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The fixed point property for some cartesian products

by

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Abstract. It is proved that the cylinder $X \times I$ over a planar λ-dendroid X has the fixed point property. This is a partial solution of two problems posed by R. H. Bing (cf. [1], Questions 9 and 10).

1. Introduction. The principal objective of this paper is to prove the following

Main Theorem. *If X is a planar λ-dendroid then $X \times I$ has the fixed point property.*

(For definitions of undefined terms see the following section.)

Remarks. 1. It will be clear from the proof that the conclusion still holds if I is replaced by any compact absolute retract. Also, we can replace X by any λ-dendroid each of whose subcontinua has at most one dense arc component. This is satisfied, for instance, for planar λ-dendroids (see 2.1) and for λ-dendroids with countably many arc components (see [4]).

2. In a forthcoming paper [11] we shall extend this result to all λ-dendroids. The principal results and methods of this paper will be included as the starting point of the proof.

3. There is a contractible 2-dimensional continuum with the fixed point property whose product with the unit interval does not have the fixed point property ([5], cf. [2]).

Now we outline the strategy of our argument. We fix a mapping $f : X \times I \to X \times I$. Our task is to find a point $(x_0, t_0) \in X \times I$ such that $f(x_0, t_0) = (x_0, t_0)$. To this end, we first observe that there is a subcontinuum X_0 of X minimal with respect to the property $f(X_0 \times I) \subset X_0 \times I$. Without loss of generality we can assume that $X_0 = X$. Under this assumption,

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$(*)$ $\quad f(Y\times I)$ is not contained in $Y\times I$ for any proper subcontinuum Y of X.

Then we notice that $f=(f_1,f_2)$ where $f_1 : X\times I\to X$ and $f_2 : X\times I\to I$. Thus we can define a continuous continuum-valued transformation $F : X\to X$ setting

$$F(x)=f_1(\{x\}\times I)\qquad \text{for each } x\in X.$$

This function, canonically determined by f, is said to be *induced*

by f. Applying the fixed point property for multi-valued transformations of λ-dendroids (see 4.2), we get a point $y_0\in X$ such that $y_0\in F(y_0)$. Let Y_0 denote the arc component of X that contains y_0. It follows that $F(Y_0)\subset Y_0$ (as f maps $Y_0\times I$ into itself). Then $F(\mathrm{cl}\,Y_0)\subset \mathrm{cl}\,Y_0$ (because f maps $(\mathrm{cl}\,Y_0)\times I$ into itself). By $(*)$ it follows that

(1) $$Y_0 \text{ is dense in } X.$$

Now our claim is that f has a fixed point in $Y_0\times I$. To prove this we assume, on the contrary, the claim is false: there is no fixed point for f in $Y_0\times I$; and we pursue a contradiction. By Theorem 3.1 we get a pursuit ray R in Y_0 for F such that no arc in X contains R. Since R is a pursuit ray we get $L(R)\subset F(L(R))$, where the limit $L(R)$ is determined with respect to X. Let a denote the origin of R. Then, by an important property of λ-dendroids (see Corollary 4.3), we get a point $b\in X$ such that $b\in F(b)$ and $R\subset ab$. It follows from the

inclusion that the irreducible continuum ab is not an arc. Let Y_1 denote the arc component of X that contains b. Hence $Y_1\neq Y_0$. Since b is a fixed point of F we infer, as in the case of Y_0, that $F(Y_1)\subset Y_1$ and, analogously, that

(2) $$Y_1 \text{ is dense in } X.$$

By (1) and (2) we see that X contains at least two distinct dense arc components. However this contradicts Lemma 2.1; and this contradiction concludes the argument.

REMARK 4. It may be of interest to compare our discussion of the fixed point property for products with the techniques worked out by Young [14].

2. Preliminaries. All *spaces* discussed in this paper are Hausdorff, most of them are metrizable. By a *continuum* we mean a non-void compact connected metrizable space. A continuum containing only one point is said to be *degenerate*. A continuum is said to be *irreducible* between two of its points if no proper subcontinuum contains these points. A continuum X is said to be *unicoherent* if for every representation $X=A\cup B$, where A and B are continua, the intersection $A\cap B$ is a continuum. A continuum is said to be *hereditarily unicoherent* if all its subcontinua are unicoherent.

For any two points a, b of a hereditarily unicoherent continuum X there is a unique minimal (with respect to inclusion) subcontinuum of X containing these points, denoted simply by ab. Obviously, ab is irreducible between a and b. A continuum is said to be *decomposable* if it can be represented as a union of two proper subcontinua, otherwise it is *indecomposable*. A continuum is *hereditarily decomposable* if all its non-degenerate subcontinua are decomposable. Any irreducible hereditarily decomposable continuum Z admits a canonical upper semicontinuous decomposition D into

disjoint continua with empty interiors, called *layers* of Z, such that the quotient space Z/D

is an arc ([7], cf. [8]). The layer containing a point $z \in Z$ will be denoted by $[z]_Z$; the layers corresponding to the ends of the arc will be called *end layers* or *ends* of Z; Z is irreducible between any two points lying in different ends.

By a λ-*dendroid* we mean a continuum that is both hereditarily decomposable and hereditarily unicoherent. It is well known that λ-dendroids are *tree-like* [3], hence acyclic. Combining this and standard results, one readily sees that a subcontinuum of the plane is a λ-dendroid if and only if it is hereditarily decomposable and does not separate the plane. In the proof of the Main Theorem we refer to the following property of planar λ-dendroids.

2.1. Lemma (Krasinkiewicz–Minc [6]). *Every planar λ-dendroid has at most one dense arc component.*

By a *ray* in a space X we mean the image $R = p([0,\infty))$ of a continuous injection $p : [0,\infty) \to X$ usually called a *parameterization* of R. The canonical linear order $\le$ on $[0,\infty)$ can be carried over via p to R. Since it is independent of the parameterization, each ray admits a canonical linear ordering, also denoted by $\le$. The point $p(0)$, independent of the parameterization, is called the *origin* of R. For $x \in R$, by $R(x)$ we denote the *subray* $R(x) = \{y \in R : x \le y\}$. By the *limit* (in the space X) of the ray R we mean the set $L(R) = \bigcap\{\operatorname{cl} R(x) : x \in R\}$. Clearly, $\operatorname{cl} R = R \cup L(R)$. One easily sees that $L(R) = \bigcap_n \operatorname{cl} R(x_n)$ for any sequence $\{x_n\}$ cofinal in R. If X is compact metrizable then $L(R)$ is a continuum. We need the following property of λ-dendroids:

2.2. Theorem. *Let R be a ray with origin a in a λ-dendroid X. Then $R \cap L(R) = \emptyset$ and, consequently, $R \cup L(R)$ is a continuum irreducible between a and every point of $L(R)$.*

In fact, this is a corollary to the following stronger result.

2.3. Lemma. *Let R be a ray in a hereditarily unicoherent continuum X. If $R \cap L(R) \ne \emptyset$ then $L(R)$ is an indecomposable continuum and $R \cap L(R) = R(x_0)$ for some $x_0 \in R$.*

Proof. Let $x_0 = \min(R \cap L(R))$. First note that $R(x_0) \subset L(R)$ because $R(x) \cup L(R)$ is a continuum for each $x \in R$. Hence $R \cap L(R) = R(x_0)$. It follows that $\operatorname{cl} R(x) = L(R)$ for each $x \geq x_0$. Now suppose $L(R) = A \cup B$, where A and B are proper subcontinua of $L(R)$. Then there are points $x_0 \leq x_1 < x_2 < x_3$ on R such that $x_1, x_3 \in A$ and $x_2 \in B \setminus A$. Then one easily sees that $A \cup x_1x_3$ is a non-unicoherent subcontinuum of $L(R)$, which contradicts our hypothesis. Hence $L(R)$ is indecomposable, which completes the proof.

A transformation F assigning to each point $x \in X$ a subset $F(x) \subset X$, often written $F : X \to X$, is called a *multi-valued transformation* of X. Obviously, any ordinary function $X \to X$ can be treated as a multi-valued transformation. For any set $A \subset X$ we define $F(A) = \bigcup\{F(x) : x \in A\}$. We are interested only in multi-valued transformation with some regularity properties. Let us recall some interesting classes. First, for any set S in X, define

$$F^{-1}(S) = \{x \in X : F(x) \cap S \neq \emptyset\}.$$

Then define F to be *upper semicontinuous* if $F^{-1}(B)$ is closed for each closed set B in X; *lower semicontinuous* if $F^{-1}(U)$ is open for each open set U in X; *continuous* if it is both upper and lower semicontinuous. We call F *continuum-valued*, briefly: *c-valued*, if each $F(x)$ is a continuum in X. Transformations of the latter kind can be treated as functions $F : X \to C(X)$, where $C(X)$ is the hyperspace of subcontinua with the Vietoris topology (but we must take care because $F(A)$ and F^{-1} have a different meaning in this setting). Then $F : X \to X$ is continuous if and only if $F : X \to C(X)$ is a mapping. $F : X \to X$ is said to *preserve locally* (resp., *arcwise*) *connected continua* if $F(A)$ is locally (arcwise, resp.) connected continuum for each locally (arcwise, resp.) connected continuum A in X. A point $x_0 \in X$ is said to be a *fixed point* for a

multi-valued transformation $F : X \to X$ if $x_0 \in F(x_0)$. A space X has the *fixed point property* if every mapping of X into itself has a fixed point.

By a *pursuit ray* of a multi-valued transformation $F : X \to X$ we mean a ray $R = p([0,\infty))$ such that there are arbitrarily large numbers $t \in [0,\infty)$ with $p([0,t])$ contained in an arc $p(0)q_t \subset X$ for some $q_t \in F(p(t))$.

An arcwise connected space is said to be *uniquely arcwise connected* if for any two distinct points there is a unique arc in this space having these points as its endpoints.

By $\mathbb{N}$ we denote the natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$; I stands for the unit interval $[0,1]$.

3. Pursuit rays in uniquely arcwise connected spaces. In this section we discuss a *uniquely arcwise connected space Y and a mapping f :*

$Y \times I \to Y \times I$ *with no fixed point*. Then $f = (f_1, f_2)$, where $f_1 : Y \times I \to Y$ and $f_2 : Y \times I \to I$. Let $F : Y \to Y$ denote the c-valued transformation induced by f,

$$F(x) = f_1(\{x\} \times I) \qquad \text{for each } x \in Y.$$

Clearly, F is continuous. We are going to prove the following

3.1. THEOREM. *There is a pursuit ray R of F contained in no arc in Y.*

REMARKS. 1. It will be clear from the proof that this theorem is still valid if I is replaced by any compact absolute retract.

2. By a *B-space* we mean a uniquely arcwise connected space such that each ray in it is a subset of an arc in this space. The above yields the following theorem of Okhezin [12]: *the product of a B-space and a compact absolute retract has the fixed point property.*

Proof of Theorem 3.1. First we construct a sequence $(x_0, s_0), (x_1, s_1), \ldots$ in $Y \times I$ satisfying conditions (1) and (2) below:

$$(1) \qquad f_2(x_n, s_n) = s_n, \quad \text{i.e.} \quad f(x_n, s_n) = (x'_n, s_n) \quad \text{for each } n \geq 0.$$

Consequently, $x'_n = f_1(x_n, s_n)$ is different from x_n as f has no fixed point.

$$(2) \qquad (x_0x'_0 \cup \ldots \cup x_{n-1}x'_{n-1}) \cap x_nx'_n = \{x_n\} \qquad \text{for each } n \geq 1.$$

(Here ab, for distinct points a, b in Y, stands for the unique arc in Y with these endpoints.)

To start the construction, we pick an arbitrary point $x_0 \in Y$. Then s_0 is defined to be a fixed point of the mapping $I \ni s \mapsto f_2(x_0, s) \in I$. Clearly, $f_2(x_0, s_0) = s_0$; hence (1) is satisfied for $n = 0$. Now assume $(x_0, s_0), \ldots, (x_{m-1}, s_{m-1})$, $m - 1 \geq 0$, have been defined so that (1) and (2) hold for $n = 0, \ldots, m-1$. Then one easily sees that $T_{m-1} = x_0x'_0 \cup \ldots \cup x_{m-1}x'_{m-1}$ is a tree. It remains to construct a point (x_m, s_m) satisfying the conditions:

$$(*) \qquad f_2(x_m, s_m) = s_m,$$

$$(**) \qquad (x_0x'_0 \cup \ldots \cup x_{m-1}x'_{m-1}) \cap x_mx'_m = \{x_m\}.$$

To this end, note that $f_1(T_{m-1} \times I)$ is a locally connected continuum which contains $x'_{m-1} = f_1(x_{m-1}, s_{m-1})$. Since this point also belongs to T_{m-1} the union $T_{m-1} \cup f_1(T_{m-1} \times I)$ is also a locally connected continuum. Hence this continuum is a dendrite because it contains no simple closed curve. Therefore, by a classical result (see e.g. [13, 4]), there is a monotone retraction $r : T_{m-1} \cup f_1(T_{m-1} \times I) \to T_{m-1}$. Then $(r \circ f_1, f_2)$ maps $T_{m-1} \times I$ into itself. Since the product is an absolute retract the mapping has a fixed point. Define $(x_m, s_m) \in T_{m-1} \times I$ to be a fixed point of this mapping, $(r \circ f_1(x_m, s_m),\ f_2(x_m, s_m)) = (x_m, s_m)$. Obviously, $(*)$ is satisfied. We have already noticed that $x'_m = f_1(x_m, s_m)$ must be different from

x_m. Since $r(x'_m) = x_m$ and r is monotone, both points x_m, x'_m belong to the continuum $r^{-1}(x_m)$ and this continuum meets T_{m-1} only at x_m. Since $r^{-1}(x_m)$ is a dendrite the arc $x_m x'_m$ is its subset. Thus $(**)$ is also satisfied. This completes the construction of the sequence $(x_0, s_0), (x_1, s_1), \ldots$

Notice the following corollary to (2):

(3) $\quad$ if $\quad x_i x'_i \cap x_j x'_j \neq \emptyset$ and $i < j \quad$ then $\quad x_i x'_i \cap x_j x'_j = \{x_j\}$.

We have the following important property:

(4) for any locally connected continuum $A \subset Y$ the set $\mathbb{N}(A) = \{j \in \mathbb{N} : x_j \in A\}$ is finite.

Indeed, suppose $\mathbb{N}(A)$ is infinite. Then there are $k(1) < k(2) < \ldots$ in $\mathbb{N}(A)$ such that the sequence $(x_{k(1)}, s_{k(1)}), (x_{k(2)}, s_{k(2)}), \ldots$ in $A \times I$ converges to a point $(y_0, t_0) \in A \times I$. It follows from (1) that $f_2(y_0, t_0) = t_0$ because $f_2(x_{k(n)}, s_{k(n)}) = s_{k(n)}$ for each $n \geq 1$. Hence we must have $f_1(y_0, t_0) \neq y_0$. Consequently, there exists a small enough locally connected continuum $K \subset A$ and an arc $L \subset I$ such that $K \times L$ is a neighbourhood of (y_0, t_0) in $A \times I$ and $K \cap f_1(K \times L) = \emptyset$. Therefore, both $(x_{k(n)}, s_{k(n)})$ and $(x_{k(n+1)}, s_{k(n+1)})$ belong to $K \times L$ for sufficiently large $n \geq 1$. Then both $x_{k(n)}$ and $x_{k(n+1)}$ belong to K, and both $x'_{k(n)} = f_1(x_{k(n)}, s_{k(n)})$ and $x'_{k(n+1)} = f_1(x_{k(n+1)}, s_{k(n+1)})$ belong to $f_1(K \times L)$. Since $x_{k(n)} x'_{k(n)} \cup x_{k(n+1)} x'_{k(n+1)} \cup K \cup f_1(K \times L)$ is a locally connected continuum in Y it is a dendrite. On the other hand, one can easily construct a simple closed curve in this continuum, because the arcs are disjoint off the point $x_{k(n+1)}$. This contradiction completes the proof of (4).

Now we are going to study a function $\mu : \mathbb{N} \to \mathbb{N}$ reflecting a relation between the arcs $x_0 x'_0, x_1 x'_1, x_2 x'_2, \ldots$ pertaining their intersections. Let us define

$$\mu(j) = \min\{i : x_j \in x_i x'_i\}.$$

Clearly, by (2) we have

(5) $$\mu(0) = 0 \quad \text{and} \quad \mu(j) < j \quad \text{for } j > 0.$$

Consequently, for each $j \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that $\mu^k(j) = 0$. It follows that

(6) $$\{0\} \subset \mu^{-1}(0) \subset \mu^{-2}(0) \subset \ldots \quad \text{and} \quad \{0\} \cup \mu^{-1}(0) \cup \mu^{-2}(0) \cup \ldots = \mathbb{N}.$$

(Here $\mu^{-k} = (\mu^k)^{-1}$.) We shall see that (4) implies the following:

(7) $$\mu^{-1}(k) \text{ is finite for each } k \geq 0.$$

In fact, since $x_j \in x_k x'_k$ for each $j \in \mu^{-1}(k)$ (because $\mu(j) = k$) hence $\mu^{-1}(k) \subset \mathbb{N}(x_k x'_k)$. As the latter set is finite by (4), the proof is complete.

Using (7), we can now show by simple induction that

(8) $$\mu^{-k}(0) \text{ is finite for each } k \geq 0.$$

It follows that

(9) $\quad \mu^{-k}(0) \setminus \mu^{-(k-1)}(0) \neq \emptyset \quad$ for each $k \geq 0 \quad (\mu^{-0}(0) = \mathrm{id}^{-1}(0) = \{0\})$.

In fact, otherwise $\mu^{-k}(0) = \mu^{-(k-1)}(0)$ by (6). This implies $\mu^{-n}(0) = \mu^{-(n-1)}(0) = \ldots = \mu^{-(k-1)}(0)$ for each $n \geq k$, contrary to (6) and (8). Finally, we have

(10) $\quad \mu$ maps $\mu^{-(k+1)}(0) \setminus \mu^{-k}(0)$ into $\mu^{-k}(0) \setminus \mu^{-(k-1)}(0)$ for each $k \geq 0$.

Indeed, let $j \in \mu^{-(k+1)}(0) \setminus \mu^{-k}(0)$. Then $\mu^{k+1}(j) = 0$ and $\mu^k(j) \neq 0$. Since $\mu^{k+1}(j) = \mu^k(\mu(j))$ and $\mu^k(j) = \mu^{k-1}(\mu(j))$ we infer that $\mu(j) \in \mu^{-k}(0) \setminus \mu^{-(k-1)}(0)$.

These properties show that we can construct an inverse sequence

$$\mu^{-1}(0) \setminus \{0\} \leftarrow \mu^{-2}(0) \setminus \mu^{-1}(0) \leftarrow \mu^{-3}(0) \setminus \mu^{-2}(0) \leftarrow \ldots$$

of non-void finite sets with the bonding maps induced by μ. Hence the limit of this inverse sequence is non-void. Let $(j(1), j(2), \ldots)$ be an element of the limit. Hence

(11) $\quad j(n) \in \mu^{-n}(0) \setminus \mu^{-(n-1)}(0) \quad$ and $\quad \mu(j(n+1)) = j(n) \quad$ for $n \geq 1$.

Now we can prove the conclusive fact of this proof:

(12) The union $x_{j(1)}x_{j(2)} \cup x_{j(2)}x_{j(3)} \cup x_{j(3)}x_{j(4)} \cup \ldots$ is a pursuit ray in Y and it is contained in no locally connected subcontinuum of Y.

Indeed, first note that $0 < j(1) < j(2) < \ldots$ by (5) and (11). Hence the second assertion of (12) follows from (4). Now we shall prove the first one. Since $j(1) > 0$ and the sequence of $j(n)$'s is strictly increasing, (11) implies $x_{j(n)} \neq x_{j(n+1)}$ for each $n \geq 1$. Hence each $x_{j(n)}x_{j(n+1)}$ is an arc. According to (3) and the definition of μ we also get $x_{j(n)}x_{j(n+1)} \cap x_{j(n+1)}x_{j(n+2)} = \{x_{j(n+1)}\}$ and $x_{j(n)}x_{j(n+1)} \cap x_{j(n+i)}x_{j(n+i+1)} = \emptyset$ for each $i \geq 2$. Hence the union in (12) is a ray. Since $x_{j(n)}x'_{j(n)} = x_{j(n)}x_{j(n+1)} \cup x_{j(n+1)}x'_{j(n)}$ and $x_{j(n)}x_{j(n+1)} \cap x_{j(n+1)}x'_{j(n)} = \{x_{j(n+1)}\}$, similar reasoning shows that $x_{j(1)}x'_{j(n)} = x_{j(1)}x_{j(2)} \cup \ldots \cup x_{j(n)}x_{j(n+1)} \cup x_{j(n+1)}x'_{j(n)}$ is an arc containing the arc $x_{j(1)}x_{j(n)} = x_{j(1)}x_{j(2)} \cup \ldots \cup x_{j(n-1)}x_{j(n)}$. Since $x'_{j(n)} \in F(x_{j(n)})$ for each $n \geq 1$, the ray is a pursuit ray. This proves (12), and completes the entire proof.

3.2. Theorem. *Let $R \subset Y$ be a pursuit ray for F which is contained in no arc in Y. Then for each $x_0 \in R$ there is $x_1 \in R$ such that $R(x_1) \subset F(R(x_0))$.*

Proof. Let a be the origin of R and let $\leq$ be the canonical order on R. By the hypothesis there is $s_0 \in R$ and an arc aq_0, with $q_0 \in F(s_0)$, such that $x_0 \leq s_0$ and $as_0 \subset aq_0$. Since Y is uniquely arcwise connected and R is not contained in the arc aq_0, the set $R \cap aq_0$ is an arc at_0, $t_0 \geq s_0$, because it is a compact subset of R. Put $x_1 = t_0$. We are going to verify that this point satisfies the conclusion of our theorem. First notice that $x_1q_0 \cap R = \{x_1\}$. Let us note that we will get the conclusion of our theorem once we show that for each $x > x_1$,

$$(*) \qquad x_1x \setminus \{x_1\} \subset F(s_0s_1) \quad \text{for some } s_1 > x.$$

In fact, suppose this holds. Then $x_1x \subset F(s_0s_1)$, so x_1, $x \in F(s_0s_1)$. Since $s_0s_1 \subset R(x_0)$ we infer that $x_1, x \in F(R(x_0))$ for each $x > x_1$. Hence it remains to prove $(*)$.

Using the hypothesis again, we can find $s_1 > x$ on R such that $as_1 \subset aq_1$ for some $q_1 \in F(s_1)$. This s_1 satisfies $(*)$. Indeed, repeating the above reasoning we get a point $t_1 \geq s_1$ such that $t_1q_1 \cap R = \{t_1\}$. Since $t_0q_0 \cup F(s_0s_1) \cup t_1q_1$ is an arcwise connected continuum we have $t_0t_1 \subset t_0q_0 \cup F(s_0s_1) \cup t_1q_1$. It follows that $t_0t_1 \setminus \{t_0, t_1\} \subset F(s_0s_1)$ because $t_0t_1 \cap (t_0q_0 \cup t_1q_1) = \{t_0, t_1\}$. Since $x_1x \setminus \{x_1\} \subset t_0t_1 \setminus \{t_0\} \subset t_0t_1 \setminus \{t_0, t_1\}$, the proof is complete.

4. Some properties of λ-dendroids. In this section *we denote by X a λ-dendroid and by F an upper semicontinuous continuum-valued transformation of X into itself.* We need the following important property of λ-dendroids.

4.1. LEMMA [9, Th. 1]. *Let $u \neq v$ be two points of X such that $v \in F(u)$ and $[u]_{uv} \cap F([u]_{uv}) = \emptyset$. Then there is a point $b \in X$ such that $b \in F(b)$ and $[u]_{uv} = [u]_{ub}$.*

In the following section we shall exploit the following two corollaries to this lemma.

4.2. COROLLARY [10]. *F has a fixed point.*

Proof. Let K be a subcontinuum of X minimal with respect to the property $K \cap F(K) \neq \emptyset$. If K is a one-point set the proof is finished. So, assume K is non-degenerate, and select a point $v \in K \cap F(K)$. Then $v \in F(u)$ for some $u \in K$. Clearly, by the minimality of K we get $K = uv$ and $[u]_{uv} \cap F([u]_{uv}) = \emptyset$. Now we can apply 4.1 to get a fixed point $b \in X$ for F.

4.3. COROLLARY. *Let R be a ray in X with the origin a. If $L(R) \cap F(L(R)) \neq \emptyset$ then there is a point $b \in X$ such that $b \in F(b)$ and $R \subset ab$.*

Proof. As in the proof of 4.2, there is a subcontinuum K of $L(R)$ minimal with respect to the property $K \cap F(K) \neq \emptyset$. Let $v \in K \cap F(K)$; then $v \in F(u)$ for some $u \in K$. If $u = v$ then $b = u$ satisfies the conclusion of the corollary because $ab = \operatorname{cl} R$ (see 2.2). Now consider the case $u \neq v$. By the minimality of K we have $K = uv$ and $[u]_K \cap F([u]_K) = \emptyset$, because $[u]_K$ is a proper subcontinuum of K. Consequently, by Lemma 4.1 there is a point $b \in X$ such that

(1) $$b \in F(b),$$

(2) $$[u]_K = [u]_{ub}.$$

Notice that

(3) no subray of R is contained in ub.

Indeed, otherwise $L(R)$ would be a subcontinuum of ub with empty interior in ub which would imply $L(R) \subset [u]_{ub}$, contrary to (2) because $[u]_K$ is a proper subset of $L(R)$. Hence

(4) $$ub \cap R = \emptyset.$$

In fact, suppose it is not true. Then, by (3), $ub \cap \operatorname{cl} R$ is not connected because $\operatorname{cl} R = R \cup L(R)$ and $R \cap L(R) = \emptyset$. This is impossible as X is hereditarily unicoherent. Finally we get

(5) $$R \subset ab.$$

In fact, $ab \subset (\operatorname{cl} R) \cup ub$. Since $\operatorname{cl} R = R \cup L(R)$ we have $ab \subset R \cup L(R) \cup ub$. Since $R \cap L(R) = \emptyset$, by (4) we get $R \cap (L(R) \cup ub) = \emptyset$. Thus each continuum in $R \cup L(R) \cup ub$ connecting the origin of R to $b \in L(R) \cup ub$ must contain R. This yields (5).

5. Proof of the Main Theorem. Let X be a planar λ-dendroid and let $f : X \times I \to X \times I$ be a mapping. We are going to prove that f has a fixed point. Without loss of generality we can assume that X is minimal with respect to the property $f(X \times I) \subset X \times I$. The multi-valued map F induced by f is continuous and preserves locally connected continua. Moreover, each $F(x)$ is a dendrite because it is a locally connected continuum containing no simple closed curve. From the assumption it follows that X is minimal with respect to the property $F(X) \subset X$. Suppose, to the contrary, that f has no fixed point.

By 4.2 there is a fixed point y_0 for F. Let Y_0 be the arc component of X containing y_0. It follows that $F(Y_0) \subset Y_0$ because the image is arcwise connected and contains y_0. Since F is continuous we infer that $F(\operatorname{cl} Y_0) \subset \operatorname{cl} Y_0$. By the minimality of X we get

(1) $$\operatorname{cl} Y_0 = X.$$

By 3.1 and 3.2 there is a ray $R \subset X$ satisfying the conditions:

(2) no arc in X contains R,

(3) for each $x \in R$ there is $x' \in R$ such that $R(x') \subset F(R(x))$.

By upper semicontinuity of F and by (3) we infer that

$$L(R) \subset F(L(R)). \tag{4}$$

Let a denote the origin of R. By Corollary 4.3 there is a point $b \in X$ such that

$$b \in F(b), \tag{5}$$

$$R \subset ab. \tag{6}$$

Combining (2) and (6) we infer that

$$ab \text{ is not an arc.} \tag{7}$$

This implies that b does not belong to Y_0. Let Y_1 be the arc component of X which contains b. By (5) we infer that $F(Y_1) \subset Y_1$. As in the case of Y_0 we get $\operatorname{cl} Y_1 = X$. Thus Y_0 and Y_1 are different dense arc components in X, which contradicts 2.1. This contradiction completes the proof.

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