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# $G$-functors, $G$-posets and homotopy decompositions of $G$-spaces

by

**Stefan Jackowski** and **Jolanta Słomińska** (Warszawa)

**Abstract.** We describe a unifying approach to a variety of homotopy decompositions of classifying spaces, mainly of finite groups. For a group $G$ acting on a poset $\mathbf{W}$ and an isotropy presheaf $d : \mathbf{W} \to \mathcal{S}(G)$ we construct a natural $G$-map $\operatorname{hocolim}_{\mathcal{W}_d} G/d(-) \to |\mathbf{W}|$ which is a (non-equivariant) homotopy equivalence, hence $\operatorname{hocolim}_{\mathcal{W}_d} EG\times_G F_d \to EG\times_G |\mathbf{W}|$ is a homotopy equivalence. Different choices of $G$-posets and isotropy presheaves on them lead to homotopy decompositions of classifying spaces. We analyze higher limits over the categories associated to isotropy presheaves $\mathcal{W}_d$; in some important cases they vanish in dimensions greater than the length of $\mathbf{W}$ and can be explicitly calculated in low dimensions. We prove a cofinality theorem for functors $F : \mathcal{C} \to \mathcal{O}(G)$ into the category of $G$-orbits which guarantees that the associated map $\alpha_F : \operatorname{hocolim}_{\mathcal{C}} EG \times_G F(-) \to BG$ is a mod-$p$-homology decomposition.

**Introduction.** Let $G$ be a discrete group and let $\mathcal{C}$ be a small category equipped with a $G$-action. We introduce the notion of a $G$-structure on a functor from $\mathcal{C}$ to the category of spaces. Here a space is either a topological space or a simplicial set. The key property of a *$G$-functor* $X : \mathcal{C} \to \mathcal{S}p$ (where $\mathcal{S}p$ is a category of simplicial sets or topological spaces) is that there is a natural action of $G$ on the homotopy colimit $\operatorname{hocolim}_{\mathcal{C}} X$. Of course such an action exists whenever $X$ takes values in the category of $G$-spaces since we can take the trivial action on $\mathcal{C}$. But the notion of a $G$-functor provides additional flexibility. Many examples fit into the framework of $G$-functors, e.g. the geometric realization of a small category equipped with a $G$-action (i.e. the homotopy colimit of a constant functor) with induced $G$-space.

We note that classical cofinality theorems for homotopy colimits carry over to $G$-functors. The maps establishing the relevant homotopy equivalences turn out to be $G$-maps, although in general not $G$-homotopy equiv-

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alences (cf. Prop. 2.5, 2.6). We show that the language of $G$-functors and cofinality theorems provides a unifying approach to a variety of questions involving decompositions of classifying spaces of groups into a homotopy colimit of classifying spaces of subgroups.

We denote by $\mathbf{S}(G)$ a $G$-poset of subgroups of $G$ on which the group acts by conjugation. Let $\mathbf{W}$ be a partially ordered set (poset) equipped with a $G$-action. An *isotropy presheaf* on $\mathbf{W}$ is a morphism of $G$-posets $d : \mathbf{W} \to \mathbf{S}(G)$ such that $d(w)$ is a subgroup of the isotropy group $G_w$, for every $w \in \mathbf{W}$.

Any isotropy presheaf on $\mathbf{W}$ leads to a $G$-functor and then to a homotopy decomposition of the space $|\mathbf{W}|$. More precisely, (following [Sł1]) to an isotropy presheaf $d$ we assign a small category $\mathcal{W}_d$ whose objects are the elements of $\mathbf{W}$ and morphisms $w \to w'$ are equivalence classes modulo $d(w')$ of elements $g \in G$ such that $w \le gw'$. There is an obvious inclusion of categories $\iota : \mathbf{W} \subset \mathcal{W}_d$. We observe that the left homotopy Kan extension along $\iota$ of a constant $G$-functor on $|\mathbf{W}|$ is equivalent (as a $G$-functor) to the functor $F_d(w) := G/d(w)$. Hence we obtain a natural $G$-map

$$\operatorname{hocolim}_{\mathcal{W}_d} F_d \to |\mathbf{W}|$$

which is is a (non-equivariant) homotopy equivalence (Theorem 4.1). This decomposition leads to a homotopy decomposition of the homotopy orbit space (Borel construction) of $|\mathbf{W}|$; the induced map

$$\operatorname{hocolim}_{\mathcal{W}_d} EG \times_G F_d \to EG \times_G |\mathbf{W}|$$

is a homotopy equivalence. Extending the terminology introduced by Dwyer [Dw1] for posets of subgroups, a $G$-poset $\mathbf{W}$ will be called *$h^*$-ample* if the fibration $EG \times_G |\mathbf{W}| \to BG$ induces an isomorphism in the cohomology theory $h^*$. Thus every isotropy presheaf on an $h^*$-ample $G$-poset $\mathbf{W}$ provides an $h^*$-cohomology decomposition of the classifying space $BG$:

$$(*) \qquad \operatorname{hocolim}_{\mathcal{W}_d} EG \times_G F_d \to BG.$$

All known homology decompositions of classifying spaces ([JM], [JMO1], [Dw1-2]) fit into this framework, at least for discrete groups. In particular the present approach provides a clear picture of the relations between the subgroup, centralizer and normalizer decompositions considered by Dwyer in [Dw1-2]; the decompositions correspond to different choices of an isotropy presheaf on a poset of subgroups or its barycentric subdivision. Indeed, the inclusion of a subposet $\iota : \mathbf{W} \subset \mathbf{S}(G)$ is obviously an isotropy presheaf and the associated category $\mathcal{W}_\iota$ is a subcategory of the category of $G$-orbits $\mathcal{O}_{\mathbf{W}}(G) \subset \mathcal{O}(G)$ consisting of orbits $G/H$ such that $H \in \mathbf{W}$; the functor $F_\iota$ is just the inclusion functor. Taking intersections of the normalizers provides an isotropy presheaf on the barycentric subdivision of $\mathbf{W}$. Another important example is the centralizer isotropy presheaf. It is a contravariant functor $c : \mathbf{W}^{\mathrm{op}} \to \mathbf{S}(G)$ such that $c(H) := C_G(H)$. If $\mathbf{W} = \mathbf{A}_p(G)$ is the Quillen poset of non-trivial elementary abelian subgroups [Q2] then $\mathcal{A}_p(G)_c$ is precisely the category of elementary abelian $p$-subgroups introduced by Quillen [Q3].

Decompositions of spaces are useful in homotopy theory when we can control higher limits over the indexing category. Again following Dwyer, we call an $h^*$-decomposition $(*)$ of $BG$ *sharp* if the Bousfield–Kan spectral sequence [BK], which computes the cohomology of $\mathrm{hocolim}_{\mathcal{W}_d} EG \times_G F_d$, collapses at the $E_2$-term; i.e., $\lim^q_{\mathcal{W}_d} h^*(EG \times_G F_d) = 0$ for all $q > 0$ and consequently

$$h^*(BG) \simeq \lim_{\mathcal{W}_d} h^*(EG \times_G F_d).$$

Properties of higher limits of functors on the categories $\mathcal{W}_d$ can often be read off more directly from the poset $\mathbf{W}$ and the isotropy presheaf $d$. We begin the discussion of higher limits in a more general setting. For a pair $\mathcal{C}' \subset \mathcal{C}$ of small categories and a contravariant functor $M : \mathcal{C} \to \mathsf{Ab}$ we introduce (Sec. 5) the relative higher inverse limit $H^*(\mathcal{C}, \mathcal{C}'; M)$ which fits into an exact sequence

$$\ldots \to H^i(\mathcal{C}; M) \to H^i(\mathcal{C}'; M) \to H^{i+1}(\mathcal{C}, \mathcal{C}'; M) \to H^{i+1}(\mathcal{C}; M) \to \ldots$$

We calculate the relative groups in the case when $\mathcal{C}'$ is obtained from an ordered category $\mathcal{C}$ (a category in which all endomorphisms are isomorphisms) by removing a single object $b \in \mathcal{C}$. We prove (Prop. 5.8) that there is a natural isomorphism

$$H^*(\mathcal{C}, \mathcal{C}'; M) \simeq H^*(\mathrm{Aut}_{\mathcal{C}}(b); L^*(b; M)),$$

where the right hand side denotes the hypercohomology of the automorphism group of $b$ with coefficients in a certain cochain complex $L^*(b; M)$. This complex depends only on the category of morphisms in $\mathcal{C}$ originating and terminating at $b$. The above result provides a useful description of higher limits of an atomic functor concentrated on $b$, i.e. that it vanishes on all objects non-isomorphic to the object $b$. This generalizes a similar result of Oliver [O].

Next we specialize to higher limits over categories associated with isotropy presheaves. For an arbitrary $G$-poset morphism $f : \mathbf{W} \to \mathbf{W}'$ and isotropy presheaves $d : \mathbf{W} \to \mathbf{S}(G)$, $d' : \mathbf{W}' \to \mathbf{S}(G)$ such that $d \le d' \circ f$ the left fiber $f_*/w'$ (cf. Sec. 1) of the induced functor $f_* : \mathcal{W}_d \to \mathcal{W}'_{d'}$ over $w' \in \mathbf{W}'$ is expressed (Thm. 3.4) in terms of the poset $f/w'$, and if $d = d' \circ f$ then the right fibers $w'\backslash f_* \simeq w'\backslash f$ are equivalent. This result is crucial for translating homotopy properties of posets of subgroups into results concerning higher limits and homotopy decompositions of classifying spaces (cf. Sec. 7).

For a pair of isotropy presheaves $d_1 \le d_2 : \mathbf{W} \to \mathbf{S}(G)$ and an arbitrary functor $M : \mathcal{W}_{d_2} \to \mathsf{Ab}$ the Leray spectral sequence of the induced functor

$\iota_* : \mathcal{W}_{d_1} \to \mathcal{W}_{d_2}$ has the form

$$H^m(\mathcal{W}_{d_2}; H^n(d_2(-)/d_1(-), M(-))) \Rightarrow H^{n+m}(\mathcal{W}_{d_1}; M)$$

where for any $w \in \mathbf{W}$ the group $H^n(d_2(w)/d_1(w), M(w))$ is just the group cohomology. The spectral sequence can be used to calculate $H^*(\mathcal{W}_d; M)$ in terms of the cohomology of the quotient space $|\mathbf{W}|/G$ with coefficients in a sheaf. More precisely, under mild finiteness assumptions on $\mathbf{W}$, there is a spectral sequence

$$(**)\qquad H^m(\mathrm{sd}(\mathbf{W})/G; H^n(G_-/d(-), M(-))) \Rightarrow H^{n+m}(\mathcal{W}_d; M)$$

where the $E_2$-term is the cohomology of the quotient poset of the barycentric subdivision with coefficients in a simplicial sheaf.

At this point we should mention that higher limits on the categories $\mathcal{W}_d$ associated to isotropy presheaves can be interpreted as equivariant Bredon cohomology groups [Br1]. For every contravariant functor $N : \mathcal{O}_G \to \mathsf{Ab}$ there is a natural isomorphism

$$H^*(\mathcal{W}_d; N \circ F_d) \simeq H^*_G(\mathrm{hocolim}_{\mathcal{W}_d} F_d; N).$$

The groups appearing in the $E_2$-term in $(**)$ can be interpreted as equivariant Bredon cohomology with local coefficients.

It turns out that under certain assumptions (which are often satisfied by various important posets of subgroups) the spectral sequence $(**)$ degenerates at the $E_2$-term and $H^*(\mathcal{W}_d; M) \simeq H^*(\mathrm{sd}(\mathbf{W})/G; H^0(G_-/d(-), M(-))$, hence higher limits vanish above the length of the poset $\mathbf{W}$ (cf. Theorem 6.6), and in low dimensions only slightly depend on $d$! This result should be compared with the homotopy equivalence $\mathrm{hocolim}_{\mathcal{W}_d} F_d \xrightarrow{\simeq} |\mathbf{W}|$, which means that the homotopy type of the homotopy colimit associated to an isotropy presheaf $d$ does not depend upon $d$.

As an illustration of the abstract results we describe a uniform approach to many results on decompositions of classifying spaces, in particular those of Dwyer [Dw1-2]. Further applications can be obtained by considering other isotropy presheaves on posets of subgroups as well as by looking at other $G$-posets—this work is in progress.

We conclude the paper with a cofinality theorem (Thm. 8.1), appropriate for constructing $H^*(-;\mathbb{F}_p)$-decompositions of classifying spaces. Let $\mathbf{D}_p(G)$ be the poset of subgroups whose order is divisible by $p$; we denote the corresponding orbit category by $\mathcal{O}_{(p)}(G)$ for short. The poset $\mathbf{D}_p(G)$ is $\mathbb{F}_p$-ample, and the associated subgroup decomposition $\mathcal{O}_{(p)}(G) \ni G/H \rightsquigarrow EG \times_G G/H \in \mathcal{S}p$ of the classifying space $BG$ is $\mathbb{F}_p$-sharp. An arbitrary functor $F : \mathcal{C} \to \mathcal{O}_{(p)}(G)$ defines a map

$$\alpha_F : \mathrm{hocolim}_{\mathcal{C}}\, EG \times_G F(-) \to BG.$$

Theorem 8.1 describes conditions on $F$ under which the map $\alpha_F$ is an $H^*(-;\mathbb{F}_p)$-decomposition, and other conditions for the decomposition to be sharp.

The present paper grew out of the second author's preprint [Sł4]. Sections 3, 4 and 6–8 contain most of the results of that preprint, usually with more elaborate proofs.

*Notation.* Posets are denoted by bold capitals and their morphisms by small roman characters; general categories are denoted by script capitals; functors by roman capitals and natural transformations by Greek characters.

**1. Small categories and their geometric realizations.** We describe basic constructions related to homotopy theory of small categories which we shall be using throughout the paper. Our notation follows [Q1], [Q2].

Let $F : \mathcal{C}' \to \mathcal{C}$ be a functor between small categories. For any object $c \in \operatorname{ob}\mathcal{C}$ one defines the *left fiber of $F$ over $c$* as the category $F/c$ whose objects are pairs $(c', F(c') \to c)$. A morphism $(c_1', F(c_1') \to c) \to (c_2', F(c_2') \to c)$ in $F/c$ is a morphism $c_1' \to c_2'$ in $\mathcal{C}'$ for which the corresponding triangle over $c$ commutes. The *right fiber of $F$ under $c$*, denoted by $c\backslash F$, is defined dually; its objects are pairs $(c', c \to F(c'))$. If $\mathcal{C}' \subset \mathcal{C}$ is an inclusion functor then we denote the corresponding fiber categories by $\mathcal{C}'/-$ and $-\backslash\mathcal{C}'$ respectively. There are obvious projection functors from the left and right fiber categories to $\mathcal{C}'$. Also note that assigning to an object $c \in \mathcal{C}$ the corresponding left (resp. right) fiber extends to a covariant (resp. contravariant) functor from $\mathcal{C}$ to the category $\mathcal{C}at$ of small categories.

We shall also need the notion of semi-fibers. For every $c \in \operatorname{ob}\mathcal{C}$ the *right semi-fiber of $F$ under $c$* is defined to be the full subcategory $c\backslash\backslash F \subset c\backslash F$ consisting of those pairs $(c', c \to F(c'))$ where the arrow is not an isomorphism. The *left semi-fiber $F//c$* is defined analogously.

Fiber categories are involved in the definitions of Kan extensions of a functor. Let $\mathcal{S}$ be a category with colimits and limits of diagrams defined over any small category. For a functor $F : \mathcal{C}' \to \mathcal{C}$, the pull-back functor $F^* : \operatorname{Hom}(\mathcal{C}, \mathcal{S}) \to \operatorname{Hom}(\mathcal{C}', \mathcal{S})$ between categories of covariant functors into $\mathcal{S}$ has a left adjoint $F_\sharp$ and a right adjoint $F_\flat$,

$$F_\sharp, F_\flat : \operatorname{Hom}(\mathcal{C}', \mathcal{S}) \to \operatorname{Hom}(\mathcal{C}, \mathcal{S}),$$

defined as follows: $F_\sharp M(d) := \operatorname{colim}_{F/d} M$ and $F_\flat M(d) := \lim_{d\backslash F} M$. For more details cf. [HS, IX.5] or [GZ, App. 2, Sec. 3; JMO2].

Let us recall that there is a classifying space functor from the category of small categories to the category of simplicial sets. It assigns to every functor $F : \mathcal{C}' \to \mathcal{C}$ a simplicial map $BF : B\mathcal{C}' \to B\mathcal{C}$ between the nerves of the categories. The *nerve* of a category $\mathcal{C}$ is a simplicial set whose $q$-dimensional simplices are sequences $c_0 \to c_1 \to \ldots \to c_q$ of morphisms in $\mathcal{C}$;

boundary maps are defined by composition of morphisms (for details cf. [S]). On the subcategory consisting of partially ordered sets it coincides with the geometric realization functor $|-|$, considered by Quillen [Q2]; thus we shall use his notation for posets and $B\mathcal{C}$ for general small categories.

1.1. Proposition. *Every natural transformation $\Phi : F' \to F$ between functors $F, F' : \mathcal{C}' \to \mathcal{C}$ induces a simplicial homotopy between the induced simplicial maps $BF', BF : B\mathcal{C}' \to B\mathcal{C}$.*

*Proof.* A simplicial homotopy $h_i : (B\mathcal{C}')_q \to (B\mathcal{C})_{q+1},\ 0 \le i \le q$ (cf. [May], Sect. 5), is defined by the formula

$$h_i([c_0 \to c_1 \to \ldots \to c_q]) := [F'(c_0) \to \ldots \to F'(c_i) \xrightarrow{\Phi(c_i)} F(c_i) \to \ldots \to F(c_q)]. \ \blacksquare$$

We shall often identify the nerve of a category with its geometric realization. We shall say that a functor $F : \mathcal{C}' \to \mathcal{C}$ is a *homotopy equivalence* if the map $BF$ is a homotopy equivalence. Similarly, other topological notions can be assigned to small categories and functors.

Recall that for any simplicial set $X = (X_n, \partial_*, \sigma_*)$ its chain complex $C_*(X)$ consists of free abelian groups $C_n(X)$ generated by the sets $X_n$ of $n$-simplices and boundary maps given by the alternating sums of the homomorphisms defined by the face maps $\partial_*$ in $X$. In particular, for an arbitrary small category $\mathcal{C}$ we shall denote by $C_*(\mathcal{C})$ the chain complex of its nerve $B\mathcal{C}$. It is clearly a functor from the category of small categories to the category of chain complexes. According to Proposition 1.1 a natural transformation $\Phi : F' \to F$ defines a chain homotopy between the induced chain homomorphisms $F_*, F'_* : C_*(\mathcal{C}') \to C_*(\mathcal{C})$ (cf. [May, 5.3]). Note that if $\mathcal{C}$ is non-empty then its chain complex is augmented by a map $\varepsilon : C_0(\mathcal{C}) \to \mathbb{Z}$ sending each object to 1. The chain complex $C_*(\mathcal{C})$ is chain homotopy equivalent to the singular chain complex of the classifying space $B\mathcal{C}$.

**2. $G$-functors and homotopy colimits.** We recall briefly the definition of homotopy colimit. For every diagram of spaces, i.e. a functor $X : \mathcal{C} \to \mathcal{S}p$ where $\mathcal{S}p$ denotes a category of simplicial sets (or topological spaces), one defines

$$\operatorname{hocolim}_{\mathcal{C}} X := B(-\backslash\mathcal{C}) \times_{\mathcal{C}} X$$

where the right hand side is a coequalizer of the diagram

$$\coprod_{c'' \to c'} B(c'\backslash\mathcal{C}) \times X(c'') \rightrightarrows \coprod_{c} B(c\backslash\mathcal{C}) \times X(c).$$

For a functor into a category of topological spaces we can identify the homotopy colimit with the realization of a topological category $\mathcal{C}_X$ whose object space is the disjoint union of $X(c)$ over $c \in \mathcal{C}$. The homotopy colimit of a constant functor is simply the geometric realization of the indexing category considered in Section 1. For details cf. [HV], [JMO2] etc.

Recall also that the cohomology of a homotopy colimit is related to the cohomology of spaces and maps occurring in the diagram via the Bousfield–Kan spectral sequence [BK]:

$$E_2^{p,q} = H^p(\mathcal{C}; h^q(X)) \Rightarrow h^{p+q}(\operatorname{hocolim}_{\mathcal{C}} X)$$

where $H^p(\mathcal{C}; h^q(X)) := \lim^p_{\mathcal{C}}(h^q \circ X)$ denotes the $p$-th higher inverse limit of the functor $h^q(X) : \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$ defined as $h^q(X)(c) := h^q(X(c))$. Higher limits will be discussed in detail in Section 5.

Let $G$ be a discrete group. We shall define an additional structure on a diagram $X : \mathcal{C} \to \mathcal{S}p$ which provides a $G$-action on its homotopy colimit. We shall be particularly interested in the behavior of homotopy colimits equipped with such an action with respect to pulling back and pushing forward a diagram by a functor between small categories.

2.2. Definition. Let $\mathcal{C}$ be a small category and $\mathcal{D}$ an arbitrary category. A (left) *$G$-functor* is a functor $X : \mathcal{C} \to \mathcal{D}$ equipped with the following structure:

(a) a left $G$-action on $\mathcal{C}$,

(b) for each $g \in G$ a natural transformation $\Phi_g : X \circ g \to X$ such that $\Phi_{gg'} = \Phi_g \circ \Phi_{g'}$.

Morphisms (natural transformations) of $G$-functors are defined in the obvious way.

Alternatively $G$-functors can be described in terms of the Grothendieck construction (introduced in [T] and extensively used in [Sł1] and [Dw1-2].) Recall that for a small category $\mathcal{C}$ equipped with a $G$-action the *Grothendieck construction* is defined as a category whose objects are objects of $\mathcal{C}$ and a morphism $c \to c'$ is any pair $(g, c \to gc')$. We shall denote this category by $\mathcal{C}_{hG}$ to emphasize that $B\mathcal{C}_{hG} \simeq EG \times_G B\mathcal{C}$ is the homotopy orbit space of the action of $G$ on $B\mathcal{C}$. There is an obvious inclusion functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}_{hG}$ which is clearly a $G$-functor. Moreover, $G$-structures on a functor $X : \mathcal{C} \to \mathcal{D}$ correspond to its extensions $\widetilde{X} : \mathcal{C}_{hG} \to \mathcal{D}$.

2.3. Proposition. *If $\mathcal{C}'$, $\mathcal{C}$ are categories equipped with $G$-actions, $\mathcal{C}' \xrightarrow{F} \mathcal{C}$ is an $G$-equivariant functor and $\mathcal{D} \xrightarrow{Y} \mathcal{D}'$ is any functor then the composition $Y \circ X \circ F : \mathcal{C}' \to \mathcal{D}'$ is a $G$-functor.* ∎

The notion of a $G$-functor generalizes functors to $G$-spaces. Indeed, if the action on $\mathcal{C}$ is trivial then a $G$-functor is precisely a functor to the category whose objects are objects of $\mathcal{D}$ equipped with a $G$-action. An important example of a $G$-functor is the following: let $\mathbf{S}(G)$ be the poset of all subgroups of $G$, on which $G$ acts via conjugation, and let $\mathcal{O}_G$ denote the category of

$G$-orbits and equivariant maps. Then the functor $\mathbf{S}(G) \to \mathcal{O}_G$ sending each subgroup $H$ to the orbit $G/H$ is a $G$-functor; other examples will appear in later sections.

2.4. Proposition. *If $X : \mathcal{C} \to \mathcal{S}p$ is a left $G$-functor then* $\operatorname{hocolim}_{\mathcal{C}} X$ *is equipped with a natural left $G$-action such that the projection* $\operatorname{hocolim}_{\mathcal{C}} X \to B\mathcal{C}$ *is a $G$-map. Moreover, natural transformations of $G$-functors induce $G$-equivariant maps of the corresponding homotopy colimits.*

*Proof.* A $G$-structure on a functor $X$ defines a $G$-action on the spaces which occur in the diagram defining the homotopy colimit. Indeed, for every $g \in G$ and a morphism $c' \to c''$ we have a map $g : B(c'\backslash\mathcal{C}) \times X(c'') \to B(gc'\backslash\mathcal{C}) \times X(gc'')$ defined as $g(c' \to c_0 \to \ldots \to c_n, x) := (gc' \to gc_0 \to \ldots \to gc_n, \Phi_g(c'')(x))$. ■

If we identify $\operatorname{hocolim}_{\mathcal{C}} X$ with the realization of the category $\mathcal{C}_X$ then a $G$-action on it is defined on objects by the formula $g(c,x) := (gc, \Phi_g(c)(x))$ and it extends in the obvious way to morphisms.

Let $X' : \mathcal{C}' \to \mathcal{S}p$ be any diagram in the category of spaces. We recall the definition of the left homotopy Kan extension of $X'$ to $\mathcal{C}$ along a functor $F : \mathcal{C}' \to \mathcal{C}$ denoted by $F_{h\sharp}X'$ and defined as follows:

$$F_{h\sharp}X'(c) := \operatorname{hocolim}_{F/c} X'\pi_{\mathcal{C}'}.$$

There is a natural transformation $F_{h\sharp} \to F_\sharp$, thus for every functor $Y : \mathcal{C} \to \mathcal{S}p$ there is a natural map

$$\operatorname{Hom}_{\mathcal{C}'}(X, Y \circ F) \to \operatorname{Hom}_{\mathcal{C}}(F_{h\sharp}X, Y).$$

Assume now that $F : \mathcal{C}' \to \mathcal{C}$ and $X' : \mathcal{C}' \to \mathcal{S}p$ are $G$-functors. For each object $c \in \mathcal{C}$ the group $G$ acts on the fiber $F/c : g(c', F(c') \to c) := (gc', F(gc') \to F(c') \to c)$, and the projection $\pi_{\mathcal{C}'} : F/c \to \mathcal{C}'$ is clearly $G$-equivariant. Thus the composition $X'\pi_{\mathcal{C}'} : F/c \to \mathcal{S}p$ is a $G$-functor, hence by Proposition 2.4 the functor $\mathcal{C} \ni c \to \operatorname{hocolim}_{F/c} X'\pi_{\mathcal{C}'}$ takes values in the category of $G$-spaces. Thus we have the following extension to $G$-functors of *the push down theorem for homotopy colimits* (cf. [DK, Thm. 9.8], [HV, Thm. 5.5]).

2.5. Proposition. *For any $G$-functors $X' : \mathcal{C}' \to \mathcal{S}p$ and $F : \mathcal{C}' \to \mathcal{C}$ the collection of maps $F_{h\sharp}X'(c) \to \operatorname{hocolim}_{\mathcal{C}'} X'$ extends to a $G$-equivariant map*

$$\operatorname{hocolim}_{\mathcal{C}} F_{h\sharp}X' \xrightarrow{\simeq} \operatorname{hocolim}_{\mathcal{C}'} X'$$

*which is a homotopy equivalence* (*although not necessarily a $G$-homotopy equivalence.*) ■

Now we turn to the *reduction theorem* (cf. [DK, Thm. 9.6] and [HV, Thm. 4.4]), i.e. a situation when we "pull back" functors from $\mathcal{C}$ to $\mathcal{C}'$ along a functor $F$. If $F : \mathcal{C}' \to \mathcal{C}$ is a $G$-equivariant functor between categories equipped with $G$-actions, then as we noticed in Proposition 2.3 for any $G$-functor $X : \mathcal{C} \to \mathcal{D}$ the composition $X \circ F : \mathcal{C}' \to \mathcal{S}p$ is again a $G$-functor.

The reduction map $\varrho : \operatorname{hocolim}_{\mathcal{C}'} X \circ F \to B(-\backslash F) \times_{\mathcal{C}} X$ is induced by a map from the diagram defining the homotopy colimit to the diagram defining the target coequalizer:

$$\varrho : \coprod_{c'' \to c'} B(c'\backslash\mathcal{C}) \times X(F(c'')) \to \coprod_{c_2 \to c_1} B(c_1\backslash F) \times X(c_2).$$

The map $\varrho$ sends the summand indexed by a morphism $c'' \to c'$ to the summand indexed by the morphism $F(c'') \to F(c')$ and it is given by the formula $\varrho(c' \to c'_0 \to \ldots, x) := (F(c') \to F(c'_0) \to \ldots, x)$.

Since the functor $F$ is $G$-equivariant the target space admits a $G$-action and the map $\varrho$ is $G$-equivariant. Note that the forgetful map $B(-\backslash F) \times_{\mathcal{C}} X \to B(-\backslash\mathcal{C}) \times_{\mathcal{C}} X = \operatorname{hocolim}_{\mathcal{C}} X$ is also $G$-equivariant.

2.6. Proposition. *For any $G$-equivariant functor $F : \mathcal{C}' \to \mathcal{C}$ and a $G$-functor $X : \mathcal{C} \to \mathcal{S}p$ the natural map*

$$\operatorname{hocolim}_{\mathcal{C}'} X \circ F \to B(-\backslash F) \times_{\mathcal{C}} X$$

*is $G$-equivariant and a (non-equivariant) homotopy equivalence.*

We shall introduce several variants of cofinality. A functor $F$ is called

(1) *conically right cofinal*,
(2) *right cofinal*,
(3) *right $h^*$-cofinal* (where $h^*$ is a cohomology theory)

if for every object $c \in \mathcal{C}$ the right fiber $c\backslash F$ is respectively:

(1) *conical*, i.e. it contains either an initial or a terminal object,
(2) contractible,
(3) $h^*$-acyclic.

Obviously (1)⇒(2)⇒(3).

2.7. Corollary. *If a $G$-equivariant functor $F : \mathcal{C}' \to \mathcal{C}$ is right cofinal (resp. right $h^*$-cofinal) then for any $G$-functor $X : \mathcal{C} \to \mathcal{S}p$ the natural $G$-map*

$$\operatorname{hocolim}_{\mathcal{C}'} X \circ F \to \operatorname{hocolim}_{\mathcal{C}} X$$

*is a (non-equivariant) homotopy equivalence (resp. $h^*$-equivalence.)* ■

We shall introduce a useful notion of a quasi-terminal object in a category.

2.8. Definition. An object $c_0 \in \mathcal{C}$ is called *quasi-terminal* if for every object $c \in \mathcal{C}$ there is a morphism $c \to c_0$ and for any two morphisms $\phi_i$ :

$c \to c_0$, $i = 1, 2$, there exists a unique endomorphism $\psi : c_0 \to c_0$ such that $\psi\phi_1 = \phi_2$.

A motivating example is the category $\mathcal{O}_{G,\le H}$ of $G$-orbits whose isotropy subgroups are contained in a given normal subgroup $H \subset G$. The object $G/H$ is obviously quasi-terminal in $\mathcal{O}_{G,\le H}$.

2.9. Proposition. *If $c_0 \in \mathcal{C}$ is a quasi-terminal object then the full subcategory of $\mathcal{C}$ with the single object $c_0$ is conically right cofinal in $\mathcal{C}$.*

*Proof.* For any $c \in \operatorname{ob}\mathcal{C}$ consider the right fiber over $c$ of the inclusion functor. Its objects are pairs $(c, c \to c_0)$. By definition every object in this category is terminal. ∎

We conclude the section by recalling an interesting equivariant version of 2.5 and 2.7 applied to posets and constant functors, due to Thévenaz and Webb [TW].

Recall that a *G-poset* is a partially ordered set on which a group $G$ acts preserving the partial order. The $G$-posets form a category in which morphisms are order preserving equivariant maps.

If $f : \mathbf{V} \to \mathbf{W}$ is a morphism of $G$-posets then for every $w \in \mathbf{W}$ the left fiber $f/w$ and the right fiber $w\backslash f$ are equipped with an action of the isotropy group $G_w$ and the action clearly preserves the corresponding semi-fibers contained in the fibers.

2.10. Theorem ([TW, Thm. 1]). *Let $f : \mathbf{V} \to \mathbf{W}$ be a morphism of $G$-posets. Suppose that either all left fibers $f/w$ or all right fibers $w\backslash f$ are $G_w$-contractible, where $w \in \mathbf{W}$. Then $f$ induces a $G$-homotopy equivalence $|f| : |\mathbf{V}| \to |\mathbf{W}|$.*

2.11. Corollary ([TW, Prop. 1.7]). *Let $\mathbf{W}$ be a $G$-poset of finite length and $\iota : \mathbf{V} \subset \mathbf{W}$ a $G$-subposet such that for each $w \in \mathbf{W}\backslash\mathbf{V}$, the left semi-fiber $\iota//w$ (resp. the right semi-fiber $w\backslash\backslash i$) is $G_w$-contractible. Then the inclusion $\iota$ is a $G$-homotopy equivalence.*

**3. Isotropy presheaves on $G$-posets and related categories.** Let $G$ be a discrete group. Clearly a $G$-poset can be considered as a small category with $G$-action. Recall that by $\mathbf{S}(G)$ we denote the poset of all subgroups of $G$, equipped with the $G$-action by conjugation. There is a $G$-functor $Q : \mathbf{S}(G) \to \mathcal{O}_G$ to the category of $G$-orbits sending a subgroup $H$ to the orbit $G/H$. Recall the definition of an isotropy presheaf on a $G$-poset.

3.1. Definition. Let $\mathbf{W}$ be a $G$-poset. An *isotropy presheaf* on $\mathbf{W}$ is a morphism $d : \mathbf{W} \to \mathbf{S}(G)$ of $G$-posets such that for each $w \in \mathbf{W}$, $d(w) \subset G_w := \{g \in G : gw = w\}$.

Remark. Such morphisms of $G$-posets were introduced in [Sł3] and they were called admissible functions, but the present terminology seems to be more intuitive. Most of the results in this section were contained, with different proofs, in [Sł1-2].

Note that if $d : \mathbf{W} \to \mathbf{S}(G)$ is an isotropy presheaf then $d(w) \subset G_w$ is a normal subgroup. Indeed, since the map $d$ is $G$-equivariant, $gd(w)g^{-1} = d(gw) = d(w)$ for each $w \in \mathbf{W}$ and $g \in G_w$.

For an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$ one defines a small category $\mathcal{W}_d$ as a quotient category of the Grothendieck construction $\mathbf{W}_{hG}$. We set $\mathrm{ob}(\mathcal{W}_d) := \mathbf{W}$ and $\mathrm{Mor}_{\mathcal{W}_d}(w, w') := \mathrm{Mor}_{\mathbf{W}_{hG}}(w, w')/d(w')$. (Note that the isotropy group $G_{w'}$ acts on $\mathrm{Mor}_{\mathbf{W}_{hG}}(w, w')$.) Equivalently, a morphism $w \to w'$ in $\mathcal{W}_d$ is an equivariant map $f : G/d(w) \to G/d(w')$ such that $w \le gw'$ where $g \in f([e])$. Note that since $d(w) \subset G_w$ the element $gw'$ does not depend upon the choice of $g \in f([e])$.

Note that composition of an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$ with the canonical $G$-functor $Q : \mathbf{S}(G) \to \mathcal{O}_G$ is a $G$-functor $Q_d : \mathbf{W} \to \mathcal{O}_G$ and thus it has an extension $\widetilde{Q}_d : \mathbf{W}_{hG} \to \mathcal{O}_G$ to the Grothendieck construction. The functor $Q_d$ clearly factors through the category $\mathcal{W}_d$ and we denote the corresponding functor by $F_d : \mathcal{W}_d \to \mathcal{O}_G$. The functor $F_d$ assigns to every object $w$ the orbit $G/d(w)$ and to a morphism $w \to w'$ the corresponding $G$-map.

Remark. If a $G$-poset has a largest element $w_{\max}$ then $w_{\max}$ is a quasi-terminal object in $\mathcal{W}_d$ (cf. 2.8) and its automorphism group is $G/d(w_{\max})$.

Note that an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$ on a $G$-poset can be restricted to any subgroup $H \subset G$. We define an $H$-poset map $d|H : \mathbf{W} \to \mathbf{S}(H)$ by the formula $(d|H)(w) := d(w) \cap H$. The restriction is a right adjoint functor to the extension functor which assigns to an isotropy presheaf $t : \mathbf{V} \to \mathbf{S}(H)$, defined on an $H$-poset $\mathbf{V}$, its extension $\mathrm{id} \times_H t : G \times_H \mathbf{V} \to \mathbf{S}(G)$. There is an obvious functor $\iota_H^G : \mathcal{W}_{d|H} \to \mathcal{W}_d$. If $H = e$ is the trivial subgroup then $\mathcal{W}_{d|e} = \mathbf{W}$ and the functor coincides with the inclusion $\iota : \mathbf{W} \to \mathcal{W}_d$. It is easy to identify the left fibers of $\iota_H^G$:

3.2. Proposition. *There is a natural transformation from the functor defined by $\mathcal{W}_d \ni w \rightsquigarrow \iota_H^G/w \in \mathcal{C}at$ to the functor $\mathcal{W}_d \ni w \rightsquigarrow H\backslash G/d(w) \in \mathcal{C}at$ (where sets of double cosets are considered as discrete categories) which for every object $w \in \mathcal{W}_d$ is a homotopy equivalence of categories.*

*Proof.* We define a functor $H\backslash G/d(w) \to \iota_H^G/w$ assigning to a coset $Hgd(w)$ the object $(gw, G/d(gw) \cap H) \xrightarrow{[g]} (w, G/d(w))$. Since the right fibers of this functor are contractible, the conclusion follows from 2.7. ∎

We shall consider morphisms $f : (\mathbf{W}_1, d_1) \to (\mathbf{W}_2, d_2)$ between $G$-posets equipped with isotropy sheaves. Such a morphism is a $G$-poset map $f$ :

$\mathbf{W}_1 \to \mathbf{W}_2$ such that for every $w \in \mathbf{W}_1$, $d_1 w \le d_2 f w$. Clearly $G$-posets equipped with isotropy presheaves and their morphisms form a category denoted by $\mathcal{P}re_G^{\mathrm{iso}}$.

Isotropy presheaves on a given $G$-poset form a poset with the unique minimal object $d_e(w) := e$. If for any $w \le w'$ we have the inclusion $G_w \subset G_{w'}$ of isotropy groups then we define the maximal isotropy presheaf $d_{\mathrm{iso}}(w) := G_w$. Note that for every $G$-poset $\mathbf{W}$ its barycentric subdivision $\mathrm{sd}(\mathbf{W})$ (cf. [Q2, 1.4] and Sec. 4) has the above property.

To summarize: for a fixed group $G$ we consider the category $\mathcal{F}un_{\mathcal{O}_G}$ of functors from small categories into the orbit category $\mathcal{O}_G$. A morphism in $\mathcal{F}un_{\mathcal{O}_G}$ between two functors $F_i : \mathcal{C}_i \to \mathcal{O}_G$, $i = 1, 2$, consists of a functor $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ and a natural transformation $T : F_1 \to F_2\phi$.

3.3. Proposition. *Assigning to every $G$-poset equipped with an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$ the functor $F_d : \mathbf{W}_d \to \mathcal{O}_G$ extends to a functor $\mathcal{P}re_G^{\mathrm{iso}} \to \mathcal{F}un_{\mathcal{O}_G}$.*

*Proof.* It remains to define natural transformations induced by morphisms of $G$-posets equipped with isotropy presheaves. For a morphism $f : (\mathbf{W}_1, d_1) \to (\mathbf{W}_2, d_2)$ we set $f_*(w) := f(w)$; now the definition of the natural transformation $F_{d_1} \to F_{d_2} f_*$ is obvious. ∎

Let $f_* : \mathbf{W}_d \to \mathbf{W}'_{d'}$ be a functor induced by a morphism $f$. Since we are going to apply to such functors various cofinality arguments (cf. 2.5, 2.7) we need to identify the fiber categories $f_*/w'$ and $w'\backslash f_*$ in terms of the corresponding subposets $f/w'$ and $w'\backslash f$.

Note that the poset $f/w'$ is equipped with an action of the group $d'(w') \subset G_{w'}$ and with an isotropy presheaf $f/w' \to \mathbf{S}(d'w')$ defined by $d$ and denoted with the same letter: $d(w, f(w) \to w') := d(w)$. Assigning to $w' \in \mathbf{W}'$ the small category $(f/w')_d$ defines a functor on the poset $\mathbf{W}'$, but *not* on the category $\mathcal{W}'_{d'}$.

3.4. Theorem. *For an arbitrary morphism $f : (\mathbf{W}, d) \to (\mathbf{W}', d')$ there are natural transformations of functors defined on the poset $\mathbf{W}' \subset \mathcal{W}'_{d'}$:*

$$(f/w')_d \to f_*/w'$$

*and, if $d = d'f$,*

$$w'\backslash f \to w'\backslash f_*,$$

*which for each $w \in \mathbf{W}'$ are equivalences of categories.*

*Proof.* To prove the first assertion we construct an equivalence of categories $\Phi : (f/w')_d \to f_*/w'$. An object of $(f/w')_d$ is an element $w \in \mathbf{W}$ such that $f(w) \le w'$, thus we set $\Phi(w) := [e] : f(w) \to w'$. Every object of $f_*/w'$ is up to isomorphism in the image of $\Phi$ since if $f(w) \le gw'$ for some $g \in G$ then $f(g^{-1}w) \le w'$. It remains to define $\Phi$ on the morphism sets and prove that it is bijective. If $[g] : w \to v$ where $g \in d'(w')$ is a morphism in $(f/w')_d$ then $\Phi([g]) := [g]$ is clearly a well defined morphism $f(w) \to f(v)$ over $w'$ and $\Phi$ is clearly injective. It is also surjective since for every morphism $g : v \to w$ which defines a morphism $f(w) \to f(v)$ over $w'$ the element $g$ must belong to $d'(w')$.

The proof of the second assertion is even more straightforward. ∎

The next corollary turns out to be a very practical tool for comparing colimits over categories related to $G$-posets.

3.5. Corollary. *Let $i : (\mathbf{W}, d_1) \to (\mathbf{W}, d_2)$ be a comparison morphism (i.e. $i = \mathrm{id}$ and $d_1 \le d_2$). Then for every $w \in \mathbf{W}$,*

(a) *the category $i_*/w$ has a quasi-terminal object $(w, w = w)$ whose group of endomorphisms is $d_2(w)/d_1(w)$,*

(b) *the natural transformation $i_{h\sharp}F_{d_1}(w) \to F_{d_2}(w)$ of $G$-functors, adjoint to $F_{d_1} \to F_{d_2} i_*$, is a homotopy equivalence.*

*Proof.* Part (a) is a very special case of 3.4. Part (b) follows from the definition of the homotopy left Kan extension given in Section 2, part (a) and Proposition 2.9. ∎

**4. Homotopy decompositions defined by isotropy presheaves.** We shall explain how isotropy presheaves on a $G$-poset lead to homotopy decompositions of its geometric realization. Recall that to distinguish geometric realizations of posets from classifying spaces of general small categories we shall keep denoting the former by $|-|$.

4.1. Theorem. *For any $G$-poset $\mathbf{W}$ equipped with an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$ there exists a natural $G$-equivariant map $\alpha : \mathrm{hocolim}_{\mathcal{W}_d} F_d \to |\mathbf{W}|$. For any subgroup $H \subset G$ its restriction to the fixed point set of $H$ is a homotopy equivalence $\alpha^H : (\mathrm{hocolim}_{\mathcal{W}_d} F_d)^H \xrightarrow{\simeq} |H\backslash d| \subset |\mathbf{W}|^H$. In particular $\alpha$ is a homotopy equivalence.*

*First proof.* Consider the embedding $\imath : \mathbf{W} \subset \mathcal{W}_d$, which is a $G$-functor, and a constant functor on the poset $c : \mathbf{W} \to \mathcal{S}p$ sending each object to a point, considered as a $G$-functor. We apply the push down theorem for homotopy colimits (Proposition 2.5). Proposition 3.2 implies that there is a natural homotopy equivalence of $G$-functors $\imath_{h\sharp} c \xrightarrow{\simeq} F_d$. Thus 2.5 implies that there exists a $G$-map $\mathrm{hocolim}_{\mathcal{W}_d} \imath_{h\sharp} c \to \mathrm{hocolim}_{\mathbf{W}} c = |\mathbf{W}|$ which is a homotopy equivalence. For every subgroup $H \subset G$ we have a restriction $\alpha^H : (\mathrm{hocolim}_{\mathcal{W}_d} F_d)^H \simeq \mathrm{hocolim}_{\mathcal{W}_d} F_d^H \to |W|^H$. The same argument as in 2.5 shows that the homotopy Kan extension of the constant functor along the inclusion functor $H\backslash d \subset \mathcal{W}_d$ is equivalent to $(F_d)^H$, hence $\alpha^H : (\mathrm{hocolim}_{\mathcal{W}_d} F_d)^H \xrightarrow{\simeq} |H\backslash d|$.

*Second proof.* The homotopy colimit $\mathrm{hocolim}_{\mathcal{W}_d} F_d$ is by definition homeomorphic to the classifying space of the category $(\mathbf{W}_d)_{F_d}$ whose objects are pairs $([g], w)$ and a single morphism $([g], w) \to ([g'], w')$ exists if $w \le hw'$ where $gh = g' \bmod d(w')$. The action $\mu : G \times \mathbf{W} \to \mathbf{W}$ defines a functor $M_\mu : (\mathcal{W}_d)_{F_d} \to \mathbf{W}$ which is clearly a $G$-map and an equivalence of categories, thus it induces a homotopy equivalence of their classifying spaces. For every subgroup $H \subset G$ the restriction $M_\mu^H : (\mathcal{W}_d)_{F_d}^H \to d\backslash H$ is also an equivalence of categories. ■

Recall that we call a $G$-poset *$h^*$-ample* if the associated bundle map $\pi : EG \times_G |\mathbf{W}| \to BG$ is an $h^*$-equivalence. Note that the Borel construction $EG \times_G |\mathbf{W}|$ is homotopy equivalent to the classifying space $B\mathcal{W}_e$, and the projection $\pi$ corresponds to the map induced on classifying spaces by the obvious projection $\mathcal{W}_e \to G$.

4.2. Corollary. *Let $d : \mathbf{W} \to \mathbf{S}(G)$ be an isotropy presheaf. Assume that $X$ is a $G$-CW-complex such that for every $x \in X$ the poset $\mathbf{W}$ is $h^*$-ample as a $G_x$-poset. Then the functor $\mathcal{W}_d \ni w \rightsquigarrow EG \times_{d(w)} X$ is an $h^*$-decomposition of the Borel construction $EG \times_G X$, i.e. the map*

$$\alpha_G(X) : \mathrm{hocolim}_{\mathcal{W}_d} EG \times_{d(-)} X \to EG \times_G X$$

*is an $h^*$-equivalence.*

*Proof.* Consider the functor $F_d \times X : \mathcal{W}_d \to G\text{-}\mathcal{S}p$. Theorem 4.1 implies that we have a homotopy equivalence $\mathrm{hocolim}_{\mathcal{W}_d} EG \times_{d(-)} X \to EG \times_G (|\mathbf{W}| \times X)$. The standard Leray spectral sequence argument applied to the projection onto the second factor $EG \times_G (|\mathbf{W}| \times X) \to EG \times_G X$ implies that it is an $h^*$-equivalence if for every isotropy group $G_x$ the projection $EG \times_{G_x} |\mathbf{W}| \to BG_x$ is an $h^*$-equivalence. ■

A $G$-poset equipped with an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$ will be called *$h^*$-ample and sharp* if $\mathbf{W}$ is $h^*$-ample and the associated $h^*$-decomposition of $BG$ is sharp, i.e. $H^i(\mathcal{W}_d; h^*(EG \times_G F_d)) = 0$ for $i > 0$. The above terminology extends the terminology introduced by Dwyer [Dw1] beyond posets of subgroups.

Note that some standard constructions on posets preserve ampleness: The opposite poset of an $h^*$-ample poset is $h^*$-ample. If $f : \mathbf{V} \to \mathbf{W}$ is an equivariant map between posets and an $h^*$-equivalence then $\mathbf{W}$ is $h^*$-ample if and only if $\mathbf{V}$ is $h^*$-ample. In particular $\mathbf{W}$ is $h^*$-ample if its barycentric subdivision $\mathrm{sd}(\mathbf{W})$ is $h^*$-ample. (Recall for further reference that $\mathrm{sd}(\mathbf{W})$ is a $G$-poset whose points are simplices of $\mathbf{W}$ (i.e. ascending sequences of elements of $\mathbf{W}$), ordered by inverse inclusion. Obviously, a $G$-action on $\mathbf{W}$ induces an action on $\mathrm{sd}(\mathbf{W})$. The barycentric subdivision is equipped with two canonical projections $p_0 : \mathrm{sd}(\mathbf{W}) \to \mathbf{W}$ and $p_e : \mathrm{sd}(\mathbf{W}) \to \mathbf{W}^{\mathrm{op}}$ which induce $G$-homotopy equivalences of the corresponding geometric realizations.)

Sharpness is an even more delicate property. In general it is not preserved by homotopy equivalence. However if a morphism of isotropy presheaves $f : (\mathbf{V}, d_{\mathbf{V}}) \to (\mathbf{W}, d_{\mathbf{W}})$, where $d_{\mathbf{V}} = d_{\mathbf{W}} \circ f$, is right cofinal then the decomposition associated to $(\mathbf{V}, d_{\mathbf{V}})$ is sharp if and only if the decomposition associated to $(\mathbf{W}, d_{\mathbf{W}})$ is sharp (cf. 2.5, 3.4, 5.4). In particular an isotropy presheaf $(\mathbf{W}, d_{\mathbf{W}})$ is $h^*$-ample and sharp if and only if $(\mathrm{sd}(\mathbf{W}), d_{\mathbf{W}} \circ p_0)$ is $h^*$-ample and sharp.

Let $\mathbf{W}' \subset \mathbf{W}$ be a $G$-subposet and $d : \mathbf{W} \to \mathbf{S}(G)$ an isotropy presheaf. We introduce another $G$-subposet $\mathrm{Cl}_{h^*}(\mathbf{W}', d) \subset \mathbf{W}$, called the $h^*$-*closure* of $\mathbf{W}$. Note that for every $w \in \mathbf{W}$ the isotropy group $G_w$ acts on the poset $\mathbf{W}'_{\leq w} := \{w' \in \mathbf{W}' \,|\, w' \leq w\}$. The $G$-subposet $\mathrm{Cl}_{h^*}(\mathbf{W}', d) \subset \mathbf{W}$ consists of $w \in \mathbf{W}$ for which the $d(w)$-poset $\mathbf{W}_{\leq w}$ is $h^*$-ample. Obviously $\mathbf{W}' \subset \mathrm{Cl}_{h^*}(\mathbf{W}') \subset \mathbf{W}$.

Similarly we define a collection $\mathrm{En}_{h^*}(\mathbf{W}, d)$ of subgroups, called the $h^*$-*envelope* of $\mathbf{W}$, consisting of those $w \in \mathbf{W}$ for which $(\mathbf{W}'_{\leq w}, d_w)$, where $d_w : \mathbf{W}'_{\leq w} \to \mathbf{S}(d(w))$ is the restriction of $d$, is $h^*$-ample and sharp.

4.3. Proposition. *Let $\mathbf{W}' \subset \mathbf{W}'' \subset \mathbf{W}$ be a triple of $G$-posets equipped with an isotropy presheaf $d : \mathbf{W} \to \mathbf{S}(G)$. If $\mathbf{W}' \subset \mathbf{W}'' \subset \mathrm{Cl}_{h^*}(\mathbf{W}', d)$ then the induced map $EG \times_G |\mathbf{W}'| \to EG \times_G |\mathbf{W}''|$ is an $h^*$-equivalence. If also $\mathbf{W}' \subset \mathbf{W}'' \subset \mathrm{En}_{h^*}(\mathbf{W}', d)$ then $H^*(\mathcal{W}''_d; h^* \circ F_d) \to H^*(\mathcal{W}'_d; h^* \circ F_d)$ is an isomorphism.*

*Proof.* As we have noticed the Borel construction $EG \times_G |\mathbf{W}|$ is homotopy equivalent to $B\mathcal{W}_e$. Thus it is enough to notice that the fibers of the inclusion functor $\iota : \mathcal{W}'_e \subset \mathcal{W}''_e$ are $h^*$-acyclic. Indeed, Theorem 3.4 tells us that for any $w'' \in \mathbf{W}''$ we have an equivalence $\iota/w'' \simeq (\mathbf{W}'_{\leq w''})_e \simeq \mathbf{W}'_{\leq w''}$. The Leray spectral sequence (5.3) implies the desired conclusion. Similarly to prove the second assertion we apply the spectral sequence 5.3 to the inclusion functor $\iota : \mathcal{W}'_d \subset \mathcal{W}''_d$. ∎

We shall describe some decompositions of the Borel construction on a $G$-space $X$ associated to posets of subgroups of $G$. For any given $G$-subposet $\mathbf{W} \subset \mathbf{S}(G)$ and any isotropy presheaf on it we shall construct a decomposition of the $\mathbf{W}$-fixed point set $X^{\mathbf{W}} := \bigcup_{H \in \mathbf{W}} X^H$. Note that $X^{\mathbf{W}}$ is a $G$-subspace of $X$. A $G$-space $X$ defines the fixed point $G$-functor $\mathrm{Fix}_X : \mathbf{W}^{\mathrm{op}} \to \mathcal{S}p$ on the opposite poset where $\mathrm{Fix}_X(H) := X^H$. We write $X^{\mathrm{t}\mathbf{W}} := \mathrm{hocolim}_{\mathbf{W}^{\mathrm{op}}} \mathrm{Fix}_X$ and call it the *thick* $\mathbf{W}$-*fixed point set.* According to Proposition 2.4 the thick $\mathbf{W}$-fixed point set is also a $G$-space, the action being induced by conjugation on $\mathbf{W}$ and the original action on $X$. There are two natural $G$-maps

$$|\mathbf{W}^{\mathrm{op}}| \xleftarrow{\pi} X^{\mathrm{t}\mathbf{W}} \xrightarrow{\eta} X^{\mathbf{W}} \subset X;$$

the map $\pi$ is the canonical projection from the colimit over an indexing category while $\eta$ is the canonical map from the homotopy colimit to the ordinary colimit. As usual it is interesting to compare the homotopy colimit to the ordinary colimit. The thick fixed point set is the geometric realization of the category whose objects are pairs $(H, x)$ where $x \in X^H$; the morphisms correspond to inverse inclusions of subgroups; the $G$-action on the category is given by the formula $g(H, x) := (H^g, gx)$. Thus $X^{\mathrm{t}\mathbf{W}}$ is homotopy equivalent to a subspace of $X \times |\mathbf{W}|$ which is the union of simplices $(x, H_0 \supset \ldots \supset H_q)$ such that $x \in X^{H_0}$. The structure maps $\eta,\ \pi$ correspond to projections onto factors.

4.4. Proposition. *Let $X$ be a $G$-space and $\mathbf{W} \subset \mathbf{S}(G)$.*

($\pi$) *If for each $H \in \mathbf{W}$ the fixed point set $X^H$ is $h^*$-acyclic (resp. contractible) then $\pi : X^{\mathrm{t}\mathbf{W}} \to |\mathbf{W}|$ is an $h^*$-equivalence (resp. a homotopy equivalence).*

($\eta$) *If for each $x \in X$ the poset $\mathbf{W}_{\le G_x}$ is $h^*$-acyclic (resp. contractible) then the map $\eta : X^{\mathrm{t}\mathbf{W}} \to X^{\mathbf{W}}$ is an $h^*$-equivalence (resp. a homotopy equivalence).*

($\eta'$) *If for each $x \in X$ the poset $\mathbf{W}_{\le G_x}$ is $h^*$-ample then the map* $\mathrm{id} \times_G \eta : EG \times_G X^{\mathrm{t}\mathbf{W}} \to EG \times_G X^{\mathbf{W}}$ *is an $h^*$-equivalence.*

*Proof.* ($\pi$) follows directly from the Bousfield–Kan spectral sequence calculating cohomology of the homotopy colimit (or equivalently the Leray spectral sequence of the map $\pi$ in the cohomology theory $h^*$).

$(\eta), (\eta')$. Note that the fiber of the map $\eta$ over a point $x \in X$ is the union of simplices of the form $(x; H_0 \supset H_1 \supset \ldots \supset H_n)$ where $G_x \supset H_0$, i.e. it is homotopy equivalent to the subposet $\mathbf{W}_{\le G_x} \subset \mathbf{W}$ consisting of all elements of $\mathbf{W}$ which are contained in the isotropy subgroup $G_x$. ■

The following proposition extends the decomposition of Theorem 4.1 to $G$-spaces.

4.5. Proposition. *Let $\mathbf{W}$ be a $G$-subposet of $\mathbf{S}(G)$ and $d : \mathbf{W}^{\mathrm{op}} \to \mathbf{S}(G)$ be an isotropy presheaf. Then for any $G$-space $X$ there exists a natural $G$-map*

$$\mathrm{hocolim}_{\mathcal{W}_d^{\mathrm{op}}}\, G \times_{d(-)} \mathrm{Fix}_X \to X^{\mathrm{t}\mathbf{W}},$$

*which is a (non-equivariant) homotopy equivalence.*

*Proof.* The proof is similar to that of 4.1.

*First proof.* We consider an embedding $\imath : \mathbf{W}^{\mathrm{op}} \subset \mathcal{W}_d^{\mathrm{op}}$. The left homotopy Kan extension of the functor $\mathrm{Fix}_X : \mathbf{W}^{\mathrm{op}} \to \mathcal{S}p$ is equivalent to $G \times_{d(-)} \mathrm{Fix}_X$ (cf. 3.5) and the homotopy equivalence defined in Proposition 2.5 is a $G$-map.

*Second proof.* We define a functor between the categories defining the two colimits. The first one is the geometric realization of the category whose objects are triples $(H,[g,x])$ where $x\in X^H$, and a morphism $(H',[g',x'])\to (H,[g,x])$ is an element $a\in G$ such that $H^a\subset H'$, $g'=ga$, $x=a^{-1}x'$. The action of $G$ is given on objects by the formula $g''(H,[g,x]):=(H,[g''g,x])$ and it extends in an obvious way to morphisms. Objects of the second category are pairs $(H,x)$ where $x\in X^H$; morphisms correspond to inverse inclusions of subgroups; a $G$-action on the category is given by the formula $g(H,x):=(H^g,gx)$. To a triple $(H,g,x)$ we assign the pair $(H^g,gx)$; this is clearly a $G$-map. Its inverse is given by sending a pair $(H,x)$ to $(H,e,x)$, which is an equivalence of categories (but not a $G$-map!). ∎

4.6. Corollary. *If the map $EG\times_G X^{t\mathbf{W}}\to EG\times_G X^{\mathbf{W}}$ is an $h^*$-equivalence* (cf. 4.4) *then the functor $\mathcal{W}_d\ni H\rightsquigarrow EG\times_{d(H)}X^H$ is an $h^*$-decomposition of the space $EG\times_G X^{\mathbf{W}}$, i.e. the map induced by the inclusions $X^H\subset X^{\mathbf{W}}$ induces an $h^*$-equivalence*

$$\operatorname{hocolim}_{\mathcal{W}_d^{\mathrm{op}}} EG\times_{d(-)}\mathrm{Fix}_X \xrightarrow{\simeq} EG\times_G X^{\mathbf{W}}. \ \blacksquare$$

The last corollary generalizes Henn's result [H].

We shall discuss in detail the decomposition defined by the maximal isotropy presheaf on a $G$-poset. Recall that if $G_w\subset G_{w'}$ for all $w\le w'\in\mathbf{W}$ then the formula $d_{\mathrm{iso}}(w):=G_w$ defines a maximal isotropy presheaf. Observe that the category $\mathbf{W}_{d_{\mathrm{iso}}}$ is itself equivalent to a poset if the action satisfies some further regularity condition (cf. [Br2, III.1]).

4.7. Definition. Let $\mathbf{W}$ be a $G$-poset.

(a) $\mathbf{W}$ is *regular* if for any $w,w'\in\mathbf{W}$, $g\in G$ the inequalities $w\le w'$ and $w\le gw'$ imply that $gw'=w'$.

(b) $\mathbf{W}$ is *normal* if $G_w\subset G_{w'}$ for all $w\le w'\in\mathbf{W}$.

(c) $\mathbf{W}$ satisfies *condition* (EI) if $gw=w$ for all $g\in G$ and $w\in\mathbf{W}$ such that $w\le gw$.

Clearly a regular $G$-poset must be normal and satisfy condition (EI). Unfortunately, many interesting examples of $G$-posets are not regular, even not normal. However we shall see that the barycentric subdivision of every $G$-poset $\mathbf{W}$ is normal and, if $\mathbf{W}$ satisfies condition (EI), it is even regular.

Observe that for a $G$-poset $\mathbf{W}$ satisfying condition (EI) the set $\mathbf{W}/G$ of orbits is also a poset and the canonical projection $\pi:\mathbf{W}\to\mathbf{W}/G$ is a quotient map with the obvious universality property.

4.8. Proposition. *If $\mathbf{W}$ is a regular $G$-poset then the projection $\mathbf{W}\to\mathbf{W}/G$ induces a canonical equivalence of categories $\Pi:\mathcal{W}_{d_{\mathrm{iso}}}\to\mathbf{W}/G$.*

*Proof.* It is easy to see that the formula $\Pi(w):=[w]$ defines an equivalence. ∎

4.9. Examples. A $G$-poset $\mathbf{W}$ satisfies condition (EI) if one of the following conditions holds:

(1) the group $G$ is finite,

(2) all strictly ascending sequences $w_0 < w_1 < \ldots < w_n$ of elements of $\mathbf{W}$ have bounded length (i.e. the geometric realization of $\mathbf{W}$ is finite-dimensional),

(3) $\mathbf{W}$ is the barycentric subdivision of some $G$-poset.

The following proposition corresponds to Proposition III.1.1 in [Br2].

4.10. Proposition. *For every $G$-poset $\mathbf{W}$ its barycentric subdivision* $\mathrm{sd}(\mathbf{W})$ *is normal. Moreover*, *if* $\mathbf{W}$ *satisfies condition* (EI) *then* $\mathrm{sd}(\mathbf{W})$ *is regular.*

*Proof.* Elements of the barycentric subdivision $\mathrm{sd}(\mathbf{W})$ are simplices of $\mathbf{W}$, ordered by inverse inclusion. Note that for every $w_. := (w_0, w_1, \ldots, w_n)$ its isotropy group $G_{w_.}$ is equal to $G_{w_0} \cap \ldots \cap G_{w_n}$, thus if $w_. \leq v_.$ then $G_{w_.} \leq G_{v_.}$. Hence $\mathrm{sd}(\mathbf{W})$ is normal.

Assume now that in $\mathbf{W}$ an inequality $w \leq gw$ implies that $gw = w$. Let $w_. \leq v_.$ and $w_. \leq gv_.$ for some $g \in G$, thus $gv_. \cup v_. \subset w_.$ Since $w_.$ is a chain, every $v_i \in v_.$ is comparable with $gv_i$. Since we have assumed that $\mathbf{W}$ satisfies (EI) we obtain $g \in G_{v_i}$, hence $g \in G_{v_.}$. ∎

Note that for a regular $G$-poset $\mathbf{W}$ Theorem 4.1 implies that

$$\alpha : \mathrm{hocolim}_{\mathbf{W}/G}\, \pi^{-1} \to |\mathbf{W}|$$

is indeed a $G$-homotopy equivalence. Thus we obtain

4.11. Corollary. *For every $G$-poset* $\mathbf{W}$ *which satisfies condition* (EI) *the natural map*

$$\mathrm{hocolim}_{[s_.]\in \mathrm{sd}(\mathbf{W})/G}\, G/G_{s_.} \to |\mathbf{W}|$$

*is a $G$-homotopy equivalence. Thus if* $\mathbf{W}$ *is $h^*$-ample then the map induces an $h^*$-equivalence*

$$\mathrm{hocolim}_{[s_.]\in \mathrm{sd}(\mathbf{W})/G}\, EG \times_G G/G_{s_.} \to BG. \ ∎$$

A generalized homotopy push-out is the homotopy colimit of a functor defined on a poset. The fundamental group of the generalized homotopy push-out is given by a generalization of the van Kampen theorem. Note that the last corollary provides a way of converting general homotopy colimits which appear in Proposition 4.2 into generalized homotopy push-outs (cf. [Sł1]).

4.12. Example. Let a discrete group $G$ act simplicially on a tree $T$. Then the simplicial barycentric subdivision $\mathrm{sd}(T)$ is a simplicial complex defined by a regular $G$-poset which we denote by $T'$. The elements of $T'$ are the vertices and the edges of $T$, ordered by inclusion. The isotropy group of an edge is the intersection of the isotropy groups of its ends. The resulting decomposition

$$\operatorname{hocolim}_{T'} EG \times_G G/G_- \to BG$$

is precisely the one considered by Serre and others in combinatorial group theory and leads to the presentation of $G$ as the amalgamated product of the isotropy groups of the vertices along their intersections corresponding to the edges. The example generalizes to actions on higher dimensional contractible simplicial complexes.

**5. Higher limits and pruning of ordered categories.** Let $\mathcal{C}$ be an arbitrary small category. We write $\mathcal{C}$-$\mathsf{mod}$ for the category of $\mathcal{C}$-modules, i.e. the abelian category of contravariant functors $M : \mathcal{C} \to \mathsf{Ab}$. There is a functor $\varprojlim_{\mathcal{C}} : \mathcal{C}\text{-}\mathsf{mod} \to \mathsf{Ab}$ which assigns to every $M$ its inverse limit

$$\varprojlim_{\mathcal{C}}(M) := \operatorname{Hom}_{\mathcal{C}}(\mathbb{Z}, M)$$

where $\mathbb{Z}$ denotes the constant functor.

We want to study the derived functors $\varprojlim^i_{\mathcal{C}}$ of the inverse limit. In order to emphasize the analogy with group cohomology, we shall denote the functors $\varprojlim^i_{\mathcal{C}}$ by

$$H^i(\mathcal{C}; -) : \mathcal{C}\text{-}\mathsf{mod} \to \mathsf{Ab}.$$

Recall the basic long exact sequence of derived functors:

5.1. Proposition. *For any small category $\mathcal{C}$, and any short exact sequence of $\mathcal{C}$-modules $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}$-$\mathsf{mod}$, there exists a functorial long exact sequence*

$$\ldots \to H^i(\mathcal{C}; M') \to H^i(\mathcal{C}; M) \to H^i(\mathcal{C}; M'') \to H^{i+1}(\mathcal{C}; M) \to \ldots \ \blacksquare$$

From general nonsense in homological algebra it follows that instead of taking an injective resolution for $M$ when computing higher inverse limits, one can choose a single projective resolution $P_*$ of the constant functor $\mathbb{Z}$, and then $H^*(\mathcal{C}; M) \simeq H^*(\operatorname{Hom}_{\mathcal{C}}(P_*, M))$ for all $\mathcal{C}$-modules $M$. We shall describe how fiber categories lead to such resolutions. Consider the relative case: for a subcategory $\mathcal{C}' \subset \mathcal{C}$ we define the relative chain complex functor which assigns to an object $c \in \mathcal{C}$ the chain complex

$$C_*(c\backslash\mathcal{C}, c\backslash\mathcal{C}') := C_*(c\backslash\mathcal{C})/C_*(c\backslash\mathcal{C}').$$

(The chain complex of a small category was defined in Section 1.)

5.2. Proposition. *For every $n \geq 0$ the functor $C_n(-\backslash\mathcal{C}, -\backslash\mathcal{C}')$ is a projective $\mathcal{C}$-module. If $\mathcal{C}'$ is empty, then the augmented chain complex $C_*(-\backslash\mathcal{C})$ is a projective resolution in $\mathcal{C}$-$\mathsf{mod}$ of the constant functor $\mathbb{Z}$.*

*Proof.* We shall prove that for an arbitrary epimorphism $M' \to M$ of $\mathcal{C}$-modules the induced homomorphism

$$\mathrm{Hom}_{\mathcal{C}}(C_n(-\backslash\mathcal{C}, -\backslash\mathcal{C}'), M') \to \mathrm{Hom}_{\mathcal{C}}(C_n(-\backslash\mathcal{C}, -\backslash\mathcal{C}'), M)$$

is also an epimorphism. For every $\mathcal{C}\text{-}\mathsf{mod}$ $M$ we have a bijection

$$\mathrm{Hom}_{\mathcal{C}}(C_n(-\backslash\mathcal{C}, -\backslash\mathcal{C}'), M) \to \prod_{[c_0\to\ldots\to c_n]\notin(B\mathcal{C}')_n} M(c_0)$$

sending each natural transformation $\Phi$ to $\Phi(c_0 = c_0 \to \ldots \to c_n)$ . The cartesian product preserves epimorphisms. To prove that the augmented complex $C_*(c\backslash\mathcal{C})$ is acyclic it is enough to note that the category $c\backslash\mathcal{C}$ has an initial object $(c, c \xrightarrow{\mathrm{id}} c)$, thus Proposition 1.1 provides a contracting homotopy. ∎

We introduce the relative cohomology of categories. Recall that for a homomorphism $f : C_* \to D_*$ of non-negative chain complexes of modules over a ring $R$ one defines the mapping cone by $\Delta_n(f) := C_{n-1} \oplus D_n$ and $\partial_{\Delta_*}(c, d) := (-\partial_{C_*}(c), \partial_{D_*}(d) + f(c))$. The mapping cone fits into a short exact sequence

$$0 \to D_* \to \Delta_*(f) \to \Sigma C_* \to 0$$

where $\Sigma C_*$ denotes the suspension of $C_*$, i.e. $(\Sigma C_*)_n := C_{n-1}$. If $F = \iota : C_* \subset D_*$ is a map of complexes such that for every $n$, $\iota_n : C_n \subset D_n$ is a split inclusion of projective $R$-modules, then the obvious projection $\Delta_*(\iota) \to D_*/C_*$ is a chain homotopy equivalence (cf. [Bro, I.8.4]). For a functor $F : \mathcal{C}' \to \mathcal{C}$ we define its cone $\Delta_*(F)$ as the chain complex in $\mathcal{C}\text{-}\mathsf{mod}$ which for each $c \in \mathcal{C}$ is the mapping cone of the chain map $F_* : C_*(c\backslash F) \to C_*(c\backslash\mathcal{C})$ sending an object $(c', c \to F(c'))$ to $(F(c'), c \to F(c'))$.

Let $\mathcal{C}' \subset \mathcal{C}$. For an arbitrary functor $M : \mathcal{C} \to \mathsf{Ab}$ the relative cohomology $H^*(\mathcal{C}, \mathcal{C}'; M)$ is defined as the cohomology of the relative cochain complex $C^*(\mathcal{C}, \mathcal{C}'; M)$, or equivalently the cohomology of the mapping cone $\Delta_*(\iota_n)$. Hence the relative groups fit into an exact sequence relating higher limits over $\mathcal{C}'$ and $\mathcal{C}$:

$$\ldots \to H^i(\mathcal{C}; M) \to H^i(\mathcal{C}'; M) \to H^{i+1}(\mathcal{C}, \mathcal{C}'; M) \to H^{i+1}(\mathcal{C}; M) \to \ldots$$

The following proposition generalizes Shapiro's lemma (the subgroup theorem), often used in calculating group cohomology (cf. [Bro, III.6.2]). For general categories it takes the form of a spectral sequence (cf. [GZ, Appendix 2, Thm. 3.6] for the dual statement involving homology groups).

5.3. Proposition. *Let $F : \mathcal{C}' \to \mathcal{C}$ be a functor between small categories. For any contravariant functor $M' : \mathcal{C}' \to \mathsf{Ab}$ there exists a spectral sequence*

$$H^m(\mathcal{C}; H^n(F/-; M'\pi_{\mathcal{C}'})) \Rightarrow H^{n+m}(\mathcal{C}'; M'). \ ∎$$

Note that $H^0(F/c; M'\pi_{\mathcal{C}'}) = F_\flat M'(c)$ is the right Kan extension of $M'$.

5.4. Proposition. *Let $F : \mathcal{C}' \to \mathcal{C}$ be a functor between small categories. For any contravariant functor $M : \mathcal{C} \to R$-mod there is a canonical isomorphism of chain complexes*

$$\mathrm{Hom}_{\mathcal{C}'}(C_*(-\backslash\mathcal{C}'), M \circ F) \xrightarrow{\approx} \mathrm{Hom}_{\mathcal{C}}(C_*(-\backslash F), M)$$

*and of cohomology groups*

$$H^m(\mathcal{C}', M \circ F) \simeq H^m(\mathrm{Hom}_{\mathcal{C}}(C_*(-\backslash F), M)).$$

*Moreover if the functor $F$ is right $R$-cofinal then $F^* : H^*(\mathcal{C}'; M \circ F) \xrightarrow{\simeq} H^*(\mathcal{C}; M)$.*

*Proof.* It suffices to note that the functor $C_*(-\backslash F)$ is the left Kan extension of the functor $C_*(-\backslash\mathcal{C}')$ along $F$. The isomorphism of cohomology follows since, by Proposition 5.2, $C_*(-\backslash\mathcal{C}')$ provides a projective resolution that can be used to calculate higher limits. Note that $C_*(-\backslash F)$ as the left Kan extension of a projective object in $\mathcal{C}'$-$\mathsf{mod}$ is a projective object in $\mathcal{C}$-$\mathsf{mod}$, hence the last assertion follows. ■

Let $\mathcal{C}$ be an *ordered category*, i.e. a small category in which all endomorphisms are isomorphisms. Fix an object $d \in \mathrm{ob}\,\mathcal{C}$. We denote by $\mathcal{C} - [d]$ the full subcategory of $\mathcal{C}$ consisting of objects *not* isomorphic to $d$. Following Dwyer [Dw2] we say that $\mathcal{C} - [d]$ is obtained from $\mathcal{C}$ by *pruning* the object $d$. The aim of the section is to compare higher limits over $\mathcal{C}$ and $\mathcal{C}' := \mathcal{C} - [d]$ by expressing the relative cohomology groups $H^*(\mathcal{C}, \mathcal{C}'; M)$ as cohomology of the automorphism group $A(d)$ with coefficients in a local cochain complex at $d$.

For any object $d \in \mathcal{C}$ consider the forgetful functor $\mathcal{C}/d \to \mathcal{C}$. Its right fiber under an object $c \in \mathcal{C}$, denoted by $c\backslash\mathcal{C}/d$, is the category whose objects are pairs of morphisms $c \to d' \to d$ and morphisms those morphisms $d' \to d''$ which make the corresponding diagrams commute. Note that composition of morphisms defines an action of the group $A(d) := \mathrm{Mor}_{\mathcal{C}}(d, d)$ on $c\backslash\mathcal{C}/d$. Varying $c$ we obtain a functor from $\mathcal{C}$ to the category of small categories equipped with an $A(d)$-action.

We use the notation for semi-fibers introduced in Section 2. Recall that $\mathcal{C}//d$ (resp. $d\backslash\backslash\mathcal{C}$) denotes the subcategory of $\mathcal{C}/d$ (resp. $d\backslash\mathcal{C}$) consisting of the morphisms $c \to d$ (resp. $d \to c$) which are not isomorphisms. Thus $c\backslash\mathcal{C}//d \subset c\backslash\mathcal{C}/d$ consists of the sequences $c \to d' \to d$ such that $d' \to d$ is not an isomorphism. The following main theorem of the present section describes explicitly the effect of pruning on chain complex functors.

5.5. Theorem. *Let $\mathcal{C}'$ be the subcategory of an ordered category $\mathcal{C}$, obtained by pruning an object $d \in \mathcal{C}$. Then there exists a natural homotopy equivalence of functors defined on the category $\mathcal{C}$:*

$$\Phi : C_*(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d) \otimes_{A(d)} C_*(d\backslash\mathcal{C}, d\backslash\backslash\mathcal{C}) \xrightarrow{\simeq} C_*(-\backslash\mathcal{C}, -\backslash\mathcal{C}').$$

For the proof we can assume that $d$ is the only object in its isomorphism class. To construct a homomorphism $\Phi$ we need to replace the tensor product with some other homotopy equivalent chain complex. This is done in the next lemmas.

5.6. Lemma. (a) *For each* $p \geq 0$ *the functor* $C_p(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d)$ *is a projective* $\mathcal{C}$*-module.*

(b) *For every* $q \geq 0$ *the* $A(d)$*-module* $C_q(d\backslash\mathcal{C}, d\backslash\backslash\mathcal{C})$ *is free.*

*Proof.* (a) Indeed, as in Proposition 5.2, for any $\mathcal{C}$-module $M$ we have

$$\mathrm{Hom}_{\mathcal{C}}(C_p(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d), M) = \prod_{[c=c_0\to\ldots\to c_p=d\to d]} M(c).$$

(b) A free $A(d)$-basis consists of the elements of the form $[d \xrightarrow{\mathrm{id}} d = c_0 \to \ldots \to c_q]$. ■

Let us consider the set of morphisms $\mathrm{Mor}_{\mathcal{C}}(c,d)$ as a discrete category (i.e. with identity morphisms only) equipped with a right $A(d)$-action. Then composition of morphisms yields an $A(d)$-equivariant functor $\mu : c\backslash\mathcal{C}/d \to \mathrm{Mor}_{\mathcal{C}}(c,d)$ which is a homotopy equivalence. The homotopy inverse is induced by the functor which sends a morphism $c \to d$ to the pair $c \to d \xrightarrow{\mathrm{id}} d$. Varying $c$ we obtain a natural (non-equivariant) homotopy equivalence between functors into the category of small categories with $A(d)$-action.

Let $\iota(c)_* : C_*(c\backslash\mathcal{C}//d) \subset C_*(c\backslash\mathcal{C}/d)$ be the inclusion and let

$$\eta := \mu \circ \iota : C_*(c\backslash\mathcal{C}//d) \to \mathrm{Mor}(c,d).$$

We shall shorten notation by defining

$$C_* := C_*(d\backslash\mathcal{C}, d\backslash\backslash\mathcal{C}) \quad \text{and} \quad C'_*(-) := C_*(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d).$$

5.7. Lemma. *There are natural homotopy equivalences of functors on* $\mathcal{C}$:

$$\Delta_*(\eta) \otimes_{A(d)} C_* \overset{\simeq}{\leftarrow} \Delta_*(\iota) \otimes_{A(d)} C_* \overset{\simeq}{\rightarrow} C_*(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d) \otimes_{A(d)} C_*.$$

*Proof.* Observe first that the natural maps

$$\Delta_*(\eta) \overset{\simeq}{\leftarrow} \Delta_*(\iota) \overset{\simeq}{\rightarrow} C_*(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d)$$

preserve $A(d)$-actions and they are natural chain homotopy equivalences. To see this note that in the composition $\eta := \mu \circ \iota$ the first map is a natural chain equivalence and all the functors involved are projective $\mathcal{C}$-modules (Lemma 5.6(a)). The maps remain natural homotopy equivalences after tensoring with $C_*$. Indeed, by Lemma 5.6(b), $C_*$ is a free $A(d)$-module, thus tensoring over $A(d)$ with $C_*$ amounts to taking a direct sum of an appropriate number of the tensored module. And direct sums clearly preserve chain homotopy equivalences. ■

*Proof of Theorem 5.5.* To define a natural homotopy equivalence we first replace the chain complex $C_*(c\backslash\mathcal{C}/d, c\backslash\mathcal{C}//d) \otimes C_*$ by $\Delta_*(\eta) \otimes C_*$. Now we shall exhibit an isomorphism

$$\bigoplus_{p+q=n} \Delta_p(\eta) \otimes_{A(d)} C_q(d\backslash\mathcal{C}, d\backslash\backslash\mathcal{C}) \to C_n(-\backslash\mathcal{C}, -\backslash\mathcal{C}').$$

Recall that $\Delta_p(\eta) = C_{p-1}(c\backslash\mathcal{C}//d)$ for $p > 0$ and $C_0(\eta) = \mathbb{Z}[\mathrm{Mor}_{\mathcal{C}}(c,d)]$. Since $C_{p-1}(c\backslash\mathcal{C}//d)$ is the free abelian group generated by sequences $c \to c_0 \to \ldots \to c_{p-1} \xrightarrow{\not\simeq} d$, it follows (cf. Lemma 5.6(b)) that the tensor product $C_{p-1}(c\backslash\mathcal{C}/d) \otimes C_q$ is the free abelian group generated by the tensors $[c \to c_0 \to \ldots \to c_{p-1} \xrightarrow{\not\simeq} d] \otimes [d \xrightarrow{\mathrm{id}} d = c'_0 \to \ldots \to c'_q]$. To such a generator we assign the simplex $[c \to c_0 \to \ldots \to c_{p-1} \to d = c'_0 \to \ldots \to c'_q]$. The resulting homomorphism establishes a bijection of bases (over $\mathbb{Z}$) of the source and target groups. ∎

We shall express the relative cohomology $H^*(\mathcal{C}, \mathcal{C}'; M)$ as the hypercohomology of the group of endomorphisms $A(d)$ with coefficients in a certain chain-cochain complex depending on "neighborhoods" of $d$ in $\mathcal{C}$.

We define a cochain complex of $A(d)$-modules with coefficients in a contravariant functor $M : \mathcal{C} \to \mathsf{Ab}$ as follows:

$$L^*(d; M) := \mathrm{Hom}_{\mathcal{C}}(C'_*(-) \otimes C_*, M)) \simeq \mathrm{Hom}(C_*, D^*(\mathcal{C}, d; M))$$

where $D^*(\mathcal{C}, d; M) := \mathrm{Hom}_{\mathcal{C}}(C'_*(-), M)$.

5.8. Proposition. *Let $\mathcal{C}'$ be the subcategory of a category $\mathcal{C}$, obtained by pruning an object $d \in \mathcal{C}$, and $M : \mathcal{C} \to \mathsf{Ab}$ be a contravariant functor. Then there is a natural isomorphism*

$$H^*(\mathcal{C}, \mathcal{C}'; M) \simeq H^*(A(d); L^*(d; M)).$$

*Proof.* We have to calculate $H^*(\mathrm{Hom}_{\mathcal{C}}(C'_*(-) \otimes_{A(d)} C_*, M))$; the desired expression follows from the adjunction properties of tensor product. Indeed $\mathrm{Hom}_{\mathcal{C}}(C'_*(-) \otimes_{A(d)} C_*, M) \simeq \mathrm{Hom}_{A(d)}(C_*, \mathrm{Hom}_{\mathcal{C}}(C'_*(-), M))$. Let $C_*(EA(d)) = C_*(d\backslash d)$ be the chain complex of the universal contractible free $A(d)$-space. The augmentation homomorphism $C_*(EA(d)) \to \mathbb{Z}$ induces a weak equivalence $C_*(EA(d)) \otimes C_* \to C_*$. Since $C_*$ is a free $A(d)$-module it is a chain homotopy equivalence and therefore we have a cochain homotopy equivalence

$$\begin{aligned}&\mathrm{Hom}_{A(d)}(C_*, \mathrm{Hom}_{\mathcal{C}}(C'_*(-), M))\\ &\qquad\simeq \mathrm{Hom}_{A(d)}(C_*(EA(d)) \otimes C_*, \mathrm{Hom}_{\mathcal{C}}(C'_*(-), M));\end{aligned}$$

since the last cochain complex is isomorphic to

$$\mathrm{Hom}_{A(d)}(C_*(EA(d)), \mathrm{Hom}(C_*, \mathrm{Hom}_{\mathcal{C}}(C'_*(-), M)))$$

the assertion follows. ∎

We define several full subcategories of $\mathcal{C}$ playing the role of "neighborhoods of $d$ in $\mathcal{C}$". The *star-categories* are defined as full subcategories of $\mathcal{C}$:

$$\mathrm{st}^+_{\mathcal{C}}(d) := \{d' \in \mathcal{C} : \exists d \to d'\}, \qquad \mathrm{st}^-_{\mathcal{C}}(d) := \{d' \in \mathcal{C} : \exists d' \to d\},$$
$$\mathrm{st}_{\mathcal{C}}(d) := \mathrm{st}^+_{\mathcal{C}}(d) \cup \mathrm{st}^-_{\mathcal{C}}(d).$$

Analogously we define the *link-categories*

$$\mathrm{lk}_{\mathcal{C}}(d) := \mathrm{lk}^+_{\mathcal{C}}(d) \cup \mathrm{lk}^-_{\mathcal{C}}(d)$$

as full subcategories of the corresponding star-categories consisting of objects not isomorphic to $d$.

We shall see that the star-subcategories play a crucial role in comparing higher limits over $\mathcal{C}'$ and $\mathcal{C}$. Recall that we have defined the chain complex $D^*(\mathcal{C}/d, \mathcal{C}//d; M) := \mathrm{Hom}_{\mathcal{C}}(C_*(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d), M)$.

5.9. Proposition. *The restriction maps*

$$D^*(\mathcal{C}/d, \mathcal{C}//d; M) \to D^*(\mathrm{st}^-_{\mathcal{C}}(d)/d, \mathrm{lk}^-_{\mathcal{C}}(d)/d; M),$$
$$C_*(d\backslash\mathcal{C}, d\backslash\backslash\mathcal{C}) \to C_*(d\backslash\mathrm{st}^+_{\mathcal{C}}(d), d\backslash\mathrm{lk}^+_{\mathcal{C}}(d))$$

*are isomorphisms.*

*Proof.* The first map is clearly injective since $C_*(-\backslash\mathcal{C}/d, -\backslash\mathcal{C}//d)$ vanishes outside $\mathrm{st}_{\mathcal{C}}(d)$. It remains to observe that it is surjective, i.e. to check that the obvious extension of a natural transformation over $\mathrm{st}^-_{\mathcal{C}}(d)$ to objects in $\mathcal{C}$ is also a natural transformation. The assertion about the second restriction map is even more straightforward. ■

We conclude the section by expressing the "local cohomology" $H^*(A(d); L(d; M))$ in simpler terms if the object $d$ is either maximal (i.e. every morphism $d \to c$ is an isomorphism) or minimal (i.e. every morphism $c \to d$ is an isomorphism). This is a typical situation in many applications. Note that the element $d$ is minimal (resp. maximal) in $\mathrm{st}^+_{\mathcal{C}}(d)$ (resp. $\mathrm{st}^-_{\mathcal{C}}(d)$).

5.10. Corollary. *If $d = d_{\max}$ (resp. $d_{\min}$) is a maximal (resp. minimal) object in a category $\mathcal{C}$ then there are natural isomorphisms*

$$H^*(A(d_{\max}); L(d_{\max}; M)) \simeq H^*(A(d_{\max}); D^*(\mathcal{C}/d_{\max}, \mathcal{C}//d_{\max}; M)),$$
$$H^*(A(d_{\min}); L(d_{\min}; M))$$
$$\simeq H^*(A(d_{\min}); \mathrm{Hom}(C_*(d_{\min}\backslash\mathcal{C}, d_{\min}\backslash\backslash\mathcal{C}), M(d_{\min}))),$$
$$H^*(\mathcal{C}, \mathcal{C}\backslash[d_{\min}]; M) \simeq H^*(\mathrm{Hom}_{A(d_{\min})}(C_*, M(d_{\min}))).$$

*Proof.* If $d_{\max}$ is a maximal object then

$$C_*(d_{\max}\backslash\mathcal{C}, d_{\max}\backslash\backslash\mathcal{C}) \simeq C_*(d_{\max}\backslash\mathcal{C}) \simeq C_*(EA(d_{\max})),$$

hence by adjunction we obtain the first isomorphism.

Let now $d_{\min}$ be a minimal object. Then $\mathrm{st}^-_{\mathcal{C}}(d_{\min}) = \{d_{\min}\}$. Hence, again by adjunction, we obtain the second isomorphism. The last isomorphism follows directly from 5.6. ∎

As an application of the last proposition we shall calculate higher limits of atomic functors. A functor $M : \mathcal{C} \to \mathsf{Ab}$ is called *atomic*, *concentrated on* $c$ if $M(c') = 0$ for all objects not isomorphic to $c$. Note that atomic functors concentrated on $c$ can be identified with $A(c)$-modules.

Note that every functor $M : \mathcal{C} \to \mathsf{Ab}$ defined on an ordered category admits a filtration by subfunctors $0 = M_0 \subset M_1 \subset \ldots \subset M$ such that the subquotients $M_i/M_{i-1}$ are sums of atomic functors and moreover for every object $c \in \mathcal{C}$ one of the three terms of the short exact sequence

$$0 \to M_{i-1}(c) \to M_i(c) \to M_i(c)/M_{i-1}(c) \to 0$$

vanishes (cf. [JMO2]).

A subcategory $\mathcal{C}' \subset \mathcal{C}$ is called *upward saturated* if all morphisms in $\mathcal{C}$ originating from objects of $\mathcal{C}'$ belong to $\mathcal{C}'$. In particular $\mathcal{C}'$ is a full subcategory.

5.11. Proposition. *Let $\mathcal{C}' \subset \mathcal{C}$ be an upward saturated subcategory and $M : \mathcal{C} \to \mathsf{Ab}$ a functor such that $M(c) = 0$ for all $c \notin \mathcal{C}'$. Then the inclusion $\mathcal{C}' \subset \mathcal{C}$ induces an isomorphism $H^*(\mathcal{C}; M) \xrightarrow{\simeq} H^*(\mathcal{C}'; M)$.*

*Proof.* It follows from the definitions that $\mathrm{Hom}_{\mathcal{C}}(C_*(-\backslash\mathcal{C}, -\backslash\mathcal{C}'), M) = 0$, and thus the relative groups vanish. ∎

The object $d$ is minimal in $\mathrm{st}^+_{\mathcal{C}}(d)$ so the last proposition implies that for an atomic functor concentrated on $d$ the cohomology $H^*(\mathrm{st}^+_{\mathcal{C}}(d); M)$ can be calculated using Proposition 5.10.

Finally, we shall observe that Proposition 5.10 is closely related to a result of Oliver [O] who studied higher limits over *ordered categories with subobjects*. Since we consider contravariant functors whereas Oliver considered covariant functors, we give the dual definition of a *category with quotients.*

5.12. Definition. A *category with quotients* is a pair of ordered categories $\mathcal{D} \subset \mathcal{C}$ such that $\mathrm{ob}\,\mathcal{D} = \mathrm{ob}\,\mathcal{C}$ and

(a) $|\mathrm{Mor}_{\mathcal{D}}(x, y)| \le 1$ for any pair of objects,

(b) each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(x, y)$ can be written in a unique way as a composite $x \xrightarrow{p} y' \xrightarrow{a} y$ where $a \in \mathrm{Iso}_{\mathcal{C}}(y', y)$ and $p \in \mathrm{Mor}_{\mathcal{D}}(x, y')$.

Note that (a) & (b) implies that the category $\mathcal{D}$ is defined by a partially ordered set which we shall denote by $(\mathbf{D}, \le)$. Condition (b) implies that for every object $c \in \mathcal{D}$ the automorphism group $\mathrm{Mor}_{\mathcal{C}}(c, c)$ acts on $\mathbf{D}_{c\le} := \{c' \in \mathbf{D} : c \le c'\}$.

5.13. Remark. Observe that a category with quotients $(\mathcal{C}, \mathcal{D})$ is not only ordered but also all morphisms are epimorphisms. Indeed, it suffices to show that all morphisms which belong to $\mathcal{D}$ are epimorphisms. Suppose that $x \xrightarrow{p} y$ is in $\mathcal{D}$ and for two morphisms $y \xrightarrow{f_i} z$ we have $f_1 \circ p = f_2 \circ p$. According to (b) each $f_i$ decomposes into $y \xrightarrow{q_i} z_i \xrightarrow{a_i} z$ where $q_i \in \mathcal{D}$ and $a_i$ is an isomorphism, thus $a_1 \circ q_1 \circ p = a_2 \circ q_2 \circ p$. Since $p \in \mathcal{D}$ and the decomposition in (b) is unique we infer that $z_1 = z_2$, $q_1 \circ p = q_2 \circ p$ and $a_1 = a_2$. Since there are no more than one morphism between objects in $\mathcal{D}$ we conclude that $q_1 = q_2$, hence $f_1 = f_2$. ■

Now we are ready to show a generalization of Oliver's [O, Proposition 3].

5.14. Theorem. *If $d_{\min}$ is a minimal object in an ordered category with quotients $\mathcal{C} \supset \mathbf{D}$ then for any functor $M : \mathcal{C} \to \mathsf{Ab}$ there is a natural isomorphism*

$$H^*(A(d); L(d_{\min}; M)) \simeq H^*(EA \times_{A(d_{\min})} (|\mathbf{D}_{d_{\min}\le}|, |\mathbf{D}_{d_{\min}<}|); M(d_{\min})).$$

*Proof.* The isomorphism follows easily from 5.10. We have to observe that the complex $C_*(d_{\min}\backslash\mathcal{C}, d_{\min}\backslash\backslash\mathcal{C})$ of $A(d_{\min})$-modules is isomorphic to $C_*(|\mathbf{D}_{d_{\min}\le}|, |\mathbf{D}_{d_{\min}<}|)$. Indeed the condition (b) in Definition 5.12 implies that there is an equivalence of categories $d_{\min}\backslash\mathcal{C} \simeq d_{\min}\backslash\mathcal{D} \simeq \mathbf{D}_{d_{\min}\le}$ and by definition it preserves the $A(d_{\min})$-action. ■

**6. Higher limits on categories associated to isotropy presheaves.** Let $d : \mathbf{W} \to \mathbf{S}(G)$ be an isotropy presheaf on a $G$-poset $\mathbf{W}$, and $M : \mathcal{W}_d \to R$-mod a coefficient system on the associated category. We shall prove several results concerning the cohomology groups (higher limits) $H^*(\mathcal{W}_d; M)$.

Let $f : (\mathbf{W}, d_{\mathbf{W}}) \to (\mathbf{V}, d_{\mathbf{V}})$ be a morphism of isotropy presheaves. As we noticed in Theorem 3.4 fibers of the induced functor $f_* : \mathcal{W}_{d_{\mathbf{W}}} \to \mathcal{V}_{d_{\mathbf{V}}}$ can be expressed in terms of the fibers of the poset map $f$. This observation implies certain relations between higher limits on $\mathcal{W}_{d_{\mathbf{W}}}$ and $\mathcal{V}_{d_{\mathbf{V}}}$. In particular if $d_{\mathbf{W}} = d_{\mathbf{V}} \circ f$ and the right fibers $v\backslash f$ of the poset map are $R$-acyclic for all $v \in \mathbf{V}$ then for every coefficient system $N : \mathcal{V}_{d_{\mathbf{V}}} \to R$-mod Proposition 5.4 provides an isomorphism

$$f^* : H^*(\mathcal{V}_{d_{\mathbf{V}}}; N) \xrightarrow{\simeq} H^*(\mathcal{W}_{d_{\mathbf{W}}}; N \circ f). \tag{6.A}$$

Note that in general in Propositions 5.3 and 5.4 applied to the functor $f_*$ the fiber categories $f_*/-$ and $-\backslash f_*$ cannot be replaced by $(f/-)_{d_{\mathbf{W}}}$ and (if $d_{\mathbf{W}} = d_{\mathbf{V}} \circ f$) $-\backslash f$ respectively, since the equivalences of categories described in Proposition 3.4 are natural on the poset $\mathbf{V}$ but not on the category $\mathcal{V}_{d_{\mathbf{V}}}$.

In certain cases calculations of higher limits over $\mathcal{W}_d$ can be reduced to more familiar higher limits over orbit categories. Recall that for $\mathbf{V} \subset \mathbf{S}(G)$ we denote by $\mathcal{O}_G(\mathbf{V})$ the category of orbits $G/H$ such that $H \in \mathbf{V}$.

6.1. Proposition. *Let $d : \mathbf{W} \to \mathbf{S}(G)$ be an isotropy presheaf such that $d(\mathbf{W}) \subset \mathbf{V}$, where $\mathbf{V} \subset \mathbf{S}(G)$ is a $G$-subposet. Assume moreover that for any $H \in \mathbf{V}$ the right fiber $H\backslash d$ is $R$-acyclic. Then*

(a) *the induced map $|d| : |\mathbf{W}| \to |\mathbf{V}|$ is an $R$-equivalence,*

(b) *the induced functor $F_d : \mathcal{W}_d \to \mathcal{O}_G(\mathbf{V})$ is right $R$-cofinal,*

(c) *for every coefficient system $M : \mathcal{O}_G(\mathbf{V}) \to R$*-mod *the induced homomorphism $d^* : H^*(\mathcal{O}_G(\mathbf{V}); M) \to H^*(\mathcal{W}_d; M \circ F_d)$ is an isomorphism.*

*Proof.* Let $\iota : \mathbf{S}(G) \to \mathbf{S}(G)$ be the identity isotropy presheaf. Assertions (a)–(c) follow immediately from the isomorphism (6.A) applied to $d : (\mathbf{W}, d) \to (\mathbf{V}, \iota|\mathbf{V})$, interpreted as a map of isotropy presheaves. ■

In the next proposition we specify an important situation when the restriction homomorphism $H^*(\mathcal{W}_d; M) \to H^*(\mathcal{W}'_d; M)$ induced by an inclusion $\mathbf{W}' \subset \mathbf{W}$ is an isomorphism; the proposition extends to higher limits a result of Thévenaz and Webb (cf. 2.11).

6.2. Proposition. *Let $\iota : \mathbf{W}' \subset \mathbf{W}$ be a $G$-subposet such that for every $w \in \mathbf{W} \setminus \mathbf{W}'$ the right fiber $|\mathbf{W}_{w\le}|$ of the inclusion is finite-dimensional and the right semi-fiber $|\mathbf{W}_{w<}|$ is $R$-acyclic. Then*

(a) *the inclusion $|\iota| : |\mathbf{W}'| \subset |\mathbf{W}|$ is an $R$-equivalence,*

(b) *the inclusion functor $\iota : \mathcal{W}'_d \subset \mathcal{W}_d$ is right $R$-cofinal,*

(c) *for every coefficient system $M : \mathcal{W}_d \to R$*-mod *the induced homomorphism $\iota^* : H^*(\mathcal{W}_d; M) \to H^*(\mathcal{W}'_d; M)$ is an isomorphism.*

*Proof.* Thévenaz–Webb ([TW, Prop. 1.7], quoted as 2.11 above) constructed a chain $\mathbf{W}' = \mathbf{W}_n \subset \ldots \subset \mathbf{W}_0 = \mathbf{W}$ of $G$-subposets such that the right fibers of every inclusion $\mathbf{W}_i \subset \mathbf{W}_{i+1}$ are $R$-acyclic. (A similar argument can be applied to prove that all fibers $|\mathbf{W}_{w\le}|$ are $R$-acyclic.) Now assertions (a)–(b) follow from the isomorphism (6.A). ■

We shall establish a relation between higher limits over categories associated to different isotropy presheaves on the same $G$-poset. Let $d_1 \le d_2 : \mathbf{W} \to \mathbf{S}(G)$ be a pair of isotropy presheaves. Then we have a functor $i_* : \mathbf{W}_{d_1} \to \mathbf{W}_{d_2}$. According to Corollary 3.4 for every $w \in \mathbf{W}$ the right fiber $i_*/w$ is equivalent to the category defined by the group $d_2(w)/d_1(w)$. Moreover the equivalence is natural up to inner automorphisms on $\mathcal{W}_{d_2}$. Thus for a coefficient system $M : \mathcal{W}_{d_1} \to \mathsf{Ab}$ the Leray spectral sequence 5.3 has the form

$$H^m(\mathcal{W}_{d_2}; H^n(d_2(-)/d_1(-); M(-))) \Rightarrow H^{n+m}(\mathcal{W}_{d_1}; M) \tag{6.B}$$

where the coefficient system in the $E_2$-term is given by assigning to $w \in \mathcal{W}_{d_2}$ the group cohomology $H^*(d_2(w)/d_1(w); M(w))$; note that since $M$ has been defined on $\mathcal{W}_{d_1}$, for every $w \in \mathbf{W}$ the group $M(w)$ is a $G_w/d_1(w)$-module. The spectral sequence is clearly natural with respect to morphisms of isotropy presheaves. If $d_1 = e$ is a minimal isotropy presheaf then the spectral sequence is convergent to the cohomology of the poset $\mathbf{W}$.

If $\mathbf{W}$ is a regular $G$-poset (cf. 4.7 and 4.8), then for $d_2 = d_{\max}$ and an arbitrary isotropy presheaf $d : \mathbf{W} \to \mathsf{Ab}$ we have a spectral sequence

$$(6.C) \qquad H^m(\mathbf{W}/G; H^n(G_-/d(-); M(-))) \Rightarrow H^{n+m}(\mathcal{W}_d; M).$$

Here the $E_2$-term is the cohomology of the quotient poset with coefficients in the simplicial presheaf, defined by the functor on the poset $\mathbf{W}/G$.

Recall that for any $G$-poset $\mathbf{W}$ satisfying condition (EI) its barycentric subdivision $\mathrm{sd}(\mathbf{W})$ is regular (cf. 4.9). The projection $p_0 : \mathrm{sd}(\mathbf{W}) \to \mathbf{W}$ is right cofinal. The composition $d_0 := d \circ p_0$ is an isotropy presheaf on $\mathrm{sd}(\mathbf{W})$, hence for any $M : \mathcal{W}_d \to \mathsf{Ab}$ we have an isomorphism $p_0^* : H^*(\mathcal{W}_d; M) \xrightarrow{\simeq} H^*(\mathrm{sd}(\mathbf{W})_{d_0}; M \circ p_0)$. Therefore applying (6.C) to $(\mathrm{sd}(\mathbf{W}), d_0)$ we obtain an important spectral sequence

$$(6.D) \qquad H^m(\mathrm{sd}(\mathbf{W})/G; H^n(G_-/d_0(-); M(-))) \Rightarrow H^{n+m}(\mathcal{W}_d; M).$$

It turns out that under certain assumptions (which are often satisfied by various important posets of subgroups) the $E_2$-term of the spectral sequence reduces to the first row. Thus $H^*(\mathcal{W}_d; M)$ almost does not depend on the particular choice of the isotropy presheaf $d$ (cf. Theorem 6.6)! This result should be compared with (4.1) where we noticed the homotopy equivalence $\mathrm{hocolim}_{\mathcal{W}_d} F_d \xrightarrow{\simeq} |\mathbf{W}|$; thus the homotopy type of the homotopy colimit associated to an isotropy presheaf $d$ does not depend on $d$.

For the proof of Theorem 6.6 we shall need an interpretation of higher limits over $\mathcal{W}_d$ as equivariant Bredon cohomology of certain $G$-spaces. Recall that the *Bredon cohomology* of a $G$-space with coefficients in a contravariant functor $N : \mathcal{O}_G \to \mathsf{Ab}$ is defined as the cohomology of the cochain complex $\mathrm{Hom}_{\mathcal{O}_G}(C_*(X), N)$ where $C_*(X)(G/H) := C_*(X^H)$ and $C_*(-)$ is the singular (or simplicial) chain complex functor (cf. [Br1]).

6.3. Lemma. *Let $\mathcal{C}$ be an arbitrary small category equipped with a functor $F : \mathcal{C} \to \mathcal{O}_G$. Then for any coefficient system $N : \mathcal{O}_G \to \mathsf{Ab}$ there is a natural isomorphism*

$$H^*(\mathcal{C}; N \circ F) \simeq H^*_G(B(G\backslash F); N).$$

*($G\backslash F$ denotes the right fiber of $F$ under the free $G$-orbit equipped with the left $G$-action.)*

*Proof.* We have an isomorphism of the corresponding chain complexes provided by the obvious adjointness property:

$$\mathrm{Hom}_{\mathcal{C}}(C_*(-\backslash\mathcal{C}), N \circ F) \simeq \mathrm{Hom}_{\mathcal{O}_G}(C_*(-\backslash F), N).$$

Now it suffices to notice the natural equivalence of functors on the orbit category: $(G/H)\backslash F \simeq (G\backslash F)^H$. ■

6.4. Corollary. *Let $d : \mathbf{W} \to \mathbf{S}(G)$ be an isotropy presheaf on a $G$-poset $\mathbf{W}$. Then for every coefficient system $N : \mathcal{O}_G \to \mathsf{Ab}$ there is a natural isomorphism*

$$H^*(\mathcal{W}_d; N \circ F_d) \simeq H^*_G(\operatorname{hocolim}_{\mathcal{W}_d} F_d; N).$$

*In particular for a regular $G$-poset $\mathbf{W}$ there is a natural isomorphism*

$$H^*(\mathcal{W}_{\mathrm{iso}}; N \circ F_{\mathrm{iso}}) \simeq H^*_G(|\mathbf{W}|; N).$$

*Proof.* The functor $F_d : \mathcal{W}_d \to \mathcal{O}_G$ was defined in Section 3. A direct inspection exhibits an isomorphism $G\backslash F_d \simeq (\mathcal{W}_d)_{F_d}$ of categories equipped with $G$-action, and by definition $\operatorname{hocolim}_{\mathcal{W}_d} F_d$ coincides with the (geometric realization of) the nerve of $(\mathcal{W}_d)_{F_d}$ (cf. Sec. 2). For a regular $G$-poset the map $\operatorname{hocolim}_{\mathbf{W}_{\mathrm{iso}}} F_{\mathrm{iso}} \to |\mathbf{W}|$ (cf. 4.11) is a $G$-homotopy equivalence, thus the second isomorphism follows. ■

The interpretation of higher limits as equivariant cohomology turns out to be often more flexible and geometrically more intuitive (cf. [JMO1]). Following [Sł3] we shall describe briefly a construction of the transfer map in Bredon cohomology which will be used to prove the vanishing of certain higher limits in the next theorem. Another, more categorical approach to transfers was described in [J].

Let $G$ be a finite group and $M : \mathcal{O}_G \to \mathsf{Ab}$ be a coefficient system which extends to an additive functor on the Hecke category $\mathbb{Z}\mathcal{S}_G$ of finite $G$-sets. We call $M$ a *Hecke functor* (cf. [Wa, Def. 2.1]); in particular $M$ is a Mackey functor as defined in [JM]. Recall that a morphism $S \to T$ in $\mathbb{Z}\mathcal{S}_G$ is a $G$-homomorphism $\mathbb{Z}S \to \mathbb{Z}T$ of the free abelian groups generated by $S$ and $T$ respectively. For any $G$-space $X$, a Hecke functor $M$ and an orbit $G/K$ one defines a transfer map $\mathrm{tr} : H^*_K(X; M) \to H^*_G(X; M)$ as the composition

$$H^*_K(X; M) \simeq H^*_G(X; M(G/K \times -)) \to H^*_G(X; M)$$

where the first isomorphism is induced by the adjunction isomorphism of the corresponding chain complexes and the second map is induced by the fundamental transfer homomorphism $\mathbb{Z}[S] \to \mathbb{Z}[G/K \times S]$ defined for any $G$-set $S$. The restriction homomorphism $\mathrm{res} : H^*_G(X; M) \to H^*_K(X; M)$ is induced simply by the projection $G/K \times S \to S$. As usual we need to understand the composition of the restriction and the transfer.

6.5. Lemma. *For any Hecke coefficient system $M$ the composition*

$$H^*_G(X; M) \xrightarrow{\mathrm{res}} H^*_K(X; M) \xrightarrow{\mathrm{tr}} H^*_G(X; M)$$

*is multiplication by the index $|G : K|$.*

*Proof.* The composition $\mathrm{tr} \circ \mathrm{res}$ is induced by the natural transformation $\mathbb{Z}[S] \xrightarrow{\mathrm{tr}} \mathbb{Z}[G/K \times S] \xrightarrow{\mathrm{pr}} \mathbb{Z}[S]$ of coefficient systems, which is clearly multiplication by $|G : K|$. ■

Group cohomology is an important example of the Hecke functor. More precisely for an arbitrary $G$-module $A$ consider the functor $H^*_{[G]}(-;A) : \mathcal{O}_G \to \mathsf{Ab}$ defined as $H^*_{[G]}(S;A) := H^*(EG \times_G S; A)$. By the Shapiro lemma $H^*_{[G]}$ easily extends to a Hecke functor.

6.6. THEOREM. *Let $d : \mathbf{W} \to \mathbf{S}(G)$ be an isotropy presheaf defined on a $G$-poset $\mathbf{W}$ of finite length. Assume moreover that for every $w \in \mathbf{W}$ the group $A(w) := G_w/d(w)$ is finite and that for every non-trivial $p$-subgroup $P \subset A(w)$, $H^*(|\mathbf{W}_{w\le}|^P, |\mathbf{W}_{w<}|^P; \mathbb{F}_p) = 0$. Then for any coefficient system $M : \mathcal{W}_d \to \mathbb{F}_p$-mod in the spectral sequence (6.D), $E_2^{p,q} = 0$ for $q > 0$, hence there is a natural isomorphism*

$$H^*(\mathrm{sd}(\mathbf{W})/G; \overline{M}^0) \xrightarrow{\cong} H^*(\mathcal{W}_d; M)$$

*where for each $w_. = (w_0, \ldots, w_n)$, $\overline{M}^0(w_.) := H^0(G_{w_.}; M(w_0)) = M(w_0)^{G_{w_.}}$ and $G_{w_.} = G_{w_0} \cap G_{w_1} \cap \ldots \cap G_{w_n}$.*

*Proof.* Note that a $G$-poset of finite length must satisfy condition (EI) (cf. Ex. 4.9). Thus its barycentric subdivision $\mathrm{sd}(\mathbf{W})$ is regular (4.10) and we can consider the spectral sequence (6.D). We have to prove that for $n > 0$, $H^m(\mathrm{sd}(\mathbf{W})_{\mathrm{iso}}; \overline{M}^n) = 0$, where $\overline{M}^n(w_.) := H^n(G_{w_.}/d(w_0), M(w_0))$. We begin by proving it in the case when $M$ is an atomic functor on $\mathcal{W}_d$ concentrated on an object $w \in \mathcal{W}_d$. We write $A := A(w)$ for short. We have the following isomorphisms in cohomology:

$$\begin{aligned} H^*(\mathrm{sd}(\mathbf{W})_{\mathrm{iso}}; \overline{M}^n) &\overset{(1)}{\to} H^*(\mathrm{sd}(G\mathbf{W}_{w\le})_{\mathrm{iso}}; \overline{M}^n) \\ &\overset{(2)}{\to} H^*(\mathrm{sd}(G\mathbf{W}_{w\le}, G\mathbf{W}_{w<})_{\mathrm{iso}}; \overline{M}^n) \\ &\overset{(3)}{\to} H^*((G \times_{G_w} \mathrm{sd}(\mathbf{W}_{w\le}, \mathbf{W}_{w<}))_{\mathrm{iso}}; \overline{M}^n) \\ &\overset{(4)}{\to} H^*(\mathrm{sd}(\mathbf{W}_{w\le}, \mathbf{W}_{w<})_{\mathrm{iso}}; \overline{M}^n|A) \\ &\overset{(5)}{\to} H^*(\mathrm{sd}(\mathbf{W}_{w\le}, \mathbf{W}_{w<})_{\mathrm{iso}}; \widetilde{M}^n). \end{aligned}$$

The isomorphism (1) is induced by the inclusion of $G$-posets $G\mathbf{W}_{w\le} \subset \mathbf{W}$ ($G\mathbf{W}_{w\le}$ denotes the union of $G$-orbits of points from $\mathbf{W}_{w<}$) and follows from Proposition 5.11. The isomorphism (2) follows from the exact sequence of a pair (cf. Sec. 5) since $\overline{M}^n$ vanishes on the subcategory $\mathrm{sd}(G\mathbf{W}_{w<})_{\mathrm{iso}}$. The isomorphism (3) is induced by the canonical map $\alpha : G \times_{G_w} \mathrm{sd}(\mathbf{W}_{w\le}, \mathbf{W}_{w<}) \to (G\mathbf{W}_{w\le}, G\mathbf{W}_{w<})$ of $G$-posets since it induces a bijection on the corresponding chain complexes.

For (4) note that $(\mathbf{W}_{w\le}, \mathbf{W}_{w<})$ is a pair of $A$-posets. Indeed, the subgroup $d(w)$ acts trivially on every element $w' > w$ since from the definition of an isotropy presheaf we obtain $d(w) \subset d(w') \subset G_{w'}$. Moreover for any subgroup $H \subset G$, a regular $H$-poset $\mathbf{V}$ and a coefficient system $N : (G\times_H \mathbf{V})_{\mathrm{iso}} \to \mathsf{Ab}$ we have an obvious isomorphism $H^*((G\times_H \mathbf{V})_{\mathrm{iso}}; N) \simeq H^*(\mathcal{V}_{\mathrm{iso}}; N|H)$ where $N|H$ is the restriction of $N$ to $\mathcal{V}_{\mathrm{iso}} \subset (G\times_H \mathbf{V})_{\mathrm{iso}}$. Note that the system $\overline{M}^n|A$ is given by $\overline{M}^n|A(w_.) = H^n(G_{w_.}/d(w_0); M(w_0)) = H^n(A_{w_.}/(d(w_0)/d(w)); M(w_0))$.

In the last cohomology group (the target of (5)) the coefficient system is defined as $\widetilde{M}^n(w_.) = H^n(A_{w_.}; M(w))$ and the isomorphism (5) is given by the morphism of the coefficient systems induced by the homomorphisms $A_{w_.} \to A_{w_.}/(d(w_0)/d(w))$. (Note that the morphism is an isomorphism outside $\mathrm{sd}(\mathbf{W}_{w<})$.)

We shall use the transfer to prove that $H^*(\mathrm{sd}(\mathbf{W}_{w\le}, \mathbf{W}_{w<})_{\mathrm{iso}}; \widetilde{M}^n) = 0$ for $n > 0$. It is convenient to identify the latter groups with equivariant Bredon cohomology $H^*_A(\mathrm{sd}(\mathbf{W}_{w\le}, \mathbf{W}_{w<}); \widetilde{M}^n)$ with coefficients in the coefficient system $\widetilde{M}^n : \mathcal{O}_A \to \mathbb{F}_p$-mod given by

$$\widetilde{M}^n(A/K) := H^n(EA\times_A A/K; M(w)).$$

Since Bredon equivariant cohomology is a topological invariant we shall denote it by $H^*_A(|\mathbf{W}_{w\le}|, |\mathbf{W}_{w<}|; \widetilde{M}^n)$ for short. Let $P \subset A$ be a Sylow $p$-subgroup. According to Lemma 6.5 and the remarks following it the restriction homomorphism $H^*_A(|\mathbf{W}_{w\le}|, |\mathbf{W}_{w<}|; \widetilde{M}^n) \to H^*_P(|\mathbf{W}_{w\le}|, |\mathbf{W}_{w<}|; \widetilde{M}^n)$ is a monomorphism.

For an arbitrary $G$-space $X$ we denote by $\mathrm{Sing}(X)$ the union of fixed point sets of all non-trivial subgroups. Thus we have an isomorphism

$$H^*_P(|\mathbf{W}_{w\le}|, |\mathbf{W}_{w<}|; \widetilde{M}^n) \xrightarrow{\simeq} H^*_P(\mathrm{Sing}(|\mathbf{W}_{w\le}|, |\mathbf{W}_{w<}|); \widetilde{M}^n) = 0.$$

Indeed restriction to the singular set is an isomorphism since on the free orbit $\widetilde{M}^n(P/e) = 0$ for $n > 0$. The target group vanishes by general properties of equivariant cohomology since we have assumed that the relative cohomology of the fixed point sets of all non-trivial subgroups of $P$ vanishes (cf. [JMO1, Lemma A.10] and [We]).

Now we shall pass from atomic functors to arbitrary coefficient systems $M : \mathcal{W}_d \to \mathbb{F}_p$-mod. There exists a filtration $0 = M_0 \subset M_1 \subset \ldots \subset M_n = M$ such that for every $i > 0$ the quotient functor $M_i/M_{i-1}$ is a direct sum of atomic functors and in the short exact sequence $0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0$ one of the three terms vanishes. This guarantees that the induced sequence of functors $0 \to \overline{M}^n_{i-1} \to \overline{M}^n_i \to \overline{M}^n_i/\overline{M}^n_{i-1} \to 0$ is also exact, hence $H^*(\mathrm{sd}(\mathbf{W})_{\mathrm{iso}}; \overline{M}^n) = 0$ for an arbitrary functor $M$ and $n > 0$. ∎

6.7. Corollary. *Let $d : \mathbf{W} \to \mathbf{S}(G)$ be an isotropy presheaf satisfying the assumptions of Theorem* 6.6. *Then $H^q(\mathcal{W}_d; M) = 0$ for all $q$ greater than the length of the poset* $\mathbf{W}$ (*thus independent of* $d$). ∎

The last corollary generalizes some vanishing results proved in [JMO1] and [O]. A result closely related to 6.6 in the special case of some orbit categories has recently been proved by J. Grodal [G].

**7. Posets of subgroups.** In this section we specialize our considerations to posets of subgroups equipped with the conjugation action; note that the isotropy group of a subgroup is its normalizer. Let $p$ be a fixed prime. For a finite group $G$ recall the following important examples of $G$-subposets of $\mathbf{S}(G)$:

- $\mathbf{D}_p(G)$ — subgroups of order divisible by $p$,
- $\mathbf{N}_p(G)$ — subgroups which contain a non-trivial normal $p$-subgroup,
- $\mathbf{Z}_p(G)$ — subgroups $H \subset G$ such that $p$ divides the order $|Z(H)|$ of the center,
- $\mathbf{S}_p(G)$ — $p$-subgroups,
- $\widetilde{\mathbf{S}}_p(G)$ — non-trivial $p$-subgroups,
- $\mathbf{R}_p(G)$ — $p$-stubborn subgroups (i.e. $p$-subgroups $P \subset G$ such that $P$ is the maximal normal $p$-subgroup in its normalizer $N_G(P)$),
- $\mathbf{B}_p(G)$ — non-trivial $p$-stubborn subgroups (Bouc's collection),
- $\mathbf{C}_p(G)$ — centric $p$-subgroups (i.e. $P \subset G$ such that $C_G(P) \simeq Z(P) \times Q$ where $(p, |Q|) = 1$,
- $\mathbf{A}_p(G)$ — non-trivial elementary abelian $p$-subgroups,
- $\mathbf{A}_p^{\mathrm{dist}}(G)$ — distinguished, non-trivial elementary abelian $p$-subgroups (i.e. those which are maximal elementary abelian $p$-subgroups of their centralizers; cf. [Be]).

There are three important isotropy presheaves on an arbitrary $G$-subposet $\mathbf{W} \subset \mathbf{S}(G)$ which have been studied by many authors:

- (sub) the *subgroup presheaf* $\iota_G : \mathbf{W} \subset \mathbf{S}(G)$,
- (norm) the *normalizer presheaf* $n_G : \mathrm{sd}(\mathbf{W}) \to \mathbf{S}(G)$; it is the maximal isotropy function on the barycentric subdivision of $\mathbf{W}$, given by the intersections of the normalizers: $n_G(H_0, \ldots, H_q) := NH_0 \cap \ldots \cap NH_q$.
- (cen) the *centralizer presheaf* $c_G : \mathbf{W}^{\mathrm{op}} \to \mathbf{S}(G)$, $c_G(K) := C_G(K)$.

Other examples can be produced by replacing subgroups, centralizers and normalizers with other group-theoretical constructions.

The following lemma due to Quillen [Q2] and Thévenaz–Webb [TW] is crucial for comparing homotopy types of $G$-posets and higher limits over associated categories:

7.1. Lemma (Quillen, Thévenaz–Webb). *Let $P \in \widetilde{\mathbf{S}}_p(G)$. Then*

(a) *if $G$ contains a normal non-trivial $p$-subgroup then $\widetilde{\mathbf{S}}_p(G)$ is $G$-contractible*,

(b) *$\mathbf{S}_p(G)_{P<}$ is $N_G(P)$-homotopy equivalent to $\widetilde{\mathbf{S}}_p(N_G(P)/P)$*,

(c) $\mathbf{S}_p(G)_{P<}$ *is* $N_G(P)$*-contractible if and only if* $P \notin \mathbf{B}_p(G)$,

(d) $\widetilde{\mathbf{S}}_p(G)_{<P}$ *is* $N_G(P)$*-contractible if and only if* $P \notin \mathbf{A}_p(G)$. ∎

Lemma 7.1 combined with Theorem 3.4 gives a variety of cofinality results for functors induced by morphisms of posets equipped with isotropy presheaves. Before we state the next theorem recall that a subposet $\mathbf{W}' \subset \mathbf{W}$ is called *upward saturated* if for each $w' \in \mathbf{W}'$ every element $w \geq w'$ also belongs to $\mathbf{W}'$.

7.2. Theorem. (a) *Suppose that* $\mathbf{W} \subset \mathbf{S}_p(G)$ *is an upward saturated* $G$*-subposet. Let* $\mathbf{W}' \subset \mathbf{W}$ *be a* $G$*-subposet such that* $\mathbf{W} \cap \mathbf{R}_p(G) \subset \mathbf{W}'$ *and let* $d : \mathbf{W} \to \mathbf{S}(G)$ *be an arbitrary isotropy presheaf. Then the inclusion functor* $I_* : \mathcal{W}'_d \subset \mathcal{W}_d$ *is right cofinal.*

(b) *Let* $\mathbf{A}_p(G) \subset \mathbf{W} \subset \mathbf{N}_p(G)$ *and let* $d : \mathbf{W}^{\mathrm{op}} \to \mathbf{S}(G)$ *be an isotropy presheaf. Then the inclusion functor* $I_* : \mathcal{A}_p(G)_d^{\mathrm{op}} \subset \mathbf{W}_d^{\mathrm{op}}$ *is right cofinal.*

*Proof.* (a) We have to prove that for every $P \in \mathbf{W}$ the right fiber $P\backslash I_*$ is contractible. From Theorem 3.4 we obtain an equivalence of categories $P\backslash I_* \simeq P\backslash I \simeq \mathbf{W}'_{P\leq}$. Since we have assumed that $\mathbf{W} \subset \mathbf{S}_p(G)$ is upward saturated and $\mathbf{W} \cap \mathbf{R}_p(G) \subset \mathbf{W}'$ we obtain the inclusions $\mathbf{R}_p(G)_{P\leq} = \mathbf{W} \cap \mathbf{R}_p(G)_{P\leq} \subset \mathbf{W}'_{P\leq}$. For every $P \in \mathbf{W}'$ the fiber $\mathbf{W}'_{P\leq}$ is obviously contractible. For each $P \in \mathbf{W}\backslash\mathbf{W}'$ Lemma 7.1(c) implies that $\mathbf{R}_p(G)_{P\leq} = \mathbf{R}_p(G)_{P<}$ is contractible and the inclusion $\mathbf{R}_p(G)_{P\leq} \subset \mathbf{W}'_{P\leq}$ is right cofinal.

(b) Let $H \in \mathbf{N}_p(G)$. Then $H\backslash I_* \simeq H\backslash I \simeq \{E \subset H \mid E \in \mathbf{A}_p(G)\} = \mathbf{A}_p(H) \simeq *$, by Lemma 7.1(a). ∎

Note that the last theorem generalizes the well known homotopy equivalences of posets of subgroups.

The next result reduces in some cases calculations of higher limits over all non-trivial elementary abelian subgroups to distinguished subgroups (cf. definition at the beginning of the section). Note that the inclusion $I : \mathbf{A}_p^{\mathrm{dist}}(G) \subset \mathbf{A}_p(G)$ admits a retraction $r : \mathbf{A}_p(G) \to \mathbf{A}_p^{\mathrm{dist}}(G)$ given by $r(E) := Z_pC_G(E)$, where $Z_p(H)$ denotes the maximal elementary abelian $p$-subgroup of the center of $H$.

7.3. Proposition. *For any isotropy presheaf* $d : \mathbf{A}_p^{\mathrm{dist}}(G)^{\mathrm{op}} \to \mathbf{S}(G)$ *the induced functor* $r_* : \mathcal{A}_p(G)_{d'}^{\mathrm{op}} \to \mathcal{A}_p^{\mathrm{dist}}(G)_d^{\mathrm{op}}$, *where* $d' := d \circ r$, *is right cofinal.*

*Proof.* For every distinguished elementary abelian $p$-subgroup $E$ we have the equivalences $E\backslash r_* \simeq E\backslash r \simeq \{E' \in \mathbf{A}_p(G) \mid E \supset Z_pC_G(E')\} \simeq \mathbf{A}_p(E) \simeq *$. ∎

Note that the last proposition does not imply that the inclusion of categories $I : \mathcal{A}_p^{\mathrm{dist}}(G)_d \subset \mathcal{A}_p(G)_d$ is cofinal. However, applied to the minimal isotropy presheaf, together with Theorem 4.1, it implies that $I : \mathbf{A}_p^{\mathrm{dist}}(G) \subset \mathbf{A}_p(G)$ is a homotopy equivalence (cf. [Be]).

The next consequence of 7.1 is a sort of duality theorem. Note that the centralizer presheaf $c_G : \mathbf{A}_p(G)^{\mathrm{op}} \to \mathbf{S}(G)$ has values in the subposet $\mathbf{Z}_p(G) \subset \mathbf{S}(G)$, since for every non-trivial elementary abelian $p$-subgroup $E$ obviously $E \subset Z(C_G(E))$.

7.4. Proposition. *For any isotropy presheaf $d : \mathbf{Z}_p(G) \to \mathbf{S}(G)$ the induced functor $c_{G*} : \mathcal{A}_p(G)^{\mathrm{op}}_{d\circ c} \to \mathcal{Z}_p(G)_d$ is right cofinal.*

*Proof.* For every object $H \in \mathcal{Z}_p(G)_d$ we have $H\backslash c_* \simeq H\backslash c = \{E \in \mathbf{A}_p(G) \mid H \subset C_G(E)\} = \mathbf{A}_p(C_G(H))$ and $Z_p(H)$ is a normal subgroup of the centralizer, thus by 7.1(a), $H\backslash c$ is contractible. ∎

Observe that up to now we have just been comparing the homotopy properties of various posets and associated categories. The rest of the section will be devoted to showing that many interesting posets are ample with respect to a cohomology theory associated to an $\mathbb{F}_p[G]$-module.

We recall the definition of functors $\Lambda^*$ [JMO1]. Let $G$ be finite group and let $M$ be an $\mathbb{F}_p[G]$-module. The module $M$ defines an atomic functor on the orbit category $\mathcal{O}_p(G) := \mathbf{S}_p(G)_{\mathrm{id}}$ given by $\widetilde{M}_e(G/e) = M$. In [JMO1] the authors defined the functor

$$\Lambda^*(G; M) := H^*(\mathcal{O}_p(G); M_e).$$

These functors play an important role in calculating higher limits over orbit categories due to the following

7.5. Theorem. (a) ([JMO1, Lemma 5.4]) *For any atomic functor $F : \mathcal{O}_p(G) \to \mathbb{F}_p$-mod concentrated on the orbit $G/P$,*

$$H^*(\mathcal{O}_p(G); F) \simeq \Lambda^*(NP/P; F(G/P)).$$

(b) ([JMO1, Thm. 5.5]) *If $M$ is an $\mathbb{F}_p[G]$-module such that $\ker\{G \to \mathrm{Aut}(M)\}$ is divisible by $p$ then $\Lambda^*(G; M) = 0$.* ∎

If an $\mathbb{F}_p[G]$-module satisfies the assumption of (b) then we call it *$p$-non-faithful*; an important example is the trivial $G$-module $\mathbb{F}_p$ if $p$ divides $|G|$.

For a $G$-module $M$ we define an equivariant Borel cohomology theory

$$H^*_{[G]}(X; M) := H^*(EG \times_G X; \pi^* M)$$

where $\pi : EG \times_G X \to BG$. Recall that a $G$-poset $|\mathbf{W}|$ is *$M$-ample* if $\pi^* : H^*(BG; M) \xrightarrow{\cong} H^*(EG \times_G |\mathbf{W}|; \pi^* M)$ is an isomorphism.

As an application of general considerations in the earlier sections we shall give a proof of a theorem of Dwyer [Dw2]:

7.6. Theorem. *Let $M$ be a $p$-non-faithful $\mathbb{F}_p[G]$-module. Then the $G$-poset $\widetilde{\mathbf{S}}_p(G)$ equipped with the subgroup isotropy presheaf $\iota : \widetilde{\mathbf{S}}_p(G) \subset \mathbf{S}(G)$ is $M$-ample and sharp. Moreover if every cyclic subgroup of order $p$ acts* *trivially on $M$ then any $G$-subposet $\widetilde{\mathbf{S}}_p(G) \subset \mathbf{W} \subset \mathbf{D}_p(G)$ equipped with the subgroup isotropy presheaf is $M$-ample and sharp.*

*Proof.* The poset $\mathbf{S}_p(G)$ of all $p$-subgroups contains the smallest element (trivial subgroup), thus it is contractible and by [J], [Mi], [JM] the associated subgroup decomposition is sharp. Thus to prove that the subgroup isotropy presheaf $\widetilde{\mathbf{S}}_p(G) \subset \mathbf{S}(G)$ is $M$-ample and sharp it suffices (cf. 5.8, 5.14) to show that $H^*(EG \times_G (|\mathbf{S}_p(G)|, |\widetilde{\mathbf{S}}_p(G)|); M) = 0$. By 5.14 and 5.11 the latter group can be identified with $\Lambda^*(G; M)$ defined above and it vanishes by 7.5. The last assertion of the theorem follows from 4.3 and the observation that $\mathrm{En}_{H^*(-;M)}(\widetilde{\mathbf{S}}_p(G), \mathrm{id}) = \mathbf{D}_p(G)$. ∎

The last theorem together with 7.2 implies easily that the posets $\mathbf{R}_p(G)$, $\mathbf{B}_p(G)$ are $M$-ample. Moreover we prove another result of Dwyer's:

7.7. Corollary. *Let $\mathbf{CR}_p(G) := \mathbf{C}_p(G) \cap \mathbf{R}_p(G) \subset \mathbf{S}_p(G)$ be the $G$-subposet consisting of the $p$-subgroups which are both centric and stubborn. Then the subgroup isotropy presheaf $\iota : \mathbf{CR}_p(G) \subset \mathbf{S}(G)$ is $\mathbb{F}_p$-ample and sharp.*

*Proof.* Note that the poset of $p$-centric subgroups is upward saturated, thus Theorem 7.2(a) implies that for every isotropy presheaf $d : \mathbf{C}_p(G) \to \mathbf{S}(G)$ the inclusion functor $\mathcal{CR}_p(G)_d \subset \mathcal{C}_p(G)_d$ is right cofinal. It remains to prove that the restriction map $H^*(\mathcal{O}_p(G); H) \to H^*(\mathcal{C}_p(G); H)$ is an isomorphism for $H(G/P) := H^*(EG \times_G G/P; \mathbb{F}_p)$. The functor $H$ can be filtered by a sequence of functors with atomic quotients (cf. discussion following 5.10), thus it is enough to observe that $\Lambda^*(N(P)/P; H(G/P)) = 0$ for every $P \notin \mathbf{C}_p(G)$. Indeed, if $P$ is not centric then $PC(P)/P \subset N(P)/P$ is a non-trivial subgroup of order divisible by $p$ acting trivially on $H(G/P)$, hence an application of 7.5 concludes the proof. ∎

We conclude the section with a result concerning the centralizer isotropy presheaf.

7.8. Corollary. *Let $G$ be a group of order divisible by $p$ and let $M$ be an $\mathbb{F}_p[G]$-module such that every cyclic subgroup of order $p$ acts trivially on $M$. Any $G$-poset $\mathbf{W}$ satisfying $\mathbf{A}_p(G) \subset \mathbf{W} \subset \mathbf{D}_p(G)$ is $M$-ample. If moreover $\mathbf{W} \subset \mathbf{N}_p(G)$ then $\mathbf{W}$ equipped with the centralizer isotropy presheaf $c_G : \mathbf{W}^{\mathrm{op}} \to \mathbf{S}(G)$ is also sharp.*

*Proof.* Since $\widetilde{\mathbf{S}}_p(G) \subset \mathbf{Z}_p(G) \subset \mathbf{D}_p(G)$ we know from 7.6 that $\mathbf{Z}_p(G)$ is $M$-ample, hence (by 7.4) $\mathbf{A}_p(G)$ is also $M$-ample. Now for an arbitrary $G$-subposet $\mathbf{A}_p(G) \subset \mathbf{W} \subset \mathbf{D}_p(G)$ the same conclusion follows from 4.3 because $\mathbf{D}_p(G) = \mathrm{Cl}_{H^*(-;M)}(\mathbf{A}_p(G))$. To see this, note that for any subgroup $H \subset G$ we have $\mathbf{A}_p(G)_{\leq H} = \mathbf{A}_p(H)$.

Next we prove that $c_G : \mathbf{A}_p(G)^{\mathrm{op}} \to \mathbf{S}(G)$ is $M$-sharp. This is a well known theorem for constant coefficients (cf. [JM]), extended by Dwyer [Dw2] to twisted coefficients. Here we shall give a different proof based on 7.4 and 7.6. Indeed by 7.4 we have an isomorphism

$$H^*(\mathbf{A}_p(G)^{\mathrm{op}}_c; H^*(BC_G(-); M)) \xrightarrow{\simeq} H^*(\mathbf{Z}_p(G)_\iota; H^*(B(-); M)).$$

To prove that the last higher limits vanish we notice that $\widetilde{\mathbf{S}}_p(G) \subset \mathbf{Z}_p(G) \subset \mathbf{D}_p(G)$ and apply Theorem 7.6. Now the conclusion follows immediately from 7.2(b). ■

**8. Cofinality in orbit categories.** Let $G$ be a finite group, $p$ a fixed prime and $\mathcal{O}_{(p)}(G)$ the category of $G$-orbits whose order of isotropy group is divisible by $p$.

8.1. Theorem. *Let $F : \mathcal{C} \to \mathcal{O}_{(p)}(G)$ be a functor.*

(1) *If for every non-trivial elementary abelian $p$-subgroup $E \subset G$ the category $(G/E)\backslash F$ is $\mathbb{F}_p$-acyclic then the map*

$$\operatorname{hocolim}_{\mathcal{C}} EG \times_G F(-) \to BG$$

*is an $\mathbb{F}_p$-cohomology equivalence.*

(2) *If moreover the categories $(G/P)\backslash F$ are $\mathbb{F}_p$-acyclic for all non-trivial $p$-subgroups $P \subset G$ then the above decomposition is sharp.*

*Proof of (1).* Let $X := \operatorname{hocolim}_{\mathcal{C}} F = B(G\backslash F)$. We apply Proposition 4.4($\eta'$) to the space $X$ and the poset $\mathbf{A}_p(G)$. Indeed, for every point $x \in X$ the order of the isotropy group is divisible by $p$, hence the poset $\mathbf{A}_p(G)_{\leq G_x} = \mathbf{A}_p(G_x)$ is $\mathbb{F}_p$-ample. Therefore the map

$$\mathrm{id} \times_G \eta : EG \times_G X^{\mathrm{t}\mathbf{A}_p(G)} \to EG \times_G X$$

is an $\mathbb{F}_p$-equivalence. Now Proposition 4.4($\pi$) and Theorem 7.6 imply that $EG \times_G X^{\mathrm{t}\mathbf{A}_p(G)} \xrightarrow{\simeq} EG \times_G |\mathbf{A}_p(G)| \xrightarrow{\simeq} BG$, where $\simeq$ denotes $\mathbb{F}_p$-equivalence. ■

The second assertion could be proved along the same lines, replacing Borel cohomology with Bredon cohomology $H^*_G(-; H^*_{[G]})$ with coefficients in the Borel cohomology functor. We give a different proof based on the next lemma which compares higher limits in a more general categorical setting.

Suppose $F_i : \mathcal{C}_i \to \mathcal{D}$, $i = 0, 1$, are functors into the same category and let $N$ be a $\mathcal{D}$-module. We shall compare higher limits of the pull-backs $N \circ F_0$ and $N \circ F_1$. We denote by $\mathcal{D}_i$ the image of $F_i$.

8.2. Lemma. *Suppose that for every $d_0 \in \mathcal{D}_0$ the right fiber $d_0\backslash F_1$ is acyclic and for every $d_1 \in \mathcal{D}_1$ the canonical map $N(d_1) \to H^*(F_0/d_1; N \circ F_0)$ is an isomorphism. Then there is a natural isomorphism $H^*(\mathcal{C}_0; N \circ F_0) \simeq H^*(\mathcal{C}_1; N \circ F_1)$.*

*Proof.* We define a new category $\mathcal{C}$ with two full inclusions $I_k : \mathcal{C}_k \subset \mathcal{C}$ and a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F|\mathcal{C}_k = F_k$, $k = 0, 1$. The category $\mathcal{C}$ is defined as follows: $\operatorname{ob}\mathcal{C} := \operatorname{ob}\mathcal{C}_0 \cup \operatorname{ob}\mathcal{C}_1$; $\operatorname{Mor}_{\mathcal{C}}(c_0, c_1) := \operatorname{Mor}_{\mathcal{D}}(F_0(c_0), F_1(c_1))$ and $\operatorname{Mor}_{\mathcal{C}}(c_1, c_0) := \emptyset$. Let $M := N \circ F$. We prove that the inclusions induce isomorphisms $H^*(\mathcal{C}_0; M) \overset{\simeq}{\leftarrow} H^*(\mathcal{C}; M) \overset{\simeq}{\rightarrow} H^*(\mathcal{C}_1; M)$. For the functor $I_0 : \mathcal{C}_0 \subset \mathcal{C}$ and the $\mathcal{C}_0$-module $M$ consider the Leray spectral sequence (Prop. 5.3)

$$H^*(\mathcal{C}; H^*(\mathcal{C}_0/-; M)) \Rightarrow H^*(\mathcal{C}_0; M).$$

Observe that the second term degenerates off the first column. Indeed, we have $H^*(\mathcal{C}_0/c_0; M) = M(c_0)$ since the category $\mathcal{C}_0/c_0$ has a terminal object. For an object $c_1 \in \mathcal{C}_1 \subset \mathcal{C}$ we have isomorphisms $H^*(\mathcal{C}_0/c_1; M) \simeq H^*(F_0/c_1; M) = M(c_1)$, the last one by assumption. Therefore we have shown that $I_0^* : H^*(\mathcal{C}; M) \overset{\simeq}{\rightarrow} H^*(\mathcal{C}_0; M)$. To show that $I_1^*$ is an isomorphism we apply Proposition 5.4 to the inclusion $I_0 : \mathcal{C}_0 \subset \mathcal{C}$. We obtain an isomorphism

$$H^*(\mathcal{C}_1; M) \simeq H^*(\operatorname{Hom}_{\mathcal{C}}(C_*(-\backslash\mathcal{C}_1), M)).$$

Now we have equivalences of categories $c_0\backslash\mathcal{C}_1 \simeq F_0(c_0)\backslash F_1$ and $c_1\backslash\mathcal{C}_1 \simeq *$, therefore in both cases they are acyclic. Thus we conclude that $I_1^* : H^*(\mathcal{C}; M) \overset{\simeq}{\rightarrow} H^*(\mathcal{C}_1; M)$. ∎

*Proof of 8.1(2).* To prove that the decomposition is sharp we need to show that $H^p(\mathcal{C}; H^*(EG \times_G F)) = 0$ for $p > 0$. We shall prove that $H^p(\mathcal{C}; H^*(EG \times_G F)) \simeq H^*(\widetilde{\mathcal{O}}_p(G); H^*(EG \times_G -))$ where $\widetilde{\mathcal{O}}_p(G)$ denotes the category of orbits whose isotropy groups are non-trivial $p$-groups. The vanishing of higher limits on that category follows from Theorem 7.6. We apply Lemma 8.2 to the pair of functors $I : \widetilde{\mathcal{O}}_p(G) \subset \mathcal{O}_{(p)}(G)$ and $F : \mathcal{C} \to \mathcal{O}_{(p)}(G)$. We have assumed that for every orbit $G/P \in \widetilde{\mathcal{O}}_p(G)$ the fiber $(G/P)\backslash F$ is $\mathbb{F}_p$-acyclic. For any object $G/K \in \mathcal{O}_{(p)}(G)$ the category $I/(G/K)$ is equivalent to the category of $K$-orbits $\widetilde{\mathcal{O}}_{(p)}(K)$ and the functor on $I/(G/K)$ we are interested in corresponds to the functor $H^*(EG \times_K -)$. Thus by Theorem 7.5 the functor is acyclic and therefore the assumptions of Lemma 8.2 are satisfied. ∎

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Institute of Mathematics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: S.Jackowski@mimuw.edu.pl

Faculty of Mathematics and Information Sciences
Warsaw Technical University
Pl. Politechniki 1
00-661 Warszawa, Poland
E-mail: jolslom@snowman.impan.gov.pl



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