archimedean* if there is a dense set of translators $\{x\}$ for which $A + x = A$. (This term was coined in [17].)

It is easy to prove (as in [10, Thm. 6.1]; see also Corollary 5.4 in the present paper) that for every Archimedean set A and every $\mu \in \mathfrak{B}$, the set A

has constant μ -shade. (See [14] for more results concerning Banach measures and Archimedean sets.)

DEFINITION 3.2. Let G be a group of transformations on \mathbb{R} . A set $A \subset \mathbb{R}$ is called *almost G -invariant* if $|A \triangle f(A)| < c$ for every $f \in G$. The set A is simply called *almost invariant* when G is the group of all isometries on \mathbb{R} . (See, e.g., [7, p. 9] or [8, p. 76].)

It is easy to prove (Corollary 5.5) that if $A \subset \mathbb{R}$ is almost invariant, then A has constant μ -shade for any fixed $\mu \in \mathcal{B}$. (It suffices for A to be almost G -invariant when G contains a dense set of translations.)

So Archimedean sets and almost invariant sets are “uniformly dense” with respect to any *fixed* $\mu \in \mathcal{B}$. But it turns out that in either of these two classes of sets, there exist a set A and Banach measures μ_1, μ_2 for which $\text{sh}_{\mu_1}(A) \neq \text{sh}_{\mu_2}(A)$, so neither Archimedean sets nor almost invariant sets are guaranteed to be shadings (although they can be). Hence one cannot reasonably associate any specific, fixed density (in our sense, at least) with such A . Indeed, this sort of situation can be extreme—in [6] an almost invariant set $C \subset \mathbb{R}$ is constructed, along with a family $\{\nu_t\}_{t \in [0,1]}$ of isometry-invariant countably additive extensions of λ , for which $\nu_t(C \cap E) = t\lambda(E)$ for each t and for any bounded $E \in \mathfrak{L}$. Since each such ν_t can be extended to a Banach measure μ_t , we have $\text{sh}_{\mu_t}(C) = t$ for each $t \in [0, 1]$. (For the set C , take any of the sets denoted by A_γ in [6]; see the final paragraphs in that paper.) Although not originally constructed to be so, this C can easily be made Archimedean. Indeed, for any set $B \subset \mathbb{R}$, the set $B + \mathbb{Q}$ is Archimedean with dense set of translators \mathbb{Q} . If B is almost invariant then so is $B + \mathbb{Q}$, and furthermore, $|(B + \mathbb{Q}) \triangle B| < c$. The shade of any noncontinuum set is zero ([10, Lemma 4.5]) so $\text{sh}_\mu(B + \mathbb{Q}) = \text{sh}_\mu(B)$ for any $\mu \in \mathcal{B}$. Therefore, if C is the example in [6], the set $C + \mathbb{Q}$ is both Archimedean and almost invariant and has the property that for each $t \in [0, 1]$ there exists a $\mu_t \in \mathcal{B}$ for which $\text{sh}_{\mu_t}(C + \mathbb{Q}) = t$.

At the other extreme, Neu proves in [15, Thm. 2.2] that if S is an almost isometry-invariant shading, then $\text{sh}(S) = 0$ or 1 .

4. The main problem. Keith Neu asked (private communication) the following, circa 2004: Is the μ -shade of a μ -shading necessarily preserved under a similarity transformation? That is, is it true that $\text{sh}_\mu(g(A)) = \text{sh}_\mu(A)$ whenever A is a μ -shading and g is a similarity transformation? We answer this question in the negative (Theorem 10.1). Our counterexample will be Archimedean but not almost invariant. A few years after reading an earlier draft of this paper, Neu (private communication) devised a slick counterexample that is almost invariant.

Specifically, we will show the existence of a Banach measure μ , an Archimedean set $A \subset \mathbb{R}$ and $h > 0$ such that $\text{sh}_\mu(A) \neq \text{sh}_\mu(hA)$. We shall see that the pair of numbers $\text{sh}_\mu(A)$ and $\text{sh}_\mu(hA)$ can be any values at all in $[0, 1]$ for fairly arbitrary values of h .

First, in the opposite direction, we prove an easy but revealing result, namely that if a set A has constant shade t , then so does any similarity transformation of A . We prove a slightly stronger result because it is not much extra trouble.

THEOREM 4.1. *If g is a similarity transformation on \mathbb{R} and F is a φ -shading of \mathbb{R} , where $\varphi : \mathbb{R} \rightarrow [0, 1]$, then $\text{sh}(g(F)) = \varphi \circ g^{-1}$. In particular, if $\varphi(x) = t$ is a constant function, then $\text{sh}(g(F)) = \text{sh}(F) = t$.*

Proof. Let $\mu \in \mathcal{B}$ and let $g(x) = ax + b$ for reals b and $a \neq 0$. For $X \subset \mathbb{R}$, consider the set function μ_g defined by

$$\mu_g(X) = \frac{1}{|a|} \mu(g(X)) = \frac{1}{|a|} \mu(aX + b) = \frac{1}{|a|} \mu(aX).$$

It is easy to see that μ_g extends Lebesgue measure, since μ itself does: if $E \in \mathcal{L}$, then $g(E) \in \mathcal{L}$ and $\lambda(g(E)) = |a|\lambda(E)$, hence $\mu_g(E) = (1/|a|)\mu(aE) = (1/|a|)\lambda(aE) = \lambda(E)$. Also, the isometry-invariance of μ easily yields the isometry-invariance of μ_g , so $\mu_g(f(X)) = \mu_g(X)$. The fact that μ is defined for all subsets of \mathbb{R} yields the same for μ_g , hence μ_g is itself a Banach measure. As such, it must be that (2) holds with μ_g in place of μ , since F is a φ -shading. Fixing a bounded interval J and $x \in J$, we have

$$\begin{aligned} \frac{\mu(g(F) \cap J)}{\mu(J)} &= \frac{\mu(g(F \cap g^{-1}(J)))}{\mu(J)} = \frac{|a|\mu_g(F \cap g^{-1}(J))}{\mu(J)} \\ &= \frac{|a|\mu_g(F \cap g^{-1}(J))}{|a|\mu_g(g^{-1}(J))}. \end{aligned}$$

Clearly, $g^{-1}(x) \in g^{-1}(J)$ and $\mu(g^{-1}(J)) \rightarrow 0$ as $\mu(J) \rightarrow 0$. Since F is a φ -shading, the last term approaches $\varphi(g^{-1}(x))$ as $\mu(J) \rightarrow 0$, and that finishes the proof. ■

The rest of this paper is devoted to showing that in the theorem just proved, shade cannot be replaced with μ -shade. We will need some preliminaries first.

5. Shade-almost invariance. Denote by \mathcal{Z} the class of sets having shade zero in \mathbb{R} . It is clear that \mathcal{Z} is an ideal in \mathbb{R} , and that \mathcal{Z} is G -invariant when G is the group of isometries on \mathbb{R} . We also note that \mathcal{Z} is not a σ -ideal. One can see this via a set $W(h_0)$, defined later (Definition 7.1), having the following property: there is a positive number h_0 such that for any $q \in \mathbb{Q}$, $qh_0 + W(h_0) \in \mathcal{Z}$ but $\bigcup_{q \in \mathbb{Q}} (qh_0 + W(h_0)) = \mathbb{R} \notin \mathcal{Z}$.

DEFINITION 5.1. For subsets A and B of \mathbb{R} , we write $A \doteq B$ if $A \Delta B \in \mathcal{Z}$. We will then say that A and B are *shade-almost equal*. (One could also say that A and B are “not a shade different”.) Two sets A and B will be called *shade-almost disjoint* if $A \cap B \doteq \emptyset$.

Clearly, $A \doteq \emptyset$ iff $A \in \mathcal{Z}$. We do not make use of it, but we note that the relation \doteq is an equivalence relation on \mathbb{R}

DEFINITION 5.2. For a group G of isometries on \mathbb{R} , we say that A is *shade-almost G -invariant* if $A \doteq g(A)$ for each isometry $g \in G$. If G is the group of all isometries on \mathbb{R} , we simply say that A is *shade-almost invariant*.

We observe that shade-almost invariance is weaker than almost invariance: as already noted, in [10, Lemma 4.5] it is shown that if $|Y| < c$ then $\text{sh}(Y) = 0$, but that the set $W(h_0)$ (Def. 7.1 again) has shade zero, while $|W(h_0)| = c$. We also note that for any shade-almost invariant set A and Banch measure μ , the μ -shade of A is constant. For completeness, we include a short, easy proof of this, using slightly weaker assumptions.

THEOREM 5.3. *Suppose G is a group of isometries on \mathbb{R} that includes a dense set of translations. Let $\mu \in \mathcal{B}$ and suppose that A is a shade-almost G -invariant subset of \mathbb{R} . Then A has constant μ -shade.*

Proof. Recall that \uplus denotes a disjoint union. Suppose J_1 and J_2 are any two nonempty bounded intervals in \mathbb{R} . We first assume that $\lambda(J_1) = \lambda(J_2)$. Let $0 < \varepsilon < \lambda(J_1)$. Choose a translation $g \in G$ such that $\lambda(g(J_1) \Delta J_2) < \varepsilon$. Then

$$\begin{aligned} \mu(A \cap J_1) &= \mu(g(A \cap J_1)) = \mu(g(A) \cap g(J_1)) \\ &= \mu([(g(A) \cap A) \uplus (g(A) \setminus A)] \cap g(J_1)) \\ &= \mu((g(A) \cap A) \cap g(J_1)) + \mu((g(A) \setminus A) \cap g(J_1)) \\ &\leq \mu(A \cap g(J_1)) + \mu((g(A) \Delta A) \cap g(J_1)) = \mu(A \cap g(J_1)) \\ &\leq \mu(A \cap (J_2 \cup g(J_1))) = \mu(A \cap [J_2 \uplus (g(J_1) \setminus J_2)]) \\ &= \mu(A \cap J_2) + \mu(A \cap (g(J_1) \setminus J_2)) \\ &\leq \mu(A \cap J_2) + \mu(g(J_1) \setminus J_2) \leq \mu(A \cap J_2) + \varepsilon, \end{aligned}$$

and it follows that $\mu(A \cap J_1) \leq \mu(A \cap J_2)$. A symmetric argument (obtained by reversing the roles of J_1 and J_2 and replacing g with g^{-1}) shows that $\mu(A \cap J_2) \leq \mu(A \cap J_1)$. Therefore, we have $\mu(A \cap J_1) = \mu(A \cap J_2)$ when $\lambda(J_1) = \lambda(J_2)$.

In the case that the intervals J_1 and J_2 are of different sizes, let $\delta > 0$ and cover each J_i by letting $\biguplus_{k=1}^{N_i} I_i^{(k)} \subset J_i \subset \biguplus_{k=0}^{N_i} I_i^{(k)}$, where for each fixed $i = 1, 2$, the $I_i^{(k)}$ are all half-open intervals of size δ . From the previous case concerning intervals of equal length, we may choose $t \in [0, 1]$ such that $t\delta = \mu(A \cap I_i^{(k)})$ for each $i = 1, 2$ and $0 \leq k \leq N_i$. We obtain the following

two inequalities:

$$N_i t \delta \leq \mu(A \cap J_i) \leq (N_i + 1) t \delta, \quad N_i \delta \leq \mu(J_i) \leq (N_i + 1) \delta$$

Dividing these gives

$$\frac{N_i}{N_i + 1} t \leq \frac{\mu(A \cap J_i)}{\mu(J_i)} \leq \frac{N_i + 1}{N_i} t,$$

and now letting $\delta \rightarrow 0$ (so that $N_i \rightarrow \infty$ for each $i = 1, 2$) leads to

$$\frac{\mu(A \cap J_i)}{\mu(J_i)} = t,$$

and that proves the claim. ■

COROLLARY 5.4. *If A is Archimedean and $\mu \in \mathcal{B}$, then A is a μ -shading.*

COROLLARY 5.5. *If A is almost invariant and $\mu \in \mathcal{B}$, then A is a μ -shading.*

The proofs of these two corollaries follow easily by choosing the appropriate G in Theorem 5.3. In the case of Archimedean sets it is the dense set of translations corresponding to the definition of an Archimedean set. (As already mentioned, Corollary 5.4 is given in [10, Thm 6.1].) For almost invariant sets, G is arbitrary, so long as it contains a dense set of translations; “almost invariant” can then be replaced by “almost G -invariant” in Corollary 5.5.

In Section 6 we use the notion of shade-almost invariance to generate Banach measures with some convenient properties.

6. Shady extensions. One way to construct countably additive-isometry invariant extensions of λ is to add to the domain \mathfrak{L} of λ those sets generated by \mathfrak{L} and a fixed almost invariant set having inner measure zero and full outer measure. Such measures can then be extended to Banach measures. As we will see, something similar can be done using shade-almost invariant sets. But Banach measures already do so much that we need not always construct new ones from scratch when we want them to have certain properties—we can often start with “existing” ones.

LEMMA 6.1. *Suppose B is a shade-almost invariant subset of \mathbb{R} that does not have shade zero. Then there exists a $\nu \in \mathcal{B}$ for which $\text{sh}_\nu(B) = 1$.*

Proof. The assumptions on B , along with Theorem 5.3, allow us to choose $\mu' \in \mathcal{B}$ such that $b := \text{sh}_{\mu'}(B) > 0$. Consider the algebra $\mathfrak{A}_{\text{bdd}}$ of all bounded subsets of \mathbb{R} . Define the set function μ on $\mathfrak{A}_{\text{bdd}}$ by means of the equality $\mu(S) = (1/b)\mu'(S \cap B)$ for each $S \in \mathfrak{A}_{\text{bdd}}$. Extend μ to a measure ν defined

on all subsets of \mathbb{R} by means of the equality

$$(3) \quad \nu(X) = \sum_{i \in \mathbb{Z}} \mu(X \cap [i, i + 1)).$$

Clearly, ν is well defined, since μ' is defined on all subsets of \mathbb{R} and $b > 0$. To see that ν extends Lebesgue measure, first note that for any bounded λ -measurable set E , we have

$$(4) \quad \nu(E) = \frac{1}{b} \mu'(E \cap B) = \frac{1}{b} (\text{sh}_{\mu'} B) \mu(E) = \lambda(E)$$

so ν and λ agree on the bounded λ -measurable sets. Using (3) and the σ -finiteness of λ , we see that ν and λ agree on the unbounded λ -measurable sets as well.

As to invariance, if g is any isometry on \mathbb{R} and $S \subset \mathbb{R}$ is bounded, we have $\mu(g(S)) = (1/b) \mu'(B \cap g(S)) = (1/b) \mu'(g^{-1}(B) \cap S) = (1/b) \mu'(B \cap S) = \mu(S)$, showing that μ is isometry-invariant on $\mathfrak{A}_{\text{bdd}}$. (Observe that we used the shade-almost invariance of B and the boundedness of S to assert that $\mu'(g^{-1}(B) \cap S) = \mu'(B \cap S)$.) The invariance of ν follows from its definition in terms of μ , so we have shown that $\nu \in \mathcal{B}$. Finally, as to the ν -shade of B , observe that for any bounded λ -measurable E we have $\nu(E \cap B) = \mu(E \cap B) = (1/b) \mu'(E \cap B)$, so (4) shows that $\text{sh}_{\nu}(B) = 1$, completing the proof of Lemma 6.1. ■

We only need finite sequences in the following lemma, but the proof goes through for all denumerable sequences as well.

LEMMA 6.2. *Let $\{t_0, t_1, t_2, \dots\} \subset [0, 1]$ be a sequence (finite or denumerable) such that $\sum_i t_i = 1$ and let B_0, B_1, B_2, \dots be shade-almost invariant subsets of \mathbb{R} such that the following conditions hold:*

- (i) *the B_i are pairwise shade-almost disjoint;*
- (ii) *none of the B_i has shade zero.*

Then there exists a $\mu \in \mathcal{B}$ for which each of the B_i is a μ -shading with $\text{sh}_{\mu}(B_i) = t_i$, $i = 0, 1, 2, \dots$

Proof. For each i , apply Lemma 6.1 and condition (ii) to obtain $\mu_i \in \mathcal{B}$ such that $\text{sh}_{\mu_i}(B_i) = 1$ for each i . The shade-almost disjointness of the B_i implies that $\text{sh}_{\mu_i}(B_j) = 0$ for each $i \neq j$. It follows that if we define the function μ on $2^{\mathbb{R}}$ by means of $\mu(S) = \sum_i t_i \mu_i(S)$ for each $S \subset \mathbb{R}$, then μ is in \mathcal{B} and satisfies the conclusion of the lemma. ■

7. Shadings from Hamel bases. Let H be any Hamel basis for \mathbb{R} over \mathbb{Q} , injectively well-ordered as $\{h_{\alpha} : \alpha < c\}$. Consider H to be fixed until otherwise stated.

Each $x \in \mathbb{R}$ has a unique representation as

$$(5) \quad x = \sum_{\alpha < c} q_\alpha(x)h_\alpha,$$

where $q_\alpha(x) \in \mathbb{Q}$ and all but finitely many of the $q_\alpha(x)$ are zero. We may think of q_α as the projection of x onto the α th axis of the infinite-dimensional vector space \mathbb{R} over \mathbb{Q} (with basis H). (Few would call q_α a true projection, as it is not linear in the strict sense. The function $q_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is, however, an additive function.)

The next definition plays a key role in what follows.

DEFINITION 7.1. For $K \subset H$, let $W(K)$ denote the set of all reals whose expansion (5) with respect to H involves no members of K , i.e.,

$$(6) \quad W(K) = \{x \in \mathbb{R} : q_\alpha(x) = 0 \ \forall h_\alpha \in K\}.$$

Another way to put it is that $W(K)$ is the linear span or hull (over the rationals) of the basis elements $H \setminus K$. For K given explicitly as a finite sequence $\{k_0, k_1, \dots, k_n\}$ we drop the braces and simply write $W(k_0, k_1, \dots, k_n)$ for $W(K)$. We will employ a few other notations for convenience. For a sequence Q_0, Q_1, \dots, Q_n of subsets of rationals, let

$$A(Q_0, Q_1, \dots, Q_n) = W(h_0, h_1, \dots, h_n) + Q_0h_0 + Q_1h_1 + \dots + Q_nh_n.$$

Thus $x \in A(Q_0, Q_1, \dots, Q_n)$ if and only if $q_\alpha(x) \in Q_\alpha$ for each $\alpha \in [0, n]$. Let $A^+ = A(\mathbb{Q}^+)$, where \mathbb{Q}^+ is the set of (strictly) positive rationals, and similarly define $A^- = A(\mathbb{Q}^-)$, where $\mathbb{Q}^- = -\mathbb{Q}^+$. (Our sequences $\{Q_\alpha\}$ are finite, though transfinite ones still make sense.)

LEMMA 7.2. *The set $W(K)$ has shade zero whenever K is nonempty.*

The preceding lemma follows from [10, Example 4.8]. Something a bit stronger is proved next, in Lemma 7.3.

While it is not needed here, it is interesting to note that for each $K \subset H$, the set $W(K)$ is *homogeneous* in the sense of Emil Borel [1]: $X \subset \mathbb{R}$ is homogeneous if whenever $x, y \in X$, then $X + x - y = X$. Erdős and Marcus show in [5, Lemma 2] that every nontrivial homogeneous set has shade zero, from which Lemma 7.2 follows. (Of course, the term “shade” was not used in [5].)

It is easy to show that A^+ has shade $1/2$. (This is mentioned in [10, Example 4.8]; the proof is easy.) Another useful fact is the following

LEMMA 7.3. *If $Q \subset \mathbb{Q}$ is bounded and $h_\alpha \in H$, the set $\{x : q_\alpha(x) \in Q\}$ has shade zero.*

Proof. It suffices to prove this for $\alpha = 0$. Then $\{x : q_\alpha(x) \in Q\} = A(Q)$. We may assume $Q \subset (r, s) \cap \mathbb{Q}$ for rationals $r < s$. Let $Q_r = (-\infty, r) \cap \mathbb{Q}$ and $Q_s = (s, \infty) \cap \mathbb{Q}$. Then $A(Q_s) = sh_0 + A(\mathbb{Q}^+)$ and $A(Q_r) = rh_0 - A(\mathbb{Q}^+)$,

so the invariance properties of Banach measures imply that $\text{sh}(A(Q_r)) = \text{sh}(A(Q_s)) = 1/2$. Therefore, since the sets $A(Q)$, $A(Q_r)$ and $A(Q_s)$ are disjoint, it follows that for any $\mu \in \mathcal{B}$ and any bounded $E \in \mathcal{L}$, we have

$$\begin{aligned} \mu(E \cap A(Q)) &\leq \mu(E) - \mu(E \cap A(Q_r)) - \mu(E \cap A(Q_s)) \\ &= \mu(E) - \frac{1}{2}\mu(E) - \frac{1}{2}\mu(E) = 0. \blacksquare \end{aligned}$$

We have used translation and reflection invariance of Banach measures in the preceding proof, but we note that the result holds using only translation invariance: Since $Q \subset \mathbb{Q}$ is bounded, there exist countably many disjoint translates $Q + q_k$ of Q with each translator q_k rational. The sets $q_k h_0 + A(Q)$ are disjoint, congruent and Archimedean, hence each has shade zero.

Intuitively, one might think of A^+ (and similarly, $A(Q_r)$, $A(Q_s)$) as being analogous to a “half-plane”, or perhaps more aptly, the space of reals lying on one side of a “hyperplane” in an infinite-dimensional *Hamelbild* of the reals. In this way, it makes some sense that the reals are “evenly split”, forming a 1/2-shading. Likewise, when Q is bounded, the reals in $A(Q)$ might be thought of as lying in a “strip” bounded by two such parallel hyperplanes, and so have an arbitrarily small measure relative to the entire space. (We shall subsequently see that further extension of such geometric analogies can be unwise.)

Continuing along these lines, let us employ notations of the form A^{*+} , A^{+--} , A^{+++} , etc., so that $+$, $-$, $*$ correspond to the sets of rationals \mathbb{Q}^+ , \mathbb{Q}^- , \mathbb{Q} , respectively. So, for instance, A^{+*-} represents $A(\mathbb{Q}^+, \mathbb{Q}, \mathbb{Q}^-)$, the set of real x whose coefficients $q_0(x)$ and $q_2(x)$ in (5) are, respectively, positive and negative, and the coefficient $q_1(x)$ is arbitrary. Clearly, trailing asterisks can be omitted: $A^{+*-} = A^{+*-} = A^{+*-}$, etc.

Now consider the set A^{++} . One might suspect that this set must have shade 1/4, as it is the intersection of A^{+*} ($= A^+$) and A^{*+} , each of which has shade 1/2. In terms of hyperplanes, one might be tempted to imagine this set as a “quadrant” of a Hamel-based vector space. However, to dispel such a notion, let us be reminded not to imagine Hamel basis vectors as being in any way “mutually orthogonal” ⁽¹⁾. The same can be said of the eight sets $A^{\pm\pm\pm}$ —one might think that they should each have shade 1/8. Each of these eight sets is Archimedean with a dense set of translators given by $W(h_0, h_1, h_2)$, so each has some constant μ -shade for each fixed $\mu \in \mathcal{B}$. For such μ the μ -shade of any of $A^{\pm\pm}$, $A^{\pm\pm\pm}$, etc., can be no greater than 1/2, however, other than that restriction, we will see in Section 8 that any μ -shades for these sets are possible.

⁽¹⁾ This point was made to the author by Alexander Kharazishvili during the 24th Summer Symposium in Real Analysis at the University of North Texas.

Meanwhile, it is immediately clear that it is impossible that *all eight* of the sets $A^{\pm\pm\pm}$ have shade zero, or even μ -shade zero for some particular μ . To assume otherwise, it follows by finite additivity that the union of all eight sets has μ -shade zero as well. But this union is the complement in \mathbb{R} of $W(h_0, h_1, h_2)$, which does have shade zero, and we would be forced to conclude that the reals have shade zero, contrary to the obvious fact that $\text{sh}(\mathbb{R}) = 1$. What is less obvious (but intuitively reasonable) is that *none* of the eight sets $A^{\pm\pm\pm}$ has shade zero. We will prove this in Section 8.

DEFINITION 7.4. Fix a set of ordinals $M \subset [0, c)$ and to each $\alpha \in M$ associate a sign $\sigma(\alpha) \in \{-1, 1\}$. Denote by $A_{M, \sigma}$ the set of all $x \in \mathbb{R}$ for which, in the Hamel expansion (5) of x , $\sigma(\alpha)q_\alpha(x) > 0$ for all $\alpha \in M$. Notice that $A_{M, -\sigma} = -A_{M, \sigma}$; let $A_{M, \pm\sigma} = A_{M, \sigma} \cup A_{M, -\sigma}$.

This is simply a generalization of sets we have introduced before, more precisely defined (even if less intuitively notated). As an example, let $M = \{1, 3\}$, $\sigma(1) = 1$, and $\sigma(3) = -1$. Then $A_{M, \sigma} = A^{*+*-} = A(\mathbb{Q}, \mathbb{Q}^+, \mathbb{Q}, \mathbb{Q}^-)$, which is the set of reals whose Hamel basis expansions have positive coefficients of h_1 and negative coefficients of h_3 , and $A_{M, \pm\sigma} = A^{*+*-} \cup A^{*-+*}$.

Notice that, in general, $-A_{M, \pm\sigma} = A_{M, \pm\sigma}$ and that $A_{M, \sigma}$ and $A_{M, -\sigma}$ are Archimedean with a dense set of translators being $W(\{h_\alpha : \alpha \in M\})$. But it is translations by the rest of the reals that give such sets interesting properties, as we see in the next result.

LEMMA 7.5. *The sets $A_{M, \pm\sigma}$ in Definition 7.4 are shade-almost invariant.*

Proof. Fix M and σ . The result is trivial if $M = \emptyset$, so we may assume otherwise. Let A and W denote $A_{M, \sigma}$ and $W(\{h_\alpha : \alpha \in M\})$, respectively. We first show that that $t + A \dot{\div} A$ for any $t \in \mathbb{R}$. We already noted that if $t \in W$, then we have $t + A = A$, so suppose that $t \notin W$. It suffices to show that $(t + A) \triangle A$ is contained in a finite union of the “strips” alluded to previously. Specifically, since $t \notin W$ we can write $t = \sum_{\alpha \in M'} r_\alpha h_\alpha + w$ for some $w \in W$ and rational $r_\alpha \neq 0$, where M' is a *finite* subset of M (since the Hamel expansion of t involves only finitely many α). Then $(t + A) \triangle A \subset \bigcup_{\alpha \in M'} \{x : q_\alpha(x) \in Q_\alpha\}$, where $Q_\alpha = \mathbb{Q} \cap (0, r_\alpha]$ if $r_\alpha > 0$ and $Q_\alpha = \mathbb{Q} \cap [r_\alpha, 0)$ if $r_\alpha < 0$. It follows from Lemma 7.3 that $(t + A) \triangle A \dot{\div} \emptyset$.

That takes care of translations. Looking at reflections, it is now clear that for any $t \in \mathbb{R}$, $t - A = -((-t) + A) \dot{\div} -A$. This means that for any isometry g on \mathbb{R} we have $g(A \uplus (-A)) \dot{\div} A \uplus (-A)$. ■

8. Some custom-built Banach measures. The results in this section can probably be generalized a great deal, but we settle for the situation at

hand pertaining to our particular Archimedean sets. Refer to Definitions 7.1 and 7.4 for what follows.

In this section G denotes the group of all isometries on \mathbb{R} , M is a fixed subset of $[0, c)$ with $2 \leq |M| \leq \aleph_0$, and $A = A_{M, \pm\sigma}$.

Obviously A and A^c cannot both have shade zero, but the main hurdle in what follows is to show that *neither* can have shade zero. To prove this we will find a Banach measure μ with respect to which each of A and A^c has positive shade, i.e., $\text{sh}_\mu(A) > 0$ and $\text{sh}_\mu(A^c) > 0$.

DEFINITION 8.1. For $\alpha \in M$, consider sets of the form $W(h_\alpha) + Q_\alpha h_\alpha$, where each Q_α is a *bounded* subset of \mathbb{Q} . Define \mathcal{T}_M to be the ideal generated by all such $W(h_\alpha) + Q_\alpha h_\alpha$, $\alpha \in M$.

Each $W(h_\alpha) + Q_\alpha h_\alpha$ in Definition 8.1 is a ‘‘Hamel cylinder’’ of sorts, contained in one of our ‘‘strips’’ mentioned earlier. It is clear by Lemma 7.3 that each $W(h_\alpha) + Q_\alpha h_\alpha$ has shade zero, so $\mathcal{T}_M \subset \mathcal{Z}$.

THEOREM 8.2. *Let $\theta \in [0, 1]$. Then there exists a Banach measure μ_θ for which $\text{sh}_\mu(A) = \theta$.*

Proof. Note that $-A = A$, since $-A_{M, \sigma} = A_{M, -\sigma}$. The following clearly hold:

- (i) $g(A) \triangle A = g(A^c) \triangle A^c \in \mathcal{T}_M$ for any $g \in G$;
- (ii) $A \notin \mathcal{T}_M$ and $A^c \notin \mathcal{T}_M$.

The proof of Lemma 7.5 implies property (ii), which says that A and (therefore) A^c are \mathcal{T}_M -almost invariant. Property (ii) holds since any member of \mathcal{T}_M differs from A and A^c by sets of reals x having arbitrarily large values of $|q_\alpha(x)|$ for each $\alpha \in M$. It is straightforward to verify that the family $G(A)$ of all translates and reflections of A generates a G -invariant algebra $\mathfrak{A}(A)$ whose members are sets having one of the forms

$$N_1, \quad (A \setminus N_1) \cup N_2, \quad (A^c \setminus N_1) \cup N_2, \quad \mathbb{R} \setminus N_1 \quad (N_1, N_2 \in \mathcal{T}_M).$$

This allows us to define a finitely additive G -invariant probability measure ν_θ on $\mathfrak{A}(A)$ by means of the assignments

$$\nu_\theta(N_1) = 0, \quad \nu_\theta((A \setminus N_1) \cup N_2) = \theta, \quad \nu_\theta((A^c \setminus N_1) \cup N_2) = 1 - \theta, \quad \nu_\theta(\mathbb{R} \setminus N_1) = 1,$$

where θ is any fixed number in $[0, 1]$. That ν_θ is well defined can be verified directly using the aforementioned properties of \mathcal{T}_M .

For the convenience of directly applying a result of Marczewski concerning products of probability measures, we define the probability measure space $(\mathfrak{L}_0, \lambda_0)$ as follows. Let $\mathfrak{L}_0 = \{\mathbb{Z} + E_0 : E_0 \in \mathfrak{L} \text{ and } E_0 \subset [0, 1)\}$, and define $\lambda_0(\mathbb{Z} + E_0) = \lambda(E_0)$. (The space $(\mathfrak{L}_0, \lambda_0)$ is, in essence, the Lebesgue measure space on $[0, 1)$.) We now show that the probability measure spaces $(\mathfrak{A}(A), \nu_\theta)$

and $(\mathfrak{L}_0, \lambda_0)$ are *almost independent* in the sense of Marczewski [11], meaning that if $X \in \mathfrak{A}(A)$ and $E_0 \in \mathfrak{L}_0$ with $\nu_\theta(X) \neq 0$ and $\lambda_0(E_0) \neq 0$, then $X \cap E_0 \neq \emptyset$.

To see this, let $\beta \in [0, c) \setminus M$ and let $H_M = \{h_\beta\} \cup \{h_\alpha : \alpha \in M\}$. (The added element h_β will be used only to simplify a portion of the proof ahead.) First notice that if $X \in \mathfrak{A}(A)$ with $\nu_\theta(X) > 0$, then $X = P_M + W_M$, where W_M is the span (over the rationals) of $H \setminus H_M$, P_M is countable (formed of certain linear combinations, over \mathbb{Q} , of the set of basis elements H_M), and P_M is dense in \mathbb{R} (obvious, thanks to the addition of h_β). Notice also that $\mathbb{R} = Q_M + W_M$, where Q_M is the set (still countable) of *all* linear combinations, over \mathbb{Q} , of the basis elements H_M . This shows that the reals are covered by countably many translates of W_M , so we know that W_M cannot have measure zero. We will use the following well-known fact (it follows from the Lebesgue density theorem).

LEMMA 8.3. *Let $E \subset \mathbb{R}$ be a set of positive Lebesgue measure. Then $\lambda(\mathbb{R} \setminus (E + T)) = 0$ whenever T is a dense set.*

Now suppose that $E \cap X = \emptyset$, where X is as postulated just earlier and $E = \mathbb{Z} + E_0$ is a set of positive measure in \mathfrak{L}_0 (equivalently, E_0 is a set of positive Lebesgue measure in $[0, 1)$). Then $E \cap (x + W_M) = \emptyset$ for each $x \in P_M$, hence $(E - x) \cap W_M = \emptyset$ for each such x . But this means that $(E - P_M) \cap W_M$ is empty. Since E has positive Lebesgue measure, it follows from Lemma 8.3 that $E - P_M = \mathbb{R} \setminus N$ for some set N of Lebesgue measure zero. But then, since $(E - P_M) \cap W_M = \emptyset$, it follows that $W_M \subset N$, implying W_M has Lebesgue measure zero, which we already know not to be the case. This contradiction finally establishes that $E \cap X \neq \emptyset$, and in turn that $(\mathfrak{A}(A), \nu_\theta)$ and $(\mathfrak{L}_0, \lambda_0)$ are almost independent measure spaces.

This enables us to define a measure μ_0 on the algebra $\mathfrak{A}_0(A)$ generated by $\mathfrak{A}(A) \cup \mathfrak{L}_0$ that is the *multiplicative extension* of ν_θ and λ_0 . (See Theorem I in [11].) In particular, we have

$$\mu_0(A \cap (\mathbb{Z} + E_0)) = \nu_\theta(A)\lambda_0(\mathbb{Z} + E_0) = \theta\lambda(E_0)$$

when $\mathbb{Z} + E_0 \in \mathfrak{L}_0$.

Note that $(\mathfrak{A}_0(A), \mu_0)$ is a G -invariant measure space. The algebra $\mathfrak{A}'(A)$ generated by $\mathfrak{A}(A) \cup \mathfrak{L}$ consists of sets of the form $\biguplus_{k \in \mathbb{Z}} Y_k \cap [k, k + 1)$, where $Y_k \in \mathfrak{A}_0(A)$ for each k . We can then define the G -invariant measure μ' on $\mathfrak{A}'(A)$ by means of the equality

$$\mu' \left(\biguplus_{k \in \mathbb{Z}} (Y_k \cap [k, k + 1)) \right) = \sum_{k \in \mathbb{Z}} \mu_0(Y_k).$$

Clearly, the measure μ' is well defined, extends the Lebesgue measure, is G -invariant, and has the property that $\mu'(A \cap E) = \theta\lambda(E)$ for any $E \in \mathfrak{L}$. Since μ' is G -invariant, it can be extended (via the *Invariant Extension*

Theorem [19, Thm. 10.8]) to a Banach measure μ . This, at last, proves that the set A is a μ -shading with $\text{sh}_\mu(A) = \theta$. ■

COROLLARY 8.4. *The set $A_{M,\sigma}$ cannot have shade zero for any choice of σ . In particular, none of the eight sets $A^{\pm\pm\pm}$ has shade zero.*

Proof. By choosing $\theta \in (0, 1)$, we see that neither A nor A^c can be a set of shade zero. Since $\text{sh}_\mu(A) = 2 \text{sh}_\mu(A_{M,\sigma})$, the conclusion follows. ■

We note that if $|M| = 1$, then $\text{sh}(A_{M,\sigma}) = 1/2$. Also, despite warnings that, for example, $\text{sh}_\mu(A^{++} \cup A^{--})$ need not equal $1/2$, it is clear that this does indeed happen for certain μ . Likewise, toward the other extreme, if $|M| = \aleph_0$, then despite the seemingly thinned-out nature of any resulting $A_{M,\sigma}$ (being akin to the set \mathbb{R} cut by half denumerably many times), such sets still do not have shade zero, though they can have μ -shade zero (or one). In fact, if one wishes to consider even larger M , even with $|M| = c$ (or to simply shorten the proof of Theorem 8.2, avoiding the paragraphs surrounding Lemma 8.3), one can consider suitably tailored Hamel bases that are Bernstein sets for obtaining the independence needed for applying Marczewski's theorem. (Such H do exist and are called Burstin sets [12].)

9. A Hamel basis with a weak closure property. To prove our main result we find a way to control how our A_i are stretched. We do so by controlling certain multiplicative properties of the underlying Hamel basis H . (See [4, Thm. 2.1/Lemma 2.2] for another application of products in Hamel bases to invariant extensions of Lebesgue measure.) Up to now, no special properties of H have been required, but in this section we impose restrictions on H that we leave in place until the end of the paper.

We will use a result ([16, Theorem 2.1.1]) that is possibly well known and whose proof is straightforward enough to omit. We note first that if \mathbb{E} is an algebraic subfield of a field \mathbb{F} , then \mathbb{F} always has a Hamel basis over \mathbb{E} . (The proof of this fact is also routine.)

LEMMA 9.1. *For $n \geq 2$ let $\mathbb{F}_0 \subsetneq \mathbb{F}_1 \subsetneq \cdots \subsetneq \mathbb{F}_n$ be a properly inclusive sequence of algebraic fields (i.e., a tower), and let H_i be a Hamel basis for the field \mathbb{F}_i over the proper subfield \mathbb{F}_{i-1} for $i = 1, \dots, n$. Then the set of products*

$$H = \prod_{i=1}^n H_i = \{h_1 \cdots h_n : h_i \in H_i\}$$

is a Hamel basis for \mathbb{F}_n over \mathbb{F}_0 .

We apply the lemma with $n = 2$ as follows. From now on, let $\mathbb{F}_2 = \mathbb{R}$, $\mathbb{F}_1 = \mathbb{Q}[\sqrt[3]{2}]$, and $\mathbb{F}_0 = \mathbb{Q}$. (\mathbb{F}_1 is the smallest field containing \mathbb{Q} and $\sqrt[3]{2}$.)

Then

$$H_1 = \{1, \sqrt[3]{2}, \sqrt[3]{4}\}$$

is a basis for \mathbb{F}_1 over \mathbb{Q} ; for each member e of \mathbb{F}_1 there are unique rationals q_0, q_1, q_2 for which $e = q_0 \cdot 1 + q_1 \cdot \sqrt[3]{2} + q_2 \cdot \sqrt[3]{4}$. Let H_2 be any basis for \mathbb{R} over \mathbb{F}_1 for which $1 \in H_2$ (this is always possible). We will henceforth let

$$H = H_1 H_2$$

denote the Hamel basis for $\mathbb{F}_2 = \mathbb{R}$ over $\mathbb{F}_0 = \mathbb{Q}$ whose existence is asserted in Lemma 9.1. Notice that we have $H_1 \subset H$. In our injective well-ordering $\{h_\alpha\}_{\alpha < c}$ of H , we may and do assume that

$$h_0 = 1, \quad h_1 = \sqrt[3]{2}, \quad h_2 = \sqrt[3]{4},$$

which we desire for later convenience of notation.

The basis H_1 has the following convenient property, a weak sort of multiplicative closure:

$$k, k' \in H_1 \Rightarrow kk' = qk'' \text{ for some } q \in \mathbb{Q} \text{ and } k'' \in H_1.$$

We are particularly interested in the products $h_1 h$ for $h \in H$. If $h \in H_1 = \{h_0, h_1, h_2\}$, this amounts to the cyclicity:

$$h_1 h_0 = h_1, \quad h_1^2 = h_2, \quad h_1 h_2 = 2h_0.$$

Otherwise, for $h \in H \setminus H_1$, we see that $h_1 h$ is itself a member of $H \setminus H_1$, so that the expansion of $h_1 h$ involves no members of H_1 . That is,

$$q_\alpha(h_1 h) = 0 \quad \text{for } \alpha \in \{0, 1, 2\} \text{ and } h \in H \setminus H_1.$$

(Recall that q_α is defined by (5).) What must be emphasized is that the expansion of $\sqrt[3]{2} h$ involves no member of H_1 for $h \notin H_1$, while the expansion of $\sqrt[3]{2} h$ involves only members of H_1 when $h \in H_1$. Furthermore, $q_\alpha(\sqrt[3]{2} h) \geq 0$ for $\alpha \in \{0, 1, 2\}$ and $h \in H_1$.

For the rest of this paper, let

$$(7) \quad A_0 = A^{+++} \cup A^{*--}, \quad A_1 = A^{*++} \cup A^{-**}, \quad A_2 = A^{+++} \cup A^{-**}.$$

(See Section 7 for the relevant definitions.)

By regarding the properties just mentioned, with this basis H our sets A_0, A_1, A_2 enjoy the following cyclical properties under stretching by the factor of $\sqrt[3]{2}$:

$$(8) \quad \sqrt[3]{2} A_0 = A_1, \quad \sqrt[3]{2} A_1 = A_2, \quad \sqrt[3]{2} A_2 = A_0.$$

We establish some other useful properties of A_0, A_1, A_2 in Section 10.

10. A Banach measure that is not scale-invariant. The main motivation for this paper is the following theorem.

THEOREM 10.1. *There exists a Banach measure μ , an Archimedean set A (thereby a μ -shading), and $h \in \mathbb{R}$ such that $\text{sh}_\mu(hA) \neq \text{sh}_\mu(A)$.*

The proof of Theorem 10.1 will be easy once we establish some useful properties of the sets A_0, A_1, A_2 defined by (7) in Section 9. By Lemma 7.5, each of these three sets is shade-almost invariant. Each is of the form $A_{M,\pm\sigma}$ with $|M| = 2$, so by Theorem 8.2 none has shade zero.

We can now construct a measure having convenient properties on A_0, A_1, A_2 .

LEMMA 10.2. *Let t_0, t_1, t_2, t_3 be any real numbers chosen from $[0, 1]$ such that $t_0 + t_1 + t_2 + t_3 = 1$. Then there exists a Banach measure μ for which $\text{sh}_\mu(A_i) = t_i + t_3$ for $i = 0, 1, 2$.*

Proof. Let $B_0 = A^{-++} \cup A^{+--}$, $B_1 = A^{+-+} \cup A^{-+-}$, $B_2 = A^{+--} \cup A^{-++}$, $B_3 = A^{+++} \cup A^{---}$. Then B_0, B_1, B_2, B_3 have the following properties.

- (i) The sets B_0, B_1, B_2, B_3 are pairwise shade-almost disjoint. (They are pairwise disjoint.)
- (ii) None of B_0, B_1, B_2, B_3 has shade zero. (This follows from Corollary 8.4.)

Therefore, by Lemma 6.2, there is a Banach measure μ for which $\text{sh}_\mu(B_i) = t_i$ for $i = 0, 1, 2, 3$. Finally, for $i = 0, 1, 2$, the set A_i is the shade-almost disjoint union of B_i and B_3 , hence $\text{sh}_\mu(A_i) = \text{sh}_\mu(B_i \cup B_3) = t_i + t_3$. ■

We do not need it here, but the following answers a question posed in [10, Problem 3].

COROLLARY 10.3. *There exist Archimedean sets which are not shadings.*

Proof. The sets A_0, A_1, A_2 serve as examples, since Lemma 10.2 shows that their μ -shades depend on the particular μ . ■

The set $C + \mathbb{Q}$ mentioned earlier also serves as such an example, as would any shade-almost invariant Archimedean set that does not have shade zero or one.

Proof of Theorem 10.1. With H the Hamel basis constructed in Section 9, let $A = A_2$ and $h = h_1 = \sqrt[3]{2}$. Letting μ be the measure obtained in Lemma 10.2 and noting that $hA_2 = A_0$, we find that $\text{sh}_\mu(A_2) = t_2 + t_3$ but $\text{sh}_\mu(hA_2) = \text{sh}_\mu(A_0) = t_0 + t_3$, which need not equal $t_2 + t_3$. ■

It is instructive (or at least, amusing) to consider some extreme examples of Theorem 10.1. For instance, we can let $t_0 = 1$ and $t_1 = t_2 = t_3 = 0$, so that for the corresponding μ constructed via Theorem 10.1, we have $\text{sh}_\mu(A_2) = 0$ while $\text{sh}_\mu(hA_2) = 1$. Also, it is not difficult to see that for any nonzero rational q , the μ -shade of the A_i is equal to that of qA_i (for any μ). Therefore, qh can be any size, so that qhA_2 can be either a dilation or a contraction of A_2 . And qh can be arbitrarily close to 1, showing that a stretch (or contraction)

by the smallest amount can still give an extreme change in the μ -shade density. Also, using the same example, we can consider the set $A = A_2 \cap (0, \infty)$, which has (constant) μ -shade zero, whereas hA has μ -shade equal to 1 on $(0, \infty)$ but 0 on $(-\infty, 0)$. Scandalous!

Such possibilities were alluded to in the abstract: a similarity transformation of a set of constant μ -shade might not have constant μ -shade. How far can this behavior be stretched? Employing other extension fields of \mathbb{Q} , such as $\mathbb{Q}[\sqrt[n]{2}]$ for large n , more fun (or mockery) can certainly be had, by allowing greater varieties of μ -shades to result from a single transformation (with different μ). In this way, given an arbitrary step function f with values in $[0, 1]$ and only finitely many jumps, one can construct a set $B \subset \mathbb{R}$ and Banach measure μ for which $\text{sh}_\mu(B) = 1/2$ (say) but $\text{sh}_\mu(xB) = f$ for some $x > 0$. By considering different fields and/or building “taller towers” in the construction of the Hamel bases via Lemma 9.1, one can get infinitely many basis elements into the act (since Lemma 6.2 holds for countably many B_i). It should then be possible to pass from step functions to continuous functions, using techniques similar to those used for creating continuously varying shadings in [10, Thm. 5.7] or [14, Thm. 4.1].

How far can this go? Is it possible that given two different continuous $f_1, f_2 : \mathbb{R} \rightarrow [0, 1]$ there exists a set $B \subset \mathbb{R}$, a Banach measure μ , and $x > 0$ for which $\text{sh}_\mu(B) = f_1$ and $\text{sh}_\mu(xB) = f_2$? For such f_1 and f_2 , do there exist Banach measures μ_1 and μ_2 for which $\text{sh}_{\mu_1}(B) = f_1$ and $\text{sh}_{\mu_2}(B) = f_2$? If these sorts of examples are possible, can one go further and devise sets that are more universal, such as a single set B for which there is a Banach measure μ such that for each continuous f there is an x for which $\text{sh}_\mu(xB) = f$?

In all of this fun, we must bear in mind that these sets have μ -shades for particular $\mu \in \mathcal{B}$, as opposed to all Banach measures. It merely shows that *these* Banach measures are not able to agree on the density of these sets. In that respect, these measures are “not very good” at measuring such sets, one (e.g., [3]) might say.

Aside from such diverting examples, we more importantly have shown the following main result.

COROLLARY 10.4. *There exists a Banach measure on \mathbb{R} that is not scale-invariant.*

Proof. Our measure μ , constant h , and set A constructed in Theorem 10.1 witness the statement of the corollary. To see this more explicitly, consider the set $S = A \cap I$, where I is any bounded interval. The corollary follows if $\mu(hS) \neq h\mu(S)$. Indeed, on the one hand we have $\mu(hS) = \mu((hA) \cap (hI)) = \text{sh}_\mu(hA)\lambda(hI) = (\text{sh}_\mu(hA))h\lambda(I)$. But on the other hand we have $h\mu(S) = h\mu(A \cap I) = h\text{sh}_\mu(A)\lambda(I)$. The equality of these quantities would imply that $\text{sh}_\mu(A) = h\text{sh}_\mu(A)$, and would then contradict Theorem 10.1. ■

11. Conclusion. It now must be pointed out that if μ is an “improved” Banach measure on \mathbb{R} , i.e., one that is scale-invariant, then the μ -shade of a μ -shading is unaffected by a similarity transformation. More precisely, we have the following theorem, the proof of which we leave as an easy exercise for the reader.

THEOREM 11.1. *If μ is an improved Banach measure on \mathbb{R} and A is a μ -shading of \mathbb{R} , then $g(A)$ is also a μ -shading and $\text{sh}_\mu(g(A)) = \text{sh}_\mu(A)$ for every similarity transformation g on \mathbb{R} .*

Speaking of “improved measures” (cf. [2]), it might seem reasonable to incorporate scale-invariance into the definition of a shading, especially in view of Theorem 11.1. After all, reflection-invariance is included even though a similar definition can be given using only translation-invariance. But using translation-invariance alone yields results that are in some fundamental ways unsatisfying. For example, in spite of the fact that $A^+ \uplus (-A^+) \doteq \mathbb{R}$, one cannot claim, without making use of reflection-invariance, that A^+ should be viewed as “half” of the line in any measure-theoretic sense considered here, because translation-invariant total extensions of λ can be constructed with respect to which A^+ has any shade (in a similar way to Lemma 6.2, using the fact that $x + A^+ \doteq A^+$ for all $x \in \mathbb{R}$). Fortunately, we have reflection-invariance available as another intuitively appealing condition. We want our extensions of λ to have as many as possible of the properties we consider desirable in λ , so we include the reflection-invariance in our extensions because—well, because we can. So, if the result of Theorem 10.1 seems unreasonable (especially in view of the remarks following it), we have Theorem 11.1 to settle the matter, and perhaps we should indeed insist upon this improvement in the definition of shadings (because we can). On the other hand, in neither [2] nor [3] is this improvement mentioned, so perhaps tastes will differ in this regard!

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