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The class of 2-dimensional neat reducts is not elementary

by

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Abstract. SC, CA, QA and QEA stand for the classes of Pinter's substitution algebras, Tarski's cylindric algebras, Halmos' quasipolyadic algebras and Halmos' quasipolyadic algebras with equality, respectively. Generalizing a result of Andréka and Németi on cylindric algebras, we show that for $K \in \{SC, QA, CA, QEA\}$ and any $\beta > 2$ the class of 2-dimensional neat reducts of β-dimensional algebras in K is not closed under forming elementary subalgebras, hence is not elementary. Whether this result extends to higher dimensions is open.

0. Introduction. Neat reducts and related notions like neat embeddings play a central role in Algebraic Logic. One of the main representation methods used in the theory of cylindric algebras and variants thereof like polyadic algebras is based on the *Neat Embedding Theorem*, to be recalled below. (See also [15, p. 400] and [16, Thm. 3.2.10]). It is known (cf. [29] or [31]) that the Neat Embedding Theorem, or NET for short, proved by Henkin in the fifties is an algebraization of Henkin's celebrated proof of the completeness of first order logic. Ever since, variants of the NET have been successfully applied to (algebraically) prove completeness of other versions of quantifier logics (e.g. Keisler's logics, cf. [12]). Other works adressing the notion of neat reducts, in one way or another, include [1], [29] and [27]. In [1] and [29], neat reducts are studied in connection to (isomorphism types of) algebras of sentences of first order logic. In [27] a NET is formulated and proved which implies the completeness of certain finitary fragments of Keisler's logics that are also expansions of first order logic without equality. This provides a solution to the so-called *finitization problem* in Algebraic Logic. In fact, the NET has proved to be a successful strategy to address different versions of the finitization problem (see e.g. [35]). Very

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roughly the finitization problem is: Find a *structural* (in the sense of [10]) extension or modification of first order logic that admits a *strongly complete* finite Hilbert-style axiomatization in the sense of [6]. The finitization problem, traced back at least to the works of Alfred Tarski on relation algebras in the forties (cf. [36]), is currently a very active and rich research direction in Algebraic Logic (cf. [24, the subsections on finitization], [27], [30] and [26]). The connection of neat reducts to other metalogical properties like the Beth-definability property and omitting types can be found in [30]. This (and other) connection(s) are investigated more thoroughly in [25], [20], [21], [23], [13], [14], [5], [8], [31] and [34]. We should point out that this list of references addressing the important notion of neat reducts in Algebraic Logic is far from being complete.

History (*Previous results*). Let $\alpha < \beta$ be ordinals. Solving problem 2.11 of [15], Németi proves in [23] that the class $\mathbf{N}r_\alpha CA_\beta$, of α-dimensional neat reducts of β-dimensional cylindric algebras, though closed under homomorphic images and products, is not a variety (i.e. it is not closed under forming subalgebras). Independently, Maddux [21] obtained a partial solution of problem 2.11 of [15] proving that $\mathbf{N}r_3 CA_\beta$ is not closed under forming subalgebras for $\beta \geq 5$. The significance of the closure of the class of neat reducts under forming subalgebras in connection to proving amalgamation results was discovered by Pigozzi [25, Lemma 2.2.12], and this tie is further emphasized and pursued in [34] and [31]. In [34] Németi's result—concerning closure of the class of neat reducts under forming subalgebras—is generalized to other classes frequently studied in Algebraic Logic. These include Halmos' quasipolyadic algebras and Pinter's substitution algebras and various reducts thereof. In particular, it is shown in [34] that for $K \in \{SC, QA, QEA\}$, the class $\mathbf{N}r_\alpha K_\beta$ of α-dimensional neat reducts of β-dimensional algebras in K is not closed under forming subalgebras for $\alpha \geq 2$ and $\beta > \alpha$. In particular, this class—though closed under homomorphic images and products and hence under ultraproducts ([1])—cannot be axiomatized by any set of equations (cf. [34]). Motivated to give some (hopefully first order) characterization of such classes, it is thus natural to ask whether the class of neat reducts is perhaps closed under forming *elementary* subalgebras, so that it is an elementary (i.e. first order axiomatizable) class. In [17, II.8.6, p. 266], Andréka and Németi provide a negative answer for the cylindric case but only for the lowest value of α. They show that the class $\mathbf{N}r_2 CA_\beta$ for $\beta > 2$ is not even closed under forming elementary subalgebras. That $\mathbf{N}r_1 CA_\beta$ is a variety is proved by Németi in [23]. $\mathbf{N}r_0 CA_\beta$ is just the class of boolean algebras and thus is also a variety. In [30] this result is generalized to higher dimensions, but only for cylindric

([1]) Recall that an ultraproduct is a homomorphic image of a product.

algebras. Thus for any pair of ordinals $1 < \alpha < \beta$ the class $\mathbf{N}r_\alpha CA_\beta$ is not closed under forming elementary subalgebras, hence this class is not elementary. This confirms a conjecture of Németi formulated in [23] and solves problem 4.4 of [16]. Here, as indicated in the abstract, we extend this result to Halmos' quasipolyadic algebras and Pinter's substitution algebras, but again only for the lowest values of α, namely when $\alpha = 2$. It seems that the generalization of this result to higher dimensions is not trivial. We conjecture, though, that it can proved—using a combination of the ideas presented herein with those in [30]—that for any pair of ordinals $2 < \alpha < \beta$ the class $\mathbf{N}r_\alpha K_\beta$ is not first order axiomatizable. As it is closed under ultraproducts, this amounts to showing that it is not closed under elementary equivalence or equivalently—by the celebrated Keisler–Shelah ultrapower theorem (cf. [11, Theorem 6.15])—under ultraroots.

Organization. In Section 0, we review the basic notions and formulate the main theorems. The proofs are given in Section 1.

0. Basic notions and main results. To formulate our results we review some basic notions from [16] and [24]. We start by defining the algebras we shall be dealing with. From now on, $K \in \{SC, CA, QA, QEA\}$, where SC, CA, QA and QEA abbreviate the classes of substitution, cylindric, quasipolyadic, and quasipolyadic equality algebras, respectively. K_α, α an ordinal, stands for the class of all algebras in K of dimension α, sometimes referred to as α-dimensional algebras. Algebras will be denoted by calligraphic letters, and when we write $\mathcal{A}$ then we will be tacitly assuming that A will denote the universe of $\mathcal{A}$. Let $\mathcal{A} \in K_\alpha$. Then $\mathcal{A}$ is a boolean algebra with extra operations. In fact, $\mathcal{A}$ will be a boolean algebra with operators. In particular, the non-boolean extra operations are required to distribute over the boolean join. The basic boolean operations of $\mathcal{A}$ will be denoted by $\cup$, $\cap$, $\smallsetminus$, 0, 1, standing for join, meet, complementation, least and greatest elements, respectively.

An algebra in SC_α is of the form $\mathcal{A} = \langle A, \cup, \cap, \smallsetminus, 0, 1, c_i, s_i^j\rangle_{i,j\in\alpha}$ where

(1) $\langle A, \cup, \cap, \smallsetminus, 0, 1\rangle$ is a boolean algebra (BA), hereafter denoted by $\mathrm{Bl}\,\mathcal{A}$.

(2) c_i and s_i^j $(i, j \in \alpha)$ are unary operations on A satisfying a finite schema of equations. As we shall be mostly dealing with the concrete versions of SC_α's, the so-called representable ones, we do not have to remember all those schemas; we recall from [24] only those we need.

PROPOSITION 0.1. *Let $\mathcal{A} \in SC_\alpha$. Then the following equations hold for any $i, j, k \in \alpha$ and any $x, y \in A$:*

(E_1) $\quad c_j0 = 0,\ x \le c_ix,\ c_i(x \cap c_iy) = c_ix \cap c_iy,$ *and* $c_ic_jx = c_jc_ix.$

In other words the c_i's are complemented closure operators and c_i, c_j commute.

(E_2) $s_i^i x = x$.
(E_3) s_j^i *are BA endomorphisms.*
(E_4) $s_j^i c_i x = c_i x$.
(E_5) $c_i s_j^i x = s_j^i x$ *whenever* $i \neq j$.
(E_6) $s_j^i c_k x = c_k s_j^i x$ *whenever* $k \notin \{i, j\}$.
(E_7) $c_i s_i^j x = c_j s_j^i x$.
(E_8) $s_j^i s_i^k c_i x = s_j^k c_i x$.

In our treatment of CA_α's, we follow the monographs [15], [16], while in our treatment of QA_α's and QEA_α's, we rather follow [28]. In more detail, we think of QA_α's as algebras of the form $\langle \mathcal{A}, p_{ij} \rangle_{i,j \in \alpha}$, where $\mathcal{A} \in SC_\alpha$, and for all $i, j \in \alpha$, p_{ij} are unary operations on A, which also happen to be boolean endomorphisms. QEA_α's, on the other hand, are expansions of QA_α's by diagonal elements $(d_{ij},\ i, j \in \alpha)$. For an explicit finite schema of equations axiomatizing the variety K_α, we refer the reader to the appendix of [24] ([2]). For the purposes of the present paper, however, it is, enough to know that all algebras considered are expansions of SC_α's; that is, if $\mathcal{A} \in K_\alpha$, its SC reduct obtained by deleting the p_{ij}'s and d_{ij}'s, if any, in symbols $\mathrm{Rd}_{SC}\, \mathcal{A}$, is (term equivalent to) an SC_α. In particular, if $\mathcal{A} \in K_\alpha$, then $\mathrm{Rd}_{SC}\, \mathcal{A} \models (E_1)$–$(E_8)$. Here by reduct we understand a generalized reduct. That is, the SC operations may not be basic in the algebras considered, but are term definable. For example, in the standard formalism of CA's [15] and QEA's [28], [24], s_j^i is not a basic operation but is term definable by $s_j^i = c_i(x \cap d_{ij})$. However when dealing with QEA's, and for that matter with CA's, we shall often treat the term definable operation s_j^i for all i, j, as a basic operation.

We now recall from [15, Def. 2.6.28] a notion that prevails throughout this paper, namely that of neat reducts.

Definition. 0.2. (i) Let $\mathcal{B} \in K_\beta$ and let $x \in B$. Then Δx, the *dimension set* of x, is defined to be the set $\{i \in \beta : c_i x \neq x\}$.

(ii) Let $\alpha < \beta$ be ordinals. Let $\mathcal{B} \in K_\beta$. Then the *neat α-reduct of* $\mathcal{B}$, in symbols $\mathcal{N}r_\alpha \mathcal{B}$, is the K_α with universe $Nr_\alpha B = \{b \in B : \Delta b \subseteq \alpha\}$, and whose operations are those of the similarity type of K_α restricted to $Nr_\alpha B$.

([2]) It is known that $\langle K_\alpha : \omega \leq \alpha \rangle$ forms a *system of varieties definable by schemas*, a concept originating with Monk from cylindric algebras, and further investigated in its own right by Andréka and Németi (cf. [16, pp. 260–263] for a sample of Andréka and Németi's results).

It is not hard to see that $\mathcal{N}r_\alpha\mathcal{B}$ is closed under the indicated operations, so the definition of $\mathcal{N}r_\alpha\mathcal{B}$ is sound. It is also easy to see that $\mathcal{N}r_\alpha\mathcal{B} \in K_\alpha$. We should point out that neat reducts are subreducts (subalgebras of reducts) in the universal algebraic sense: when forming the neat α-reduct of a K_β, we throw away not only operations, but also some elements of the algebra. We refer the reader to [7] where neat reducts were investigated in a (more general) "universal algebraic" setting.

NOTATION. For $L \subseteq K_\beta$, $\mathbf{N}r_\alpha L$ denotes the class of all neat α-reducts of algebras in L. That is, $\mathbf{N}r_\alpha L = \{\mathcal{N}r_\alpha\mathcal{B} : \mathcal{B} \in L\}$.

It is known (see e.g. [15] or [29, Chapter 4] that $S\mathbf{N}r_\alpha K_\beta$ is a variety. Here S stands for the operation of forming subalgebras. Borrowing terminology from [15, Remark 2.6.26], algebras in the class $S\mathbf{N}r_\alpha K_{\alpha+\omega}$ are said to have the *neat embedding property*, or NEP for short. $\mathcal{A} \in K_\alpha$ has the NEP iff $\mathcal{A}$ is neatly embeddable in a $K_{\alpha+n}$ for every $n \in \omega$. A central theorem in the representation theory of such algebras, the so-called *neat embedding theorem* of [15], states that the class of algebras in K_α having the NEP is precisely the class of representable K_α's (RK_α). In symbols (see [16, Theorem 3.2.10], [29, Sec. 5.2])

$$RK_\alpha = S\mathbf{N}r_\alpha K_{\alpha+\omega} = \bigcap_{n\in\omega} S\mathbf{N}r_\alpha K_{\alpha+n}.$$

According to the conventions of the monograph [15], [16], an RK_α is a K_α that is isomorphic to a subdirect product of (*weak*) *set algebras* of dimension α. To unify notation, we write Ks and Ks_α to denote K set algebras and K set algebras of dimension α, respectively. By the same token, WKs_α stands for weak K set algebras of dimension α, that is, $\mathcal{A} \in WKs_\alpha$ if A is a boolean field of sets with greatest element (unit) a *weak cartesian space*, i.e. a set of the form

$${}^\alpha U^{(p)} = \{s \in {}^\alpha U : |\{i \in \alpha : s_i \neq p_i\}| < \omega\}$$

for some U and $p \in {}^\alpha U$. In addition, A of course has to be closed under the extra non-boolean K_α operations. For the sake of completeness, we recall from [24] the concrete interpretation of the non-boolean QEA_α operations on α-ary relations over $V \subseteq {}^\alpha U$. Let $X \subseteq V$ and $i, j < \alpha$. Then

$$c_i(X) = \{s \in V : \text{there exists } t \in X \text{ such that } s_j = t_j \text{ for all } j \neq i\},$$
$$s_i^j(X) = \{s \in V : s \circ [i|j] \in X\}.$$

Here, $[i|j]$ denotes the transformation on α that sends i to j and is the identity on $\alpha \smallsetminus \{i\}$. Next

$$p_{ij}(X) = \{s \in V : s \circ [i,j] \in X\}$$

where $[i,j]$ denotes the transposition on α that interchanges i and j. And finally,

$$d_{ij} = \{s \in V : s_i = s_j\}.$$

Assume that $\mathcal{A} \in QEAs_2$ with base U, i.e. $\mathcal{A}$ has greatest element $U \times U$. Then $p_{01}R$ for a binary relation $R \in A$, $R \subseteq U \times U$, is just forming the converse of R. d_{01}, on the other hand, is simply the identity relation on U. For their geometric meaning, c_i is called the ith *cylindrification* while d_{ij} is called a *diagonal element*. $s_i^j(p_{ij})$ is called the *substitution* operation corresponding to the transformation $[i|j]([i,j])$, or even simply a *substitution*. This stems rather from their metalogical interpretation, the latter being substitution of the variable x_i for x_j (interchanging the free occurrences of x_i and x_j); cf. [24] or [29, Chapter 2], for further elaboration on such connections.

Unlike boolean algebras, not every K_α, $\alpha > 2$, is representable. In fact, the following is known. Let $\alpha < \beta$. Then $S\mathbf{N}r_\alpha K_\beta = K_\alpha = RK_\alpha$ if $\alpha \leq 1$. Here S denotes the operation of forming subalgebras. For $\alpha > 2$, the sequence $K_\alpha \supseteq S\mathbf{N}r_\alpha K_{\alpha+1} \supseteq S\mathbf{N}r_\alpha K_{\alpha+2} \supseteq \ldots$ is not eventually constant. This was proved and used by Monk for CA's to show that RCA_α, for $\alpha > 2$, is not finite schema axiomatizable ([3]). The analogous result for SC's, QA's and QEA's was proved by Johnson (the finite-dimensional case) [19], and Sain and Thompson (the infinite-dimensional case) [28]. When $\alpha = 2$, then all diagonal free algebras, i.e. algebras in SC_2 and QA_2, are representable (see [16, Theorem 5.4.33]). Though there are non-representable algebras in K_2 ($K \in \{CA, QEA\}$), RK_2 is finitely axiomatizable. Also $RK_2 = S\mathbf{N}r_2K_\beta$ for all $\beta > 2$. The latter two results are due to Henkin. The reader is referred to the comments in [16, below Theorem 4.1.44, pp. 125–126] for a fuller discussion of such results for CA's. The SC, QA and QEA analogues can be found in [3], [24] or/and [29, Chapter 4], and the references therein. We hasten to add that we shall be mostly dealing with (concrete) set algebras of dimension 2. We are now ready to formulate the main results concerning K_α's. Theorem 1 is proved in Section 1.

THEOREM 1 (Main Theorem). *Let $\beta > 2$. Let L be any class such that $WKs_\beta \subseteq L \subseteq K_\beta$. Then $\mathbf{N}r_2L$ is not closed under forming elementary subalgebras, hence is not elementary. In particular, the classes $\mathbf{N}r_2RK_\beta$ and $\mathbf{N}r_2K_\beta$ are not elementary.*

1. Proofs. In this section we prove Theorem 1. We construct an algebra $\mathcal{A} \in Ks_2 \cap \mathbf{N}r_2WKs_\beta$ for all $\beta > 2$ and $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B}$ is an elementary

([3]) Lately Hirsch, Hodkinson and Maddux [18] proved that the above sequence for $\omega > \alpha > 2$ is *strictly* decreasing, i.e. there is no $k \in \omega$ such that $\mathbf{S}Nr_\alpha CA_{\alpha+k+1} = S\mathbf{N}r_\alpha CA_{\alpha+k}$. This solves the long-standing open (neat embedding) problem 2.12 in [15]. It seems that their proof generalises to SC's, QA's and QEA's. The infinite analogue for CA's was settled by Pigozzi (unpublished).

subalgebra of $\mathcal{A}$ and $\mathcal{B}$ is not (isomorphic to an algebra) in $\mathbf{N}r_2K_\beta$ for all $\beta > 2$. We start off by fixing the notation.

Some notation. Our notation is mostly in conformity with that adopted in the monograph [15]. It is worthwhile though to review the notation mostly used.

For a set X, $\wp(X)$ denotes the set of all subsets of X, i.e. the powerset of X. Ordinals will be identified with the set of smaller ordinals. In particular, for finite n we have $n = \{0, \ldots, n-1\}$. ω denotes the least infinite ordinal, which is the set of all finite ordinals.

${}^AB$ denotes the set of functions from A to B. If $f \in {}^AB$ and $X \subseteq A$ then $f{\restriction}X$ denotes the restriction of f to X. We denote by $\operatorname{Do} f$ and $\operatorname{Rg} f$ the domain and range of a given function f, respectively. We frequently identify f with the sequence $\langle f_x : x \in \operatorname{Do} f\rangle$. We write fx or f_x or $f(x)$ to denote the value of f at x. We define composition so that the right hand function acts first, thus for given functions f, g, $f \circ g(x) = f(g(x))$ whenever the left hand side is defined, i.e.c when $g(x) \in \operatorname{Do} f$.

Let X be a set. Then $f(X)$ denotes the image of X under f, i.e. $f(X) = \{f(x) : x \in X\}$. $|X|$ denotes the cardinality of X and Id_X, or simply Id when X is clear from context, denotes the identity function on X. A set X is countable if $|X| \leq \omega$. Let Y be a set. Then $X \subseteq_\omega Y$ denotes that X is a finite subset of Y. For a given class K of algebras, $\mathbf{I}K$ denotes the class obtained by taking all isomorphic copies of algebras in K. Finally, given algebras $\mathcal{A}$ and $\mathcal{B}$ having the same signature, we let $\operatorname{Ism}(\mathcal{A}, \mathcal{B})$ denote the set of all isomorphisms of $\mathcal{A}$ into $\mathcal{B}$.

1.1. *Construction of the algebras.* R denotes the set of real numbers. Let $U = R \times 3$. Note that U is simply 3 disjoint copies of R. $\mathcal{C}$ denotes the *full* polyadic set algebra with unit ${}^2U$, that is,

$$\mathcal{C} = \langle \wp({}^2U), \cap, \cup, \smallsetminus, \emptyset, {}^2U, c_i, d_{ij}, p_{ij}, s_i^j\rangle_{i,j<2}.$$

Let $u = \langle u_0, u_1\rangle \in {}^23$. For $r \in R$, we define $p(u, r)$ to be the following binary relation on U:

$$p(u, r) = \{\langle\langle a_0, u_0\rangle, \langle a_1, u_1\rangle\rangle \in {}^2U : a_0 = a_1 + r\}$$

and we let

$$P(u) = \{p(u, r) : r \in R\}.$$

Note that $P(u)$ is an uncountable set of binary relations on U. Let N be a fixed countably infinite subset of R (4). That is, $N \subseteq R$ and $|N| = \omega$. We set

$$P_\omega(u) = \{p(u, r) : r \in N\}.$$

(4) We point out that countability of N is not essential. In fact any infinite subset N of R with $|N| < |R|$ will do just as well.

The subscript ω indicates that $P_\omega(u)$ has countably many binary relations. Finally we let

$$S = \{u \in {}^23 : u_0 \leq u_1\}.$$

For an algebra $\mathcal{A}$ and $X \subseteq A$, $\mathrm{Sg}^{\mathcal{A}} X$ or simply $\mathrm{Sg}\, X$ denotes the subalgebra of $\mathcal{A}$ generated by X.

Now we define $\mathcal{A} \in QEAs_2$ and $\mathcal{B} \subseteq \mathcal{A}$ as follows:

$$\mathcal{A} = \mathrm{Sg}^{\mathcal{C}}(\bigcup\{P(u) : u \in S\})$$

and

$$\mathcal{B} = \mathrm{Sg}^{\mathcal{A}}(\bigcup\{P(u) : u \in S \smallsetminus \{\langle 0, 2\rangle\}\} \cup P_\omega(\langle 0, 2\rangle)).$$

Plan of the proof. We will show that:

(1) $\mathcal{A} \in \mathbf{N}r_2 WQEAs_\beta$ for all $\beta > 2$.
(2) $\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$.
(3) $\mathrm{Rd}_{SC}\, \mathcal{B} \notin \mathbf{IN}r_2 SC_\beta$ for all $\beta > 2$.

From (1)–(3), Theorem 1 will immediately follow by passing to reducts. In more detail, let $K \in \{SC, QA, CA, QEA\}$ and let $\beta > 2$. Given $\mathcal{D} \in QEA_\alpha$, α an ordinal, let $\mathrm{Rd}_K\, \mathcal{D} \in K_\alpha$ denote the K_α reduct of $\mathcal{D}$. Then it follows from (1) that $\mathrm{Rd}_K\, \mathcal{A} \in Nr_2 WKs_\beta$, from (3) that $\mathrm{Rd}_K\, \mathcal{B} \notin \mathbf{I}Nr_2 K_\beta$ and from (2) that $\mathrm{Rd}_K\, \mathcal{B}$ is an elementary subalgebra of $\mathrm{Rd}_K\, \mathcal{A}$ (in the first order language of K_2).

1.2. *$\mathcal{A}$ is a neat reduct.* In this section we show that $\mathcal{A} \in \mathbf{N}r_2 WQEAs_\beta$ for all $\beta > 2$. This will be done in two steps. First we show that $\mathcal{A} \in \bigcap_{2 \leq k < \omega} \mathbf{N}r_2 QEAs_k$, then we show that $\mathcal{A} \in \mathbf{N}r_2 WQEAs_\beta$ for infinite β using a limiting construction. Throughout, unless otherwise specified, $n = \{0, \ldots, n-1\}$ denotes a finite ordinal > 1.

Notation. $\mathbf{R} = \langle R, +, -, r\rangle_{r \in R}$ denotes the group of real numbers expanded with constants. L denotes the first order language of $\mathbf{R}$. For ϕ an L-formula, we let $\mathrm{var}(\phi)$ denote the set of variables occurring in ϕ, and we let $\mathrm{fr}(\phi)$ denote the set of variables occurring free in ϕ.

Let $u = \langle u_0, \ldots, u_{n-1}\rangle \in {}^n3$. For ϕ an L-formula with $\mathrm{fr}(\phi) \subseteq \{x_0, \ldots, x_{n-1}\}$, we let $E(u, \phi)$ denote the following n-ary relation on U:

$$E(u, \phi) = \{\langle\langle a_0, u_0\rangle, \ldots, \langle a_{n-1}, u_{n-1}\rangle\rangle \in {}^nU : \mathbf{R} \models \phi[a_0, \ldots, a_{n-1}]\}.$$

We let

$$P(n) = \{E(u, x_i = x_j + r) : u \in {}^n3,\ r \in R \text{ and } i, j \in n,\ i < j\},$$
$$\mathcal{A}(n) = \mathrm{Sg}^{\mathcal{C}(n)}(P(n)),$$

where $\mathcal{C}(n)$ is the full $QEAs_n$ with unit ${}^nU$.

It is easy to see that $\mathcal{A} = \mathcal{A}(2)$.

We will show that for $1 < l < m < \omega$, $\mathcal{A}(l) \cong \mathcal{N}r_l\mathcal{A}(m)$. For that we need some more definitions:

$$F(n) = \{x_i = x_j + r,\ x_i \neq x_j + r : i, j \in n \text{ and } r \in R\}.$$

Forming finite conjunctions of formulas in $F(n)$, we let

$$F(n)^* = \{\bigwedge J : J \subseteq_\omega F(n)\}.$$

Forming finite disjunctions of formulas in $F(n)^*$, we let

$$F(n)^{**} = \{\bigvee J : J \subseteq_\omega F(n)^*\}.$$

Now set

$$G(n) = \{E(u, \phi) : u \in {}^n3 \text{ and } \phi \in F(n)^{**}\};$$

and finally

$$G(n)^* = \{\bigcup J : J \subseteq_\omega G(n)\}.$$

FACT 1.2.1. *Let $n > 1$. Then $A(n) = G(n)^*$.*

Proof. First we prove that $A(n) \subseteq G(n)^*$. Since $P(n) \subseteq G(n)^*$ and $\mathcal{A}(n)$ is generated by $P(n)$, it suffices to show that $G(n)^*$ is the universe of a QEA_n, i.e. is closed under the polyadic set operations.

(1) $G(n)^*$ *is closed under the boolean operations.* Indeed, let $u, v \in {}^n3$ and $\phi_1, \phi_2 \in F(n)^{**}$. Then

$$E(u, \phi_1) \cap E(v, \phi_2) = E(u, \phi_1 \wedge \phi_2) \quad \text{if } u = v$$

and is zero otherwise, while

$${}^nU \smallsetminus E(u, \phi_1) = \bigcup\{E(v, \top) : v \in {}^n3 \smallsetminus \{u\}\} \cup E(u, \neg\phi_1).$$

From the definition of $G(n)^*$, and by noting that for ϕ_1 and ϕ_2 in $F(n)^{**}$, there exist η and θ in $F(n)^{**}$ such that $\mathbf{R} \models \phi_1 \wedge \phi_2 \leftrightarrow \eta$ and $\mathbf{R} \models \neg\phi_1 \leftrightarrow \theta$, we get the desired conclusion.

(2) $G(n)^*$ *contains the diagonal elements.* Let $i < j < n$. Then

$$d_{ij} = \bigcup\{E(u, x_i = x_j) : u \in {}^n3,\ u_i = u_j\}.$$

and so by the definition of $F(n)$ we get $d_{ij} \in G(n)^*$.

(3) $G(n)^*$ *is closed under cylindrifications.* First a piece of useful notation. For $u, v \in {}^n3$ and $i < n$ we write $u \equiv_i v$ iff u and v agree off i, i.e. $u(j) = v(j)$ for all $j \neq i$. Now let $g \in G(n)^*$ and $i < n$. Since the c_i's ($\exists$) distribute(s) over joins ($\vee$) we can and will assume that $g = E(u, \phi)$ with $u \in {}^n3$ and $\phi \in F(n)^{**}$. Computing we get

$$c_i g = c_i E(u, \phi) = \bigcup\{E(v, \exists x_i\phi) : v \equiv_i u\}.$$

Now we make use of the following consequence of the fact that $\mathbf{R}$ is in a sense "saturated", which we abbreviate by $(*)$:

$(*)$ Given $\phi \in F(n)^{**}$ and $i < n$, there exists $\psi \in F(n)^{**}$ such that

$$x_i \notin \text{var}(\psi) \quad \text{and} \quad \mathbf{R} \models \exists x_i\phi \leftrightarrow \psi.$$

Note that $(*)$ is a form of elimination of quantifiers. Now applying $(*)$ we see that $c_i g \in G(n)^*$.

(4) *$G(n)^*$ is closed under substitutions.* Since $s^i_j(x) = c_i(x \cap d_{ij})$ for $i \neq j$ and $s^i_i(x) = x$, so that the s^i_j's are term definable, it suffices to check only those substitutions of the form p_{ij}. So let $\phi \in F(n)^{**}$, $u \in {}^n3$ and $i, j < n$ be distinct. Abusing notation slightly, we let $p_{ij}\phi$ stand for the formula obtained from ϕ by interchanging the free occurrences of x_i and x_j, and we let $p_{ij}u = u \circ [i, j]$. Since p_{ij} is a boolean endomorphism, it suffices to show that for all $u \in {}^n3$ and all $\phi \in F(n)$ we have $p_{ij}E(u, \phi) \in G(n)^{**}$. But this follows from the facts that

$$p_{ij}E(u, \phi) = E(p_{ij}u, p_{ij}\phi)$$

and that $p_{ij}\phi \in F(n)$ for $\phi \in F(n)$.

Next we show that $G(n)^* \subseteq A(n)$. It clearly suffices to show that for all $u \in {}^n3$ and $\phi \in F(n)$, we have $E(u, \phi) \in A(n)$. Let u and ϕ be as indicated. Suppose that ϕ is $x_i = x_j + r$ for some $i, j \in n$ and some $r \in R$. If $i < j$, then $E(u, \phi) \in P(n)$. If $i > j$ then $E(u, \phi) = E(u, x_j = x_i + (-r))$, which also belongs to $P(n)$. Thus we are left with the case $i = j$. We have two subcases.

Case a. $r \neq 0$ and so $E(u, \phi) = 0 \in A(n)$.

Case b. $r = 0$, then $E(u, \phi) = E(u, \top)$. In order to show that the latter belongs to $A(n)$, we let $k \in n \smallsetminus \{i\}$. This is possible because $n > 1$. Then a straightforward computation gives

$$c_i E(u, x_i = x_k) \cap c_k E(u, x_i = x_k) = E(u, \top),$$

thus the latter is in $A(n)$. Finally by noting that $E(u, \neg\phi) = E(u, \top) \smallsetminus E(u, \phi)$ we infer (using the above reasoning) that $E(u, x_i \neq x_j + r)$ is also in $A(n)$, for all $u \in {}^n3$ and $i, j \in n$. ∎

Having shown that $G(n)^* = A(n)$, and in particular that $G(n)^*$ is closed under the QEA_n operations, we let $\mathcal{G}(n)^*$ denote the $QEAs_n$ with universe $G(n)^*$.

For further use we shall need

Definition. For $m \geq 1$, let $\mathcal{C}(m)$ denote the full $QEAs_m$ with unit ${}^mU$, i.e. the universe of $\mathcal{C}(m)$ is $\wp({}^mU)$. Now let $1 \leq l < k$. Let $i(l, k)$ be the following (neat embedding) function:

$$i(l, k) : C(l) \to C(k), \quad a \mapsto \{t \in {}^kU : t{\restriction}l \in a\}.$$

Fact 1.2.2. *Let $1 < l < k < \omega$. Then $i(l, k)\mathcal{G}(l)^* = \mathcal{N}r_l\mathcal{G}(k)^*$. In particular, $\mathcal{A} \in \bigcap_{2 \leq k < \omega} \mathbf{N}r_2 QEAs_k$.*

Proof. Let $1 \leq l < k$. Assume that k, hence l, is finite. Then it is easy to check that

$$i(l,k)\upharpoonright G(l)^* \in \mathrm{Ism}(\mathcal{G}(l)^*, \mathcal{N}r_l\mathcal{C}(k)).$$

We claim that, in fact, we have $i(l,k)G(l)^* \subseteq G(k)^*$. To see this, let $u \in {}^l3$ and $\phi \in F(l)$. Then

$$\begin{aligned} i(l,k)E(u,\phi) &= \{s \in {}^kU : s\upharpoonright l \in E(u,\phi)\} \\ &= \bigcup\{E(v,\phi) : v \in {}^k3 \text{ and } v\upharpoonright l \subseteq u\}. \end{aligned}$$

The latter is in $G(k)^*$ because $F(l) \subseteq F(k)$, $\phi \in F(l)$ and $k < \omega$. Since $\mathcal{G}(k)^*$ is a $QEAs_k$, we find that $i(l,k)E(u,\phi) \in G(k)^*$ for any $u \in {}^l3$ and $\phi \in F(l)^{**}$. As $Nr_lG(k)^* = G(k)^* \cap Nr_lC(k)$, we deduce that

$$i(l,k)\upharpoonright G(l)^* \in \mathrm{Ism}(\mathcal{G}(l)^*, \mathcal{N}r_l\mathcal{G}(k)^*).$$

Next we show that $i(l,k)$ is actually onto $\mathcal{N}r_l\mathcal{G}(k)^*$, i.e. $i(l,k)G(l)^* = Nr_lG(k)^*$. This will be proved by using the fact that the structure $\mathbf{R}$ admits elimination of quantifiers as expressed in $(*)$ above. In more detail, since the c_i's are additive, it suffices to show that for all $g \in G(k)$, there exists $a \in G(l)^*$ such that

$$i(l,k)a = c_lc_{l+1}\dots c_{k-1}g.$$

So let $g = E(u,\phi)$ be as specified with $u \in {}^k3$ and $\phi \in F(k)^{**}$. Suppose further that $\phi = \phi_0 \vee \phi_1 \vee \dots \vee \phi_m$ with $\phi_j \in F(k)^*$ for $0 \leq j \leq m$. For brevity set $v = u\upharpoonright l$. Note that $v \in {}^l3$. Then an easy computation gives, for every $j \leq m$,

$$c_lc_{l+1}\dots c_{k-1}E(u,\phi_j) = i(l,k)E(v, \exists x_l\exists x_{l+1}\dots\exists x_{k-1}\phi_j).$$

Now $(*)$ guarantees the existence of $\psi_j \in F(l)^{**}$ for every $j \leq m$ such that

$$\mathbf{R} \models \exists x_l\exists x_{l+1}\dots\exists x_{k-1}\phi_j \leftrightarrow \psi_j.$$

Now computing we get

$$\begin{aligned} &c_lc_{l+1}\dots c_{k-1}E(u,\phi) = c_lc_{l+1}\dots c_l\dots c_{k-1}E(u,\phi_0 \vee \dots \vee \phi_m) \\ &\quad = c_lc_{l+1}\dots c_{k-1}E(u,\phi_0) \cup \dots \cup c_lc_{l+1}\dots c_{k-1}E(u,\phi_m) \\ &\quad = i(l,k)E(v,\psi_0) \cup \dots \cup i(l,k)E(v,\psi_m) = i(l,k)E(v,\psi_0 \vee \dots \vee \psi_m). \end{aligned}$$

Since by definition $\psi_0 \vee \dots \vee \psi_m \in F(l)^{**}$, we are done. ■

FACT 1.2.3. *Let $2 \leq n < \omega$ and β be infinite. Then $\mathcal{G}(n)^* \in \mathbf{N}r_nQEA_\beta$. In particular, $\mathcal{A} \in \mathbf{N}r_2QEA_\alpha$ for every $\alpha > 2$.*

Proof. Fix $\beta \geq \omega$. Let

$$\mathcal{G}(\beta) = \mathrm{Sg}^{\mathcal{C}(\beta)}(\bigcup\{i(k,\beta)G(k)^* : n \leq k < \omega\}).$$

Here, as in Fact 1.2.2, $\mathcal{C}(\beta)$ denotes the full Cs_β with unit ${}^\beta U$; and for $a \in G(k)^*$ recall that $i(k,\beta)a = \{s \in {}^\beta U : s\upharpoonright k \in a\}$. Let $a \in G(\beta)$. We will show that $a \in Nr_nG(\beta)$ iff $a \in i(n,\beta)G(n)^*$, from which it readily follows that $i(n,\beta)G(n)^* = Nr_nG(\beta)$, and so $\mathcal{G}(n)^* \cong \mathcal{N}r_n\mathcal{G}(\beta)$. First note that $\bigcup\{i(k,\beta)G(k)^* : n \le k < \omega\}$ is a subuniverse of $G(\beta)$. This is straightforward. Thus

$$G(\beta) = \bigcup\{i(k,\beta)G(k)^* : n \le k < \omega\}.$$

It then follows that $a = i(l,\beta)g$ for some $n \le l < \omega$ and $g \in G(l)^*$. Now suppose that $a \in Nr_nG(\beta)$. Then $g = c_jg$ for all $n \le j < \omega$ and so $g \in Nr_nG(l)^*$. By Fact 1.2.2 we have $Nr_nG(l)^* = i(n,l)G(n)^*$; thus $g = i(n,l)g'$ say, for some $g' \in G(n)^*$. But

$$a = i(l,\beta)g = i(l,\beta)\circ i(n,l)g' = i(n,\beta)g'.$$

This shows that $a \in i(n,\beta)G(n)^*$. Conversely if $a \in i(n,\beta)G(n)^*$, then it is easy to see that $a \in Nr_nG(\beta)$. We have shown that $i(n,\beta)G(n)^* = Nr_nG(\beta)$. Since $\mathcal{A} \cong \mathcal{G}(2)^{**}$, and $\mathcal{G}(\beta) \in QEA_\beta$, we conclude that $\mathcal{A} \in \mathbf{N}r_2QEA_\beta$ for all $\beta > 2$. ∎

In Facts 1.2.2–3, we have proved that $\mathcal{A} \in \bigcap_{2\le k<\omega}\mathbf{N}r_2QEAs_k$ and that $\mathcal{A} \in \mathbf{N}r_2QEA_\beta$ for infinite β, as well. In Fact 1.2.4 below we show that we can replace the class QEA_β by the smaller classes $WQEAs_\beta$ and $QEAs_\beta$ for infinite β. For this we need a definition.

Definition. Let β be an ordinal. We recall from [16, Def. 3.1.1(viii)] that $\mathcal{A} \in Ks_\beta$ with unit ${}^\beta U$ is *regular* if for all $x \in A$ and all $f, g \in {}^\beta U$, whenever $f{\restriction}\Delta x \subseteq g$ and $f \in X$ then $g \in X$. $\mathcal{A} \in K_\beta$ is *locally finite* if Δx is finite for every element in A.

Notation. Let β be an infinite ordinal. Then Ks_β^{reg} and $\mathrm{Lf}\,K_\beta$ stand for the classes of regular K set algebras, and locally finite K algebras of dimension β, respectively.

It is known (cf. [16, 3.1.70]) ([5]) that for infinite β we have

$$Ks_\beta^{\mathrm{reg}} \cap \mathrm{Lf}\,K_\beta \subseteq IWKs_\beta.$$

Fact 1.2.4. *$\mathcal{G}(n)^* \in \mathbf{N}r_nWQEAs_\beta$ for infinite β. In particular, $\mathcal{A} \in \mathbf{N}r_2WQEAs_\beta$ for infinite β.*

Proof. Fix $\beta \ge \omega$. Let $\mathcal{F}(\beta) = \mathrm{Sg}^{\mathcal{G}(\beta)}\, i(n,\beta)(\mathcal{G}(n)^*)$, where $\mathcal{G}(\beta)$, as usual, denotes the full quasipolyadic set algebra with unit ${}^\beta U$. Then, as is easily checked, we have $\mathcal{F}(\beta) \in QEAs_\beta^{\mathrm{reg}} \cap \mathrm{Lf}\,QEA_\beta$, and $\mathcal{G}(n)^* \cong \mathcal{N}r_n\mathcal{F}(\beta)$. Since $IQEAs_\beta^{\mathrm{reg}} \cap \mathrm{Lf}\,QEA_\beta \subseteq IWQEAs_\beta$, we are done. ∎

Remarks. (1) The class $\mathrm{Lf}\,K_\beta \cap Ks_\beta^{\mathrm{reg}}$ is the algebraic counterpart of first order models. For more on such connections the reader is referred to [24],

([5]) We should point out that Theorem 3.1.70 in [16] is formulated only for cylindric algebras. However the proof easily adapts to SC's, QA's and QEA's, as shown for example in [29] or [3].

[16, Sec. 4.3], [3] and [29, Chapter 4]. In fact, one can define an isomorphism h : Models $\to \mathrm{Lf}\, K_\omega \cap Ks_\omega^{\mathrm{reg}}$ definable in ZFC by an absolute formula without parameters (cf. [23, p. 406]).

(2) Following the referee, we give a simpler description of the algebra $\mathcal{A}$. Consider a first order language L consisting of binary relation symbols δ_u^r for all $u \in {}^2 3$ and all real numbers r. Let M be the L-structure with domain consisting of three disjoint copies R_0, R_1, R_2 of the real numbers, and with

$$M \models \delta_u^r(a,b) \quad \text{iff} \quad a \in R_{u_0},\ b \in R_{u_1}, \text{ and } b - a = r.$$

Now $\mathcal{A}$ is isomorphic to the first order definable subsets of ${}^2 M$. In more detail, for $\beta > 2$, any $v \in {}^\beta M$ can be regarded as an assignment of variables x_i $(i < \beta)$ to M. Let $\mathcal{D}$ be the set of all subsets of ${}^\beta M$ of the form $\{v \in {}^\beta M : M \models \phi[v]\}$ where ϕ is a first order L-formula written with variables x_i, $i < \beta$, and the operations are defined as for set algebras. Here we are adopting the usual notion: $M \models \phi[v]$ if v satisfies ϕ in M. Then, of course, $\mathcal{D} \in QEAs_\beta$. In fact, $\mathcal{D} \in \mathrm{Lf}\, QEA_\beta \cap QEAs_\beta^{\mathrm{reg}}$. On the other hand, $\mathcal{A}$ is isomorphic to the set algebra consisting of all subsets of ${}^2 M$ of the form $\{v \in {}^2 M : M \models \phi[v]\}$ where ϕ is an L-formula written with two variables x_0, x_1, and with operations defined by restricting those of $\mathcal{D}$ to the first two dimensions. By the proof of Fact 1.2.1, M has quantifier elimination, so $\mathcal{A} \cong \mathcal{N}r_2\mathcal{D}$. This also proves Fact 1.2.4.

Henceforth we only deal with the 2-dimensional polyadic set algebras $\mathcal{A}$ and $\mathcal{B}$.

1.3. *$\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$.* In this section we show that the $QEAs_2$ $\mathcal{A}$ admits a particularly rich set of automorphisms, a property we use to show that $\mathcal{B}$ is elementarily equivalent to $\mathcal{A}$.

Notation. For $u = \langle u_0, u_1\rangle \in {}^2 3$ we let $1_{u_0 u_1}$, and sometimes simply 1_u, denote $E(u, \top)$. For $a \in A$ and $X \subseteq A$, we let $\mathrm{Rl}_a X$ denote the *relativization* of X to a, that is, $\mathrm{Rl}_a X = \{x \in X : x \le a\}$.

Fact 1.3.1. *Let $u \in {}^2 3$. For brevity, set $V = E(u, \top)$. Then*:

(i) $\mathrm{Rl}_V A = \{E(u,\phi) : \phi \in F(u)^{**}\}$.

(ii) $\mathrm{Rl}_V A$ *is an atomic boolean algebra and the set of its atoms is equal to $P(u) = \{p(u,r) : r \in R\}$. In particular*, $\mathrm{Rl}_V A$ (*as a boolean algebra*) *is generated by its atoms.*

(iii) *For all non-zero $a \in \mathrm{Rl}_V A$ and for all $i < 2$ we have $c_i a = c_i V$.*

(iv) *For all $a \in A$ and for all $i < 2$ we have $c_i a \cap V \in \{0, V\}$.*

Proof. (i) follows from the fact that $\mathcal{A} = \mathcal{G}(2)$. (ii) follows from the fact that for $u \in {}^2 3$ and r_1, r_2 distinct elements of R we have

$$p(u, r_1) \cap p(u, r_2) = 0.$$

(iv) follows from (iii). Now we prove (iii).

Let $a \in \mathrm{Rl}_V A$. Then $a \le V$. By (i), $a = E(u, \phi)$ for some $\phi \in F(2)^{**}$. Now suppose that $0 \ne a$ and $i < 2$. Then by (ii) there exists an atom in $\mathrm{Rl}_V A$ below a, i.e. an $r \in R$ such that $E(u, x_0 = x_1 + r) = p(u, r) \le a$. Now we have

$$\begin{aligned} c_i a \le c_i V &= \bigcup\{E(v, \top) : v \in {}^2 3,\ v \equiv_i u\} \\ &= c_i E(u, x_0 = x_1 + r) = c_i(p(u, r)) \le c_i a. \end{aligned}$$

We have proved (iii). ∎

Remark. Notice that the set $\{1_u : u \in {}^2 3\}$ forms a partition of the unit $1 = \bigcup\{1_u : u \in {}^2 3\}$ of the algebra $\mathcal{A}$ so that the boolean structure of $\mathcal{A}$ is rather simple. It is isomorphic to the product of $\mathrm{Rl}_{1_u} A$ indexed by ${}^2 3$, the isomorphism being

$$h(a) = \langle a \cap 1_u : u \in {}^2 3 \rangle \quad \text{for } a \in A.$$

$\mathrm{Rl}_{1_u} A$ has an even simpler boolean structure. It is isomorphic to the boolean algebra (with universe) $\wp(R)$ generated by the singletons, which is the same as the finite co-finite boolean algebra on R, i.e. the algebra with universe

$$\{X \subseteq R : X \text{ or } R \smallsetminus X \text{ is finite}\}.$$

Note too that $P(u)$ constitutes further an uncountable partition of 1_u that is a *splitting* of 1_u in the sense of Andréka [2]. This means that every element in $P(u)$ is cylindrically equivalent to 1_u, i.e. $c_i x = c_i 1_u$ for every $i < 2$ and every $x \in P(u)$. So in a sense (at the least in the above one) the elements in $P(u)$ are "big". Being atoms in $\mathrm{Rl}_{1_u} A$ (and even more, as is easily checked, in $\mathcal{A}$), they are also "small". Andréka [2] refers to such elements as *big atoms*. It is precisely the "duality" in the nature of such elements (the big atoms) that will enable us, in Fact 1.3.2 below, to extend given permutations on the set $P = \{p(u, r) : u \in {}^2 3,\ u_0 \le u_1,\ r \in R\}$ to automorphisms on $\mathcal{A}$. $\mathcal{A}$ in this respect resembles the free structure on P.

Fact 1.3.2. *Let $P = \{p(u, r) : u \in {}^2 3,\ u_0 \le u_1,\ r \in R\}$. Let f be a permutation of $P(\langle 0, 2\rangle)$. Then there exists an automorphism h of $\mathcal{A}$ such that $f \subseteq h$ and $h{\restriction}(P \smallsetminus [P(\langle 0, 2\rangle) \cup P(\langle 2, 0\rangle)]) = \mathrm{Id}$.*

Proof. Let f be a given permutation of $P(\langle 0, 2\rangle)$. Define

$$f^* = p_{01} \circ f \circ p_{01}.$$

Then, as is easily checked, f^* is a permutation of $P(\langle 2, 0\rangle)$. By Fact 1.3.1, $\mathrm{Rl}_{1_{02}} A$ is generated as a boolean algebra by the set $P(\langle 0, 2\rangle)$ of its atoms. It follows that there exists an automorphism h_1 of $\mathrm{Bl}\,\mathrm{Rl}_{1_{02}} A$ such that $f \subseteq h_1$. Similarly there exists an automorphism h_2 of $\mathrm{Bl}\,\mathrm{Rl}_{1_{20}} A$ such that $f^* \subseteq h_2$. For $x \in A$ let

$$h(x) = h_1(x \cap 1_{02}) \cup h_2(x \cap 1_{20}) \cup \bigcup\{x \cap 1_u : u \in {}^2 3 \text{ and } \mathrm{Rg}\, u \ne \{0, 2\}\}.$$

We show that h is the desired automorphism. Clearly h is a boolean automorphism that extends f and is the identity on $P \smallsetminus [P(\langle 0,2\rangle) \cup P(\langle 2,0\rangle)]$. So all we have to check is that h also preserves the extra non-boolean operations.

(1) *h preserves the diagonal element.* Indeed,

$$h(d_{01}) = h_1(d_{01} \cap 1_{02}) \cup h_2(d_{01} \cap 1_{20}) \cup \bigcup\{d_{01} \cap 1_u : u \in {}^2 3 \text{ and } \operatorname{Rg} u \neq \{0,2\}\}.$$

Since $d_{01} \cap 1_{02} = d_{01} \cap 1_{20} = 0$, and h_1, h_2 are automorphisms of $\mathrm{Rl}_{1_{02}} A$ and $\mathrm{Rl}_{1_{20}} A$, respectively, so that $h_1(d_{01} \cap 1_{02}) = h_2(d_{01} \cap 1_{20}) = 0$, we have

$$h(d_{01}) = \bigcup\{d_{01} \cap 1_u : u \in {}^2 3 \text{ and } \operatorname{Rg} u \neq \{0,2\}\} = d_{01}.$$

(2) *h preserves* p_{01}. Let $Y = \{p(u,r) : u \in {}^2 3,\ r \in R\}$. Let $X = \{c_i a : a \in A,\ i < 2\} \cup \{d_{01}\}$, and let $\mathcal{D}$ be the boolean algebra generated by $X \cup Y$. We claim that D is a subuniverse of $\mathcal{A}$. Since D contains d_{01} and is closed under the boolean set operations, it suffices to show that D is closed under c_0, c_1 and p_{01}. So let $a \in D$ and $i < 2$. As $D \subseteq A$ we have $a \in A$ and so, by definition, $c_i a \in D$. Since p_{01} is a boolean endomorphism, it suffices to check that for $x \in Y \cup X$ we have $p_{01} x \in D$. But this is true because $x \in Y$ iff $p_{01} x \in Y$, $x \in X$ iff $x = 0$ or there is a subset J of ${}^2 3$ such that $x = \{1_u : u \in J\}$ and $p_{01} 1_{u_0 u_1} = 1_{u_1 u_0}$. Thus we have shown that D is a subuniverse of $\mathcal{A}$. Since $Y \subseteq D$, Y generates $\mathcal{A}$, and $D \subseteq A$, we get $D = A$. Now let $X' = \{1_u : u \in {}^2 3\} \cup \{d_{01}\}$ and let $\mathcal{D}'$ be the boolean algebra generated by $X' \cup Y$. Then $D = D'$ because every non-zero element of X is a finite union of elements in X'. Thus $D' = A$. Since p_{01} is a boolean endomorphism, to prove that h preserves p_{01}, it is enough to show that $h(p_{01} x) = p_{01} h(x)$ for all $x \in Y \cup X'$. To this end, let $x \in Y \cup X'$. If $u \in {}^2 3$, $\operatorname{Rg} u \neq \{0,2\}$ and $x \in \{p(u,r), 1_u\}$, then

$$p_{01} x \cap 1_v = 0 \qquad \text{for } v \in \{\langle 0,2\rangle, \langle 2,0\rangle\}$$

thus

$$\begin{aligned} h(p_{01} x) &= \bigcup\{p_{01} x \cap 1_u : u \in {}^2 3 \text{ and } \operatorname{Rg} u \neq \{0,2\}\} \\ &= p_{01} x = p_{01} h(x) \qquad (\text{since } h(x) = x). \end{aligned}$$

Else

$$x = p(u,r) \quad \text{or} \quad x = 1_u \quad \text{with} \quad \operatorname{Rg} u = \{0,2\}.$$

Suppose that $u = \langle 0,2\rangle$ and that $x = p(\langle 0,2\rangle, r)$. (The subcases $x = 1_{02}$ or $x = 1_{20}$ or $x = p(\langle 2,0\rangle, r)$ can be treated analogously and are left to the reader.) By noting that $p_{01} x = p(\langle 2,0\rangle, -r)$ and computing we get

$$\begin{aligned} h(p_{01} x) &= h_2(p(\langle 2,0\rangle, -r)) = f^*(p(\langle 2,0\rangle, -r)) = p_{01} \circ f \circ p_{01}[p(\langle 2,0\rangle, -r)] \\ &= p_{01} \circ f \circ p_{01} \circ p_{01}(x) = p_{01} \circ f(x) = p_{01}(h(x)). \end{aligned}$$

(3) Finally we check cylindrifications. Let $k < 2$ and $x \in A$. Then computing we get

$$(*) \quad c_k h(x)$$
$$= c_k(h_1(x \cap 1_{02}) \cup h_2(x \cap 1_{20}) \cup \bigcup_{u \in {}^2 3}\{x \cap 1_u : \operatorname{Rg} u \neq \{0,2\}\})$$
$$= c_k h_1(x \cap 1_{02}) \cup c_k h_2(x \cap 1_{20}) \cup \bigcup_{u \in {}^2 3}\{c_k(x \cap 1_u) : \operatorname{Rg} u \neq \{0,2\}\}.$$

Since h_1 is an automorphism of $\mathrm{Rl}_{1_{02}} A$ and $x \cap 1_{02} \leq 1_{02}$, we get

$$h_1(x \cap 1_{02}) \leq h_1(1_{02}) = 1_{02}.$$

Since $x \cap 1_{02} = 0$ iff $h_1(x \cap 1_{02}) = 0$, by Fact 1.3.1(iii) we have

$$c_k(h_1(x \cap 1_{02})) = c_k(x \cap 1_{02}).$$

By exactly the same reasoning it follows that

$$c_k(h_2(x \cap 1_{20})) = c_k(x \cap 1_{20}).$$

Substituting in $(*)$ we obtain

$$c_k h(x) = c_k(x \cap 1_{02}) \cup c_k(x \cap 1_{20}) \cup \bigcup_{u \in {}^2 3}\{c_k(x \cap 1_u) : \operatorname{Rg} u \neq \{0,2\}\}$$
$$= c_k(x \cap \bigcup\{1_u : u \in {}^2 3\}) = c_k x.$$

Now computing $h(c_k x)$ we get

$$(**) \qquad h(c_k x) = h_1(c_k x \cap 1_{02}) \cup h_2(c_k x \cap 1_{20})$$
$$\cup \bigcup_{u \in {}^2 3}\{c_k x \cap 1_u : \operatorname{Rg} u \neq \{0,2\}\}.$$

That $h_1(c_k x \cap 1_{02}) = c_k x \cap 1_{02}$ and $h_2(c_k x \cap 1_{20}) = c_k x \cap 1_{20}$ follows from Fact 1.3.1(iv) upon noting that h_1 and h_2 are boolean automorphisms; thus we have

$$h(c_k x) = c_k x \cap \bigcup\{1_u : u \in {}^2 3\} = c_k x.$$

We have shown that $c_k h(x) = h_c k(x) = c_k x$ (6), thus h preserves cylindrifications. ∎

FACT 1.3.3. *$\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$.*

Proof. We shall use the Tarski–Vaught criterion to show that $\mathcal{B}$ is an elementary subalgebra of $\mathcal{A}$. So let $\exists x_n \phi(x_0, \ldots, x_n)$ be any first order formula in the language of QEA_2. Suppose that $b_0, \ldots, b_{n-1} \in B$, and that $\mathcal{A} \models \phi(b_0, \ldots, b_{n-1}, a)$ for some $a \in A$. We want to find $b_n \in B$ such that $\mathcal{A} \models \phi(b_0, \ldots, b_{n-1}, b_n)$. Since $\mathcal{A}$ is generated by $P = \{p(u,r) : u \in {}^2 3,\ u_0 \leq u_1,\ r \in R\}$, there exists a finite subset H_0 of P such that $a \in \operatorname{Sg}^{\mathcal{A}} H_0$. Let

$$P' = \{p(u,r) : u \in {}^2 3,\ u_0 \leq u_1,\ u \neq \langle 0,2\rangle,\ r \in R\} \cup P_\omega(\langle 0,2\rangle).$$

(6) We used the "bigness" condition to show that h preserves cylindrifications. Loosely speaking, when a cylindric or polyadic algebra $\mathcal{A}$ is generated by a set of big atoms, then we can forget about cylindrifications when constructing automorphisms of $\mathcal{A}$. This method is also referred to as the method of eliminating cylindrifications.

Since $b_0, \ldots, b_{n-1} \in B$, and $\mathcal{B}$ is generated by P', there exists a finite subset H_1 of P' such that $(H_0 \cap B) \cup \{b_0, \ldots, b_{n-1}\} \subseteq \mathrm{Sg}^{\mathcal{B}} H_1$. Let f be a permutation of $P(\langle 0,2\rangle)$ such that

$$f{\restriction}P(\langle 0,2\rangle) \cap H_1 \subseteq \mathrm{Id} \quad \text{and} \quad f(H_0 \smallsetminus H_1) \subseteq P_\omega(\langle 0,2\rangle).$$

Such a permutation exists, since $|H_0| < \omega$, $|H_1| < \omega$ and $|P_\omega(\langle 0,2\rangle)| = \omega$. By Fact 1.3.2, f extends to an automorphism h of $\mathcal{A}$ such that

$$h{\restriction}(P \smallsetminus P(\langle 0,2\rangle)) \subseteq \mathrm{Id}.$$

But

$$\mathcal{A} \models \phi(b_0, \ldots, b_{n-1}, a)$$

and so

$$\mathcal{A} \models \phi(h(b_0), \ldots, h(b_{n-1}), h(a)).$$

Since $h{\restriction}H_1 \subseteq \mathrm{Id}$ and $\{b_0, \ldots, b_n\} \subseteq \mathrm{Sg}^{\mathcal{B}} H_1$, it follows that $h(b_i) = b_i$ for all $i < n$. Since $h(H_0) \subseteq B$ and $a \in \mathrm{Sg}^{\mathcal{A}} H_0$, we get $h(a) \in B$. This completes the proof. ■

Remark. We have shown that first order logic ($L_{\omega,\omega}$) cannot distinguish between $\mathcal{A}$ and $\mathcal{B}$. It can be proved that stronger logics like $L_{k,\omega}$ where k is an infinite cardinal also *cannot* distinguish between $\mathcal{A}$ and $\mathcal{B}$. Recall that $L_{k,\omega}$ is obtained from $L_{\omega,\omega}$ by adding k-ary conjunction to the logical connectives. In particular, $\mathcal{B}$ is a complete subalgebra of $\mathcal{A}$. That is, if $X \subseteq B$ is such that $\sup X$ (the supremum of X) exists in A, then $\sup X$ exists in B, and they are equal, in symbols $\sum^{\mathcal{A}} X = \sum^{\mathcal{B}} X$.

1.4. *$\mathcal{B}$ is not a neat reduct.* Here we show that $\mathcal{B}$ is not a neat reduct. Recall that $\mathcal{B}$ was obtained from $\mathcal{A}$ by an infinite "cardinality twist", namely by keeping only countably many atoms below 1_{Id} and deleting the rest. In this section we show that first order logic cannot "see" this infinite cardinality twist. We start off by showing that the set of *all* elements in B below 1_{02} remains countable.

Fact 1.4.1. *The set $\{b \in B : b \le 1_{02}\}$ is countable.*

Proof. Let $Y = \{p(u,r) : u \in {}^2 3 \text{ and } \mathrm{Rg}\, u \ne \{0,2\}\} \cup P_\omega(\langle 0,2\rangle) \cup P_\omega(\langle 2,0\rangle)$. (Recall that $P_\omega(\langle u_0,u_1\rangle) = \{p(\langle u_0,u_1\rangle, r) : r \in N\}$, where N is a fixed countable subset of R.) Let $X = \{c_i a : i < 2,\ a \in A\} \cup \{d_{01}\}$. Let $\mathcal{D}$ be the boolean subalgebra of $\mathcal{A}$ generated by $Y \cup X$. Then D is a subuniverse of $\mathcal{A}$; the proof is similar to that of (2) in Fact 1.3.1 and so we omit it. Since $Y \subseteq D$, we have $B \subseteq D$ (in fact, $B = D$, but we do not need that much), and so $\mathrm{Rl}_{1_{02}} B \subseteq \mathrm{Rl}_{1_{02}} D$. It clearly suffices to show that $|\mathrm{Rl}_{1_{02}} D| \le \omega$. To this end, let $\mathrm{rl}_{1_{02}}$ denote the following endomorphism of the structure $\langle A, \cup, \cap, \smallsetminus\rangle$:

$$\mathrm{rl}_{1_{02}}\, a = a \cap 1_{02}, \qquad a \in A.$$

Then $\mathrm{Rl}_{1_{02}} D = \mathrm{Sg}^{\mathrm{Bl}(\mathcal{A})} \mathrm{rl}_{1_{02}}(X \cup Y)$. But $\mathrm{rl}_{1_{02}}(X \cup Y)$ is countable because $|P_\omega(\langle 0, 2\rangle)| = \omega$ and $|X| < \omega$. Therefore $\mathrm{Rl}_{1_{02}} D$, generated as a boolean algebra by a countable set, is also countable. ∎

Next we show that, were $\mathcal{B}$ a neat reduct, $\mathrm{Rl}_{1_{02}} B$ would be an uncountable set. This contradicts Fact 1.4.1. From this we infer that $\mathcal{B}$ is not a neat reduct. For this we need:

FACT 1.4.2. *Let $\tau(x, y)$ denote the SC_2 term $c_1(c_0x \cap s^0_1c_1y) \cap c_1x \cap c_0y$. Let $\tau_3(x, y)$ denote the SC_3 term $c_2(s^1_2c_2x \cap s^0_2c_2y)$. Then*:

(i) $\tau^{\mathcal{B}}(1_{01}, 1_{12}) = 1_{02}$.

(ii) $SC_3 \models \tau_3(x, y) \le \tau(c_2x, c_2y)$.

Proof. (i) We have

$$\tau^{\mathcal{B}}(1_{01}, 1_{12}) = c_1(c_01_{01} \cap s^0_1c_11_{12}) \cap c_11_{01} \cap c_01_{12}.$$

We first compute $s^0_1c_11_{12}$:

$$\begin{aligned} s^0_1c_11_{12} &= c_0(d_{01} \cap (1_{12} \cup 1_{11} \cup 1_{10})) = c_0(d_{01} \cap 1_{11}) \\ &= c_01_{11} \qquad \text{(by Fact 1.3.1(iv))} \\ &= c_01_{01}. \end{aligned}$$

Thus

$$\tau^{\mathcal{B}}(1_{01}, 1_{12}) = c_1(c_01_{01} \cap c_01_{01}) \cap c_11_{01} \cap c_01_{12} = c_11_{01} \cap c_01_{12} = 1_{02}.$$

(ii) In our derivation we use the axiomatization given in Proposition 0.1. Now computing we get

$$\begin{aligned} \tau_3(x, y) &= c_2(s^1_2c_2x \cap s^0_2c_2y) \\ &\le c_2(s^1_2(c_0c_2x \cap c_1c_2x) \cap s^0_2(c_0c_2y \cap c_1c_2y)) \\ &= c_2(s^1_2c_0c_2x \cap s^1_2c_1c_2x \cap s^0_2c_0c_2y \cap s^0_2c_1c_2y) \\ &= c_2(s^1_2c_0c_2x \cap c_1c_2x \cap c_0c_2y \cap s^0_2c_1c_2y)) \\ &= c_2(s^1_2c_0c_2x \cap s^0_2c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= c_2(s^1_2c_0c_2x \cap s^1_2s^0_1c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= c_2s^1_2(c_0c_2x \cap s^0_1c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= c_1s^2_1(c_0c_2x \cap s^0_1c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= c_1s^2_1c_2(c_0c_2x \cap s^0_1c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= c_1c_2(c_0c_2x \cap s^0_1c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= c_1(c_0c_2x \cap s^0_1c_1c_2y) \cap c_1c_2x \cap c_0c_2y \\ &= \tau(c_2x, c_2y). \end{aligned}$$

REMARK. $\tau_3(x, y)$ is more commonly denoted in the literature by $x; y$. $x; y$ abstracts composition of binary relations (cf. [16]).

FACT 1.4.4. *Let* $\mathrm{Rd}_{SC}\,\mathcal{B}$ *denote the SC reduct of* $\mathcal{B}$.

(i) *Suppose that* $\mathrm{Rd}_{SC}\,\mathcal{B} \subseteq \mathcal{N}r_2(\mathcal{C})$ *with* $\mathcal{C} \in SC_3$. *Let*

$$D = \{\tau_3^{\mathcal{C}}(x,y) : x, y \in Nr_2(C),\ x \leq 1_{01},\ y \leq 1_{12}\}.$$

Then $|D| > \omega$. *In fact*, $P(\langle 0,2\rangle) \subseteq D$.

(ii) $\mathrm{Rd}_{SC}\,\mathcal{B} \notin Nr_2 SC_3$.

Proof. (i) Assume first that $\mathcal{C}$ is a set algebra. Then the result follows by noting that

$$\tau_3^{\mathcal{C}}(p(\langle 0,1\rangle, r_1), p(\langle 1,2\rangle, r_2)) = p(\langle 0,2\rangle, r_1 + r_2).$$

Now if $\mathcal{C}$ is not a set algebra, then replacing $\mathcal{C}$ by $\mathcal{E} = \mathrm{Sg}^{\mathcal{C}}\, B$, we deduce that $\mathcal{E}$ is isomorphic to a set algebra by [22] ([7]). This is so because $\mathcal{E}$ is an atomic algebra that is generated by a set of atoms, namely the set of atoms of B that are functions. Indeed, every $p(u,r)$ for $u \in {}^2 3$ and $r \in R$ is a binary relation that is a function. The same argument as above, but now applied to $\mathcal{E}$, works.

(ii) Assume, seeking a contradiction, that $\mathrm{Rd}_{SC}\,\mathcal{B} = \mathcal{N}r_2\mathcal{C}$, say, with $\mathcal{C} \in SC_3$. Let $x, y \in B$ be such that $x \leq 1_{01}$ and $y \leq 1_{12}$. Then x and y are 2-*closed*, i.e. $c_2 x = x$ and $c_2 y = y$. Also $\tau_3^{\mathcal{C}}(x,y) \in B$. Now

$$\begin{aligned}\tau_3^{\mathcal{C}}(x,y) &\leq \tau_3^{\mathcal{C}}(1_{01}, 1_{12}) \leq \tau^{\mathcal{C}}(c_2 1_{01}, c_2 1_{12}) = \tau^{\mathcal{B}}(1_{01}, 1_{12}) \\ &= 1_{02} \quad \text{(by Fact 1.4.2(i))}.\end{aligned}$$

In other words we have

$$D = \{\tau_3^{\mathcal{C}}(x,y) : x \leq 1_{01},\ y \leq 1_{12}\} \subseteq \{b \in B : b \leq 1_{02}\}.$$

By Fact 1.4.1, D must be countable. But this contradicts Fact 1.4.4(i). Thus, the proof of Theorem 1 is complete. ■

REMARK. We have actually proved the following: Let $\beta > 2$. Let L be any class such that $Ks_\beta^{\mathrm{reg}} \cap \mathrm{Lf}\, K_\beta \subseteq L \subseteq K_\beta$. Then $\mathbf{N}r_2 L$ is not closed under forming elementary subalgebras, hence is not elementary. We conjecture that a slight modification of our construction can lift this result to higher dimensions. In this connection, [32] and [30] might be useful.

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([7]) Strictly speaking the result of Maddux and Tarski says that if $\mathcal{A}$ is an atomic relation algebra whose atoms are functions, then A is representable. Unpublished work of Andréka and Givant generalizes this result to CA-like algebras of relations. For results of this kind—giving sufficient conditions for representability of atomic algebras of relations by imposing conditions on the atoms like *density*—we refer the reader to [4].

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