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Triangulation in o-minimal fields with standard part map

by

Lou van den Dries (Urbana, IL) and **Jana Maříková** (Macomb, IL)

Abstract. In answering questions of J. Maříková [Fund. Math. 209 (2010)] we prove a triangulation result that is of independent interest. In more detail, let R be an o-minimal field with a proper convex subring V, and let $\mathrm{st} : V \to \boldsymbol{k}$ be the corresponding standard part map. Under a mild assumption on (R, V) we show that a definable set $X \subseteq V^n$ admits a triangulation that induces a triangulation of its standard part $\mathrm{st}\, X \subseteq \boldsymbol{k}^n$.

1. Introduction. This paper is a sequel to [6], and answers some questions it raised. To discuss this we adopt the notations and conventions from [6]. In particular, R is an *o-minimal field*, that is, an o-minimal expansion of an ordered field, V is a proper convex subring of R with maximal ideal $\mathfrak{m}$, ordered residue field $\boldsymbol{k} = V/\mathfrak{m}$, and residue map (or standard part map) $\mathrm{st}\colon V \to \boldsymbol{k}$. For each n this induces $\mathrm{st}\colon V^n \to \boldsymbol{k}^n$, and for $X \subseteq R^n$ we set $\mathrm{st}\, X := \mathrm{st}(X \cap V^n) \subseteq \boldsymbol{k}^n$. The primitives of the expansion $\boldsymbol{k}_{\mathrm{ind}}$ of the ordered field $\boldsymbol{k}$ are the ordered ring primitives plus the n-ary relations $\mathrm{st}\, X$ with $X \in \mathrm{Def}^n(R)$, for all n. Throughout, k, l, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. The problem studied in [6] is the following:

> *What conditions on (R, V) guarantee that $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal, and what are the definable relations of $\boldsymbol{k}_{\mathrm{ind}}$ in that case?*

Here is the main result of [6] on this issue: *If $(R, V) \models \Sigma_{\mathrm{i}}$, then for all n the boolean algebra* $\mathrm{Def}^n(\boldsymbol{k}_{\mathrm{ind}})$ *is generated by* $\{\mathrm{st}\, X : X \in \mathrm{Def}^n(R)\}$. Here Σ_{i} is a certain first-order axiom scheme to be stated below. It is satisfied in most cases where $\boldsymbol{k}_{\mathrm{ind}}$ was known to be o-minimal: when V is $\mathrm{Th}(R)$-convex in the sense of [5], and also when R is ω-saturated and V is the convex hull of $\mathbb{Q}$ in R. (In the latter case, $\boldsymbol{k}$ is isomorphic to the real field $\mathbb{R}$.)

In particular, [6] shows that if $(R, V) \models \Sigma_i$, then $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal. Here we prove a kind of converse:

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Theorem 1.1. *If* $\mathrm{Def}^2(\boldsymbol{k}_{\mathrm{ind}})$ *is generated as a boolean algebra by its subset* $\{\mathrm{st}\, X : X \in \mathrm{Def}^2(R)\}$, *then* $(R,V) \models \Sigma$.

Here Σ is a strong version of Σ_{i}. To define Σ_{i} and Σ, put

$$I := \{x \in R : |x| \le 1\},$$

and for $X \subseteq R^{1+n}$ and $r \in R$, put $X(r) := \{x \in R^n : (r,x) \in X\}$. "Definable" means "definable in R with parameters from R" unless we specify otherwise. The conditions Σ_{i} and Σ on (R,V) are as follows:

$\Sigma_{\mathrm{i}}(n)$: for all definable $X \subseteq I^{1+n}$, if $X(r) \subseteq X(s)$ for all $r \le s$ in I, then there exists $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\mathrm{st}\, X(\epsilon_0) = \mathrm{st}\, X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$;

$\Sigma(n)$: for all definable $X \subseteq I^{1+n}$ there exists an $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\mathrm{st}\, X(\epsilon_0) = \mathrm{st}\, X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$.

Also, let $\mathcal{C}(n)$ be the condition that every closed subset of $\boldsymbol{k}^n$ definable in $\boldsymbol{k}_{\mathrm{ind}}$ equals $\mathrm{st}\, X$ for some $X \in \mathrm{Def}^n(R)$. Finally, Σ_{i}, Σ, and $\mathcal{C}$ mean "$\Sigma_{\mathrm{i}}(n)$ for all n", "$\Sigma(n)$ for all n", and "$\mathcal{C}(n)$ for all n", respectively. Here is a sharper version of Theorem 1.1, incorporating also results from [6]:

Theorem 1.2. *The following conditions on* (R,V) *are equivalent:*

(1) $\mathcal{C}(2)$;
(2) $\mathcal{C}$;
(3) Σ;
(4) Σ_{i};
(5) $\mathrm{Def}^n(\boldsymbol{k}_{\mathrm{ind}})$ *is generated by* $\{\mathrm{st}\, X : X \in \mathrm{Def}^n(R)\}$, *for all* n;
(6) $\mathrm{Def}^2(\boldsymbol{k}_{\mathrm{ind}})$ *is generated by* $\{\mathrm{st}\, X : X \in \mathrm{Def}^2(R)\}$.

In Section 2 we prove (1)⇒(3). Since (3)⇒(2) is in [6] and (2)⇒(1) is obvious, this yields the equivalence of conditions (1)–(3). The implications (3)⇒(4) and (5)⇒(6) are also obvious, and (4)⇒(5) is in [6], but (6)⇒(1) requires a new tool: V-triangulation. In Sections 3, 4, 5 we prepare this tool. The main result about it is Theorem 6.1; we need only a special case of it to derive (6)⇒(1). In triangulating we try to follow Chapter 8 of [3], but we have to respect the standard part map and this requires a lot of extra care. The last Section 7 contains two more applications of V-triangulation.

The main problem left open here is whether the o-minimality of $\boldsymbol{k}_{\mathrm{ind}}$ implies that $(R,V) \models \Sigma$. While the present paper was being refereed, the second author answered this question positively; see [7].

Triangulation respecting the standard part map. Our V-triangulation result seems of independent interest, and may be new even when R is a real closed field without further structure. In the rest of this Introduction we state it precisely, and define some notation used throughout the paper. We let r, s, t (sometimes with subscripts or accents) range over R. For points

$a_0, \dots, a_m \in R^n$ (allowing repetitions) we let $\langle a_0, \dots, a_m \rangle$ denote the affine span of $\{a_0, \dots, a_m\}$ in R^n:

$$\langle a_0, \dots, a_m \rangle = \{t_0 a_0 + \dots + t_m a_m : t_0 + \dots + t_m = 1\},$$

and let $[a_0, , \dots, a_m]$ be the convex hull of $\{a_0, \dots, a_m\}$ in R^n:

$$[a_0, \dots, a_m] = \{t_0 a_0 + \dots + t_m a_m : t_0 + \dots + t_m = 1, \text{ all } t_i \geq 0\}.$$

A *simplex* in R^n is a set $[a_0, \dots, a_m]$ with affinely independent $a_0, \dots, a_m$ in R^n, and given such a simplex $S = [a_0, \dots, a_m]$ we put

$$S^{\circ} = (a_0, \dots, a_m) := \{t_0 a_0 + \dots + t_m a_m : t_0 + \dots + t_m = 1, \text{ all } t_i > 0\},$$

so S° is the interior of S in its affine span $\langle a_0, \dots, a_m \rangle$, and S is the closure of S° ([1]). Let $a_0, \dots, a_m \in R^n$ be affinely independent. Then we call $S = [a_0, \dots, a_m]$ an m-simplex. The points $a_0, \dots, a_m$ can be recovered from S because they are exactly the extreme points of S, as defined in [3, p. 120]; they are also referred to as the *vertices* of S. A *face* of S is a simplex $[a_{i_0}, \dots, a_{i_k}]$ with $0 \leq i_0 < \dots < i_k \leq m$. A *complex* in R^n is a finite collection K of simplexes in R^n such that each face of each $S \in K$ is in K, and for all $S, S' \in K$, if $S \cap S' \neq \emptyset$, then $S \cap S'$ is a common face of S and S'. For example, the collection of faces of a simplex S in R^n is a complex $K(S)$ in R^n. Let K be a complex in R^n. Then $S^{\circ} \cap S'^{\circ} = \emptyset$ for all distinct $S, S' \in K$. Let $|K|$ be the union of the simplexes in K. Then $K^{\circ} := \{S^{\circ} : S \in K\}$ is a finite partition of $|K|$. A *triangulation* of a definable $X \subseteq R^n$ is a pair (ϕ, K) consisting of a complex K in R^n and a definable homeomorphism $\phi\colon X \to |K|$; note that then X is closed and bounded in R^n. Such a triangulation is said to be *compatible with the set* $X' \subseteq X$ if $\phi(X')$ is a union of sets S° with $S \in K$.

Up to this point this subsection does not require the presence of V and makes sense for any (not necessarily o-minimal) expansion of an ordered field in place of R, for example $\boldsymbol{k}_{\text{ind}}$.

A set $X \subseteq R^n$ is V*-bounded* if for some $r \in V^{>0}$ we have $|x| \leq r$ for all $x \in X$. Note that if $a_0, \dots, a_m \in V^n$, then $[a_0, \dots, a_m]$ is V-bounded and

$$\text{st}[a_0, \dots, a_m] = [\text{st}(a_0), \dots, \text{st}(a_m)] \subseteq \boldsymbol{k}^n,$$

but if S is a V-bounded simplex in R^n, then $\text{st}\, S$ is not necessarily a simplex in $\boldsymbol{k}^n$, and even if it is, it might be just a single point while S is not.

A complex K in R^n is said to be V-bounded if $|K|$ is V-bounded. For a V-bounded complex K in R^n we set $\text{st}\, K := \{\text{st}\, S : S \in K\}$; this is not

([1]) Our terminology here is a little different from that in [3]: there the simplexes were the sets S°, but for the present purpose it is more convenient for our simplexes to be closed. Likewise, our definition of "complex" and "triangulation" here is not exactly the same as that in [3], but it is easy to go from one setting to the other.

always a complex in $\boldsymbol{k}^n$, and even if it is we can have $\operatorname{st} S = \operatorname{st} S'$ with distinct $S, S' \in K$.

A map $f\colon X \to R^n$ with $X \subseteq R^m$ is said to be V-bounded if its graph $\Gamma f \subseteq R^{m+n}$ is V-bounded. Suppose $f\colon X \to R^n$ is definable and V-bounded (so X is definable and V-bounded). Then we say that f *induces* (2) the map $g\colon \operatorname{st} X \to \boldsymbol{k}^n$ if $\operatorname{st}(f(x)) = g(\operatorname{st}(x))$ for all $x \in X$, equivalently, $\operatorname{st}(\Gamma f) = \Gamma g$; note that then g is definable in $\boldsymbol{k}_{\mathrm{ind}}$.

A *V-triangulation* of a definable V-bounded $X \subseteq R^n$ is a triangulation (ϕ, K) of X such that K is V-bounded, ϕ induces a map $\phi_{\mathrm{st}}\colon \operatorname{st} X \to \operatorname{st}|K|$, and $(\phi_{\mathrm{st}}, \operatorname{st} K)$ is a triangulation of $\operatorname{st} X$ in the sense of $\boldsymbol{k}_{\mathrm{ind}}$.

With this terminology in place we can state our triangulation theorem:

THEOREM 1.3. *If* $\mathrm{Def}^2(\boldsymbol{k}_{\mathrm{ind}})$ *is generated by* $\{\operatorname{st} X : X \in \mathrm{Def}^2(R)\}$, *then each definable closed V-bounded set $X \subseteq R^n$ with definable $X_1, \dots, X_k \subseteq X$ has a V-triangulation compatible with $X_1, \dots, X_k$.*

We finish this introduction with some more notation and a useful fact. Let $n \geq 1$. For $x = (x_1, \dots, x_n) \in R^n$ and $y = (y_1, \dots, y_n) \in R^n$ we set

$$d(x, y) := \max\{|x_i - y_i| : i = 1, \dots, n\}.$$

Likewise for $x, y \in \boldsymbol{k}^n$.

LEMMA 1.4. *Suppose $X \subseteq R^n$ and $f\colon X \to R$ are definable and V-bounded, and f induces a function $g\colon \operatorname{st} X \to \boldsymbol{k}$. Then g is continuous.*

Proof. We can assume $n \geq 1$. Assume towards a contradiction that $a \in X$ and $\epsilon \in R^{>\mathfrak{m}}$ are such that for every $\delta \in R^{>\mathfrak{m}}$ there is $x \in X$ such that $d(\operatorname{st}(a), \operatorname{st}(x)) < \operatorname{st}(\delta)$ and $|g(\operatorname{st}(a)) - g(\operatorname{st}(x))| > \operatorname{st}(\epsilon)$. Then the set

$$\{r \in R^{>0} : \text{there is } x \in X \text{ such that } d(a, x) < r \text{ and } |f(a) - f(x)| > \epsilon)\}$$

has an element in $\mathfrak{m}$, by o-minimality of R. This contradicts that f induces a function. ∎

2. $\mathcal{C}(2) \Rightarrow \Sigma$. The conditions $\mathcal{C}(n)$ and $\Sigma(n)$ were defined in the introduction. We begin with some observations about the case $n = 1$. It is clear that the o-minimality of $\boldsymbol{k}_{\mathrm{ind}}$ is equivalent to the condition that $\mathrm{Def}^1(\boldsymbol{k}_{\mathrm{ind}})$ is generated as a boolean algebra by $\{\operatorname{st} X : X \in \mathrm{Def}^1(R)\}$. Next, we have the equivalence

$$\boldsymbol{k}_{\mathrm{ind}} \text{ is o-minimal } \Leftrightarrow \mathcal{C}(1).$$

The forward direction is obvious. For the converse, note first that (R, V) is weakly o-minimal by a result of Baizhanov in [2] (see also [1]). Next, every $Y \subseteq \boldsymbol{k}^n$ definable in $\boldsymbol{k}_{\mathrm{ind}}$ equals $\operatorname{st} X$ for some $X \subseteq V^n$ definable in (R, V);

(2) This notion is a little different from that with the same name in [6], where it was necessary to allow the possibility that domain(g) is a proper subset of $\operatorname{st} X$.

this can be proved just like Lemma 4.1 in [6]. It remains to note that if $Y \subseteq \boldsymbol{k}$ is bounded and convex in $\boldsymbol{k}$, and has neither infimum nor supremum in $\boldsymbol{k}$, then Y is closed (and open) in $\boldsymbol{k}$.

It follows easily by cell decomposition that $\Sigma(1)$ is equivalent to the condition that for all definable $f\colon I \to I$ there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\operatorname{st}(f(\epsilon)) = \operatorname{st}(f(\epsilon_0))$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. Later in this section we prove that for all n we have $\mathcal{C}(n) \Rightarrow \Sigma(n)$, and so by the above this gives

$$\boldsymbol{k}_{\text{ind}} \text{ is o-minimal} \ \Rightarrow\ \Sigma(1).$$

We have $\mathcal{C}(n) \Rightarrow \mathcal{C}(m)$ for $n > m$ since projection maps and standard part maps commute. In particular, if $\mathcal{C}(n)$ holds for some $n \geq 1$, then $\boldsymbol{k}_{\text{ind}}$ is o-minimal.

It is convenient to introduce the following weak version $\mathcal{C}'(2)$ of $\mathcal{C}(2)$:

For all continuous $\phi\colon I(\boldsymbol{k}) \to \boldsymbol{k}$ that are definable in $\boldsymbol{k}_{\text{ind}}$ there exists a set $X \in \operatorname{Def}^2(R)$ such that $\Gamma\phi = \operatorname{st} X$.

(That $\mathcal{C}(2)$ implies $\mathcal{C}'(2)$ is because the graph of a continuous $\phi\colon I(\boldsymbol{k}) \to \boldsymbol{k}$ is closed in $\boldsymbol{k}^2$.) Using as above the weak o-minimality of $\boldsymbol{k}_{\text{ind}}$ we see that $\mathcal{C}'(2) \Rightarrow \mathcal{C}(1)$, and in particular $\mathcal{C}'(2) \Rightarrow \boldsymbol{k}_{\text{ind}}$ is o-minimal. **In the rest of this section we assume that $\boldsymbol{k}_{\text{ind}}$ is o-minimal**.

LEMMA 2.1. *Let $\phi\colon I(\boldsymbol{k}) \to \boldsymbol{k}$ be continuous and definable in $\boldsymbol{k}_{\text{ind}}$, and suppose $\Gamma\phi = \operatorname{st} X$ for some $X \in \operatorname{Def}^2(R)$. Then ϕ is induced by some V-bounded continuous definable $f\colon I \to R$.*

Proof. Take a V-bounded closed $X \subseteq I \times R$ with $X \in \operatorname{Def}^2(R)$ such that $\Gamma\phi = \operatorname{st} X$. Let $p\colon R^2 \to R$ be the projection map given by $p(x, y) = x$. Then $p(X) \subseteq I$ with $\operatorname{st}(p(X)) = I(\boldsymbol{k})$, and by definable choice we have a definable $h\colon p(X) \to R$ such that $\Gamma h \subseteq X$. By the piecewise continuity of h we can shrink $p(X)$ slightly to a closed definable set $P \subseteq p(X)$ such that $\operatorname{st} P = I(\boldsymbol{k})$ and $g := h|P$ is continuous. Since $\operatorname{st}(\Gamma g) \subseteq \Gamma\phi$, it follows that $\operatorname{st}(\Gamma g) = \Gamma\phi$. In particular, for $a, b \in P$ with $a < b$ and $(a, b) \cap P = \emptyset$, we have $\operatorname{st}(a) = \operatorname{st}(b)$ and $\operatorname{st}(g(a)) = \operatorname{st}(g(b))$. It is easy to extend g to a continuous definable $f\colon I \to R$ such that for all $a, b \in P$ as before, f is monotone on $[a, b]$. It follows that f is V-bounded and $\operatorname{st}(\Gamma f) = \Gamma\phi$, so f induces ϕ. ∎

LEMMA 2.2. *Suppose $\mathcal{C}'(2)$ holds. Then every closed and bounded $Y \in \operatorname{Def}^n(\boldsymbol{k}_{\text{ind}})$ with $\dim Y \leq 1$ equals $\operatorname{st} X$ for some $X \in \operatorname{Def}^n(R)$.*

Proof. This is clear for $n = 1$, so let $n > 1$ and $Y \in \operatorname{Def}^n(\boldsymbol{k}_{\text{ind}})$ be closed and bounded with $\dim Y \leq 1$. For a permutation σ of $\{1, \ldots, n\}$ and $Z \subseteq \boldsymbol{k}^n$ we put

$$\sigma(Z) := \{(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) : (a_1, \ldots, a_n) \in Z\}.$$

The assumptions on Y imply that Y is a finite union of sets $\sigma(\Gamma\phi)$ where $\phi\colon [a,b] \to \boldsymbol{k}^{n-1}$ is continuous and definable in $\boldsymbol{k}_{\text{ind}}$, $a \le b$ in $\boldsymbol{k}$, and σ is a permutation of $\{1,\dots,n\}$. So in order to show that $Y = \operatorname{st} X$ for some definable $X \in \operatorname{Def}^n(R)$ we can assume that $Y = \Gamma\phi$ where $\phi\colon [a,b] \to \boldsymbol{k}^{n-1}$ is continuous and definable in $\boldsymbol{k}_{\text{ind}}$ and $a \le b$ in $\boldsymbol{k}$. If $a = b$, then Y is a singleton, and there is no problem, so we can assume $a < b$. Then we can reduce to the case that $a = 0$, $b = 1$, so $\phi = (\phi_1,\dots,\phi_{n-1})\colon I(\boldsymbol{k}) \to \boldsymbol{k}^{n-1}$. By the previous lemma and the hypothesis of the present lemma we have V-bounded definable continuous $f_1,\dots,f_{n-1}\colon I \to R$ that induce $\phi_1,\dots,\phi_{n-1}$, so $f = (f_1,\dots,f_{n-1})\colon I \to R^{n-1}$ induces ϕ, and thus $\operatorname{st}(\Gamma f) = \Gamma\phi = Y$, as desired. ∎

The next two definitions and Lemma 2.5 use only that R is an o-minimal field and do not require a proper convex subring V of R, so they apply also to $\boldsymbol{k}_{\text{ind}}$ (which we have assumed to be o-minimal). For results about Hausdorff limits of definable families in an o-minimal expansion of the real field, see [4]. The notion of Hausdorff distance is also useful in our setting of an arbitrary o-minimal field.

Definition 2.3. Let $n \ge 1$ and put $d(x,Y) := \inf\{d(x,y) : y \in Y\}$ for $x \in R^n$ and nonempty definable $Y \subseteq R^n$. Next, let $X, Y \subseteq R^n$ be definable, closed, bounded and nonempty. Then the *Hausdorff distance* between X and Y is defined to be

$$d_{\mathrm{H}}(X,Y) := \min\{r \ge 0 : d(x,Y), d(y,X) \le r \text{ for all } x \in X \text{ and all } y \in Y\}.$$

With X,Y as in this definition, note that $d_{\mathrm{H}}(X,Y) \in R^{\ge 0}$, $d_{\mathrm{H}}(X,Y) = 0$ iff $X = Y$, $d_{\mathrm{H}}(X,Y) = d_{\mathrm{H}}(Y,X)$, and whenever $Z \subseteq R^n$ is also definable, closed, bounded, and nonempty, then

$$d_{\mathrm{H}}(X,Z) \le d_{\mathrm{H}}(X,Y) + d_{\mathrm{H}}(Y,Z).$$

So d_{H} behaves like a metric (but takes values in $R^{\ge 0}$ rather than $\mathbb{R}^{\ge 0}$).

Definition 2.4. Let $n \ge 1$ and let $X \subseteq R^{1+n}$ be definable such that the set $X(t) \subseteq R^n$ is closed, bounded and nonempty for all sufficiently small $t > 0$. Then a *Hausdorff limit* of $X(t)$ as $t \to 0+$ is a definable, closed, bounded, and nonempty set $Q \subseteq R^n$ such that $\lim_{t\to 0+} d_{\mathrm{H}}(X(t),Q) = 0$.

Lemma 2.5. *Let $n \ge 1$, and let $X \subseteq R^{1+n}$ be definable such that $X(t)$ is closed, bounded and nonempty for all sufficiently small $t > 0$. Then there is a unique Hausdorff limit of $X(t)$ as $t \to 0+$. Moreover, if $\dim X(t) \le m$ for all $t > 0$, then $\dim Q \le m$ for this Hausdorff limit Q.*

Proof. If Q, Q' are Hausdorff limits of $X(t)$ as $t \to 0+$, then $d_{\mathrm{H}}(Q,Q') = 0$, hence $Q = Q'$. So there is at most one Hausdorff limit of $X(t)$ as $t \to 0+$.

To show existence, let $Y := \{(t,x) \in X : t > 0\}$ and $Q := \mathrm{cl}(Y)(0) \subseteq R^n$. It is clear that Q is definable, closed, bounded, and, by cell decomposition, nonempty. We claim that $\lim_{t\to 0+} d_{\mathrm{H}}(Q, X(t)) = 0$. Suppose not. Then we have $\delta > 0$ such that for every $s > 0$ there is a positive $t < s$ with $d_{\mathrm{H}}(Q, X(t)) > \delta$. By o-minimality we can take $s > 0$ such that either there is for every $t \in (0,s)$ a point $a_t \in Q$ with $d(a_t, X(t)) > \delta$, or there is for every $t \in (0,s)$ a point $b_t \in X(t)$ with $d(b_t, Q) > \delta$. In the first case we can assume by definable choice that $t \mapsto a_t \colon (0,s) \to Q$ is definable; set $a := \lim_{t\to 0+} a_t$. Then $a \in Q$ since Q is closed, but it is also easy to check that $(0,a) \notin \mathrm{cl}(Y)$, so $a \notin Q$, a contradiction. In the second case we can assume that $t \mapsto b_t \colon (0,s) \to R^n$ is definable; set $b := \lim_{t\to 0+} b_t$. Then $(0,b) \in \mathrm{cl}(Y)$, so $b \in Q$, but also $d(b,Q) \geq \delta$, a contradiction.

Suppose now that $\dim X(t) \leq m$ for all $t > 0$. Then for $Y \subseteq X$ and $Q = \mathrm{cl}(Y)(0)$ as above we have $\{0\} \times Q \subseteq \mathrm{cl}(Y) \setminus Y$, so $\dim Q \leq m$. ∎

In the rest of this section s, s', s_0, s_1, s_2 range over I.

Lemma 2.6. *Let $s_0 \in \mathfrak{m}$ and $s_1 > \mathfrak{m}$, and suppose $f \colon (s_0, s_1) \to R$ is definable and $f(s) \in \mathfrak{m}$ for all s with $\mathfrak{m} < s < s_1$. Then there is a $p \in \mathfrak{m}^{>s_0}$ such that $f(s) \in \mathfrak{m}$ for all s with $p \leq s < s_1$.*

Proof. We can assume that f is of class C^1. For $\mathfrak{m} < s < s_1$ we have

$$f(s) - f(s/2) = (s/2)f'(s')$$

with $s' \in [s/2, s]$, so $f'(s') \in \mathfrak{m}$. It follows that the definable set

$$\{x \in (s_0, s_1) : |f'(x)| \leq 1\}$$

contains a set $[p,q]$ with $s_0 < p \in \mathfrak{m}$ and $\mathfrak{m} < q < s_1$. Let s with $p \leq s \in \mathfrak{m}$ be fixed, and take a variable s' with $\mathfrak{m} < s' \leq q$. Then $f(s') - f(s) = (s'-s)f'(x)$ for some x with $s \leq x \leq s'$, so $|f(s') - f(s)| \leq s' - s \leq s'$. Since $f(s') \in \mathfrak{m}$ and we can take s' arbitrarily small, subject to $s' > \mathfrak{m}$, we obtain $f(s) \in \mathfrak{m}$. ∎

More than the result itself, the *proof* of the following is crucial.

Lemma 2.7. *For all $n \geq 1$ we have $\mathcal{C}(n) \Rightarrow \Sigma(n)$.*

Proof. Let $n \geq 1$, and assume (R, V) satisfies $\mathcal{C}(n)$. Let $X \subseteq I^{1+n}$ be definable. Our job is to show the existence of $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\mathrm{st}\, X(\epsilon_0) = \mathrm{st}\, X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. We can assume that $X(s) \neq \emptyset$ for all $s \in I$. Put $Y := \mathrm{st}\, X \subseteq I(\boldsymbol{k})^{1+n}$. Let $Q \subseteq I(\boldsymbol{k})^n$ be the Hausdorff limit of $Y(t)$ as $t > 0$ tends to 0 in $\boldsymbol{k}$, so Q is definable in $\boldsymbol{k}_{\mathrm{ind}}$, and Q is closed, bounded, and nonempty. Using $\mathcal{C}(n)$ we take a closed definable $P \subseteq I^n$ such that $\mathrm{st}\, P = Q$.

Claim. *Let $\delta \in R^{>\mathfrak{m}}$. Then there is an $s' > \mathfrak{m}$ such that $d_{\mathrm{H}}(\mathrm{cl}(X(s)), P) < \delta$ for all s with $\mathfrak{m} < s < s'$.*

Suppose the claim is false. This gives s_0, s_1 with $s_0 \in \mathfrak{m} < s_1$ such that

$$\begin{aligned} &d_{\mathrm{H}}(\mathrm{cl}(X(s)), P) \geq \delta \quad &&\text{for all } s \in (s_0, s_1), \\ &d_{\mathrm{H}}(Y(t), Q) < \mathrm{st}(\delta) \quad &&\text{for all } t \text{ with } 0 < t \leq t_1 := \mathrm{st}(s_1). \end{aligned}$$

Let $\mathfrak{m} < s < s_1$. Then $d_{\mathrm{H}}(\mathrm{cl}(X(s)), P) \geq \delta$ gives either an $x \in \mathrm{cl}(X(s))$ with $d(x, P) \geq \delta$, or a point $p \in P$ with $d(X(s), p) \geq \delta$. But $x \in \mathrm{cl}(X(s))$ with $d(x, P) \geq \delta$ would give $\mathrm{st}(x) \in Y(\mathrm{st}(s))$ with $0 < \mathrm{st}(s) \leq t_1$ and $d(\mathrm{st}(x), Q) \geq \mathrm{st}(\delta)$, a contradiction. Thus $d(X(s), p) \geq \delta$ for some $p \in P$.

By increasing s_0 we can therefore arrange that for each $s \in (s_0, s_1)$ there is $p \in P$ with $d(X(s), p) \geq \delta$. Definable choice gives a definable function $p\colon (s_0, s_1) \to P$ such that $d(X(s), p(s)) \geq \delta$ for all $s \in (s_0, s_1)$. Definable choice in the structure $\boldsymbol{k}_{\mathrm{ind}}$ then gives a function $q\colon (0, t_1) \to Q$, definable in $\boldsymbol{k}_{\mathrm{ind}}$, such that $(t, q(t)) \in \mathrm{st}(\Gamma p)$ for all $t \in (0, t_1)$. Let $q_0 := \lim_{t \to 0} q(t)$, so $q_0 \in Q$. Take $p_0 \in P$ with $\mathrm{st}(p_0) = q_0$. Since $d(q(t), q_0) < \mathrm{st}(\delta)/2$ for all sufficiently small $t > 0$, there is for each s' with $\mathfrak{m} < s' < s_1$ an s such that $\mathfrak{m} < s < s'$ and $d(p(s), p_0) < \delta/2$, and thus $d(X(s), p_0) > \delta/2$. Then by the o-minimality of R we can decrease s_1 to arrange that $d(X(s), p_0) > \delta/2$ for all s with $\mathfrak{m} < s < s_1$ and $d(Y(t), q_0) < \mathrm{st}(\delta)/2$ for all t with $0 < t < t_1$. For such t, take $y \in Y(t)$ with $d(y, q_0) < \mathrm{st}(\delta)/2$; then $(t, y) \in Y$, so $(t, y) = \mathrm{st}(s, x)$ with $(s, x) \in X$, so $d(x, p_0) < \delta/2$ with $\mathfrak{m} < s < s_1$ and $x \in X(s)$, a contradiction. This concludes the proof of the claim.

Define $f\colon I \to R$ by $f(s) = d_{\mathrm{H}}(\mathrm{cl}(X(s)), P)$, so f is definable. By changing s_0 and s_1, if need be, we arrange that the restriction of f to $[s_0, s_1]$ is continuous and monotone. If this restriction of f is increasing, then it follows from the Claim that $f(\epsilon) \in \mathfrak{m}$ for all $\epsilon \in \mathfrak{m}$ with $\epsilon \geq s_0$, and thus $d_{\mathrm{H}}(\mathrm{cl}(X(\epsilon)), \mathrm{cl}(X(s_0))) \in \mathfrak{m}$ for all such ϵ, that is, $\mathrm{st}\, X(\epsilon) = \mathrm{st}\, X(s_0)$ for all such ϵ, and we are done.

So we can assume for the rest of the proof that f is decreasing on (s_0, s_1). Then $f(s) \in \mathfrak{m}$ for all s with $\mathfrak{m} < s < s_1$ by the Claim. Then Lemma 2.6 gives $\epsilon_0 \in \mathfrak{m}$, $\epsilon_0 > s_0$, such that $f(s) \in \mathfrak{m}$ for all s with $\epsilon_0 \leq s < s_1$. As before, this yields $\mathrm{st}\, X(\epsilon) = \mathrm{st}\, X(\epsilon_0)$ for all $\epsilon \geq \epsilon_0$ in $\mathfrak{m}$. ∎

Next a reduction to 1-parameter families of 1-dimensional sets:

LEMMA 2.8. *Let $n \geq 1$, and suppose that for all definable $X \subseteq I^{1+n}$ with $\dim X(r) \leq 1$ for all $r \in I$ there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\mathrm{st}\, X(\epsilon_0) = \mathrm{st}\, X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. Then $(R, V) \models \Sigma(n)$.*

Proof. Let $X \subseteq I^{1+n}$ be definable such that $X(r)$ is nonempty for all $r \in I$. Definable choice gives a definable map $f\colon I^2 \to I^n$ such that for all $r, s \in I$ we have $f(r, s) \in \mathrm{cl}(X(r))$ and

$$d(f(r, s), X(s)) = \sup\{d(x, X(s)) : x \in X(r)\}.$$

Define $Y \subseteq I^{1+n}$ by $Y(r) = \mathrm{cl}(\{f(r,s) : s \in I\})$ for $r \in I$. Then

$$Y(r) \subseteq \mathrm{cl}(X(r)), \quad \dim Y(r) \le 1, \quad d_{\mathrm{H}}(Y(r), Y(s)) \ge d_{\mathrm{H}}(\mathrm{cl}(X(r)), \mathrm{cl}(X(s))),$$

for all $r, s \in I$. The hypothesis of the lemma gives $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\mathrm{st}\, Y(\epsilon_0) = \mathrm{st}\, Y(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$, hence $d_{\mathrm{H}}(Y(\epsilon_0), Y(\epsilon)) \in \mathfrak{m}$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$, so $d_{\mathrm{H}}(\mathrm{cl}(X(\epsilon_0)), \mathrm{cl}(X(\epsilon))) \in \mathfrak{m}$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$, and thus $\mathrm{st}\, X(\epsilon_0) = \mathrm{st}\, X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. ∎

COROLLARY 2.9. $\mathcal{C}'(2) \Rightarrow \Sigma$.

Proof. Assume (R, V) satisfies $\mathcal{C}'(2)$. Towards proving $(R, V) \models \Sigma$, consider a definable $X \subseteq I^{1+n}$, $n \ge 1$, with $X(r) \ne \emptyset$ and $\dim X(r) \le 1$ for all $r \in I$; by Lemma 2.8 it suffices to show that then there is $\epsilon_0 \in \mathfrak{m}^{>0}$ such that $\mathrm{st}\, X(\epsilon_0) = \mathrm{st}\, X(\epsilon)$ for all $\epsilon \in \mathfrak{m}^{>\epsilon_0}$. Towards this goal we use the proof of Lemma 2.7. The present X satisfies $\dim X \le 2$, so the set $Y = \mathrm{st}\, X$ in that proof has also dimension at most 2 in the sense of the o-minimal structure $\boldsymbol{k}_{\mathrm{ind}}$, by Corollary 2.8 in [6]. So for all but finitely many $t \in I(\boldsymbol{k})$ the section $Y(t)$ has dimension at most 1. As in that proof, let $Q \subseteq I(\boldsymbol{k})^n$ be the Hausdorff limit of $Y(t)$ as $t > 0$ tends to 0 in $\boldsymbol{k}$. Then $\dim Q \le 1$, by Lemma 2.5. In the proof of Lemma 2.7 we only used the assumption $\mathcal{C}(n)$ to provide a $P \in \mathrm{Def}^n(R)$ with $\mathrm{st}\, P = Q$. Since $\dim Q \le 1$ and $\mathcal{C}'(2)$ holds, we can appeal here instead to Lemma 2.2 to provide such a P. With this P the rest of the proof of Lemma 2.7 goes through to give an ϵ_0 as desired. ∎

In particular, we have $\mathcal{C}(2) \Rightarrow \Sigma$, and together with the results from [6] this gives the equivalence of conditions (1)–(3) of Theorem 1.2. Of course, these conditions are also equivalent to $\mathcal{C}'(2)$.

3. Construction of a complex. As we mentioned in the introduction, we shall adapt to our purpose the proof of the o-minimal triangulation theorem in Chapter 8 of [3]. The first nontrivial issue that comes up in doing this is of a purely semilinear nature, and consists of finding a version of Lemma (1.10) in that chapter that is compatible with the standard part map. That lemma constructs a complex in R^{n+1} based on a simplex in R^n, and to make this construction compatible with the standard part map we need to linearly order the vertices of the simplex in a special way.

In more detail, recall that if S is a simplex in R^n, then $K(S)$ is the complex in R^n whose elements are the faces of S. Define a *V-simplex* in R^n to be a V-bounded simplex $S \subseteq R^n$ such that $\mathrm{st}(K(S))$ is a complex in $\boldsymbol{k}^n$. Note that if S is a V-simplex in R^n, then $\mathrm{st}\, S$ is a simplex in $\boldsymbol{k}^n$ and $K(\mathrm{st}\, S) = \mathrm{st}(K(S))$. We also define a *$V$-complex* in R^n to be a V-bounded complex K in R^n such that $\mathrm{st}\, K$ is a complex in $\boldsymbol{k}^n$; note that then the simplexes of K are V-simplexes.

Given a V-simplex S in R^n, our construction will require an ordering $a_0 < a_1 < \cdots < a_m$ of its vertices such that there are indices $0 = i_0 < i_1 < \cdots < i_k \le m$ for which $\mathrm{st}(a_{i_0}), \ldots, \mathrm{st}(a_{i_k})$ are the distinct vertices of $\mathrm{st}\, S$, and $\mathrm{st}(a_{i_\kappa}) = \mathrm{st}(a_i)$ whenever $i_\kappa \le i < i_{\kappa+1}$. This suggests the following notion (to be applied to the standard parts of points in V^n).

Let $a_0, \ldots, a_m \in \boldsymbol{k}^n$. We call the sequence $a_0, \ldots, a_m$ *simplicial* if there are indices $i_0 < \cdots < i_k$ in $\{0, \ldots, m\}$ with $i_0 = 0$ such that $a_{i_0}, \ldots, a_{i_k}$ are affinely independent in $\boldsymbol{k}^n$, and $a_{i_\kappa} = a_i$ whenever

$$0 \le \kappa \le k, \quad i_\kappa \le i < i_{\kappa+1} \quad (\text{with } i_{k+1} := m+1).$$

Suppose the sequence $a_0, \ldots, a_m$ is *simplicial* and let $i_0, \ldots, i_k$ be as above. Then $[a_0, \ldots, a_m] = [a_{i_0}, \ldots, a_{i_k}]$ is a k-simplex; if $0 \le j_0 < \cdots < j_l \le m$, then the sequence $a_{j_0}, \ldots, a_{j_l}$ is also simplicial, and $[a_{j_0}, \ldots, a_{j_l}]$ is a face of $[a_0, \ldots, a_m]$; all faces of $[a_0, \ldots, a_m]$ are obtained in this way, but different sequences $j_0, \ldots, j_l$ can give the same face.

Let $r_i, s_i \in \boldsymbol{k}$ for $i = 0, \ldots, m$ be such that $r_i \le s_i$ for all i and

$$\begin{aligned} &r_{i_\kappa} = r_i \quad \text{and} \quad s_{i_\kappa} = s_i \quad \text{whenever} \\ &0 \le \kappa \le k, \quad i_\kappa \le i < i_{\kappa+1} \quad (\text{with } i_{k+1} := m+1). \end{aligned}$$

Put $b_i := (a_i, r_i),\ c_i = (a_i, s_i)$ (points in $\boldsymbol{k}^{n+1}$). Then we have the following variant of Lemma (1.10) in Chapter 8 of [3].

Lemma 3.1. *If $0 \le j_0 < \cdots < j_p \le j_{p+1} < \cdots < j_q \le m$, $p < q$, then the sequence $b_{j_0}, \ldots, b_{j_p}, c_{j_{p+1}}, \ldots, c_{j_q}$ is simplicial. Let L be the set of all simplexes $[b_{j_0}, \ldots, b_{j_p}, c_{j_{p+1}}, \ldots, c_{j_q}]$ obtained from such sequences $j_0, \ldots, j_q$, and all faces of these simplexes. Then L is a complex with*

$$\begin{aligned} |L| &= \{t(t_0 b_0 + \cdots + t_m b_m) + (1-t)(t_0 c_0 + \cdots + t_m c_m) : \\ &\qquad\qquad 0 \le t \le 1,\ \textit{all } t_i \ge 0,\ t_0 + \cdots + t_m = 1\} \\ &= \textit{convex hull of } \{b_0, \ldots, b_m, c_0, \ldots, c_m\}. \end{aligned}$$

Proof. It is routine to check that the first statement is true. As to the rest, consider first the case that $r_i = s_i$ for all i. Then $b_i = c_i$ for all i, so L is just the set of faces of the k-simplex $[b_0, \ldots, b_m]$, and the claim about $|L|$ then holds trivially. Suppose $r_i < s_i$ for some i. Then, if $0 \le p \le k$ and $r_{i_p} < s_{i_p}$ we have a $(k+1)$-simplex $[b_{i_0}, \ldots, b_{i_p}, c_{i_p}, \ldots, c_{i_k}] \in L$. It is routine to check that L is the set of the $(k+1)$-simplexes obtained in this way and all their faces. Then our claim follows from Lemma (1.10) in Chapter 8 of [3]. ∎

Lemma 3.2. *Let S be a V-bounded m-simplex in R^n. Then the following are equivalent:*

1. *S is a V-simplex;*
2. *there is an enumeration $a_0, \ldots, a_m$ of the vertices of S such that the sequence $\mathrm{st}(a_0), \ldots, \mathrm{st}(a_m)$ is simplicial.*

Proof. Assume (1). Then every vertex a of S yields a vertex $\mathrm{st}(a)$ of the simplex $\mathrm{st}\, S$, so with $k := \dim(\mathrm{st}\, S)$ we have an enumeration of the vertices of S and indices $i_0 < \dots < i_k$ as in (2).

It is routine to check that (2) implies (1). ∎

Let S be a V-simplex. Then Lemma 3.2 yields an enumeration $a_0, \dots, a_m$ of its vertices and indices $0 = i_0 < \dots < i_k$ in $\{0, \dots, m\}$ such that $\mathrm{st}(a_{i_0}), \dots, \mathrm{st}(a_{i_k})$ are affinely independent in $\boldsymbol{k}^n$, and $\mathrm{st}(a_{i_\kappa}) = \mathrm{st}(a_i)$ whenever

$$0 \le \kappa \le k, \quad i_\kappa \le i < i_{\kappa+1} \quad (\text{with } i_{k+1} := m+1).$$

Let $r_i, s_i \in V$ for $i = 0, \dots, m$ be such that $r_i \le s_i$ for all i,

$$\begin{gathered}\mathrm{st}(r_i) = \mathrm{st}(r_{i_\kappa}) \quad \text{and} \quad \mathrm{st}(s_i) = \mathrm{st}(s_{i_\kappa}) \quad \text{whenever}\\ i_\kappa \le i < i_{\kappa+1}, \quad 0 \le \kappa \le k \quad (\text{with } i_{k+1} := m+1),\end{gathered}$$

and $r_i < s_i$ for some i. Put $b_i := (a_i, r_i)$, $c_i = (a_i, s_i)$ (points in V^{n+1}). Let L be the set of all $(m+1)$-simplexes $[b_0, \dots, b_i, c_i, \dots, c_m]$ with $b_i \ne c_i$, and all faces of these simplexes. Then by Lemma (1.10) of Chapter 8 in [3], L is a complex with

$$|L| = \text{convex hull of } \{b_0, \dots, b_m, c_0, \dots, c_m\}.$$

Corollary 3.3. *L is a V-complex.*

Proof. It follows easily from the assumptions on r_i, s_i that each simplex $[b_0, \dots, b_i, c_i, \dots, c_m]$ with $b_i \ne c_i$ is a V-simplex. A face of a V-simplex is also a V-simplex, so each simplex of L is a V-simplex. That $\mathrm{st}\, L$ is a complex follows from Lemma 3.1 with the $\mathrm{st}(b_i), \mathrm{st}(c_j)$ in the role of b_i, c_j. ∎

4. Extension lemmas. The first extension lemma below is a V-version of Lemma (2.1) in Chapter 8 of [3], but requires a very different proof. Before stating it we make some preliminary remarks and definitions.

First, let E be an affine subspace of R^n of dimension $k \ge 1$, so $E = e + L$ with $e \in R^n$ and L a linear subspace of R^n of dimension k. Let H_1 and H_2 be affine hyperplanes in E, so $H_i = e_i + L_i$ with $e_i \in E$ and a linear subspace L_i of L of dimension $k-1$, for $i = 1, 2$. Let $u \in L \setminus (L_1 \cup L_2)$. Then we have a direct sum decomposition $L = Ru \oplus L_2$, which yields a map

$$(H_1, H_2) : H_1 \to H_2, \quad \{(H_1, H_2)(x)\} = (x + Ru) \cap H_2 \quad \text{for all } x \in H_1.$$

This map is easily seen to be affine, and thus continuous, and to be a bijection with inverse (H_2, H_1).

Next, let S be a simplex in R^n. A *proper face* of S is a face F of S such that $F \ne S$. We set $\delta(S) :=$ union of the proper faces of S; this is the topological boundary of S in the affine span of its vertices.

These definitions and remarks go through for any ordered field instead of R, for example $\boldsymbol{k}$. In the rest of this section we assume that $\boldsymbol{k}_{\text{ind}}$ is o-minimal.

LEMMA 4.1. *Let $S \subseteq R^n$ be a V-bounded simplex and let $f\colon \delta(S) \to R$ be a continuous V-bounded definable function inducing a function* $\operatorname{st}\delta(S) \to \boldsymbol{k}$. *Then f has a continuous V-bounded definable extension $g\colon S \to R$ inducing a function* $\operatorname{st} S \to \boldsymbol{k}$.

Proof. Let $a_0, \dots, a_k$ be the distinct vertices of S. The lemma holds trivially for $k = 0$ since $\delta(S) = \emptyset$ in that case. So let $k \geq 1$, and let $E = \langle a_0, \dots, a_k\rangle$ be the affine span of the vertices of S. Below i ranges over $\{0, \dots, k\}$, and we set

$$H_i := \langle a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k\rangle, \qquad F_i := [a_0, \dots, a_{i-1}, a_{i+1}, \dots, a_k].$$

Let L be the linear subspace of R^n of which E is a translate, and let L_i be the proper linear subspace of L of which H_i is a translate. Take a vector $u \in L \setminus \bigcup_i L_i$. (Later in the proof we impose further restrictions on u.) For $x \in \delta(S)$ we have $(x + Ru) \cap S = [x, y]$ for a unique $y \in S$, and for this y we have $(x + Ru) \cap \delta(S) = \{x, y\}$; we define $\lambda : \delta(S) \to \delta(S)$ by $\lambda(x) = y$ for y as above. Note that $\lambda \circ \lambda = \mathrm{id}_{\delta(S)}$.

CLAIM 1. *λ is continuous.*

To see this note that the closed subsets $F_i \cap (H_i, H_j)^{-1}(F_j)$ $(0 \leq i, j \leq k)$ of $\delta(S)$ cover $\delta(S)$. By a remark at the beginning of this section, λ agrees on each such $F_i \cap (H_i, H_j)^{-1}(F_j)$ with the continuous map (H_i, H_j).

We now extend f to $g\colon S \to R$ by setting, for $x \in \delta(S)$,

$$g((1-t)x + t\lambda(x)) = (1-t)f(x) + tf(\lambda(x)).$$

CLAIM 2. *g is continuous.*

To see this, define

$$\alpha : [0,1] \times \delta(S) \to R^n, \qquad \alpha(t, x) = (1-t)x + t\lambda(x).$$

Then α is definable and continuous, $\alpha([0,1] \times \delta(S)) = S$, and $g \circ \alpha$ is continuous, so g is continuous by p. 96, Corollary 1.13 in [3].

It is easy to check that if the points $\operatorname{st}(a_0), \dots, \operatorname{st}(a_k)$ in $\boldsymbol{k}^n$ are affinely independent, then g induces a function on $\operatorname{st} S$, so in what follows we assume that $\operatorname{st}(a_0), \dots, \operatorname{st}(a_k)$ are not affinely independent. Then $\operatorname{st} S$ has dimension $d < k$, and we can assume that $\operatorname{st}(a_0), \dots, \operatorname{st}(a_d)$ are affinely independent.

CLAIM 3. $\operatorname{st} S = \operatorname{st}\delta(S)$.

To see this, note first that the affine span of $\operatorname{st} S = [\operatorname{st}(a_0), \dots, \operatorname{st}(a_k)]$ in $\boldsymbol{k}^n$ has dimension d. Then by a lemma of Carathéodory (p. 126 in [3]), each element of $\operatorname{st} S$ is in the convex hull of a subset of $\{\operatorname{st}(a_0), \dots, \operatorname{st}(a_k)\}$ of size $\leq d + 1$, and so in $\operatorname{st} F$ for some proper face F of S. This proves Claim 3.

The functions λ and g depend on u, and without further specifying u we cannot expect g to induce a function on $\operatorname{st} S$. We now restrict u as follows: $a_k \notin H_k = \langle a_0, \dots, a_{k-1} \rangle$ but $\operatorname{st}(a_k) \in \operatorname{st} H_k$, so we can take u as above such that $a_k + u \in H_k$ and $\operatorname{st}(u)$ is the zero vector of $\boldsymbol{k}^n$.

Claim 4. *$d(x, \lambda(x)) \in \mathfrak{m}$ for all $x \in \delta(S)$.*

To see this, note that S lies between H_k and $H_k + u$, that is,

$$S \subseteq \{x + tu : x \in H_k,\ 0 \le t \le 1\}.$$

This is because $\{x+tu : x \in L,\ 0 \le t \le 1\}$ is convex, and contains $a_0, \dots, a_k$. For $a \in H_k$,

$$(a + Ru) \cap \{x + tu : x \in H_k,\ 0 \le t \le 1\} = [a, a+u].$$

Given $x \in \delta(S)$ the line $x + Ru$ equals $a + Ru$ where $(x + Ru) \cap H_k = \{a\}$, and so $[x, \lambda(x)] \subseteq [a, a + u]$, so $d(x, \lambda(x)) \le d(a, a + u)$, whence the claim.

It is clear from Claims 2 and 4 that g induces a function on $\operatorname{st} S$. ∎

A *subcomplex* of a complex K in R^n is a subset L of K such that if F is a face of any $S \in L$, then $F \in L$; note that then L is also a complex in R^n.

Lemma 4.2. *Let L be a subcomplex of a V-bounded complex K in R^n, and let $f\colon |L| \to R$ be a V-bounded continuous definable function inducing a function $\operatorname{st}|L| \to \boldsymbol{k}$. Then f has a V-bounded continuous definable extension $|K| \to R$ inducing a function $\operatorname{st}|K| \to \boldsymbol{k}$.*

Proof. We can assume $L \ne K$, and it suffices to obtain a strictly larger subcomplex L' of K and a V-bounded continuous definable extension $f'\colon |L'| \to R$ of f inducing a function $\operatorname{st}|L'| \to \boldsymbol{k}$. Take a simplex $S \in K \setminus L$ of minimal dimension.

Suppose $S = \{a\}$ with $a \in R^n$. Then $L' = L \cup \{S\}$ is a subcomplex of K and $L \ne L'$. If $d(a, |L|) > \mathfrak{m}$, then $f'(a) = 0$ determines an extension of f to $|L'| \to R$ with the required properties. If $d(a, |L|) \in \mathfrak{m}$, then we can pick $b \in |L|$ such that $d(a, b) \in \mathfrak{m}$ and define an extension as desired by $f'(a) = f(b)$.

Next, assume that S is a k-simplex with $k > 0$. Then all proper faces of S are in L, so $\delta(S) \subseteq |L|$, and by the previous lemma, the function $f|\delta(S)$ extends to a V-bounded continuous definable function $g\colon S \to R$ inducing a function $\operatorname{st} S \to \boldsymbol{k}$. Also, $L' = L \cup \{S\}$ is a subcomplex of K, $L \ne L'$, f extends to a V-bounded continuous function $f'\colon |L'| \to R$ defined by $f'(x) = f(x)$ when $x \in |L|$ and $f'(x) = g(x)$ when $x \in S$, and f' induces a function $\operatorname{st}|L'| \to \boldsymbol{k}$. ∎

Good directions. In o-minimal triangulation we use extension lemmas in combination with the existence of good directions. For V-triangulation we need to sharpen this a little bit. Let

$$\mathbb{S}^n := \{(x_1, \dots, x_{n+1}) \in R^{n+1} : x_1^2 + \dots + x_{n+1}^2 = 1\}$$

and define $\mathbb{S}^n(\boldsymbol{k}) \subseteq \boldsymbol{k}^{n+1}$ likewise, with $\boldsymbol{k}$ instead of R. A unit vector $u \in \mathbb{S}^n$ is a *good direction* for a set $X \subseteq R^{n+1}$ if for each $a \in R^{n+1}$ the line $a + Ru$ intersects X in only finitely many points. Likewise we define what it means for a vector in $\mathbb{S}^n(\boldsymbol{k})$ to be a good direction for a set $X \subseteq \boldsymbol{k}^{n+1}$.

LEMMA 4.3. *Let $X \subseteq R^{n+1}$ be definable with $\dim X \le n$. Then there is a good direction $v \in \mathbb{S}^n(\boldsymbol{k})$ for $\operatorname{st} X$ such that all $u \in \mathbb{S}^n$ with $\operatorname{st}(u) = v$ are good directions for X.*

Proof. We have $\dim(\operatorname{st} X) \le n$, for example by Corollary 2.8 in [6]. Call $u \in \mathbb{S}^n$ a *bad direction* for X if u is not a good direction for X, and define bad directions for $\operatorname{st} X$ similarly. Let $B \subseteq \mathbb{S}^n$ be the set of bad directions for X, so B is definable and $\dim B < n$ by the Good Directions Lemma on p. 117 of [3]. Put

$$B' := \operatorname{st}(B) \cup \text{set of bad directions for } \operatorname{st} X \subseteq \mathbb{S}^n(\boldsymbol{k}).$$

Since $\boldsymbol{k}_{\text{ind}}$ is o-minimal, the set B' is definable in $\boldsymbol{k}_{\text{ind}}$, and $\dim B' < n$. It follows that we have a box $C \subseteq \boldsymbol{k}^{n+1}$ such that $C \cap \mathbb{S}^n(\boldsymbol{k}) \neq \emptyset$ and $\operatorname{cl}(C) \cap B' = \emptyset$. Then any $v \in C \cap \mathbb{S}^n(\boldsymbol{k})$ has the desired property. ∎

We define a *V-good direction* for a set $X \subseteq R^{n+1}$ to be a unit vector $u \in \mathbb{S}^n$ such that u is a good direction for X and $\operatorname{st}(u) \in \mathbb{S}^n(\boldsymbol{k})$ is a good direction for $\operatorname{st} X$. The above lemma yields an abundance of V-good directions for X if $X \subseteq R^{n+1}$ is definable with $\dim X \le n$.

5. The triangulation lemma. In this section we construct a V-triangulation of a definable closed V-bounded set in R^{n+1} if a suitable V-triangulation of its projection in R^n is given. First some more notation and terminology.

Let K be a complex in R^n. Let $\operatorname{Vert}(K)$ denote the set of vertices of the simplexes in K. Let (ϕ, K) be a triangulation of a definable closed $X \subseteq R^n$, and let $p = p^{n+1}_n \colon R^{n+1} \to R^n$ be the projection map given by

$$p(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n).$$

Then, given definable closed $Y \subseteq R^{n+1}$, a triangulation (ψ, L) of Y is said to be a *lifting* of (ϕ, K) if $K = \{p(T) : T \in L\}$ (so $K^{\circ} = \{p(T^{\circ}) : T \in L\}$) and the diagram

$$\begin{tikzcd} Y \arrow[r, "\psi"] \arrow[d] & {|L|} \arrow[d] \\ X \arrow[r, "\phi"] & {|K|} \end{tikzcd}$$

commutes where the vertical maps are restrictions of p (so $p(Y) = X$).

To construct liftings we use triangulated sets and multifunctions on them, and we proceed to define these notions. We set

$$\phi^{-1}(K) := \{\phi^{-1}(S) : S \in K\},$$

and call the pair $(X, \phi^{-1}(K))$ a *triangulated set*. To simplify notation, let $\mathcal{P} := \phi^{-1}(K)$. For $P, Q \in \mathcal{P}$ we call Q a *face* of P if $Q \subseteq P$ (equivalently, $\phi(Q)$ is a face of the simplex $\phi(P)$). For $P \in \mathcal{P}$, a *proper face* of P is a face $Q \in \mathcal{P}$ of P such that $P \neq Q$. For $P \in \mathcal{P}$ we put

$$P^{\mathrm{o}} := P \setminus \text{union of the proper faces of } P,$$

so $\phi(P^{\mathrm{o}}) = \phi(P)^{\mathrm{o}}$. A point $x \in X$ such that $\{x\} \in \mathcal{P}$ (that is, $\phi(x) \in \mathrm{Vert}(K)$) is said to be a *vertex* of $(X, \mathcal{P})$. A *multifunction* on $(X, \mathcal{P})$ is a finite collection F of continuous definable functions $f\colon X \to R$ such that for all $f, g \in F$ and $P \in \mathcal{P}$, either $f(x) < g(x)$ for all $x \in P^{\mathrm{o}}$, or $f(x) = g(x)$ for all $x \in P^{\mathrm{o}}$, or $g(x) < f(x)$ for all $x \in P^{\mathrm{o}}$.

Let F be a multifunction on $(X, \mathcal{P})$. For $P \in \mathcal{P}$ and $f, g \in F$ we say that g is the *successor* of f on P (in F) if $f(x) < g(x)$ for all $x \in P^{\mathrm{o}}$ (so $f(x) \leq g(x)$ for all $x \in P$), and there is no $h \in F$ such that $f(x) < h(x) < g(x)$ for all (equivalently, for some) $x \in P^{\mathrm{o}}$. We set

(a) $\Gamma F := \bigcup_{f \in F} \Gamma f \subseteq R^{n+1}$;
(b) $F|P := \{f|P : f \in F\}$ for $P \in \mathcal{P}$;
(c) $\mathcal{P}^F$ is the collection of all sets $\Gamma(f|P)$ with $f \in F$ and $P \in \mathcal{P}$, and all sets

$$[f|P, g|P] := \{(x, y) : x \in P \text{ and } f(x) \leq y \leq g(x)\}$$

with $P \in \mathcal{P}$, $f, g \in F$ and g the successor of f on P;
(d) $X^F :=$ union of the sets in $\mathcal{P}^F$, so $X^F \subseteq R^{n+1}$.

So ΓF and X^F are closed and bounded in R^{n+1}.

The above material in this section does not require the presence of V, and so makes sense and goes through for any o-minimal field instead of R, in particular, for $\boldsymbol{k}_{\mathrm{ind}}$ if the latter is o-minimal. We now bring in V again, and note that if (ϕ, K) is a V-triangulation of the definable closed V-bounded $X \subseteq R^n$, then the triangulation $(\phi_{\mathrm{st}}, \mathrm{st}\, K)$ of $\mathrm{st}\, X$ yields the triangulated set $(\mathrm{st}\, X, \mathrm{st}\, \mathcal{P})$, with $\mathcal{P} := \phi^{-1}(K)$, and

$$\mathrm{st}\, \mathcal{P} := \{\mathrm{st}\, P : P \in \mathcal{P}\} = \phi_{\mathrm{st}}^{-1}(\mathrm{st}\, K).$$

Remark. Suppose $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal. Let (ϕ, K) be a V-triangulation of the definable closed V-bounded set $X \subseteq R^n$. Let F be a multifunction on $(X, \mathcal{P})$ with $\mathcal{P} := \phi^{-1}(K)$ such that each $f \in F$ induces a function $f_{\mathrm{st}}\colon \mathrm{st}\, X \to \boldsymbol{k}$, and for all $f, g \in F$ the set $\{y \in \mathrm{st}\, X : f_{\mathrm{st}}(y) = g_{\mathrm{st}}(y)\}$ is a union of sets in $\mathrm{st}\, \mathcal{P}$. Then $F_{\mathrm{st}} := \{f_{\mathrm{st}} : f \in F\}$ is a multifunction on

$(\operatorname{st} X, \operatorname{st} \mathcal{P})$, with

$$\Gamma F_{\mathrm{st}} = \operatorname{st}(\Gamma F), \qquad (\operatorname{st} \mathcal{P})^{F_{\mathrm{st}}} = \{\operatorname{st} Q : Q \in \mathcal{P}^F\}, \qquad (\operatorname{st} X)^{F_{\mathrm{st}}} = \operatorname{st}(X^F).$$

(The middle equality requires a little thought.)

LEMMA 5.1. *Suppose $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal, and $\phi, K, X, \mathcal{P}, F$ are as in the remark above. Assume also that for all $P \in \mathcal{P}$ and all $Q \in \operatorname{st} \mathcal{P}$:*

(∗) *if $f, g \in F|P$, $f \neq g$, then $f(a) \neq g(a)$ for some vertex a of P;*
(∗∗) *if $f, g \in F_{\mathrm{st}}|Q$, $f \neq g$, then $f(a) \neq g(a)$ for some vertex a of Q.*

Then there is a V-triangulation (ψ, L) of X^F such that (ψ, L) is a lifting of (ϕ, K) compatible with the sets in $\mathcal{P}^F$, and $(\psi_{\mathrm{st}}, \operatorname{st} L)$ is a lifting of $(\phi_{\mathrm{st}}, \operatorname{st} K)$ compatible with the sets in $(\operatorname{st} \mathcal{P})^{F_{\mathrm{st}}}$.

Proof. Choose a total ordering $\leq$ on $\operatorname{Vert}(K)$ such that for all $a, b, c \in \operatorname{Vert}(K)$ with $a \leq b \leq c$ and $\operatorname{st}(a) = \operatorname{st}(c)$ we have $\operatorname{st}(a) = \operatorname{st}(b)$. This gives a total ordering $\leq$ on $\operatorname{Vert}(\operatorname{st} K)$ such that if $a, b \in \operatorname{Vert}(K)$ and $a \leq b$, then $\operatorname{st}(a) \leq \operatorname{st}(b)$. Now (ϕ, K), X, F are as in the proof of Lemma 2.8, p. 129 in [3], and we apply the construction from that proof, using the given ordering on $\operatorname{Vert}(K)$, to obtain a triangulation (ψ, L) of X^F that is a lifting of (ϕ, K) and is compatible with the sets in $\mathcal{P}^F$. We now briefly recall the construction of (ψ, L) from [3].

Let $P \in \mathcal{P}$, let $a_0, \ldots, a_m$ be the vertices of $\phi(P)$ such that in the ordering above we have $a_0 < a_1 < \cdots < a_m$, and let $f \in F|P$. Then the complex $L(f)$ in R^{n+1} consists of the m-simplex $[b_0, \ldots, b_m]$ and all its faces, where $b_i = (a_i, r_i) \in R^{n+1}$, $r_i := f(\phi^{-1}(a_i))$. Define

$$\psi_f \colon \Gamma f \to |L(f)|, \qquad \psi_f(x, f(x)) := \phi_b(x),$$

where $\phi_b(x)$ is the point of $[b_0, \ldots, b_m]$ with the same affine coordinates with respect to $b_0, \ldots, b_m$ as $\phi(x)$ has with respect to $a_0, \ldots, a_m$. Then ψ_f is a homeomorphism.

Suppose in addition that f has a successor $g \in F|P$. Then $L(f, g)$ is the complex L constructed just before Corollary 3.3, so $|L(f, g)|$ is the convex hull of $\{b_0, \ldots, b_m, c_0, \ldots, c_m\}$, where $c_i = (a_i, s_i) \in R^{n+1}$, $s_i := g(\phi^{-1}(a_i))$. Then the homeomorphism $\psi_{f,g} \colon [f, g] \to |L(f, g)|$ is given by

$$(x, tf(x) + (1-t)g(x)) \mapsto t\phi_b(x) + (1-t)\phi_c(x),$$

where $\phi_c(x)$ is defined in the same way as $\phi_b(x)$, and $0 \leq t \leq 1$.

The complex L is the union of the complexes $L(f)$ and $L(f, g)$ obtained in this way, and $\psi \colon X^F \to |L|$ extends each of the ψ_f and $\psi_{f,g}$ above.

Also, $(\phi_{\mathrm{st}}, \operatorname{st} K)$, $\operatorname{st} X$ and F_{st} are as in the proof of Lemma 2.8, p. 129 in [3], with $\boldsymbol{k}_{\mathrm{ind}}$ instead of R. Thus using the given ordering on $\operatorname{Vert}(\operatorname{st} K)$ we construct in the same way as before a triangulation (θ, M) of $(\operatorname{st} X)^{F_{\mathrm{st}}}$ that is a lifting of $(\phi_{\mathrm{st}}, \operatorname{st} K)$ and is compatible with the sets in $(\operatorname{st} \mathcal{P})^{F_{\mathrm{st}}}$.

CLAIM. *ψ induces θ.*

To prove this, let $P \in \mathcal{P}$ and let $a_0 < \cdots < a_m$ be the vertices of the simplex $\phi P \in K$. Put $Q := \operatorname{st} P \in \operatorname{st} \mathcal{P}$, and let the simplex $\phi_{\mathrm{st}}(Q) = \operatorname{st}(\phi P) \in \operatorname{st} K$ have vertices $\alpha_0 < \cdots < \alpha_k$. Then

$$\{0, \dots, m\} = I_0 \cup \cdots \cup I_k \quad \text{(disjoint union) with } I_j := \{i : \operatorname{st}(a_i) = \alpha_j\}.$$

Here and later in the proof, i ranges over $\{0, \dots, m\}$ and j over $\{0, \dots, k\}$. Let $f \in F$ and, towards showing that ψ induces θ on $\Gamma(f|P)$, put

$$b_i := (a_i, r_i) \in R^{n+1}, \qquad r_i := f(\phi^{-1}(a_i)),$$
$$\beta_j := (\alpha_j, \rho_j) \in \boldsymbol{k}^{n+1}, \qquad \rho_j := f_{\mathrm{st}}(\phi_{\mathrm{st}}^{-1}(\alpha_j)),$$

so $\operatorname{st}(b_i) = \beta_j$ for $i \in I_j$. Let $x \in P$. Then $\phi(x) = \sum_i t_i a_i$, where $t_i \geq 0$ and $\sum_i t_i = 1$, so $\phi_{\mathrm{st}}(\operatorname{st}(x)) = \sum_j \tau_j \alpha_j$ with $\tau_j = \sum_{i \in I_j} t_i$. Then

$$\psi(x, f(x)) = \sum_i t_i b_i, \qquad \theta(\operatorname{st}(x), f_{\mathrm{st}}(\operatorname{st}(x))) = \sum_j \tau_j \beta_j,$$

so $\operatorname{st}(\psi(x, f(x))) = \theta(\operatorname{st}(x, f(x)))$. Thus ψ induces θ on $\Gamma(f|P)$.

Next, assume also that f has a successor $g \in F$ on P, and put

$$c_i := (a_i, s_i) \in R^{n+1}, \qquad s_i := g(\phi^{-1}(a_i))$$
$$\gamma_j := (\alpha_j, \sigma_j) \in \boldsymbol{k}^{n+1}, \qquad \sigma_j := g_{\mathrm{st}}(\phi_{\mathrm{st}}^{-1}(\alpha_j)),$$

so $\operatorname{st}(c_i) = \gamma_j$ for $i \in I_j$. Then, with $\bar{x} := \operatorname{st}(x)$, $\bar{t} := \operatorname{st}(t)$,

$$\psi(x, tf(x) + (1-t)g(x)) = t \sum_i t_i b_i + (1-t) \sum_i t_i c_i,$$
$$\theta(\bar{x}, \bar{t} f_{\mathrm{st}}(\bar{x}) + (1-\bar{t}) g_{\mathrm{st}}(\bar{x})) = \bar{t} \sum_j \sigma_j \beta_j + (1-\bar{t}) \sum_j \sigma_j \gamma_j.$$

To obtain the second identity, note that either f_{st} and g_{st} coincide on $\operatorname{st} P$, or g_{st} is the successor of f_{st} on $\operatorname{st} P$ (in F_{st}). It follows as with $\Gamma(f|P)$ that ψ induces θ on $[f|P, g|P]$. Since $P \in \mathcal{P}$ was arbitrary, this proves the claim.

For (ψ, L) to have the property stated in the lemma it only remains to check that $\operatorname{st} L = M$. This equality follows from Section 3 in view of how we ordered $\operatorname{Vert}(K)$ and $\operatorname{Vert}(\operatorname{st} K)$ and constructed L and M. ∎

Satisfying conditions $(*)$ **and** $(**)$**.** In the situation of the remark before the triangulation lemma 5.1, condition $(*)$ might fail for some $P \in \mathcal{P}$. We can then replace K by its barycentric subdivision to satisfy $(*)$, as is done in [3], but a simplex of this barycentric subdivision is not necessarily a V-simplex, so this fails to deal with $(**)$. Fortunately, a slight generalization of the barycentric subdivision solves this problem, as we describe below.

Recall that the *barycenter* of an m-simplex $S = [a_0, \dots, a_m]$ in R^n is the point $\frac{1}{m+1}(a_0 + \cdots + a_m)$ in S^0. Let K be a complex in R^n. A *subdivision* of K is a complex K' in R^n such that $|K| = |K'|$ and each simplex of K is a union of simplexes of K'; it follows easily that then each set S^{o} with $S \in K$ is a union of sets S'^{o} with $S' \in K'$. Define a *K-flag* to be a sequence $S_0, \dots, S_k$ in K such that S_i is a proper face of S_{i+1} for all $i < k$. Given such a K-flag and a point $b_i \in S_i^{\mathrm{o}}$ for $i = 0, \dots, k$ we have a k-simplex $[b_0, \dots, b_k]$. Assume now that to each $S \in K$ is assigned a point $b(S) \in S^{\mathrm{o}}$. This yields a subdivision $b(K)$ of K whose simplexes are the $[b(S_0), \dots, b(S_k)]$ with $S_0, \dots, S_k$ a K-flag. (In Chapter 8 of [3] we took $b(S) :=$ barycenter of S, for each $S \in K$, and then $b(K)$ is the barycentric subdivision of K.)

The above paragraph uses only the semilinear structure of R, and so goes through with $\boldsymbol{k}$ instead of R. We now apply this to a V-complex K in R^n as follows. We choose for each $S \in K$ a point $b(S) \in S^{\mathrm{o}}$ such that

$$\mathrm{st}(b(S)) = \text{barycenter of } \mathrm{st}\, S.$$

We claim that then the subdivision $b(K)$ of K has the following property:

$$b(K) \text{ is a } V\text{-complex, and } \mathrm{st}(b(K)) = \text{barycentric subdivision of } \mathrm{st}\, K.$$

To see this, let $S_0, \dots, S_k$ be a K-flag and $T := [b(S_0), \dots, b(S_k)]$. Then

$$\mathrm{st}\, T = [\text{barycenter}(\mathrm{st}\, S_0), \dots, \text{barycenter}(\mathrm{st}\, S_k)]$$

is a simplex of the barycentric subdivision of $\mathrm{st}\, K$ (even if the sequence $\mathrm{st}\, S_0, \dots, \mathrm{st}\, S_k$ has repetitions), and each simplex of the barycentric subdivision of $\mathrm{st}\, K$ arises in this way from a K-flag.

LEMMA 5.2. *Assume $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal. Let K be a V-complex in R^n, $X := |K|$, and F a multifunction on (X, K) such that each $f \in F$ induces a function $f_{\mathrm{st}}\colon \mathrm{st}\,|K| \to \boldsymbol{k}$ and for all $f, g \in F$ the set*

$$\{y \in X : f_{\mathrm{st}}(y) = g_{\mathrm{st}}(y)\}$$

is a union of sets in $\mathrm{st}\, K$. Then there is a subdivision K' of K such that K' is a V-complex, and F as a multifunction on (X, K') satisfies the following conditions for all $P \in K'$ and all $Q \in \mathrm{st}\, K'$:

(∗) *if $f, g \in F|P$, $f \neq g$, then $f(a) \neq g(a)$ for some vertex a of P;*
(∗∗) *if $f, g \in F_{\mathrm{st}}|Q$, $f \neq g$, then $f(a) \neq g(a)$ for some vertex a of Q.*

Proof. Just take as K' a complex $b(K)$ as constructed in the paragraph just before the statement of the lemma. Then K' has the desired properties. ∎

Small paths. To apply the triangulation lemma in the next section we also need to construct a multifunction. This will require the extension Lemma 4.2 as well as the lemma below about the "small path" property. In the rest of this section $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal, and we consider a definable V-bounded set $X \subseteq R^n$. We say that X has *small paths* if for all $x, y \in X$ with $\mathrm{st}(x) = \mathrm{st}(y)$ there is $\epsilon \in \mathfrak{m}^{>0}$ and a definable continuous path $\gamma\colon [0, \epsilon] \to X$

such that $\gamma(0) = x$, $\gamma(\epsilon) = y$, and $\mathrm{st}(\gamma(t)) = \mathrm{st}(x)$ for all $t \in [0, \epsilon]$; such a γ will be called a *small path.* Note that if X is convex, then X has small paths. It follows that if there is a V-bounded simplex S in R^n and a definable homeomorphism $X \to S$ inducing a homeomorphism $\mathrm{st}\, X \to \mathrm{st}\, S$, then X has small paths.

LEMMA 5.3. *Assume X has small paths, and let $f\colon X \to R$ be definable, continuous, and V-bounded, such that the upward unit vector $e_{n+1} \in \boldsymbol{k}^{n+1}$ is a good direction for $\mathrm{st}(\Gamma f)$. Then f induces a function $\mathrm{st}\, X \to \boldsymbol{k}$.*

Proof. Let $x, y \in X$ be such that $\mathrm{st}(x) = \mathrm{st}(y)$; it is enough to show that then $\mathrm{st}(f(x)) = \mathrm{st}(f(y))$. Take a small path $\gamma\colon [0, \epsilon] \to X$ such that $\gamma(0) = x$ and $\gamma(\epsilon) = y$. Then the standard parts of the points $(\gamma(t), f(\gamma(t)))$ all lie on the same vertical line in $\boldsymbol{k}^{n+1}$, and since e_{n+1} is a good direction for $\mathrm{st}(\Gamma f)$, this yields $\mathrm{st}(f(x)) = \mathrm{st}(f(y))$. ∎

6. Proof of V-triangulation. Recall the V-triangulation theorem stated on page 136:

THEOREM 6.1. *Suppose the boolean algebra $\mathrm{Def}^2(\boldsymbol{k}_{\mathrm{ind}})$ is generated by its subset $\{\mathrm{st}\, X : X \in \mathrm{Def}^2(R)\}$. Then every V-bounded closed definable $X \subseteq R^n$ with definable subsets $X_1, \ldots, X_k$ has a V-triangulation compatible with $X_1, \ldots, X_k$.*

Before we start the proof, first note that the hypothesis of the theorem implies that $\mathrm{Def}^1(\boldsymbol{k}_{\mathrm{ind}})$ is generated by $\{\mathrm{st}\, X : X \in \mathrm{Def}^1(R)\}$, which in turn is equivalent to $\boldsymbol{k}_{\mathrm{ind}}$ being o-minimal. If $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal, the conclusion of the theorem clearly holds for $n = 1$. The proof will show that the conclusion of the theorem for $n = 2$ also follows just from assuming that $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal. The stronger hypothesis about $\mathrm{Def}^2(\boldsymbol{k}_{\mathrm{ind}})$ will only be used to obtain the conclusion for $n > 2$.

Proof. As already noted, $\boldsymbol{k}_{\mathrm{ind}}$ is o-minimal, and the theorem holds for $n = 1$. We proceed by induction on n, so assume inductively that for a certain $n \geq 1$:

- (i) $\mathrm{Def}^n(\boldsymbol{k}_{\mathrm{ind}})$ is generated by $\{\mathrm{st}\, X : X \in \mathrm{Def}^n(R)\}$;
- (ii) every V-bounded closed definable $X \subseteq R^n$ with definable subsets $X_1, \ldots, X_k$ has a V-triangulation compatible with $X_1, \ldots, X_k$.

CLAIM. *$\mathcal{C}(n)$ holds.*

To prove this claim, let $Z \in \mathrm{Def}^n(\boldsymbol{k}_{\mathrm{ind}})$ be closed and bounded in $\boldsymbol{k}^n$; we have to show that $Z = \mathrm{st}\, Q$ for some $Q \in \mathrm{Def}^n(R)$. Now by part (i) of the inductive assumption, Z is a boolean combination of $\mathrm{st}\, X_1, \ldots, \mathrm{st}\, X_k$ with $X_1, \ldots, X_k \in \mathrm{Def}^n(R)$, and we can assume that $X_1, \ldots, X_k$ are V-bounded. Take a V-bounded closed $X \in \mathrm{Def}^n(R)$ containing all X_i as subsets and such that $Z \subseteq \operatorname{st} X$. Then by part (ii) of the inductive assumption we have triangulated sets $(X, \mathcal{P})$ and $(\operatorname{st} X, \operatorname{st} \mathcal{P})$ such that each X_i is a union of sets P° with $P \in \mathcal{P}$. Then each $\operatorname{st} X_i$ is a union of sets $\operatorname{st} P^{\circ} = \operatorname{st} P \in \operatorname{st} \mathcal{P}$. Each $\operatorname{st} P$ with $P \in \mathcal{P}$ is a union of sets from the partition $(\operatorname{st} \mathcal{P})^{\circ} = \{(\operatorname{st} P)^{\circ} : P \in \mathcal{P}\}$ of $\operatorname{st} X$, so each $\operatorname{st} X_i$ is such a union as well, and so is their boolean combination Z. But Z is closed, so Z is then a union of closures $\operatorname{st} P$ of sets $(\operatorname{st} P)^{\circ}$ with $P \in \mathcal{P}$, and so $Z = \operatorname{st} Q$ where Q is a union of sets $P \in \mathcal{P}$. This proves the claim.

Then by Lemma 2.7 we have $(R, V) \models \Sigma(n)$. Also (i) holds with $n+1$ instead of n: for $n = 1$ this is just the hypothesis of the theorem, and if $n \geq 2$, then Σ holds by the claim and Section 2, and so we can use Theorem 2.22 from [6].

In order to prove that (ii) holds with $n+1$ instead of n, let $Y \subseteq R^{n+1}$ be V-bounded, closed, and definable, with definable subsets $Y_1, \ldots, Y_k$; our aim is then to construct a V-triangulation of Y compatible with $Y_1, \ldots, Y_k$. Put

$$T := \operatorname{bd}(Y_0) \cup \operatorname{bd}(Y_1) \cup \cdots \cup \operatorname{bd}(Y_k), \qquad Y_0 := Y.$$

Then $\dim T \leq n$, so by Lemma 4.3 we can replace $T, Y_0, \ldots, Y_k$ by their images under a suitable orthogonal transformation of R^{n+1} to arrange that e_{n+1} is a V-good direction for T as defined at the end of Section 4. (See [3], top of p. 131, for a similar argument.)

We are going to construct a V-triangulation of $X := p^{n+1}_n T = p^{n+1}_n Y \subseteq R^n$ so that we can use the triangulation lemma 5.1.

Cell decomposition gives a finite partition $\mathcal{C}$ of X into cells C such that $T \cap (C \times R)$ is the union of the graphs of definable continuous functions

$$f_{C,1} < \cdots < f_{C,l(C)} : C \to R, \qquad l(C) \geq 1,$$

such that for $i = 0, \ldots, k$ and $j = 1, \ldots, l(C)$,

$$\begin{aligned} &\text{either } \Gamma f_{C,j} \subseteq Y_i \text{ or } \Gamma f_{C,j} \cap Y_i = \emptyset, \text{ and for } 1 \leq j < l(C): \\ &\text{either } (f_{C,j}, f_{C,j+1}) \subseteq Y_i \text{ or } (f_{C,j}, f_{C,j+1}) \cap Y_i = \emptyset. \end{aligned}$$

Since e_{n+1} is a good direction for T and $T \supseteq \operatorname{cl}(\Gamma f)$ for each $f = f_{C,j}$, each $f_{C,j}$ extends continuously to a definable function $\operatorname{cl}(C) \to R$, and we denote this extension also by $f_{C,j}$. We need to extend these functions $f_{C,j}$ to all of X in a nice way, and towards this goal we note that the inductive assumption (ii) gives a V-triangulation (ϕ, K) of X compatible with all $C \in \mathcal{C}$. This gives a triangulated set $(X, \mathcal{P})$ with $\mathcal{P} := \phi^{-1}(K)$. Let $C \in \mathcal{C}$ be given. Then the set $\operatorname{cl}(C)$ is a finite union of sets $P \in \mathcal{P}$. The sets $P \in \mathcal{P}$ have small paths, and so by Lemma 5.3 each function $f_{C,j} \colon \operatorname{cl}(C) \to R$ induces a function on $\operatorname{st}(\operatorname{cl}(C))$, and thus, by Lemma 4.2, extends to a definable continuous V-bounded function $f \colon X \to R$ such that f induces a function $\operatorname{st} X \to \boldsymbol{k}$. In this way we obtain a finite set F of definable continuous

V-bounded functions $f\colon X \to R$ such that each $f \in F$ induces a function $f_{\mathrm{st}}\colon \operatorname{st} X \to \boldsymbol{k}$, each $f \in F$ extends some $f_{C,j}$, and each $f_{C,j}$ has an extension in F. To make F into a multifunction on $(X, \mathcal{P})$ that induces a multifunction on $(\operatorname{st} X, \operatorname{st} \mathcal{P})$ we may have to refine $\mathcal{P}$, and this is done as follows. Since $\Sigma(n)$ holds in (R, V), we have $\epsilon_0 \in \mathfrak{m}^{>0}$ such that for all $f, g \in F$ and $\epsilon \in \mathfrak{m}^{>\epsilon_0}$,

$$\operatorname{st}\{x \in X : |f(x) - g(x)| \le \epsilon_0\} = \operatorname{st}\{x \in X : |f(x) - g(x)| \le \epsilon\},$$

and thus for all $f, g \in F$,

$$\operatorname{st}\{x \in X : |f(x) - g(x)| \le \epsilon_0\} = \operatorname{st}\{x \in X : f(x) - g(x) \in \mathfrak{m}\}.$$

Using again the inductive assumption (ii) we arrange that our V-triangulation (ϕ, K) above is also compatible with all sets

$$\{x \in X : f(x) = g(x)\} \quad \text{and} \quad \{x \in X : |f(x) - g(x)| \le \epsilon_0\} \qquad (f, g \in F).$$

Note that then F is a multifunction on $(X, \mathcal{P})$, and that for all Y_i and $f \in F$ and $P \in \mathcal{P}$, either $\Gamma(f|P^{\circ}) \subseteq Y_i$ or $\Gamma(f|P^{\circ}) \cap Y_i = \emptyset$, and if also $g \in F$ is the successor of f on P, then either $(f|P^{\circ}, g|P^{\circ}) \subseteq Y_i$ or $(f|P^{\circ}, g|P^{\circ}) \cap Y_i = \emptyset$. Note that for all $f, g \in F$,

$$\operatorname{st}\{x \in X : |f(x) - g(x)| \le \epsilon_0\} = \{y \in \operatorname{st} X : f_{\mathrm{st}}(y) = g_{\mathrm{st}}(y)\},$$

and the set on the left is a union of sets in $\operatorname{st} \mathcal{P}$. Hence we are in the situation of the remark preceding Lemma 5.1, so $F_{\mathrm{st}} := \{f_{\mathrm{st}} : f \in F\}$ is a multifunction on $(\operatorname{st} X, \operatorname{st} \mathcal{P})$. By Lemma 5.2 we can replace K by a subdivision and $\mathcal{P}$ accordingly to arrange also that for all $P \in \mathcal{P}$ and $Q \in \operatorname{st} \mathcal{P}$ conditions $(*)$ and $(**)$ of Lemma 5.1 are satisfied. This triangulation lemma then yields a V-triangulation (ψ, L) of X^F that lifts (ϕ, K) and is compatible with the sets in $\mathcal{P}^F$, and such that $(\psi_{\mathrm{st}}, \operatorname{st} L)$ is a lifting of $(\phi_{\mathrm{st}}, \operatorname{st} K)$ compatible with the sets in $(\operatorname{st} \mathcal{P})^{F_{\mathrm{st}}}$. Let L' be the subcomplex of L for which $|L'| = \psi(Y)$, and put $\psi' := \psi|Y$. Then (ψ', L') is a V-triangulation of Y compatible with $Y_1, \ldots, Y_k$, as promised. ∎

In the course of the proof just given we have also established the implication (6)⇒(1) of Theorem 1.2, and this concludes the proof of that theorem, by remarks following its statement.

7. Two applications of V-triangulation. In this section we assume that (R, V) satisfies the (equivalent) conditions of Theorem 1.2. Here is an easy consequence of V-triangulation and Lemma 4.2:

COROLLARY 7.1. *Let $X, Y \subseteq R^n$ be closed and V-bounded definable sets with $X \subseteq Y$, and let $f : X \to R$ be a continuous V-bounded definable function inducing a function $\operatorname{st} X \to \boldsymbol{k}$. Then f extends to a continuous V-bounded definable function $Y \to R$ inducing a function $\operatorname{st} Y \to \boldsymbol{k}$.*

Here is a related open question: Does the above corollary go through if the assumption that X and Y are closed is replaced by the weaker one that X is closed in Y? That would give a V-version of the o-minimal Tietze extension result (3.10) of Chapter 8 in [3].

A finiteness result. Let $X \subseteq R^m$ and $Y \subseteq R^n$ be V-bounded and definable. Then a *V-homeomorphism* $f\colon X \to Y$ is by definition a definable homeomorphism $X \to Y$ that induces a homeomorphism $f_{\mathrm{st}}\colon \operatorname{st} X \to \operatorname{st} Y$.

For a V-bounded definable $X \subseteq R^{m+n}$ the sets $X(a) \subseteq R^n$ with $a \in R^m$ fall into only finitely many V-homeomorphism types. Towards proving this (in a stronger form), consider triples $(N, \mathcal{C}, E)$ such that $N \in \mathbb{N}$, $\mathcal{C}$ is a collection of nonempty subsets of $\{1, \dots, N\}$ with $\{i\} \in \mathcal{C}$ for $i = 1, \dots, N$ and $I \in \mathcal{C}$ whenever I is a nonempty subset of some $J \in \mathcal{C}$, and E is an equivalence relation on $\{1, \dots, N\}$. Note that for any given $N \in \mathbb{N}$ there are only finitely many such triples $(N, \mathcal{C}, E)$, so in total there are only countably many such triples.

Let $(N, \mathcal{C}, E)$ be a triple as above. We say that a V-complex K in R^n is of *type* $(N, \mathcal{C}, E)$ if there is a bijection $i : \operatorname{Vert}(K) \to \{1, \dots, N\}$ such that $\mathcal{C}$ is the collection of sets $\{i(a) : a \text{ is a vertex of } S\}$ with $S \in K$, and for all $a, b \in \operatorname{Vert}(K)$, $i(a) E i(b) \Leftrightarrow \operatorname{st}(a) = \operatorname{st}(b)$. Suppose the V-complexes K in R^n and K' in $R^{n'}$ are both of type $(N, \mathcal{C}, E)$, witnessed by the bijections $i : \operatorname{Vert}(K) \to \{1, \dots, N\}$ and $j : \operatorname{Vert}(K') \to \{1, \dots, N\}$. We claim that then $|K|$ and $|K'|$ are V-homeomorphic. To see this, note that the map $v := j^{-1} \circ i : \operatorname{Vert}(K) \to \operatorname{Vert}(K')$ is a bijection such that

(i) for all $a_0, \dots, a_k \in \operatorname{Vert}(K)$, $a_0, \dots, a_k$ are the vertices of a simplex in K iff $va_0, \dots, va_k$ are the vertices of a simplex in K';
(ii) for all $a, b \in \operatorname{Vert}(K)$, $\operatorname{st}(a) = \operatorname{st}(b)$ iff $\operatorname{st}(va) = \operatorname{st}(vb)$.

By (i) we can extend v uniquely to a homeomorphism $\phi\colon |K| \to |K'|$ that is affine on each simplex of K. Using Lemma 3.2 and the assumption that K and K' are V-complexes it then follows from (ii) that ϕ is a V-homeomorphism.

For the proof below it is convenient to fix a sequence of V-complexes $K_1, K_2, K_3, \dots$ in R^n such that every V-complex K in R^n is of the same type $(N, \mathcal{C}, E)$ as some complex in this sequence.

Corollary 7.2. *Let $Z \subseteq R^m$ be definable, and let $X \subseteq Z \times R^n \subseteq R^{m+n}$ be definable such that each section $X(a)$ with $a \in Z$ is V-bounded. Then there is a partition of Z into subsets $Z_1, \dots, Z_k$, definable in (R, V), such that if $a, b \in Z$ are in the same Z_i, then $X(a)$ and $X(b)$ are V-homeomorphic.*

Proof. We shall establish this in the stronger form that there are $M \in \mathbb{N}$ and definable sets $\Phi_1, \dots, \Phi_l \subseteq R^M \times R^{2n}$ such that for each $a \in Z$ there is $j \in \{1, \dots, l\}$ and $b \in R^M$ for which $\Phi_j(b) \subseteq R^{2n}$ is the graph of a map ϕ :

$\mathrm{cl}(X(a)) \to |K_j|$ that makes (ϕ, K_j) a V-triangulation of $\mathrm{cl}(X(a))$ compatible with $X(a)$. For simplicity, assume that $X(a)$ is closed for all $a \in Z$; the general case is very similar. To prove the stronger statement we can assume that (R, V) is κ-saturated with uncountable $\kappa > |L|$ where L is the language of $\mathrm{Th}(R)$. Consider L-formulas $\phi(u, x, y)$ where $u = (u_1, u_2, \dots)$ is an infinite sequence of variables and $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$. (Of course, in each such $\phi(u, x, y)$ only finitely many u_i occur.) By V-triangulation and saturation there are such formulas $\phi_1(u, x, y), \dots, \phi_l(u, x, y)$ with the property that for each $a \in Z$ there is $j \in \{1, \dots, l\}$ and $b \in R^{\mathbb{N}}$ for which $\phi(b, x, y)$ defines the graph of a map $\phi : X(a) \to |K_j|$ that makes (ϕ, K_j) a V-triangulation of $X(a)$. Now take $M \in \mathbb{N}$ such that no variable u_i with $i > M$ occurs in any of the ϕ_j. With this M the claim at the beginning of the proof is established. ∎

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Lou van den Dries
Department of Mathematics, UIUC
1409 W. Green Street
Urbana, IL 61801, U.S.A.
E-mail: vddries@math.uiuc.edu

Jana Maříková
Department of Mathematics, WIU
476 Morgan Hall, 1 University Circle
Macomb, IL 61455, U.S.A.
E-mail: J-Marikova@wiu.edu

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