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Z_2^k-actions fixing $\{\text{point}\} \cup V^n$

by

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Abstract. We describe the equivariant cobordism classification of smooth actions (M^m, Φ) of the group $G = Z_2^k$ on closed smooth m-dimensional manifolds M^m for which the fixed point set of the action is the union $F = p \cup V^n$, where p is a point and V^n is a connected manifold of dimension n with $n > 0$. The description is given in terms of the set of equivariant cobordism classes of involutions fixing $p \cup V^n$. This generalizes a lot of previously obtained particular cases of the above question; additionally, the result yields some new applications, namely with V^n an arbitrary product of spheres and with V^n any n-dimensional closed manifold with n odd.

1. Introduction. The goal of this paper is to describe the equivariant cobordism classification of smooth actions (M^m, Φ) of the group $G = Z_2^k$ on closed smooth m-dimensional manifolds M^m for which the fixed point set of the action is the union $F = p \cup V^n$, where p is a point and V^n is a connected manifold of dimension n with $n > 0$. Here, G is considered as the group generated by k commuting involutions $T_1, \ldots, T_k$.

According to [13], the equivariant cobordism class of (M^m, Φ) is determined by the cobordism class of the fixed point data $(F, \{\nu_\varrho\})$ consisting of the fixed point set F and a list of vector bundles over F indexed by the nontrivial irreducible real representations ϱ of G; these representations of G are all one-dimensional and may be described by homomorphisms $\varrho : G \to Z_2 = \{+1, -1\}$ which are onto, and G acts on the reals so that $g \in G$ acts as multiplication by $\varrho(g)$. Here ν_ϱ is the part of the normal bundle of F in M on which G acts as the representation ϱ. Specifically, ν_ϱ is the normal bundle of F in the fixed point set F_H of the subgroup $H = \ker(\varrho)$. Each s-dimensional component of $(F, \{\nu_\varrho\})$ may be considered as an element of $\mathcal{N}_s(\prod_{\varrho \neq 0} BO(n_\varrho))$, the bordism of s-dimensional manifolds with a

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map into a product of classifying spaces $BO(n_\varrho)$ for n_ϱ-dimensional vector bundles, where n_ϱ denotes the dimension of ν_ϱ over the component. For $p \cup V^n$, the bundle (F, ν_ϱ) is the union of two bundles (p, μ_ϱ) and $(V^n, \varepsilon_\varrho)$ over the two fixed point set components. Since every bundle over a point is trivial, the cobordism class of $(p, \{\mu_\varrho\})$ is completely determined by the list of integers $\dim(\mu_\varrho)$ given by the dimensions of the bundles. To complete the classification, it suffices to describe $(V^n, \{\varepsilon_\varrho\})$. For this one has

Proposition. *There is a vector bundle η^l over V^n and a cobordism of $(V^n, \{\varepsilon_\varrho\})$ to $(V^n, \{\varepsilon'_\varrho\})$, where each ε'_ϱ is either*

(a) *the trivial* 0*-dimensional bundle* (*and* $\dim(\mu_\varrho) = 0$),
(b) *the tangent bundle* τ_V *of* V^n (*and* $\dim(\mu_\varrho) = 0$), *or*
(c) $\eta^l \oplus (\dim(\varepsilon_\varrho) - l)$ (*and* $\dim(\mu_\varrho) = n + \dim(\varepsilon_\varrho)$).

Note. In this description, one needs to know which representations ϱ have ε'_ϱ of each type. The pattern of this correspondence is a standard pattern which was described in [7].

Note. For each ϱ with $\varepsilon'_\varrho = \eta^l \oplus (\dim(\varepsilon_\varrho) - l)$, the component of the fixed point set of $H = \ker(\varrho)$ containing p is a manifold with involution induced by the action of $Z_2 = G/H$ fixing $p \cup V^n$. These involutions form a family of involutions fixing $p \cup V^n$ for which the normal bundles of V^n are all stably cobordant.

In order to better understand this result suppose one has an involution (W, T) for which the fixed point set of T is $p \cup V^n$. For each t with $1 \le t \le k$ one may form an action of G on the product $W^{2^{t-1}} = W \times \ldots \times W$ (2^{t-1} factors) by letting $T_1(w_1, \ldots, w_{2^{t-1}}) = (T(w_1), \ldots, T(w_{2^{t-1}}))$, letting $T_2, \ldots, T_t$ be involutions which permute the factors of $W^{2^{t-1}}$ so that the points fixed by $T_2, \ldots, T_t$ are the diagonal copy of W, and letting $T_{t+1}, \ldots, T_k$ be the identity map. Denote this action by $\Gamma^k_t(W, T)$.

One notices that this action of G on $W^{2^{t-1}}$ has fixed point set $p \cup V^n$ (given by the copy of $p \cup V^n$ inside the diagonal copy of W). There are $2^k - 2^t$ bundles ε_ϱ with $\dim(\varepsilon_\varrho) = 0$ given by the representations ϱ for which $H = \ker(\varrho)$ does not contain all the involutions $T_{t+1}, \ldots, T_k$. There are $2^{t-1} - 1$ bundles ε_ϱ with $\varepsilon_\varrho = \tau_V$ and 2^{t-1} bundles ε_ϱ for which ε_ϱ is the normal bundle of V in W. These are given by the representations ϱ for which $H = \ker(\varrho)$ contains $T_{t+1}, \ldots, T_k$ and which either contain T_1 (for $\varepsilon_\varrho = \tau_V$) or do not contain T_1 (for $\varepsilon_\varrho =$ the normal bundle of V in W).

Finally, if $\sigma : G \to G$ is an automorphism one may obtain a G-action $\sigma\Gamma^k_t(W, T)$ by applying the automorphism to G and then using the action just described.

NOTE. The choice of an automorphism amounts to choosing a set of generating involutions for the action. This can change the cobordism class of the action, since in particular the subgroup of G fixing the manifold changes.

In [9] it was shown that if a G-action (N, Ψ) has fixed point data $(F, \{\nu_\varrho\})$ and one of the normal bundles ν_θ is isomorphic to $\nu'_\theta \oplus 1$, then there is an action (N', Ψ') with fixed point data $(F, \{\nu'_\varrho\})$ where $\nu'_\varrho = \nu_\varrho$ for $\varrho \neq \theta$ and ν'_θ is the subbundle. In particular, if (W, T) is an involution fixing $p \cup V^n$ and if the normal bundle of V in W has a section, then 2^{t-1} of the normal bundles of $\sigma\Gamma_t^k(W, T)$ have sections.

The proposition may then be restated

PROPOSITION. *Every G-action (M^m, Φ) fixing $p \cup V^n$ is cobordant to an action obtained from an involution (W, T) fixing $p \cup V^n$ by removing sections from the normal bundles of some $\sigma\Gamma_t^k(W, T)$.*

NOTE. There may be many sections of the bundles ν_ϱ and one may remove different numbers of sections for the various choices of ϱ.

We emphasize that the equivariant cobordism classifications obtained in [5] (for $V^n = S^n$ or $S^p \times S^q$), [7] (for $V^n = \mathbb{R}P(n)$ with n odd), [8] (for $V^n = \mathbb{R}P(n)$ with n even and $k = 2$) and [9] (for $V^n = \mathbb{R}P(n)$ with n even and any k) are particular cases of the above Proposition. In Section 4 we will include two new particular cases (Theorems 1 and 2), which we were not able to get before.

THEOREM 1. *If (M^m, Φ) is a G-action fixing $p \cup V^n$ with n odd and V^n connected, then (M^m, Φ) is equivariantly cobordant to one of the actions $\sigma\Gamma_t^k(\mathbb{R}P(n+1), T)$ where T is the involution*

$$T([x_0, x_1, \ldots, x_n, x_{n+1}]) = [x_0, x_1, \ldots, x_n, -x_{n+1}].$$

NOTE. This extends to any V^n with n odd the result for $V^n = \mathbb{R}P(2p+1)$ obtained in [7].

For a sequence $N = (n_1, \ldots, n_p)$ of natural numbers, consider the cartesian product of spheres $S^N = S^{n_1} \times \ldots \times S^{n_p}$. Denote by Ω the set formed by the sequences $N = (n_1, \ldots, n_p)$ such that $n_1 + \ldots + n_p = 2^s$ for some $s \geq 0$; if $s \geq 4$, we additionally require N to be a refinement of $(8, \ldots, 8)$ (2^{s-3} copies). From [6] one knows that for each $N = (n_1, \ldots, n_p) \in \Omega$ there is an involution (W_N^{2n}, T) fixing $p \cup S^N$, where $n = n_1 + \ldots + n_p$.

THEOREM 2. *If (M^m, Φ) is a G-action fixing $p \cup S^N$ with $N = (n_1, \ldots \ldots, n_p)$ and $n = n_1 + \ldots + n_p$, then $N \in \Omega$ and (M^m, Φ) is equivariantly cobordant to one of the actions $\sigma\Gamma_t^k(W_N^{2n}, T)$; in particular, $m = 2^t n$.*

NOTE. This extends to an arbitrary product of spheres the results for $V^n = S^n$ or $S^p \times S^q$ of [5].

Suppose (M^m, T) is an involution fixing $p \cup V^n$ with V^n not necessarily connected. Since the fixed point data of (M^m, T) is not a boundary, one sees from the work of Boardman ([1], [2]) that $m \le \frac{5}{2}n$. In [11], we showed that this bound may be improved to what is utmost generality; in fact, we established the upper bound for m, for each n. Writing $n = 2^p q$ with q odd, set

$$m(n) = \begin{cases} 2^{p+1}q + p + 1 - q & \text{if } p \le q, \\ 2^{p+1}q + 2^{p-q} & \text{if } p \ge q. \end{cases}$$

We proved in [11] that $m \le m(n)$ and there are involutions with $m = m(n)$ fixing a point and some V^n for each n. As another consequence of our result, we will generalize this fact to G-actions, assuming that V^n is connected.

THEOREM 3. *If (M^m, Φ) is a G-action fixing $p \cup V^n$ with V^n connected, then $m \le 2^{k-1} m(n)$; moreover, this bound is best possible for V^n connected.*

2. Involutions fixing $p \cup V^n$. Suppose (M^m, Φ) is a G-action with fixed point set $p \cup V^n$. Since $m = \sum \dim(\mu_\varrho)$, there is always at least one ϱ for which $\dim(\mu_\varrho) > 0$. For any such ϱ, the component of the fixed point set of $H = \ker(\varrho)$ containing p, F_ϱ, is a manifold of positive dimension on which G acts, and since H acts trivially, this is an action of $G/H \cong Z_2$, or an involution on F_ϱ. Since an involution on a manifold of positive dimension cannot fix a single point, F_ϱ must contain V^n. Thus, one obtains an involution (F_ϱ, T) fixing $p \cup V^n$, with the normal bundles being μ_ϱ and ε_ϱ.

Thus, one needs to know involutions (W, T) fixing $p \cup V^n$.

Following Conner and Floyd, the cobordism class of an involution (W^w, T) fixing $p \cup V^n$ is determined by the cobordism class of the normal bundle to the fixed point set, the trivial w-plane bundle over p, and a $(w-n)$-plane bundle ν^{w-n} over V^n. Among all the bundles over V^n cobordant to ν^{w-n} there will be a smallest l for which ν^{w-n} is cobordant to a bundle $\eta^l \oplus (w - n - l)$.

From [3; 26.4], it follows that there are involutions $(\overline{W}^{n+l+i}, T)$ fixing $p \cup V^n$ for which the normal bundle of V^n in $\overline{W}^{n+l+i}$ is $\eta^l \oplus i$ for $0 \le i \le w - n - l$, with $(\overline{W}^{n+l+(w-n-l)}, T)$ cobordant to (W^w, T).

Further, one knows how to add additional trivial bundles to the normal bundle of an involution. If (W^w, T) fixes $p \cup V^n$ with normal bundle ν^{w-n} over V^n, one may form

$$\Gamma(W, T) = \left(\frac{S^1 \times W^w}{-1 \times T}, \text{ conjugation} \times 1 \right).$$

The fixed point set of this involution consists of a copy of the fixed point set of (W^w, T) (the points $\frac{\{\pm i\}\times(p\cup V^n)}{-1\times T}$) with normal bundle $\nu^{w-n} \oplus 1$ over V^n and a copy of W^w (the points $\frac{\{\pm 1\}\times W^w}{-1\times T}$) with normal bundle a trivial line bundle. If W^w bounds as a manifold, $(W^w, 1)$ bounds as a bundle and $\Gamma(W, T)$ is cobordant to an involution fixing $p \cup V^n$ with normal bundle $\nu^{w-n} \oplus 1$ over V^n.

Thus, the involutions (W^w, T) fixing $p \cup V^n$ belong to families with ν^{w-n} cobordant to $\eta^l \oplus (w-n-l)$ with $n+l \le w \le w_0$, where W^{w_0} is nonbounding as a manifold.

NOTE. This is one of the key points in Boardman's approach to involutions [1], [2].

The assertion of the Proposition is that all the involutions (F_ϱ, T) fixing $p \cup V^n$ belong to the same family. Further, the normal bundles are simultaneously cobordant to bundles of the form $\eta^l \oplus i$.

These results for involutions have analogues for $G = Z_2^k$-actions.

In [9], it was shown that if a G-action (M^m, Φ) fixes $(F, \{\nu_\varrho\})$ and if some ν_ϱ has a section, then there is another G-action (M^{m-1}, Φ) fixing F for which the section has been removed.

If $(M^m, T_1, \ldots, T_k)$ is a manifold with G-action, one may form

$$\widetilde{M}^{m+1} = \frac{S^1 \times M^m}{-1 \times T_1}$$

with the involutions $\widetilde{T}_1 =$ conjugation $\times 1$, and $\widetilde{T}_i = 1 \times T_i$ for $i > 1$. The fixed point set of $\widetilde{T}_1$ for this action consists of a copy of the fixed point set of T_1, $\frac{(\pm i)\times F_{T_1}}{-1\times T_1}$, and a copy of M^m, $\frac{(\pm 1)\times M^m}{-1\times T_1}$. The normal bundle of $F_{\widetilde{T}_1}$ has an additional trivial line bundle added, and the normal bundle of the copy of M^m is a trivial line bundle. The fixed point set of the action of G on $\widetilde{M}^{m+1}$ is a copy of the fixed point set of the action of G on M^m ($\frac{(\pm i)\times F}{-1\times T}$), and the normal bundle in $\widetilde{M}^{m+1}$ is obtained by adding a trivial line bundle to the normal bundle ν_ϱ, where ϱ is the representation with $\ker(\varrho) = H =$ subgroup generated by $T_2, \ldots, T_k$, and a copy of the fixed point set of H acting on M^m, $\frac{(\pm 1)\times F_H}{-1\times T}$. If the restriction of M^m to H bounds equivariantly, the action of H on M^m with a trivial line bundle bounds, and also the normal bundle of the copy of F_H bounds. Thus the action of G on $\widetilde{M}^{m+1}$ is cobordant to an action having the same fixed point set as M^m but with a trivial line bundle added to ν_ϱ.

NOTE. For the action $\Gamma_t^k(W, T)$ described in the introduction, the restriction to H is $W \times \ldots \times W$ (2^{t-1} copies) with $T_2, \ldots, T_k$ acting as permutations. If W bounds as a manifold, this action bounds.

Thus, the actions of G also lie in families. The proposition says that the G-actions fixing $p \cup V^n$ lie in a family with minimal element $\sigma\Gamma_t^k(W_1, T_1)$ and maximal element $\sigma\Gamma_t^k(W_2, T_2)$, where (W_1, T_1) and (W_2, T_2) are the elements of minimal and maximal dimension of a family of involutions fixing $p \cup V^n$.

3. Proof of the main result. Denote by $\mathcal{A}$ the collection of all equivariant cobordism classes of involutions containing a representative (W, T) with $p \cup V^n$ as fixed point set; $\mathcal{A}$ is a disjoint union of families as described in the previous section. From the strengthened Boardman 5/2-theorem of [4] one deduces that $\mathcal{A}$ is always finite, and we can identify each element $[W, T]$ of $\mathcal{A}$ with the class of the component of the normal bundle over V^n, $\kappa \to V^n$, since the component over the point is determined by $\kappa \to V^n$. In this way, we can write

$$\mathcal{A} = \{[\kappa_1 \to V^n], [\kappa_2 \to V^n], \ldots, [\kappa_r \to V^n]\}.$$

We now consider (M, Φ), $\Phi = (T_1, \ldots, T_k)$, a G-action fixing $p \cup V^n$. Let $(p, \{\mu_\varrho\}) \cup (V^n, \{\varepsilon_\varrho\})$ be the fixed point data of Φ. The main result of [7] says that in this situation the list $\{\varepsilon_\varrho\}$ contains 2^{t-1} eigenbundles bordant to κ_i's, $2^{t-1} - 1$ eigenbundles bordant to τ_V and $2^k - 2^t$ zero bundles for some $1 \le t \le k$, and up to some automorphism $\sigma : G \to G$ these bundles are included in $\{\varepsilon_\varrho\}$ in the following way:

(i) if $H = \ker(\varrho)$ contains $T_{t+1}, T_{t+2}, \ldots, T_k$ and does not contain T_1, then ε_ϱ is bordant to some κ_i;

(ii) if H contains $T_1, T_{t+1}, T_{t+2}, \ldots, T_k$, then ε_ϱ is bordant to τ_V; and

(iii) if H does not contain all the involutions $T_{t+1}, T_{t+2}, \ldots, T_k$, then ε_ϱ is the zero bundle.

Moreover, when ε_ϱ is bordant to some κ_i, the corresponding μ_ϱ must be the trivial bundle $n + s_i \to p$, where $s_i = \dim(\kappa_i)$; in the other cases, $\mu_\varrho = 0$.

Now choose a nontrivial representation $\varrho_1 : G \to Z_2$ for which ε_{ϱ_1} is bordant to τ_V (we suppose $t \ge 2$, since for $t = 1$ there is nothing to prove), and take $T \notin H = \ker(\varrho_1)$. Then G is $H \times Z_2$, with the Z_2 summand being generated by T. The other nontrivial representations occur in pairs ϱ', ϱ'' which are the same homomorphism on H, with $\varrho'(T) = 1$ and $\varrho''(T) = -1$. One may consider the nontrivial homomorphisms from H into Z_2 as being indexed by the homomorphisms ϱ'.

If one considers the restriction $\Phi_{|H}$ of (M, Φ) to the subgroup H, one may let $F_0 \subset M$ be the component of the fixed point set of $\Phi_{|H}$ which contains V^n. The normal bundle of V^n in F_0 is $\varepsilon_{\varrho_1} \to V^n$, so F_0 has dimension $2n$; since in this case $\mu_{\varrho_1} \to p$ is the zero bundle, p does not belong to F_0, which means that V^n is the unique component of the fixed point set of Φ contained in F_0.

The normal bundle of F_0 in M decomposes under the action of H as the Whitney sum of subbundles $\varepsilon^0_{\varrho'}$ for the nontrivial homomorphisms $\varrho'_{|H} : H \to Z_2$. The submanifold $F_0 \subset M$ is invariant under the action of G, and the subbundles $\varepsilon^0_{\varrho'}$ are also invariant under G, with G acting by bundle maps covering the action of G on F_0. Of course, H acts trivially on F_0, so one really has only the action of T on F_0 as an involution, and T acts as an involution on $\varepsilon^0_{\varrho'}$ by bundle maps covering the action on F_0. Thus one has an object

$$(F_0, \{\varepsilon^0_{\varrho'}\})$$

given by a manifold with a list of bundles together with their involutions induced by T, which can be considered as an element of the equivariant bordism group

$$\mathcal{N}^{Z_2}_{2n}\Big(\prod_{\varrho'} BO(m_{\varrho'})\Big)$$

of a product of classifying spaces for bundles with involution, where $m_{\varrho'} = \dim(\varepsilon^0_{\varrho'})$. The fixed point set of T acting on F_0 is V^n, and when restricted to V^n each bundle $\varepsilon^0_{\varrho'}$ splits as the Whitney sum of subbundles on which T acts as $+1$ in the fibers (i.e. $\varepsilon_{\varrho'}$) and on which T acts as -1 in the fibers (i.e. $\varepsilon_{\varrho''}$).

If one now removes from F_0 the interior of a tubular neighborhood U of V^n, invariant under T, one obtains a manifold with boundary $F_1 = F^0 - \mathrm{int}(U)$ having boundary $\partial U = S(\varepsilon_{\varrho_1})$, the sphere bundle of ε_{ϱ_1}. On F_1 the involution T is free, therefore for each ϱ' one finds that T acts freely on the total space of $\varepsilon^0_{\varrho'|F_1}$. Thus

$$(S(\varepsilon_{\varrho_1}), \{\varepsilon^0_{\varrho'|S(\varepsilon_{\varrho_1})}\}),$$

the sphere bundle of ε_{ϱ_1} with a list of bundles together with their free involutions induced by T, bounds a corresponding list

$$(F_1, \{\varepsilon^0_{\varrho'|F_1}\})$$

of bundles over F_1 with free involution. This may be considered in

$$\widehat{\mathcal{N}}^{Z_2}_{2n-1}\Big(\prod_{\varrho'} BO(m_{\varrho'})\Big),$$

the equivariant bordism group of a product of classifying spaces for bundles with free involution.

This determines a bordism involving the corresponding quotient bundles, obtained from the above bordism by dividing out the free involution T. That is, the quotient $\frac{F_1}{T}$ is a manifold with boundary

$$\frac{\partial U}{T} = \frac{S(\varepsilon_{\varrho_1})}{(-1)}$$

which is the real projective space bundle $\mathbb{R}P(\varepsilon_{\varrho_1})$. Considering the double cover $F_1 \to \frac{F_1}{T}$ as a line bundle, there is a line bundle $\lambda \to \frac{F_1}{T}$ which restricts on $\mathbb{R}P(\varepsilon_{\varrho_1})$ to the line bundle of the double cover $S(\varepsilon_{\varrho_1}) \to \mathbb{R}P(\varepsilon_{\varrho_1})$, which will be denoted by ξ.

Now for each ϱ', $\varepsilon^0_{\varrho'}$ restricts over the boundary $\partial U = S(\varepsilon_{\varrho_1})$ to the pullback of the bundle $\varepsilon_{\varrho'} \oplus \varepsilon_{\varrho''}$, and T acts as 1 in $\varepsilon_{\varrho'}$ and as -1 in $\varepsilon_{\varrho''}$. Thus each quotient bundle

$$\frac{(\varepsilon^0_{\varrho'|F_1})}{T} \to \frac{F_1}{T}$$

has boundary

$$\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''}) \to \mathbb{R}P(\varepsilon_{\varrho_1}).$$

In this way,

$$(\mathbb{R}P(\varepsilon_{\varrho_1}), \xi, \{\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''})\}),$$

the projective space bundle of ε_{ϱ_1} with its standard line bundle and bundles $\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''})$, bounds the corresponding list of bundles over $\frac{F_1}{T}$ given by

$$\left(\frac{F_1}{T}, \lambda, \left\{\frac{\varepsilon^0_{\varrho'|F_1}}{T}\right\}\right).$$

This may be considered in

$$\mathcal{N}_{2n-1}\Big(BO(1) \times \prod_{\varrho'} BO(m_{\varrho'})\Big),$$

the bordism of classifying spaces for vector bundles.

The above argument is identical with that of [10; Section 2]. The crucial point is that F_0 does not contain the point fixed by Φ. Also the next lemma is similar to the lemma at the start of Section 3 of [10]; to ease the reading and mainly to establish some notations, we will rewrite it.

LEMMA 1. *$(V^n, \varepsilon_{\varrho_1}, \{\varepsilon_{\varrho'}, \varepsilon_{\varrho''}\})$ is cobordant to $(V^n, \tau_V, \{\varepsilon_{\varrho'}, \varepsilon_{\varrho''}\})$.*

Proof. One lets

$$W(V^n) = 1 + w_1 + \ldots + w_n$$

be the Stiefel–Whitney class of V^n and

$$W(\varepsilon_\varrho) = 1 + u_1^\varrho + \ldots + u_{n_\varrho}^\varrho$$

be the Stiefel–Whitney class of ε_ϱ for any ϱ, where $n_\varrho = \dim(\varepsilon_\varrho)$.

Letting $c \in H^1(\mathbb{R}P(\varepsilon_{\varrho_1}); Z_2)$ be the first Stiefel–Whitney class of the line bundle ξ for the double cover $S(\varepsilon_{\varrho_1}) \to \mathbb{R}P(\varepsilon_{\varrho_1})$, one knows that the Stiefel–Whitney class of $\mathbb{R}P(\varepsilon_{\varrho_1})$ is

$$W(\mathbb{R}P(\varepsilon_{\varrho_1})) = (1+w_1+\ldots+w_n)\{(1+c)^{n_{\varrho_1}} + u_1^{\varrho_1}(1+c)^{n_{\varrho_1}-1} + \ldots + u_{n_{\varrho_1}}^{\varrho_1}\},$$

the Stiefel–Whitney class of ξ is

$$W(\xi) = 1 + c,$$

and the Stiefel–Whitney class of the bundle $\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''})$ is

$$W(\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''})) = (1 + u_1^{\varrho'} + \ldots + u_{n_{\varrho'}}^{\varrho'}) \cdot \{(1+c)^{n_{\varrho''}} + u_1^{\varrho''}(1+c)^{n_{\varrho''}-1} + \ldots + u_{n_{\varrho''}}^{\varrho''}\}.$$

Because $(\mathbb{R}P(\varepsilon_{\varrho_1}, \xi, \{\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''})\})$ is a boundary, any class of dimension $2n-1$ given by a product of the classes

$$w_i(\mathbb{R}P(\varepsilon_{\varrho_1})), \quad c, \quad w_j(\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''}))$$

gives a zero characteristic number for $\mathbb{R}P(\varepsilon_{\varrho_1})$. We will apply this using certain special classes, which are polynomials in the above-displayed ones, and were initially introduced in [11] and also used in [10].

Specifically, for any r, one lets

$$W[r] = \frac{W(\mathbb{R}P(\varepsilon_{\varrho_1}))}{(1+c)^{n_{\varrho_1}-r}} \quad \text{and} \quad W_{\varrho'}[r] = \frac{W(\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''}))}{(1+c)^{n_{\varrho''}-r}}$$

so that

$$W[r] = (1 + w_1 + \ldots + w_n) \cdot \{(1+c)^r + u_1^{\varrho_1}(1+c)^{r-1} + \ldots + u_{n_{\varrho_1}}^{\varrho_1}(1+c)^{r-n_{\varrho_1}}\}$$

and

$$W_{\varrho'}[r] = (1 + u_1^{\varrho'} + \ldots + u_{n_{\varrho'}}^{\varrho'}) \cdot \{(1+c)^r + u_1^{\varrho''}(1+c)^{r-1} + \ldots + u_{n_{\varrho''}}^{\varrho''}(1+c)^{r-n_{\varrho''}}\}.$$

For these classes, one then has the special properties:

$$W[r]_{2r} = w_r c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+1} = (w_{r+1} + u_{r+1}^{\varrho_1})c^r + \text{terms with smaller } c \text{ powers},$$
$$W[r]_{2r+2} = u_{r+1}^{\varrho_1} c^{r+1} + \text{terms with smaller } c \text{ powers},$$

and in the same way

$$W_{\varrho'}[r]_{2r} = u_r^{\varrho'} c^r + \text{terms with smaller } c \text{ powers},$$
$$W_{\varrho'}[r]_{2r+1} = (u_{r+1}^{\varrho'} + u_{r+1}^{\varrho''})c^r + \text{terms with smaller } c \text{ powers},$$
$$W_{\varrho'}[r]_{2r+2} = u_{r+1}^{\varrho''} c^{r+1} + \text{terms with smaller } c \text{ powers}.$$

For a sequence $\omega = (i_1, \ldots, i_s)$ of integers, one lets $|\omega| = i_1 + \ldots + i_s$, and for $u = 1 + u_1 + \ldots + u_p$, one lets $u_\omega = u_{i_1} \ldots u_{i_s}$ be the product of the classes u_i.

Then given sequences $\omega = (i_1, \ldots, i_s)$ and $\omega_\varrho = (i_1^\varrho, \ldots, i_{s_\varrho}^\varrho)$, and a natural number r with

$$|\omega| + \sum_\varrho |\omega_\varrho| + r = n,$$

one may form the class

$$X = \prod_{i\in\omega} W[i]_{2i} \cdot \prod_{i\in\omega_{\varrho_1}} W[i-1]_{2i} \cdot \prod_{\varrho'} \Big\{ \Big(\prod_{i\in\omega_{\varrho'}} W_{\varrho'}[i]_{2i} \Big) \cdot \Big(\prod_{i\in\omega_{\varrho''}} W_{\varrho'}[i-1]_{2i} \Big) \Big\} \cdot W[r-1]_{2r-1}.$$

This is a characteristic class of $\mathbb{R}P(\varepsilon_{\varrho_1})$ of dimension $2n-1$, and has the form

$$X = w_\omega u^{\varrho_1}_{\omega_{\varrho_1}} \cdot \prod_{\varrho'} u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod_{\varrho''} u^{\varrho''}_{\omega_{\varrho''}} \cdot (w_r + u_r^{\varrho_1}) c^{n-1} + \text{terms with smaller powers of } c.$$

Because $H^*(\mathbb{R}P(\varepsilon_{\varrho_1}); Z_2)$ is the free $H^*(V^n; Z_2)$-module on

$$1, c, c^2, \ldots, c^{n_{\varrho_1}-1},$$

it follows that

$$0 = X[\mathbb{R}P(\varepsilon_{\varrho_1})] = w_\omega u^{\varrho_1}_{\omega_{\varrho_1}} \cdot \prod_{\varrho'} u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod_{\varrho''} u^{\varrho''}_{\omega_{\varrho''}} \cdot (w_r + u_r^{\varrho_1})[V^n]$$

or

$$w_\omega u_r^{\varrho_1} u^{\varrho_1}_{\omega_{\varrho_1}} \cdot \prod u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod u^{\varrho''}_{\omega_{\varrho''}}[V^n] = w_\omega w_r u^{\varrho_1}_{\omega_{\varrho_1}} \cdot \prod u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod u^{\varrho''}_{\omega_{\varrho''}}[V^n].$$

This says that any class $u_r^{\varrho_1}$ in a characteristic number of $(V^n, \varepsilon_{\varrho_1}, \{\varepsilon_{\varrho'}, \varepsilon_{\varrho''}\})$ may be replaced by w_r without changing the value of the characteristic number, which means that $(V^n, \varepsilon_{\varrho_1}, \{\varepsilon_{\varrho'}, \varepsilon_{\varrho''}\})$ and $(V^n, \tau_V, \{\varepsilon_{\varrho'}, \varepsilon_{\varrho''}\})$ have the same characteristic numbers. This gives the result. ■

LEMMA 2. *Let ϱ_a and ϱ_b be two different nontrivial representations of G for which* $\dim(\mu_{\varrho_a}) > 0$ *and* $\dim(\mu_{\varrho_b}) > 0$. *Then*

(i) *The representation $\varrho_1 = \varrho_a\varrho_b$ has* $\dim(\mu_{\varrho_1}) = 0$ *and* $\dim(\varepsilon_{\varrho_1}) = n$, *and if $H = \ker(\varrho_1)$ then $\varrho_{a|H} = \varrho_{b|H}$ so that ϱ_a and ϱ_b are paired with respect to ϱ_1.*

(ii) *If* $\dim(\varepsilon_{\varrho_a}) \le \dim(\varepsilon_{\varrho_b})$ *and* $s = \dim(\varepsilon_{\varrho_b}) - \dim(\varepsilon_{\varrho_a})$, *then*

$$(V^n, \varepsilon_{\varrho_a}, \varepsilon_{\varrho_b}, \{\varepsilon_\varrho\}_{\varrho\neq\varrho_a,\varrho_b})$$

is cobordant to

$$(V^n, \varepsilon_{\varrho_a}, \varepsilon_{\varrho_a} \oplus s, \{\varepsilon_\varrho\}_{\varrho\neq\varrho_a,\varrho_b}).$$

Proof. (i) Let $H_a = \ker(\varrho_a)$, $H_b = \ker(\varrho_b)$ and let F_a (respectively F_b) be the component of p in the fixed point set of H_a (respectively H_b). One has

$V^n \subset F_a$ (respectively $V^n \subset F_b$) and $\dim(\mu_{\varrho_a}) = n + \dim(\varepsilon_{\varrho_a}) = \dim(F_a)$ (respectively $\dim(\mu_{\varrho_b}) = n + \dim(\varepsilon_{\varrho_b}) = \dim(F_b)$). Choose involutions T_a and T_b where $T_a \notin H_a$ and $T_a \in H_b$ (respectively $T_b \notin H_b$ and $T_b \in H_a$).

Let F_0 be the component of the fixed point set of $H_a \cap H_b$ containing p. Then $F_a \subset F_0$ and $F_b \subset F_0$. Since G acts on F_0 with $H_a \cap H_b$ acting trivially, this gives an action of $G/H_a \cap H_b \cong Z_2 \times Z_2$ on F_0 with generators the involutions T_a and T_b. The subgroup H is the subgroup of G generated by $H_a \cap H_b$ and the involution T_aT_b, with μ_{ϱ_1} being the normal bundle of p in $F_0 \cap F_H$ and ε_{ϱ_1} being the normal bundle of V^n in $F_0 \cap F_H$.

Now one has

$$\begin{aligned} n + \dim(\varepsilon_{\varrho_a}) + \dim(\varepsilon_{\varrho_b}) + \dim(\varepsilon_{\varrho_1}) &= \dim(F_0) \\ &= \dim(\mu_{\varrho_a}) + \dim(\mu_{\varrho_b}) + \dim(\mu_{\varrho_1}) \\ &= (n + \dim(\varepsilon_{\varrho_a})) + (n + \dim(\varepsilon_{\varrho_b})) + \dim(\mu_{\varrho_1}). \end{aligned}$$

If $\dim(\mu_{\varrho_1}) > 0$ one has $\dim(\mu_{\varrho_1}) = n + \dim(\varepsilon_{\varrho_1})$ and

$$n + \dim(\varepsilon_{\varrho_a}) + \dim(\varepsilon_{\varrho_b}) + \dim(\varepsilon_{\varrho_1}) = 3n + \dim(\varepsilon_{\varrho_a}) + \dim(\varepsilon_{\varrho_b}) + \dim(\varepsilon_{\varrho_1}),$$

contradicting the assumption that $n > 0$. Thus $\dim(\mu_{\varrho_1}) = 0$ and

$$\begin{aligned} n + \dim(\varepsilon_{\varrho_a}) + \dim(\varepsilon_{\varrho_b}) + \dim(\varepsilon_{\varrho_1}) &= \dim(\mu_{\varrho_a}) + \dim(\mu_{\varrho_b}) \\ &= 2n + \dim(\varepsilon_{\varrho_a}) + \dim(\varepsilon_{\varrho_b}), \end{aligned}$$

giving $\dim(\varepsilon_{\varrho_1}) = n$.

Clearly, ϱ_a agrees with ϱ_b on $H_a \cap H_b$ for

$$H_a \cap H_b \subset H_a = \ker(\varrho_a), \qquad H_a \cap H_b \subset H_b = \ker(\varrho_b)$$

and

$$\varrho_a(T_aT_b) = \varrho_a(T_a)\varrho_a(T_b) = -1 \cdot 1 = -1$$

and similarly $\varrho_b(T_aT_b) = -1$, so $\varrho_{a|H} = \varrho_{b|H}$. For $T = T_a$, $T \notin H$ and $\varrho_a(T) = -1$, $\varrho_b(T) = 1$ and for $T = T_b$, $T \notin H$ and $\varrho_a(T) = 1$, $\varrho_b(T) = -1$. Thus the representations ϱ_a and ϱ_b are paired with respect to ϱ_1.

(ii) In the geometric discussion developed before Lemma 1 we can use the representation ϱ_1 of part (i) to conclude that

$$(\mathbb{R}P(\varepsilon_{\varrho_1}), \xi, \varepsilon_{\varrho_a} \oplus (\xi \otimes \varepsilon_{\varrho_b}), \{\varepsilon_{\varrho'} \oplus (\xi \otimes \varepsilon_{\varrho''})\}_{(\varrho', \varrho'') \neq (\varrho_a, \varrho_b)})$$

bounds as an element of $\mathcal{N}_{2n-1}(BO(1) \times \prod BO(m_{\varrho'}))$.

We use now the same arguments and notations of Lemma 1. For sequences $\omega = (i_1, \dots, i_s)$ and $\omega_\varrho = (i_1^\varrho, \dots, i_{s_\varrho}^\varrho)$, and a natural number r with

$$|\omega| + \sum |\omega_\varrho| + r = n,$$

one may form the class

$$X = \Big(\prod_{i\in\omega} W[i]_{2i}\Big)\cdot\Big(\prod_{i\in\omega_{\varrho_1}} W[i-1]_{2i}\Big) \cdot \prod_{\varrho'\neq\varrho_a}\Big\{\Big(\prod_{i\in\omega_{\varrho'}} W_{\varrho'}[i]_{2i}\Big)\cdot\Big(\prod_{i\in\omega_{\varrho''}} W_{\varrho'}[i-1]_{2i}\Big)\Big\} \cdot\Big(\prod_{i\in\omega_{\varrho_a}} W_{\varrho_a}[i]_{2i}\Big)\cdot\Big(\prod_{i\in\omega_{\varrho_b}} W_{\varrho_a}[i-1]_{2i}\Big)\cdot W_{\varrho_a}[r-1]_{2r-1}.$$

As in Lemma 1, this is a characteristic class of $\mathbb{R}P(\varepsilon_{\varrho_1})$ of dimension $2n-1$ and has the form

$$X = w_\omega u^{\varrho_1}_{\omega_{\varrho_1}} u^{\varrho_a}_{\omega_{\varrho_a}} u^{\varrho_b}_{\omega_{\varrho_b}} \cdot \prod_{\varrho'\neq\varrho_a} u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod_{\varrho''\neq\varrho_b} u^{\varrho''}_{\omega_{\varrho''}} \cdot (u^{\varrho_a}_r + u^{\varrho_b}_r)c^{n-1} + \text{terms with smaller powers of } c.$$

Then

$$0 = X[\mathbb{R}P(\varepsilon_{\varrho_1})] = w_\omega u^{\varrho_1}_{\omega_{\varrho_1}} u^{\varrho_a}_{\omega_{\varrho_a}} u^{\varrho_b}_{\omega_{\varrho_b}} \cdot \prod_{\varrho'\neq\varrho_a} u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod_{\varrho''\neq\varrho_b} u^{\varrho''}_{\omega_{\varrho''}} \cdot (u^{\varrho_a}_r + u^{\varrho_b}_r)[V^n]$$

or

$$w_\omega u^{\varrho_1}_{\omega_{\varrho_1}} u^{\varrho_a}_{\omega_{\varrho_a}} u^{\varrho_b}_{\omega_{\varrho_b}} u^{\varrho_b}_r \cdot \prod_{\varrho'\neq\varrho_a} u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod_{\varrho''\neq\varrho_b} u^{\varrho''}_{\omega_{\varrho''}}[V^n] = w_\omega u^{\varrho_1}_{\omega_{\varrho_1}} u^{\varrho_a}_{\omega_{\varrho_a}} u^{\varrho_b}_{\omega_{\varrho_b}} u^{\varrho_a}_r \cdot \prod_{\varrho'\neq\varrho_a} u^{\varrho'}_{\omega_{\varrho'}} \cdot \prod_{\varrho''\neq\varrho_b} u^{\varrho''}_{\omega_{\varrho''}}[V^n].$$

This says that any class $u^{\varrho_b}_r$ in a characteristic number of

$$(V^n, \varepsilon_{\varrho_a}, \varepsilon_{\varrho_b}, \{\varepsilon_\varrho\}_{\varrho\neq\varrho_a,\varrho_b})$$

may be replaced by $u^{\varrho_a}_r$ without changing the value of the characteristic number; in particular, for $r > \dim(\varepsilon_{\varrho_a})$, any class $u^{\varrho_b}_r$ may be replaced by the zero class. In this way,

$$(V^n, \varepsilon_{\varrho_a}, \varepsilon_{\varrho_b}, \{\varepsilon_\varrho\}_{\varrho\neq\varrho_a,\varrho_b}) \quad \text{and} \quad (V^n, \varepsilon_{\varrho_a}, \varepsilon_{\varrho_a}\oplus s, \{\varepsilon_\varrho\}_{\varrho\neq\varrho_a,\varrho_b})$$

have the same characteristic numbers, and the result follows. ■

To end the proof of our result we make the iterative use of Lemma 1 and Lemma 2(ii). First we use Lemma 1 $2^{t-1}-1$ times to conclude that $(V^n, \{\varepsilon_\varrho\})$ is cobordant to

$$(V^n, \{\tau_V\}, \{\varepsilon_\varrho\}_1, \{0\}),$$

where $\{\tau_V\}$ contains $2^{t-1}-1$ copies of τ_V, $\{\varepsilon_\varrho\}_1$ is the sublist of $\{\varepsilon_\varrho\}$ formed by the 2^{t-1} bundles ε_ϱ for which $\dim(\mu_\varrho) > 0$, and $\{0\}$ means the list of $2^k - 2^t$ zero bundles. Next choose $\eta^l \in \{\varepsilon_\varrho\}_1$ with $l = \dim(\eta_l) \le \dim(\varepsilon_\varrho)$ for any $\varepsilon_\varrho \in \{\varepsilon_\varrho\}_1$. Using Lemma 2(ii) $2^{t-1} - 1$ times, one then deduces that

$$(V^n, \{\tau_V\}, \{\varepsilon_\varrho\}_1, \{0\})$$

is cobordant to

$$(V^n, \{\tau_V\}, \eta^l, \{\gamma_\varrho\}_1, \{0\}),$$

where $\{\gamma_\varrho\}_1$ is the list obtained from $\{\varepsilon_\varrho\}_1$ by excluding η^l and replacing each remaining ε_ϱ by

$$\gamma_\varrho = \eta^l \oplus (\dim(\varepsilon_\varrho) - l).$$

Therefore $(V^n, \{\varepsilon_\varrho\})$ is cobordant to this last list and the Proposition is proved.

NOTE. With the above notation, choose a representation ϱ_0 such that $\varepsilon_{\varrho_0} \in \{\varepsilon_\varrho\}_1$ and $\dim(\varepsilon_{\varrho_0}) \geq \dim(\varepsilon_\varrho)$ for any $\varepsilon_\varrho \in \{\varepsilon_\varrho\}_1$. Take $T \in G$ so that $T \notin H = \ker(\varrho_0)$ and denote by F_{ϱ_0} the component of the fixed point set of H containing p. Then the involution (F_{ϱ_0}, T) fixes $p \cup V^n$ and (M^m, Φ) is equivariantly cobordant to an action obtained by removing sections from the normal bundles of $\sigma\Gamma_t^k(F_{\varrho_0}, T)$. This is the second formulation of our Proposition given in the introduction.

4. Applications. In this section we will prove Theorems 1–3, which are consequences of our Proposition. First suppose V^n is a connected closed n-dimensional manifold for which the set $\mathcal{A}$ of all equivariant cobordism classes of involutions containing a representative fixing $p \cup V^n$ contains a single element, say $\mathcal{A} = \{[W, S]\}$. Let $\eta \to V^n$ be the normal bundle of V^n in W.

LEMMA. *Suppose (M^m, Φ) is a G-action fixing $p \cup V^n$, with V^n as above. Then (M^m, Φ) is equivariantly cobordant to one of the actions $\sigma\Gamma_t^k(W, S)$.*

Proof. Let $(p, \{\mu_\varrho\}) \cup (V^n, \{\varepsilon_\varrho\})$ be the fixed point data of Φ. For any representation ϱ for which $\dim(\mu_\varrho) > 0$, the involution (F_ϱ, T), where $T \notin \ker(\varrho)$ and F_ϱ is the component of the fixed point set of $\ker(\varrho)$ containing p, is an involution fixing $p \cup V^n$, and from the hypothesis on $\mathcal{A}$ one finds that (F_ϱ, T) is cobordant to (W, S), so $\varepsilon_\varrho \to V^n$ is cobordant to $\eta \to V^n$. Then obviously $\varepsilon_\varrho \to V^n$ has maximal dimension in $\{\varepsilon_\varrho : \dim(\mu_\varrho) > 0\}$ (and has no section because $\mathcal{A}$ is unitary). From the Proposition it follows that (M^m, Φ) is equivariantly cobordant to one of the actions $\sigma\Gamma_t^k(F_\varrho, T)$, which in turn is equivariantly cobordant to $\sigma\Gamma_t^k(W, S)$. ∎

THEOREM 1. *If (M^m, Φ) is a G action fixing $p \cup V^n$ with n odd and V^n connected, then (M^m, Φ) is equivariantly cobordant to one of the actions $\sigma\Gamma_t^k(\mathbb{R}P(n+1), T)$ where T is the involution*

$$T([x_0, x_1, \ldots, x_n, x_{n+1}]) = [x_0, x_1, \ldots, x_n, -x_{n+1}].$$

Proof. As in the proof of the above Lemma, $p \cup V^n$ is fixed by the involutions (F_ϱ, T) for the representations ϱ with $\dim(\mu_\varrho) > 0$. Since n is odd, one then sees from [12] that each (F_ϱ, T) is cobordant to $(\mathbb{R}P(n+1), T)$; in other words, $\mathcal{A} = \{[\mathbb{R}P(n+1), T]\}$. The result then follows from the above Lemma. ∎

Theorem 2. *If (M^m, Φ) is a G-action fixing $p \cup S^N$ with $N = (n_1, \ldots \ldots, n_p)$ and $n = n_1 + \ldots + n_p$, then $N \in \Omega$ and (M^m, Φ) is equivariantly cobordant to one of the actions $\sigma\Gamma_t^k(W_N^{2n}, T)$; in particular, $m = 2^t n$.*

Proof. For any representation ϱ with $\dim(\mu_\varrho) > 0$, take the involution (F_ϱ, T) fixing $p \cup S^N$. The main result of [6] says that in this situation $N \in \Omega$, $\dim(F_\varrho) = 2n$ and (F_ϱ, T) is equivariantly cobordant to (W_N^{2n}, T); that is, $\mathcal{A} = \{[W_N^{2n}, T]\}$ in this case, and the result follows from the Lemma. ∎

Finally we prove Theorem 3, recalling from the introduction that $m(n)$ means the upper bound for the dimensions of manifolds M with involution $T : M \to M$ fixing some $p \cup V^n$, for each n (with V^n not necessarily connected).

Theorem 3. *If (M^m, Φ) is a G-action fixing $p \cup V^n$ with V^n connected, then $m \le 2^{k-1} m(n)$; moreover, this bound is best possible for V^n connected.*

Proof. The result of [11] cited in the introduction implies that each of the 2^{t-1} eigenbundles $\varepsilon_\varrho \to V^n$ of the fixed point data of (M^m, Φ) for which $\dim(\mu_\varrho) > 0$ has dimension less than or equal to $m(n) - n$, while obviously each of the $2^{t-1} - 1$ eigenbundles bordant to τ_V has dimension n. Therefore

$$\begin{aligned} m &\le n + 2^{t-1}(m(n) - n) + (2^{t-1} - 1)n \\ &\le n + 2^{k-1}(m(n) - n) + (2^{k-1} - 1)n = 2^{k-1} m(n). \end{aligned}$$

To show that this bound is best possible for V^n connected, consider the maximal involution $(M^{m(n)}, T)$ constructed in [11]. This involution fixes a $p \cup V^n$ with V^n nonconnected. Let $\eta \to V^n$ be the normal bundle of V^n in $M^{m(n)}$. Then $\eta \to V^n$ is cobordant to a bundle $\kappa \to F^n$ with F^n connected, by taking F^n to be the connected sum of the components of V^n and sewing the bundles together, and $(m(n) \to p) \cup (\kappa \to F^n)$ is the fixed point data of an involution $(W^{m(n)}, T)$ equivariantly cobordant to $(M^{m(n)}, T)$.

Then $\Gamma_k^k(W^{m(n)}, T)$ shows that $2^{k-1} m(n)$ is the desired upper bound. ∎

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