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A minimal regular ring extension of $C(X)$

by

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Abstract. Let $G(X)$ denote the smallest (von Neumann) regular ring of real-valued functions with domain X that contains $C(X)$, the ring of continuous real-valued functions on a Tikhonov topological space (X,τ). We investigate when $G(X)$ coincides with the ring $C(X,\tau_\delta)$ of continuous real-valued functions on the space (X,τ_δ), where τ_δ is the smallest Tikhonov topology on X for which $\tau \subseteq \tau_\delta$ and $C(X,\tau_\delta)$ is von Neumann regular. The compact and metric spaces for which $G(X) = C(X,\tau_\delta)$ are characterized. Necessary, and different sufficient, conditions for the equality to hold more generally are found.

1. Introduction. A ring B is said to be (von Neumann) *regular* if, for each $b \in B$, there exists an $x \in B$ such that $bxb = b$. If B is a commutative regular ring, then for each $b \in B$ there exists a unique $b^* \in B$ such that $b^2 b^* = b$ and $(b^*)^2 b = b^*$. The element b^* is called the *quasi-inverse* of b. See [La] for more background information.

Let X be a nonempty set and let $F(X)$ denote the ring of all real-valued functions with domain X (with addition and multiplication defined pointwise). Clearly, $F(X)$ is a commutative regular ring, and if $f \in F(X)$ then f^* is given by

$$f^*(x) = \begin{cases} 1/f(x) & \text{if } x \in \operatorname{coz}(f), \\ 0 & \text{if } x \in Z(f). \end{cases}$$

(Here $Z(f) = \{x \in X : f(x) = 0\}$ and $\operatorname{coz}(f) = X \setminus Z(f)$.)

All hypothesized topological spaces X are assumed to be Tikhonov. For basic information and undefined notation see [GJ] and [PW]. In particular, $C(X)$ denotes the ring of continuous members of $F(X)$, and χ_S denotes the characteristic function of the subset S of X. Boldface real numbers denote constant functions.

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If S is a commutative regular ring with a subring R, define $G_S(R)$ by

$$G_S(R) = \bigcap\{A : R \subseteq A \subseteq S \text{ and } A \text{ is regular}\}.$$

Then $G_S(R)$ is the smallest regular subring of S that contains R. The following characterization "from below" of $G_S(R)$ (see 1.1(a)) can be found by juxtaposing results in [Ke, 4.2], [O_1, Props. 5 and 6], [O_2], and [W, Theorem 1, Corollary to Theorem 6]. Clearly, 1.1(b) follows from 1.1(a).

1.1. Theorem. (a) $G_S(R) = \{\sum_{i=1}^n a_i b_i^* : a_i, b_i \in R,\ b_i^*$ *is the quasi-inverse of* b_i *in* $S\}$.

(b) $|G(R)| = |R|$.

Denote $G_{F(X)}(C(X))$ by $G(X)$. Then $G(X)$ is the smallest regular subring of $F(X)$ that contains $C(X)$. Explicitly, $G(X) = \{\sum_{i=1}^n f_i g_i^* : f_i, g_i \in C(X)\}$.

If X is a space with topology τ, we define τ_δ to be the underlying set of X re-topologized by making the G_δ-sets of (X,τ) an open base for this new topology. Denote (X,τ_δ) by X_δ. Then $C(X_\delta)$ is a regular ring and $C(X) \subseteq C(X_\delta) \subseteq F(X)$ (see [GJ, 4J] and [PW, 1W]). Hence $C(X) \subseteq G(X) \subseteq C(X_\delta)$. Furthermore, τ_δ is the smallest Tikhonov topology α on X for which $\tau \subseteq \alpha$ and $C((X,\alpha))$ is a regular ring (see [PW, 1W]). Hence $G(X) = C(X,\alpha)$ for some Tikhonov topology α on X iff $G(X) = C(X_\delta)$. If $\tau = \tau_\delta$ then X is called a *P-space*.

The purpose of this article is to describe those spaces X for which $G(X) = C(X_\delta)$. We characterize compact spaces and metric spaces with this property (see 3.4), and obtain partial results in more general cases. First we name spaces with this property.

1.2. Definition. A Tikhonov space X is called *regularly good* (or an *RG-space*) if $C(X_\delta) = G(X)$.

1.3. Definition. Let X be a space. If $f \in G(X)$, we define $\mathrm{rg}(f)$ as follows: $\mathrm{rg}(f) = \min\{n \in \mathbb{N} :$ there exist $g_i, h_i \in C(X)$ $(i = 1$ to $n)$ such that $f = \sum_{i=1}^n g_i h_i^*\}$. We define $\mathrm{rg}(X)$ to be $\sup\{\mathrm{rg}(f) : f \in G(X)\}$. Note that $\mathrm{rg}(X)$ is either a positive integer or $+\infty$.

1.4. Proposition. *Let X be a space. Then* $\mathrm{rg}(X) = 1$ *iff X is a P-space.*

Proof. If X is a P-space then $C(X) = G(X)$ so $\mathrm{rg}(h) = \mathrm{rg}(h\mathbf{1}^*) = 1$ for each $h \in G(X)$.

Conversely, suppose $\mathrm{rg}(h) = 1$ for each $h \in G(X)$, and let $f \in C(X)$. Then $\mathbf{1}+(-f)f^* \in G(X)$, so by hypothesis there exist g and k in $C(X)$ such that $\mathbf{1} + (-f)f^* = gk^*$. A straightforward calculation shows that $Z(f) = \mathrm{coz}(gk)$, and so $Z(f)$ is open in X. As $Z(f)$ was an arbitrary zero-set of X, it follows that X is a P-space (see [GJ, 4J]). ∎

Observe that, as noted above, if $h \in C(X)$ then $\mathrm{rg}(h) = 1$ as $h = h(\mathbf{1}^*)$; however, if $f \in C(X)$ then $\mathrm{rg}(ff^*) = 1$ but $ff^* \notin C(X)$ unless $\mathrm{coz}(f)$ is a clopen set of X.

2. Elementary properties of RG-spaces. We begin by collecting some basic properties of RG-spaces. Later in this section we give some sufficient conditions for a space to be an RG-space.

Recall that a space X is *scattered* if each subspace of X has an isolated point, or equivalently, if each subspace of X has a dense set of isolated points. Many of the properties of scattered spaces are summarized in Z. Semadeni's memoir [Se1], his book [Se2], and in a paper by R. Levy and M. Rice [LR].

Denote the set of isolated points of X by $I(X)$. For an ordinal α define $D_\alpha(X)$ inductively as follows: $D_0(X) = X$, $D_1(X) = X \setminus I(X)$, $D_{\alpha+1}(X) = D_1(D_\alpha(X))$, and $D_\lambda(X) = \bigcap\{D_\alpha(X) : \alpha < \lambda\}$ if λ is a limit ordinal. Clearly, X is scattered iff there exists α_0 for which $D_{\alpha_0}(X) = \emptyset$. The *Cantor–Bendixson order* of the scattered space X, denoted by $\mathrm{CB}(X)$, is defined as follows:

$$\mathrm{CB}(X) = \min\{\alpha : D_\alpha(X) = \emptyset\}.$$

($\mathrm{CB}(X)$ is called the *dispersal order* of X in [LR].)

Observe that $\mathrm{CB}(X) = \mathrm{CB}(D_1(X)) + 1$ if $\mathrm{CB}(X)$ is a successor ordinal, and that $\mathrm{CB}(X)$ is a successor ordinal if X is a compact scattered space.

2.1. PROPOSITION. *Let X be a space and let $f \in G(X)$. Then*:

(a) *f is continuous* (*re X*) *on a dense open subset of X.*

(b) *If $T \subseteq X$ then $f|T \in G(T)$ and $\mathrm{rg}(f|T) \le \mathrm{rg}(f)$.*

(c) *If T is C-embedded on X then it is G-embedded in X*; *that is*, *if $f \in G(T)$ then there exists $F \in G(X)$ such that $F|T = f$.*

Proof. (a) If $f = \sum_{i=1}^n g_i h_i^*$ then f is continuous on the dense open set $X \setminus \bigcup\{\mathrm{bd}_X Z(h_i) : i = 1 \text{ to } n\}$.

(b) If $h \in C(X)$ then $h^*|T = (h|T)^*$; the result quickly follows. ∎

2.2. LEMMA. *A dense subspace of an RG-space X cannot be written as the union of countably many nowhere dense zero-sets of X.*

Proof. Suppose that the RG-space X has the set $\bigcup\{Z_i : i \in \mathbb{N}\}$ as a dense subspace S, where each Z_i is a nowhere dense zero-set of X. Let $A_1 = Z_1$ and if $n \ge 2$, let $A_n = Z_n \setminus \bigcup_{i=1}^{n-1} A_i$. Then as zero-sets of P-spaces are clopen, it is easily seen that $\{A_i : i \in \mathbb{N}\} \cup \{X \setminus S\}$ is a partition of X into clopen subsets of X_δ. (We discard empty A_i's.) Define $f : X \to \mathbb{R}$ by $f[A_i] = i + 1$ and $f[X \setminus S] = \{1\}$. Then $f \in C(X_\delta)$, so by hypothesis $f \in G(X)$. As S is dense in X, by 2.1(a) there exist $k \in \mathbb{N}$ and $p \in A_k$ such that f is X-continuous at p. Hence there is an open set W of X such that $p \in W$ and $f[W] \subseteq (k + 2/3, k + 4/3)$. Clearly, $W \subseteq A_k \subseteq Z_k$, which contradicts the hypothesis that $\mathrm{int}_X Z_k = \emptyset$. The result follows. ■

We now investigate conditions under which a subspace of an RG-space is an RG-space.

2.3. THEOREM. *A subspace Y of an RG-space X is an RG-space if any of the following conditions hold*:

(a) *Y_δ is C^*-embedded in X_δ.*

(b) *Y_δ is Lindelöf* (*in particular*, *Y is countable*).

(c) *Y is scattered and Lindelöf.*

(d) *X_δ is normal and Y is realcompact and C^*-embedded in X.*

(e) $|X| \leq$ **c**, *the continuum hypothesis holds*, *and Y is realcompact and C^*-embedded in X.*

(f) *Y is a countable union of zero-sets or cozero-sets of X.*

(g) *X_δ is normal and Y_δ is closed in X_δ*; *in particular*, *X is paracompact and scattered*, *and Y_δ is closed in X_δ.*

Proof. It suffices to show that $C(Y_\delta) \subseteq G(Y)$.

(a) As zero-sets of X_δ are clopen in X_δ, Y_δ is completely separated in X_δ from any zero-set of X_δ that is disjoint from it. Consequently, Y_δ is C-embedded in X_δ by [GJ, 1.18]. Thus if $h \in C(Y_\delta)$, there exists $H \in C(X_\delta)$ such that $H|Y_\delta = h$. By hypothesis $H \in G(X)$. Hence by 2.1(b), $h = H|Y = G(Y)$.

(b) Zero-sets of X_δ are clopen subsets of X_δ, since X_δ is a P-space (see [GJ, 4J]); hence they are completely separated in X_δ from subsets that are disjoint from them. But by [BH, 4.2], if a Lindelöf subspace of a space S is completely separated in S from each zero-set of S disjoint from it, then that subspace is C-embedded in S. Hence if Y_δ is Lindelöf, it is C-embedded in X_δ. Now use (a).

(c) If Y is Lindelöf and scattered, then by [LR, 5.2], Y_δ is Lindelöf. The result now follows from (b).

(d) Let $T = \mathrm{cl}_X Y$. As Y is dense and C^*-embedded in T, it follows that $Y \subseteq T \subseteq \beta Y$ (see [GJ, 6.7]). Hence as Y is realcompact, if $x \in T \setminus Y$ then there is a G_δ-set $E(x)$ of T such that $x \in E(x) \subseteq T \setminus Y$ (this follows quickly from [PW, 5.11(i)]). Let $H(x)$ be a G_δ-set of X for which $E(x) = T \cap H(x)$. Then $X \setminus Y = (X \setminus T) \cup \bigcup\{H(x) : x \in T \setminus Y\}$, and so $X \setminus Y$ is open in X_δ. Thus Y_δ is closed (and hence C^*-embedded) in the normal space X_δ ([GJ, 3D]). Now use (a).

(e) If we assume the continuum hypothesis, then a P-space of cardinality no greater than **c** is paracompact (see the first paragraph of §4 of [LR]), hence normal. Now use (d).

(f) Observe that Y_δ is clopen in X_δ, and hence C-embedded in X_δ. The result now follows from (a).

(g) The first assertion follows from (a). If X is scattered and paracompact, then X_δ is paracompact and hence normal (see [LR, 5.1]). ■

Example 2.10 below, in conjunction with 3.7, shows that the C^*-embedding of Y in X cannot be dropped from the hypotheses of 2.3(e), and that "Lindelöf" cannot be dropped from the hypotheses of 2.3(c). Example 3.10 shows that "realcompact" cannot be dropped from the hypothesis of 2.3(d) or 2.3(e). Also, note that Y_δ is closed in X_δ if and only if $X \setminus Y$ is a union of G_δ-sets of X; this occurs if Y is Lindelöf, or if Y is realcompact and C^*-embedded in $\mathrm{cl}_X Y$ (see [PW, 5.11(c)(2)]).

2.4. Corollary. *Countable subsets of RG-spaces are scattered.*

Proof. If S is a countable subset of the RG-space X, but is not scattered, then S has a countable subset T without isolated points. Now T is an RG-space by 2.3(b), and the singleton sets of T are nowhere dense zero-sets of T. This contradicts 2.2, so we conclude that S must be scattered. ■

2.5. Proposition. *Compact subspaces of RG-spaces are scattered RG-spaces.*

Proof. Suppose that K is a compact non-scattered subspace of the RG-space X. Then K has a compact subspace L with no isolated points. Thus there is a continuous surjection k from L onto the Cantor set C (see [LR, 3.17], for example). There exists a compact subset M of L such that $k|M = f$ is an irreducible continuous surjection from M onto C ([PW, 6.5(c)]). Let S be a countable dense subset of C, and let T be formed by selecting one point from $f^{-1}(s)$ for each $s \in S$. Then T is a countable subspace of L, and as f is irreducible and C has no isolated points, T will also have no isolated points. This contradicts 2.4, and we conclude that each compact subset of an RG-space is scattered. It now follows from 2.3(c) that compact subspaces of RG-spaces are RG-spaces. ■

A different proof of 2.5 appears (implicitly) in [RW, 5.7]. We thank the referee for suggesting the proof used above.

Observe that 2.4 and 2.5 cannot be extended to Lindelöf spaces, as there exist Lindelöf P-spaces without isolated points (see 3.11).

Recall that a space X is called *resolvable* if it can be written as the union of two complementary dense subsets. Also recall that a space X has *countable pseudocharacter* (see [Ho]) if each singleton set of X is a G_δ-set. Since singleton G_δ-sets are zero-sets (see [GJ, 3.11(b)]), evidently X has countable pseudocharacter if and only if X_δ is discrete if and only if $C(X_\delta) = F(X)$.

2.6. THEOREM. (a) *A resolvable space of countable pseudocharacter is not an RG-space.*

(b) *If X is an RG-space of countable pseudocharacter and is either countably compact or a k-space, then X is scattered.*

(*c*) *If X is a first countable RG-space, then X is scattered. In particular, metric RG-spaces are scattered.*

Proof. (a) If X is resolvable and of countable pseudocharacter, let S be a dense subset of X whose complement is also dense. Clearly, the characteristic function χ_S is nowhere continuous, so by 2.1(a), $\chi_S \notin G(X)$. But $F(X) = G(X)$ (see above).

(b) If X were not scattered it would have a closed subspace Y without isolated points. As countable compactness and being a k-space are both closed-hereditary, Y has whichever of these properties X is assumed to have. But both k-spaces and countably compact spaces without isolated points are resolvable (see [CG, 6.9 and 8.1]), and Y has countable pseudocharacter since X has. So Y is not an RG-space by (a) above. Clearly, Y_δ is C^*-embedded in X_δ as each is discrete, so by 2.3(a), X cannot be an RG-space.

(c) First countable spaces are k-spaces (see [PW, 9D(4)]) of countable pseudocharacter. Now use (b). ■

We now investigate sufficient conditions for a space to be an RG-space. Obviously, every P-space is an RG-space. Also, spaces that are "nearly, but not quite" P-spaces are RG-spaces, as we shall show next. Recall that a point of a space X is called a *P-point* of X if each G_δ-set (equivalently, each zero-set) of X that contains it is a neighborhood of it (see [GJ, 4L]). We denote by $P(X)$ the set of all P-points of X; clearly, X is a P-space if and only if $P(X) = X$.

2.7. THEOREM. *Let X be a space with exactly one non-P-point. Then X is an RG-space and* $\mathrm{rg}(X) = 2$.

Proof. It suffices to show that $C(X_\delta) \subseteq G(X)$. Let $X \setminus P(X) = \{r\}$.

Suppose that $k \in C(X_\delta)$ and $k(r) = 0$. Because X_δ is a P-space, $Z(k)$ is an open X_δ-neighborhood of r. Since $\mathbf{Z}(X)$ is an open base for X_δ, there is a $g \in C(X)$ such that $r \in Z(g) \subseteq Z(k)$. Let $A(0) = X$ and if $0 < n < \omega$, let $A(n) = g^{\leftarrow}[[-1/n, 1/n]]$. Then each $A(n)$ contains r in its X-interior. Further, if $x \in A(n) \setminus \{r\}$, then $A(n)$ is an X-neighborhood of x as $x \in P(X)$. Thus each $A(n)$ is a clopen subset of X. We define $f, g \in F(X)$ as follows:

$$f(x) = \begin{cases} \dfrac{k(x)}{n[|k(x)|+1]} & \text{if } x \in A(n) \setminus A(n+1), \\ 0 & \text{if } x \in \bigcap\{A(n) : n \in \omega\}; \end{cases}$$

$$g(x) = \begin{cases} \dfrac{1}{n[|k(x)|+1]} & \text{if } x \in A(n) \setminus A(n+1), \\ 0 & \text{if } x \in \bigcap\{A(n) : n \in \omega\}. \end{cases}$$

It is straightforward to verify that $f, g \in C(X)$ and that $k = fg^*$.

If $k(r) \neq 0$, let $m = k - \mathbf{k}(\mathbf{r})$ and apply the above argument to m to obtain $f, g \in C(X)$ such that $m = fg^*$. Thus $k = fg^* + \mathbf{k}(\mathbf{r})$ and $\mathrm{rg}(k) \leq 2$. By 1.4, $\mathrm{rg}(X) > 1$, so $\mathrm{rg}(X) = 2$. ■

2.8. Proposition. *Let X be the direct sum of the spaces $\{X(i) : i \in I\}$. The following are equivalent*:

(a) *X is an RG-space.*

(b) *Each $X(i)$ is an RG-space and there is no one-to-one map $\lambda : \mathbb{N} \to I$ such that* $\mathrm{rg}(X(\lambda(n))) \geq n$ *for each $n \in \mathbb{N}$.*

Proof. (a)⇒(b). By 2.3(a) each $X(i)$ is an RG-space. If a λ of the sort described did exist, let $f_n \in G(X(\lambda(n)))$ be such that $\mathrm{rg}(f_n) \geq n$. Obviously, there exists $F \in C(X_\delta)$ for which $F|X(\lambda(n)) = f_n$ for each $n \in \mathbb{N}$, and by 2.1(b), $F \notin G(X)$. Hence (a) would fail.

(b)⇒(a). If (b) holds there exists $m \in \mathbb{N}$ such that $\mathrm{rg}(X(i)) \leq m$ for each $i \in I$. If $F \in C(X_\delta)$ then by hypothesis $F|X(i) = \sum_{j=1}^{m} f_{i,j} g_{i,j}^*$ for some $f_{i,j}, g_{i,j} \in C(X(i))$ (some terms in the sum may be repeated if necessary). Clearly, $F = \sum_{j=1}^{n} (\bigcup\{f_{i,j} : i \in I\})(\bigcup\{g_{i,j} : i \in I\})^*$, so $F \in G(X)$. ■

Recall that a collection of subsets of a space X is called *discrete* if each point of the space has a neighborhood meeting at most one set in the collection. A subset $\{x_i : i \in I\}$ of X is called *strongly discrete* if there is a discrete collection $\{U_i : i \in I\}$ of open sets of X such that $x_i \in U_i$ for each $i \in I$. Clearly, a strongly discrete collection of points of a space is a closed discrete subspace of the space; the converse fails, as we shall see shortly.

2.9. Proposition. *If X is a space for which $X \setminus P(X)$ is a strongly discrete subset of X* (*in particular*, *if $X \setminus P(X)$ is finite*), *then X is an RG-space.*

Proof. Let $\{U_i : i \in I\}$ be the discrete collection of open sets provided by the definition of "strongly discrete set". (We may assume this collection is nonempty.) If $U_i \cap (X \setminus P(X)) = \{r_i\}$, find $Z_i \in \mathbf{Z}(X)$ such that $r_i \in \mathrm{int}_X Z_i \subseteq Z_i \subseteq U_i$ (see [GJ, 3.11(b)]). As $Z_i \setminus \{r_i\} \subseteq P(X)$, it follows that each Z_i is clopen in X. As $\{Z_i : i \in I\}$ is a strongly discrete collection, it follows that $X \setminus \bigcup\{Z_i : i \in I\}$ is a clopen set of X as well—without loss of generality attach it to some particular $Z_{i(0)}$. By 2.7, $\mathrm{rg}(Z_i) = 2$ for each $i \in I$, so by 2.8, X is an RG-space. ■

We now produce an example to show that the word "strongly" in the statement of 2.9 cannot be replaced by the word "closed". We thank Alan Dow for bringing this example to our attention.

2.10. Example. Let X be the closed unit interval $[0,1]$ topologized as follows. For each irrational number $y \in [0,1]$ choose a sequence $s(y)$ of rational numbers in $[0,1]$ that converges to y in the usual topology of $[0,1]$. A subset V of X is decreed to be open if, for each irrational number y, V contains a cofinite subset of $s(y)$ whenever it contains y. One can verify that this defines a locally compact Hausdorff topology on X, and by [GJ, 8.18], X and all its subspaces are realcompact. As X is of countable pseudocharacter, it follows that X_δ is discrete and so $|C(X_\delta)| = 2^{\mathbf{c}}$. However, the rationals are dense in X and so $|C(X)| = \mathbf{c}$. By 1.1(b) it follows that $|G(X)| = \mathbf{c}$. Hence X is not an RG-space. Clearly, each rational number is isolated in X and $P(X)$ is the set of rationals in X, while $X \setminus P(X)$ is a closed discrete subspace of X.

We now add to our stock of RG-spaces by showing that if X is a scattered space that is either Lindelöf or perfectly normal, and if $\mathrm{CB}(X)$ is finite, then X is an RG-space.

2.11. *Let X be a scattered space that is either Lindelöf or perfectly normal. Then*:

(a) *X is an RG-space if and only if $D_1(X)$ $(= X \setminus I(X))$ is an RG-space.*
(b) *Let $j \in \mathbb{N}$. If* $\mathrm{rg}(D_1(X)) \le j$ *then* $\mathrm{rg}(X) \le 2j+1$.

Proof. (a) If X is a Lindelöf scattered RG-space, then it follows from 2.3(c) that $D_1(X)$ is an RG-space. If X is perfectly normal, this follows from 2.3(f).

Conversely, suppose that $D_1(X)$ is an RG-space, and that X is scattered and either Lindelöf or perfectly normal. Suppose that $s \in C(X_\delta)$; then $s|D_1(X) \in C(D_1(X)_\delta)$. Because $D_1(X)$ is an RG-space, there exist $k \in \mathbb{N}$ and $f_i, g_i \in C(D_1(X))$ such that $s|D_1(X) = \sum_{i=1}^{k} f_i g_i^*$. Because $D_1(X)$ is closed and hence C-embedded in the normal space X, there exist $F_i, G_i \in C(X)$ such that $F_i|D_1(X) = f_i$ and $G_i|D_1(X) = g_i$ $(i = 1$ to $k)$.

Let $H = \sum_{i=1}^{k} F_i G_i^*$; then $H \in G(X) \subseteq C(X_\delta)$. Let $r = s - H$. Then $r \in C(X_\delta)$ and $Z(r)$ is open in X_δ as X_δ is a P-space. Clearly, $H|D_1(X) = s|D_1(X)$, and so $D_1(X) \subseteq Z(r)$.

First assume that X is Lindelöf. Observe that $\mathbf{U} = \{\{x\} : x \in I(X)\} \cup \{Z(r)\}$ is an open cover of X_δ. As X is Lindelöf and scattered, X_δ is Lindelöf by [LR, 5.2]. Thus $\mathbf{U}$ has a countable subcover, and hence there is a countable subset $S = \{x_n : n \in \mathbb{N}\}$ of $I(X)$ such that $X = \{x_n : n \in \mathbb{N}\} \cup Z(r)$. As S is an open F_σ-set of the normal space X, there exists $m \in C(X)$ such that $\mathrm{coz}(m) = S$. Let $A(n) = \{x_n\}$.

Next assume that X is perfectly normal. Then $X \setminus Z(r)$, being a discrete open set of X, is a cozero-set of X and can thus be partitioned into countably many clopen sets $\{A(n) : n \in \mathbb{N}\}$ of X. Let $m \in C(X)$ such that $\mathrm{coz}(m) = X \setminus Z(r)$.

In either case, define $f, g : X \to \mathbb{R}$ by

$$f(x) = \frac{s(x_n)}{n[|s(x_n)| + 1]} \quad \text{if } x \in A(n), \quad f\Big[X \setminus \bigcup\{A(n) : n \in \mathbb{N}\}\Big] = \{0\},$$
$$g(x) = \frac{1}{n[|s(x_n)| + 1]} \quad \text{if } x \in A(n), \quad g\Big[X \setminus \bigcup\{A(n) : n \in \mathbb{N}\}\Big] = \{0\}.$$

Clearly, $f, g \in C(X)$. Now define $t : X \to \mathbb{R}$ as follows:

$$t = (mf)(mg)^* + \sum_{i=1}^{k} F_i G_i^* + \sum_{i=1}^{k} (-mF_i)(mG_i)^*.$$

By 1.1(a), $t \in G(X)$. We claim that $s = t$. If $n \in \mathbb{N}$, then $m(x_n) \neq 0$ and

$$t(x_n) = 1 \cdot \left(\frac{s(x_n)}{n[|s(x_n)| + 1]} \right) (n[|s(x_n)| + 1]) + 0 = s(x_n),$$

and if $x \in X \setminus \{x_n : n \in \mathbb{N}\}$ then $x \in Z(r)$, so

$$t(x) = 0 + \sum_{i=1}^{k} F_i(x) G_i^*(x) + 0 = H(x) = s(x).$$

Thus $s \in G(X)$ and so X is an RG-space.

(b) If $\mathrm{rg}(D_1(X)) \le j$ then $k \le j$, and clearly, from the definition of t, $\mathrm{rg}(s) = \mathrm{rg}(t) \le 2j + 1$. As s was an arbitrary member of $C(X_\delta)$, it follows that $\mathrm{rg}(X) \le 2j + 1$. ∎

2.12. THEOREM. *Let X be a Lindelöf or perfectly normal scattered space of finite Cantor–Bendixson order. Then X is an RG-space and $\mathrm{rg}(X) \le 2^{\mathrm{CB}(X)} - 1$.*

Proof. Suppose to the contrary that there exists a Lindelöf (resp. perfectly normal) scattered space X of minimal finite Cantor–Bendixson order $k > 0$ that is not an RG-space. As noted just before 2.1, $\mathrm{CB}(D_1(X)) = k - 1$. Clearly, $D_1(X)$ is a scattered Lindelöf (resp. perfectly normal) space, so by the minimality of k, $D_1(X)$ is an RG-space. But by 2.11(a) this implies that X is an RG-space, contrary to the choice of X. It follows that all Lindelöf (resp. perfectly normal) scattered spaces of finite Cantor–Bendixson order are RG-spaces.

If $\mathrm{CB}(X) = 1$ then X is discrete and hence a P-space, so $\mathrm{rg}(X) = 1 = 2^{\mathrm{CB}(X)} - 1$ (see 1.4). Now suppose that if $\mathrm{CB}(V) = j$ then $\mathrm{rg}(V) \le 2^j - 1$, and suppose that X is a scattered Lindelöf space for which $\mathrm{CB}(X) = j + 1$. Then

$\mathrm{CB}(D_1(X)) = j$ so $\mathrm{rg}(D_1(X)) \le 2^j - 1$. Hence by 2.11(b), $\mathrm{CB}(X) \le 2(2^j - 1) + 1 = 2^{j+1} - 1 = 2^{\mathrm{CB}(X)} - 1$, which completes the inductive argument. ■

We leave as an exercise:

2.13. Corollary. *If X is a finite union of scattered Lindelöf spaces of finite* CB *order*, *then X is an RG-space.*

2.14. Remarks. (a) Observe that the space of Example 2.10 is separable, scattered, and has Cantor–Bendixson order 2, but is not an RG-space. Hence the hypothesis in 2.11 that X is a Lindelöf (or perfectly normal) space may not be deleted.

(b) As metric spaces are perfectly normal, and as both Lindelöf and metric spaces are paracompact, it is natural to ask whether 2.12 holds for paracompact spaces. We have been unable to answer this question.

3. A characterization of compact or metric RG-spaces. Let X be a space that is either compact or metric. In this section we prove that it is an RG-space if and only if it is scattered and has finite Cantor–Bendixson order. By 2.12, we know that if a space is scattered and of finite Cantor–Bendixson order, and is either compact or metric, then it is an RG-space. By 2.5 and 2.6(c) we know that a compact or metric RG-space is scattered. So it suffices to prove that a compact or metric scattered space of infinite Cantor–Bendixson order cannot be an RG-space.

3.1. Definition. Let X be a scattered space. For each $j \in \omega$ inductively define $D_j(X)$ and $I_j(X)$ as follows:

$$\begin{aligned} D_0(X) &= X, & I_0(X) &= I(X),\\ D_1(X) &= X \setminus I(X) = D_0(X) \setminus I_0(X), & I_1(X) &= I(D_1(X)),\\ D_{j+1}(X) &= D_j(X) \setminus I(D_j(X)), & I_{j+1}(X) &= I(D_{j+1}(X)). \end{aligned}$$

(Note that the definition of $D_j(X)$ is consistent with that preceding 2.1.)

The proof of the following is straightforward and not included.

3.2. Proposition. *Let X be a scattered space*, *let n be a positive integer*, *and let* $\mathrm{CB}(X) = n$. *Suppose that A is a clopen subset of X. Then*:

(a) *$D_{n-1}(X) \ne \emptyset$ and $D_n(X) = \emptyset$.*
(b) *$I_{n-1}(X) \ne \emptyset$ and $I_n(X) = \emptyset$.*
(c) *$\{I_j(X) : j = 0$ to $n-1\}$ partitions X into n discrete subspaces.*
(d) *If $i \in \{0, \ldots, n-1\}$ then $\bigcup_{j=0}^{i} I_j(X)$ is a dense open subset of X.*
(e) *If $i \in \{0, \ldots, n-1\}$ then $X \setminus \bigcup_{j=0}^{i} I_j(X) = D_{i+1}(X)$, which is compact.*
(f) *$A \cap I_j(X) = I_j(A)$ if $j \in \{1, \ldots, n-1\}$.*
(g) *$A \cap D_j(X) = D_j(A)$ if $j \in \{1, \ldots, n-1\}$.*

3.3. THEOREM. *Let X be a scattered space that is either compact or metric, and assume that* $\mathrm{CB}(X) = n$*. Then there exists $s \in G(X)$ for which* $\mathrm{rg}(s) \geq n$*.*

Proof. Scattered metric spaces have scattered (in fact ordinal) compactifications (see [Se2] and [Ba]), and compact scattered spaces are zero-dimensional. Hence as $D_{n-1}(X)$ is discrete, we can find $q \in D_{n-1}(X)$ and a clopen subset A of X such that $A \cap D_{n-1}(X) = \{q\}$.

By 3.2(g), $D_{n-1}(A) = \{q\}$ and $D_n(A) = \emptyset$, so $\mathrm{CB}(A) = n$. If there exists $t \in G(A)$ such that $\mathrm{rg}(t) \geq n$, then let $T : X \to \mathbb{R}$ be defined by $T|A = t$ and $T[X \setminus A] = \{0\}$. Clearly, $T \in G(X)$ and $\mathrm{rg}(T) = \mathrm{rg}(t) \geq n$. It follows that we may assume without loss of generality that $|D_{n-1}(X)| = 1$, for if the theorem holds for such spaces it will hold for all spaces Y for which $\mathrm{CB}(Y) = n$. So we will assume that $D_{n-1}(X) = I_{n-1}(X) = \{q\}$.

Consider the following assertion about the integer n:

$(*)_n$ If Y is a metric or compact scattered space and $\mathrm{CB}(Y) = n$, if $I_{n-1}(Y) = D_{n-1}(Y) = \{q\}$, and if $\lambda > 0$, then there is $s \in G(Y)$ such that $s(q) = \lambda$ and if B is a clopen neighborhood of q in Y, then $\mathrm{rg}(s|B) \geq n$.

We will verify $(*)_n$ by induction on n. Note that if $(*)_n$ holds then the theorem follows with $X = B$ and with λ being any positive number.

If $n = 1$ then $Y = \{q\}$ and $G(Y) = C(Y)$ and $\mathrm{rg}(s) = 1$ for any $s \in G(Y)$, and so $(*)_1$ holds trivially.

Now we will assume that $(*)_n$ holds and prove $(*)_{n+1}$. So, let X be a compact or metric scattered space for which $\mathrm{CB}(X) = n+1$. As noted above, we may assume that $D_n(X) = \{q\}$. Thus $D_{n-1}(X) = I_{n-1}(X) \cup \{q\}$. If X is compact, then $D_{n-1}(X)$ is the one-point compactification of the infinite discrete space $I_{n-1}(X)$; if X is metric then $D_{n-1}(X) = \mathrm{cl}_X\, I_{n-1}(X)$. In either case there is a sequence $(p_n)_{n \in \omega}$ of distinct points of $I_{n-1}(X)$ that converges to q.

As X is a zero-dimensional T_3 space, there exists a pairwise disjoint countably infinite family $(B_i)_{i<\omega}$ of clopen subsets of X for which $D_{i-1}(X) \cap B_i = \{p_i\}$. Now $D_{n-1}(B_i) = B_i \cap D_{n-1}(X) = \{p_i\}$ while $D_n(B_i) = D_n(X) \cap B_i = \emptyset$ (see 3.2(f),(g)), so $\mathrm{CB}(B_i) = n$ and $D_{n-1}(B_i) = \{p_i\}$ for each $i < \omega$. Observe that $q \notin \bigcup_{i<\omega} B_i$.

By $(*)_n$ there exists, for each $i < \omega$, $s_i \in G(B_i)$ such that $s_i(p_i) = 1$ and such that if A_i is any clopen neighborhood of p_i in B_i, then $\mathrm{rg}(s_i|A_i) \geq n$. Evidently, $s_i \in C((B_i)_\delta)$.

Now define a function $s : X \to \mathbb{R}$ as follows: $s|B_i = s_i$ and $s[X \setminus \bigcup_{i<\omega} B_i] = \{2\}$. Now $\bigcup_{i \in \omega} B_i \in \mathrm{coz}(X)$ and hence is clopen in X_δ.

Thus $\{B_i : i < \omega\} \cup \{X \setminus \bigcup_{i<\omega} B_i\}$ partitions X_δ into countably many X_δ-clopen sets. Clearly, the restriction of s to each of these is X_δ-continuous and so $s \in C(X_\delta)$. But X is an RG-space by 2.12, so $s \in G(X)$.

Now let A be a clopen neighborhood of q in X. As $I_{n-1}(X) \cup \{q\}$ is the one-point compactification of $I_{n-1}(X)$, A contains all but finitely many p_i; re-labeling if necessary, we may assume that $(p_i)_{i<\omega} \subseteq A$.

Assume that $\mathrm{rg}(s|A) = n$. (Clearly, $\mathrm{rg}(s|A) \geq n$ by the definitions of s and s_i.) There exist $f_i, g_i \in C(A)$ $(i = 1, \ldots, n)$ such that

$$s|A = \sum_{i=1}^{n} f_i g_i^*.$$

As $q \notin \bigcup_{i<\omega} B_i$, we have $s(q) = 2$. Consequently, it cannot be true that $g_i(q) = 0$ for all $i \in \{1, \ldots, n\}$. But $s|A$ is not continuous at q, since $q \in \mathrm{cl}_X\{s(p_i) : i < \omega\}$, $s(p_i) = 1$ for each $i < \omega$, and $s(q) = 2$. Consequently, $g_i(q) = 0$ for at least one $i \in \{1, \ldots, n\}$ (for if $g_i(q) \neq 0$ then g_i^* is continuous at q). Let $J = \{i \in \{1, \ldots, n\} : g_i(q) \neq 0\}$, and set

$$h = \sum_{i \in J} f_i g_i^*.$$

Then h is continuous on a clopen neighborhood E of q. Let $I = \{1, \ldots, n\} \setminus J$. Then

$$(+) \qquad s|E = h + \sum_{i \in I} (f_i|E)(g_i|E)^*$$

and $0 < |I| \leq n - 1$. Clearly, E contains all but finitely many p_i, and $g_i(q) = 0$ for each $i \in I$. So, $s|E$ fails to be continuous at q for the same reason that $s|A$ was discontinuous at q.

We claim that there exists $j \in \omega$ such that $s(p_j) \neq h(p_j)$. To see this, observe that $(s|E)(q) = h(q)$ as $g_i^*(q) = 0$ for all $i \in I$. So, if our claim failed, we would have

$$(s|E)|(\{p_i\}_{i<\omega} \cap E) \cup \{q\} = h|(\{p_i : i < \omega\} \cap E) \cup \{q\},$$

which is a contradiction as the function on the left is discontinuous at q (for the same reason that $s|E$ was), while the function on the right is continuous at q. Our claim holds.

Suppose $p_j \in E$ and $s(p_j) \neq h(p_j)$. By (+) there exists $t \in I$ such that $g_t^*(p_j) \neq 0$. Thus g_t is continuous on a clopen neighborhood T of p_j, and we see that

$$s|T = (h + f_t g_t^*)|T + \sum_{i \in I \setminus \{t\}} (f_i|T)(g_i|T)^*$$

(without loss of generality $T \subseteq B_j \cap E$).

Now $h + f_t g_t^* | T \in C(T)$ and $|I \setminus \{t\}| \leq n-2$, so $\mathrm{rg}(s|T) \leq n-1$. Letting A_j be T, we see that this contradicts the inductive hypothesis $(*)_n$ when $i = j$.

Hence $\mathrm{rg}(s|A) \geq n+1$ so $\mathrm{rg}(s) \geq n+1$ and our proof is complete. ∎

3.4. Theorem. *A compact or metric space is an RG-space if and only if it is scattered and of finite Cantor–Bendixson order.*

Proof. The sufficiency is a special case of 2.12. To prove the converse, assume that X is a compact or metric scattered space of infinite Cantor–Bendixson order; then $I_n(X) \neq \emptyset$ for all $n \in \omega$. Choose $p_n \in I_n(X)$ for each $n \in \mathbb{N}$ and set $D = (p_n)_{n \in \mathbb{N}}$. For each $n \in \mathbb{N}$, put $V_n = \bigcup_{i=1}^{n} (I_i(X) \setminus \{p_i : i < n\})$. By 3.2(c),(d) it follows that V_n is open in X and $V_n \cap D = \{p_n\}$, so D is a countably infinite discrete subset of X.

Now argue as in the proof of 3.3. As X is a zero-dimensional T_3 space, we can find a family $(B_n)_{n \in \omega}$ of pairwise disjoint clopen subsets of X for which $B_n \cap D = \{p_n\}$ and $B_n \subseteq \bigcup_{i=1}^{n} I_i(X)$ (since $\bigcup_{i=1}^{n} I_i(X)$ is open and contains p_n). By 3.2(g), $D(B_n) = D(X) \cap B_n \supseteq \{p_n\} \neq \emptyset$ while $D(B_{n+1}) = \emptyset$, so $\mathrm{CB}(B_n) = n+1$. By 3.3 there exists $s_n \in G(B_n)$ such that $\mathrm{rg}(s_n|B_n) \geq n+1$. Define $s : X \to \mathbb{R}$ by $s|B_n = s_n$ and $s[X \setminus \bigcup_{n<\omega} B_n] = \{2\}$. As in 3.3, since $\bigcup_{n<\omega} B_n \in \mathrm{coz}(X)$ and hence is clopen in X_δ, it follows that $s \in C(X_\delta)$. But if $s \in G(X)$, there exists $k \in \mathbb{N}$ such that $\mathrm{rg}(s) = k$. Thus $\mathrm{rg}(s|B_n) \leq k$ for each $n \in \mathbb{N}$, which contradicts the choice of s whenever $n > k$. Consequently, $s \in C(X_\delta) \setminus G(X)$ and so X is not an RG-space. ∎

3.5. Corollary. *If X is an RG-space, then each compact subspace of X is an RG-space of finite Cantor–Bendixson order.*

Proof. This follows from 3.4 and 2.5. ∎

3.6. Example. There are many compact scattered (necessarily countable) metric spaces of infinite Cantor–Bendixson order. For example, if $\mathbb{N}^*$ is the one-point compactification of the countably infinite discrete space $\mathbb{N}$ and k is a positive integer, it is a straightforward exercise to prove that $\mathrm{CB}((\mathbb{N}^*)^k) = k+1$. If X is the one-point compactification of the free union of $(\mathbb{N}^*)^k$ as k varies over the positive integers, then clearly X is an example of the desired sort.

3.7. Example. Open subspaces of RG-spaces need not be RG-spaces. For example, let X be the space of 2.10 and X^* its one-point compactification. Clearly, X^* is a compact scattered space and $\mathrm{CB}(X^*) = 3$, so X^* is an RG-space by 3.4. However, its open subspace X is not.

3.8. Remark. Countable compact RG-spaces are subspaces of ordinal spaces. In fact, if X is a countable compact metric space for which $\mathrm{CB}(X) =$

n, then X is a subspace of the ordinal space $[0, \omega^{n-1}k]$, where $|I_{n-1}(X)| = k$. (See [Se1] and [Ba]).

3.9. EXAMPLE. Let $D \cup \{p\} = D^*$ be the one-point compactification of the discrete space D of cardinality $\aleph_1$. As in Example 3.6, we can easily show that $\mathrm{CB}((D^*)^k) = k + 1$ for each positive integer k.

Let $H(k)$ be the free union of $\aleph_1$ copies of $(D^*)^k$ and let X be the one-point compactification of the free union $\bigoplus\{H(k + 1) : k \in \omega\}$. Then it is straightforward to show that X is a compact almost P-space, and by 2.5 and 3.4 it fails to be an RG-space, as it has clopen subspaces of CB order k for each positive integer k. This answers Question 8.3 of [RW] in the negative.

3.10. EXAMPLE. In [Mr, 3.11] S. Mrówka gives an example of a separable locally compact pseudocompact space X of countable pseudocharacter with the following properties: (1) $I(X)$ is countable and dense in X; (2) $X \setminus I(X) = D_1(X)$ is discrete and of cardinality $\mathbf{c}$ $(= 2^{\aleph_0})$; (3) X is C^*-embedded in its one-point compactification X^*, i.e. $\beta X = X^*$. (In colloquial terms, X is a "ψ-like space"; see [GJ, 5I].)

Clearly, βX is scattered and $\mathrm{CB}(\beta X) = 3$, so by 3.4, βX is an RG-space. As X_δ is discrete and of cardinality $\mathbf{c}$, it follows that $|C(X_\delta)| = 2^{\mathbf{c}}$, while $|G(X)| = |C(X)| = \mathbf{c}$ (as X is separable). Hence X is not an RG-space. Because X is pseudocompact ([GJ, 6J]), it follows that $\beta X = \upsilon X$ ([GJ, 8A(4)]). Thus $C(X) \cong C(\upsilon X)$ (see [GJ, 8.8(a)]), X is not an RG-space, and υX is an RG-space. We conclude that whether a space X is an RG-space is not determined by the ring structure of $C(X)$.

3.11. EXAMPLE. We present a σ-compact nowhere locally compact space X for which X_δ is a Lindelöf P-space without isolated points. Although X is the union of countably many RG-spaces, it is not an RG-space and cannot be embedded in an RG-space.

Let D be an uncountable discrete space, and let $D^* = D \cup \{p\}$ be its one-point compactification. For each $i < \omega$ let D_i^* be a copy of D^*, and let $Y = \prod_{i<\omega} D_i^*$, the product of countably infinitely many copies of D^*. Points of Y will be denoted in boldface, and the ith component of a point $\mathbf{y} \in Y$ will be denoted by $y(i)$. The symbol $\mathbf{p}$ will denote the point for which $p(i) = p$ for each $i \in \omega$.

Our space X is defined to be the following subspace of Y:

$$\{\mathbf{y} \in Y : \{i < \omega : y(i) \neq p\} \text{ is finite}\}.$$

If F is a finite subset of ω, we define $X(F)$ to be $\{\mathbf{x} \in X : \text{if } i \notin F \text{ then } x(i) = p\}$. Clearly, $X(F)$ is homeomorphic to the finite product $\prod\{D_i^* : i \in F\}$, and hence is compact and scattered, as well as nowhere dense.

Clearly, $X = \bigcup\{X(F) : F \text{ is a finite subset of } \omega\}$. As ω has countably many finite subsets, it follows that X is σ-compact.

By 5.7 of [LR], $X(F)_\delta$ is Lindelöf for each finite subset F of ω. Since the topology that $X(F)$ inherits from X_δ is just the G_δ-topology $X(F)_\delta$, it follows that X_δ is a union of countably many Lindelöf subspaces and thus is Lindelöf.

To show that X_δ has no isolated points, it clearly suffices to show that no G_δ-set of Y meets X in a singleton set. Without loss of generality we can confine our attention to G_δ-sets of Y formed by intersecting countably many canonical basic open sets of Y. If $\mathbf{a} \in X$ and $\{i \in \omega : a(i) \neq p\} = F$, then the smallest possible such G_δ-set of Y that contains $\mathbf{a}$ will have the form

$$G = \prod_{i\in F}\{a_i\} \times \prod_{i\in\omega\setminus F}(D_i^* \setminus G_i),$$

where each G_i is a countable subset of D_i. Clearly, if $j \in \omega \setminus F$ and d is arbitrarily chosen in D, and if the point $\mathbf{e}$ in Y is defined by: $e(i) = a(i)$ if $i \in F$, $e(j) = d$, and $e(i) = p$ if $i \in \omega \setminus (F \cup \{j\})$, then $\mathbf{e} \in (G \cap X) \setminus \{a\}$. Thus a cannot be isolated in X and our claim is verified.

Observe that $C(X_\delta) = \mathrm{Baire}(X)$ by 5.5 of [LR]. It is easy to demonstrate that both X and $Y \setminus X$ are dense in Y, and so X is nowhere locally compact.

Finally we show that X is not an RG-space. Let $\{d(n) : n \in \omega\}$ be a faithfully indexed countable subset of D. For each $n \in \omega$ let

$$E(n) = \{d_n\} \times \Big(\prod_{i=2}^{n} D_i^*\Big) \times \prod_{i>n}\{p(i)\}.$$

Clearly, $E(n)$ is homeomorphic to $\prod_{i=2}^{n} D_i^*$, and for $n \geq 2$ it is easy to prove by induction that $\mathrm{CB}(\prod_{i=2}^{n} D_i^*) = n$. Furthermore, $E(n) \subseteq X$ for each $n \in \omega$.

Let $E = \bigcup\{E(n) : n < \omega\}$. Clearly, $\Pi_1^{\leftarrow}[\{d_n\}]$ is clopen in Y (here Π_1 denotes the projection map onto the first factor), and $(\Pi_1^{\leftarrow}[\{d_n\}] \cap X) \cap E = E(n)$. Thus each $E(n)$ is a compact clopen subset of E. As $\mathrm{CB}(E(n)) = n$, it follows from 3.3 and 2.8(b) that E is not an RG-space.

Now the argument used to prove that X_δ is Lindelöf shows that E_δ is Lindelöf. But E is not an RG-space, so by the contrapositive of 2.3(b) it follows that X is not an RG-space and in fact cannot be embedded in an RG-space. ∎

4. Questions. We have been unable to answer several reasonably obvious questions. Some are fundamental, while others are more technical. The most basic question is the following.

4.1. Question. Must every RG-space contain a P-point?

Observe that since cozero-sets of RG-spaces are also RG-spaces (see 2.3(f)), an affirmative answer to 4.1 implies that each RG-space would contain a dense set of P-points.

If the answer to 4.1 is affirmative, then so is that to 4.2 below.

4.2. Question. If a space X can be expressed as a union of nowhere dense zero-sets of X, does it follow that X is not an RG-space?

4.3. Question. Is every normal weakly Lindelöf scattered space of finite Cantor–Bendixson order an RG-space?

(Recall that a space is *weakly Lindelöf* if each open cover of the space has a countable subcollection whose union is dense in the space. Lindelöf, separable, and ccc spaces (i.e. spaces having no uncountable family of pairwise disjoint open subsets) are all weakly Lindelöf. Example 2.10 shows that the answer to 4.3 is "no" if "normal" is dropped from the list of hypotheses. We do not know whether "weakly Lindelöf" can be dropped. An affirmative answer would generalize 2.12.)

4.4. Question. Can the normality hypothesis be dropped from 2.3(d)?

4.5. Question. If a space X is an RG-space, must υX be an RG-space? (Example 3.10 shows that the converse fails.)

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