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Representations of $(1,1)$-knots

by

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Abstract. We present two different representations of $(1,1)$-knots and study some connections between them. The first representation is algebraic: every $(1,1)$-knot is represented by an element of the pure mapping class group of the twice punctured torus $\mathrm{PMCG}_2(T)$. Moreover, there is a surjective map from the kernel of the natural homomorphism $\Omega : \mathrm{PMCG}_2(T) \to \mathrm{MCG}(T) \cong \mathrm{SL}(2,\mathbb{Z})$, which is a free group of rank two, to the class of all $(1,1)$-knots in a fixed lens space. The second representation is parametric: every $(1,1)$-knot can be represented by a 4-tuple (a,b,c,r) of integer parameters such that $a,b,c \geq 0$ and $r \in \mathbb{Z}_{2a+b+c}$. The strict connection of this representation with the class of Dunwoody manifolds is illustrated. The above representations are explicitly obtained in some interesting cases, including two-bridge knots and torus knots.

1. Introduction and preliminaries. A knot K in a closed, connected, orientable 3-manifold N^3 is called a $(1,1)$-*knot* if there exists a Heegaard splitting of genus one $(N^3,K) = (H,A) \cup_\varphi (H',A')$, where H and H' are solid tori, $A \subset H$ and $A' \subset H'$ are properly embedded trivial arcs ([1]), and $\varphi : (\partial H', \partial A') \to (\partial H, \partial A)$ is an attaching homeomorphism (see Figure 1). Obviously, N^3 turns out to be a lens space $L(p,q)$, including $\mathbf{S}^3 = L(1,0)$ and $\mathbf{S}^1 \times \mathbf{S}^2 = L(0,1)$.

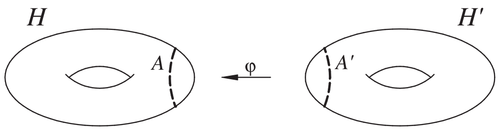

Fig. 1. A $(1,1)$-knot decomposition

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([1]) This means that there exists a disk $D \subset H$ (resp. $D' \subset H'$) with $A \cap D = A \cap \partial D = A$ and $\partial D - A \subset \partial H$ (resp. $A' \cap D' = A' \cap \partial D' = A'$ and $\partial D' - A' \subset \partial H'$).

It is well known that the family of $(1,1)$-knots contains all torus knots and all two-bridge knots in $\mathbf{S}^3$. Several topological properties of $(1,1)$-knots have recently been investigated in many papers (see references in [6]).

Two knots $K \subset N^3$ and $\overline{K} \subset \overline{N}^3$ are said to be *equivalent* if there exists a homeomorphism $f : N^3 \to \overline{N}^3$ such that $f(K) = \overline{K}$.

An n-fold cyclic covering M^3 of a 3-manifold N^3, branched over a knot $K \subset N^3$, is called *strongly-cyclic* if the branching index of K is n. This means that the fiber in M^3 of each point of K consists of a single point. Observe that a cyclic branched covering of a knot K in $\mathbf{S}^3$ is always strongly-cyclic and is uniquely determined, up to equivalence, since $H_1(\mathbf{S}^3 - K) \cong \mathbb{Z}$. Obviously, this property is no longer true for a knot in a more general 3-manifold.

Necessary and sufficient conditions for the existence and uniqueness of strongly-cyclic branched coverings of $(1,1)$-knots have been obtained in [5].

In this paper we present two different representations of $(1,1)$-knots, as developed in [5]–[7], and provide new results.

In Section 2 we show an algebraic representation, introduced in [5, 6], through the pure mapping class group of the twice punctured torus $\mathrm{PMCG}_2(T)$, where $T = \partial H$. Moreover, we give the proof that the kernel of the natural homomorphism $\Omega : \mathrm{PMCG}_2(T) \to \mathrm{MCG}(T) \cong \mathrm{SL}(2,\mathbb{Z})$ is a free group of rank two. Since there is a surjective map from $\ker \Omega$ to the class of all $(1,1)$-knots in a fixed lens space, every $(1,1)$-knot can be represented by an element of $\ker \Omega$, whose standard generators τ_m and τ_l have a nice topological meaning. A characterization of the subgroup $\mathcal{E}$ of $\mathrm{PMCG}_2(T)$, consisting of the (isotopy classes of) homeomorphisms which extend to the handlebody H, fixing A, is also given. The group $\mathcal{E}$ contains elements all producing the trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2$, so its determination appears to be important in order to produce a "more injective" representation.

In Section 3 we describe the parametric representation by 4-tuples of integers, introduced in [7]. This parametrization has a strict connection with the class of Dunwoody manifolds.

A direct connection between the two representations has been established in [7] for the interesting case of torus knots. Using this result, an explicit parametrization for a large class of torus knots is obtained (see Proposition 9) and a table with the parametrization for other torus knots is provided in the Appendix.

2. Algebraic representation of $(1,1)$-knots. The *mapping class group* of a torus T (i.e. the group of isotopy classes of orientation-preserving homeomorphisms of T) is indicated by $\mathrm{MCG}(T)$. Moreover, $\mathrm{MCG}_2(T)$ denotes the mapping class group of the twice punctured torus, with two fixed punctures P_1 and P_2.

Now, let $K \subset L(p,q)$ be a $(1,1)$-knot with $(1,1)$-decomposition $(L(p,q),K) = (H,A) \cup_\varphi (H',A')$ and let $\mu : (H,A) \to (H',A')$ be a fixed orientation-reversing homeomorphism. Then $\psi = \varphi\mu_{|\partial H}$ is an orientation-preserving homeomorphism of $(\partial H, \partial A) = (T, \{P_1, P_2\})$. Moreover, since two isotopic attaching homeomorphisms produce equivalent $(1,1)$-knots, we have a natural surjective map

$$\Theta : \psi \in \mathrm{MCG}_2(T) \mapsto K_\psi \in \mathcal{K}$$

from $\mathrm{MCG}_2(T)$ to the set $\mathcal{K}$ of all $(1,1)$-knots.

In the following, if δ is a simple closed curve in T, then t_δ denotes the right-hand Dehn twist around δ.

Let α, β, γ be the curves depicted in Figure 2. Then $\mathrm{MCG}_2(T)$ is generated by $t_\alpha, t_\beta, t_\gamma$, which fix the punctures, and a π-rotation ϱ of T, which exchanges the punctures. Observe that ϱ commutes with the other generators.

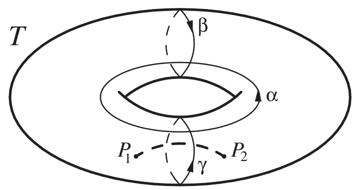

Fig. 2. Generators of $\mathrm{PMCG}_2(T)$

It is easy to see that ϱ can be extended to a homeomorphism of the pair (H,A), so K_ψ and $K_{\psi\varrho}$ are equivalent knots for each $\psi \in \mathrm{MCG}_2(T)$. Therefore, we can restrict our attention to the subgroup $\mathrm{PMCG}_2(T)$ of $\mathrm{MCG}_2(T)$, called the *pure mapping class group* of the twice punctured torus, consisting of the elements of $\mathrm{MCG}_2(T)$ fixing the punctures.

The restriction Θ' of Θ to $\mathrm{PMCG}_2(T)$ is still surjective, so every $(1,1)$-knot can be represented by elements belonging to $\mathrm{PMCG}_2(T)$.

Consider the morphism $\Omega : \mathrm{PMCG}_2(T) \to \mathrm{SL}(2,\mathbb{Z})$, obtained as the composition of the natural epimorphism from $\mathrm{PMCG}_2(T)$ to $\mathrm{MCG}(T)$ with the isomorphism between $\mathrm{MCG}(T)$ and $\mathrm{SL}(2,\mathbb{Z})$, relative to the ordered base (β,α) of $H_1(T)$. In terms of the generators of $\mathrm{PMCG}_2(T)$, Ω is given by:

$$\Omega(t_\alpha) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad \Omega(t_\beta) = \Omega(t_\gamma) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

With the above notations, if $\Omega(\psi) = \left(\begin{smallmatrix} q & s \\ p & r \end{smallmatrix}\right)$, then K_ψ is a $(1,1)$-knot in the lens space $L(|p|,|q|)$ (see [4, p. 186]).

Now we list some examples of $(1,1)$-knots given by this representation.

Example 1.

(a) If either $\psi = \psi_{0,1} = 1$ or $\psi = t_\beta$ or $\psi = t_\gamma$, then K_ψ is the trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2$.

(b) If either $\psi = t_\alpha$ or $\psi = \psi_{1,0} = t_\beta t_\alpha t_\beta$, then K_ψ is the trivial knot in $\mathbf{S}^3$.

(c) Let p, q be integers such that $0 < q < p$ and $\gcd(p,q) = 1$. If

$$\frac{p}{q} = a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_m}}},$$

then the trivial knot in the lens space $L(p,q)$ is represented by

$$\psi_{p,q} = \begin{cases} t_\alpha^{a_1} t_\beta^{-a_2} \cdots t_\alpha^{a_m} & \text{if } m \text{ is odd,} \\ t_\alpha^{a_1} t_\beta^{-a_2} \cdots t_\beta^{-a_m} t_\beta t_\alpha t_\beta & \text{if } m \text{ is even.} \end{cases}$$

(d) If $\psi = t_\alpha t_\beta t_\alpha t_\alpha t_\gamma t_\alpha$, then K_ψ is the core knot $\mathbf{S}^1 \times \{P\} \subset \mathbf{S}^1 \times \mathbf{S}^2$, where P is any point of $\mathbf{S}^2$.

The representation Θ' is not at all injective and, in general, there are infinitely many elements of $\mathrm{PMCG}_2(T)$ producing the same $(1,1)$-knot. For example, given $\psi \in \mathrm{PMCG}_2(T)$, all the elements ψt_β^c produce equivalent $(1,1)$-knots, for each $c \in \mathbb{Z}$. So a natural question arises: is it possible to decide if two elements in $\mathrm{PMCG}_2(T)$ represent the same $(1,1)$-knot? Answering this question seems to be rather hard.

A first step in this direction is given by the following result.

Theorem 1 ([6]). *Let K be a $(1,1)$-knot in $L(p,q)$. Then there exist $\psi', \psi'' \in \ker \Omega$ such that $K = K_\psi$, with $\psi = \psi' \psi_{p,q} = \psi_{p,q} \psi''$, where $\psi_{p,q}$ is the map defined in Example 1, only depending on p and q.*

As a consequence, for each lens space $L(p,q)$ we get a surjective map

$$\Theta_{p,q} : \ker \Omega \to \mathcal{K}_{p,q},$$

where $\mathcal{K}_{p,q}$ is the set of all $(1,1)$-knots in $L(p,q)$. Moreover, $\ker \Omega$ has a very simple structure, as shown in the following result, which is presented without proof in [6].

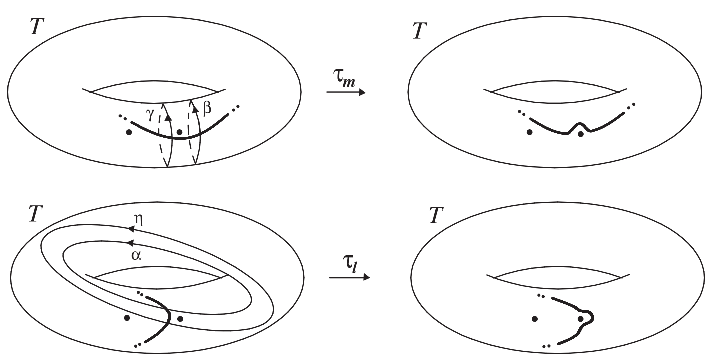

Fig. 3. Action of τ_m and τ_l

THEOREM 2. *The group* $\ker \Omega$ *is freely generated by* $\tau_m = t_\beta t_\gamma^{-1}$ *and* $\tau_l = t_\eta t_\alpha^{-1}$, *where* t_η *is the right-hand Dehn twist around the curve* η *depicted in Figure* 3, *and* $t_\eta = \tau_m^{-1} t_\alpha \tau_m$.

Proof. Let $F_2 = (T \times T) - \Delta$, where $\Delta = \{(x,x) \mid x \in T\}$ denotes the diagonal, and let $\mathcal{H}(T)$ be the group of orientation-preserving automorphisms of the torus. Moreover, let $\mathcal{H}_2$ be the subgroup of $\mathcal{H}(T)$ consisting of the elements pointwise fixing the punctures. By [3, Th. 1], the evaluation map $e : \mathcal{H}(T) \to F_2$ is a fibering with fiber $\mathcal{H}_2$ that induces the exact sequence of homotopy groups

$$\cdots \to \pi_1(\mathcal{H}(T), \mathrm{id}) \xrightarrow{e_\#} \pi_1(F_2, (P_1, P_2)) \xrightarrow{d_\#} \pi_0(\mathcal{H}_2, \mathrm{id}) \xrightarrow{i_\#} \pi_0(\mathcal{H}(T), \mathrm{id}) \to 1$$

where $i_\#$ denotes the homomorphism induced by the inclusion. Since $\pi_0(\mathcal{H}_2, \mathrm{id}) = \mathrm{PMCG}_2(T)$ and $\pi_0(\mathcal{H}(T), \mathrm{id}) = \mathrm{MCG}(T)$, we have

$$\ker \Omega \cong \ker i_\# = \operatorname{im} d_\# \cong \pi_1(F_2, (P_1, P_2))/\ker d_\#.$$

Moreover, from [2, Th. 5] we have

$$\pi_1(F_2, (P_1, P_2)) \\ = \langle \overline{\alpha}_1, \overline{\alpha}_2, \overline{\beta}_1, \overline{\beta}_2 \mid 1 = [\overline{\alpha}_1, \overline{\alpha}_2] = [\overline{\beta}_1, \overline{\beta}_2] = [\overline{\alpha}_1, \overline{\beta}_j] = [\overline{\beta}_1, \overline{\alpha}_j],\ j = 1, 2\rangle,$$

where $\overline{\alpha}_1 = (\alpha_1, \alpha_2)$, $\overline{\beta}_1 = (\beta_1, \beta_2)$, $\overline{\alpha}_2 = (P_1, \alpha_2)$, $\overline{\beta}_2 = (P_1, \beta_2)$ where α_i and β_i are the loops depicted in Figure 4 and P_1 denotes the constant loop based at the point P_1. From [3, Cor. 1.3], $\ker d_\#$ is freely generated by $\overline{\alpha}_1$ and $\overline{\beta}_1$. So $\ker \Omega$ is the free group generated by $d_\#(\overline{\alpha}_2)$ and $d_\#(\overline{\beta}_2)$, which are respectively τ_l and τ_m. ∎

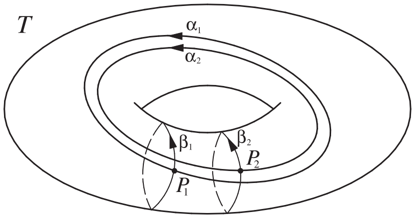

Fig. 4

The standard generators τ_m and τ_l of $\ker \Omega$ have a concrete topological meaning: the effect of τ_m and τ_l is to slide one puncture (say P_2) respectively along a meridian and along a longitude of the torus (see Figure 3).

Since every two-bridge knot admits a Conway presentation with an even number of even parameters (see [12, Exercise 2.1.14]), the following result gives a representation for all two-bridge knots in $\mathbf{S}^3$. An analogous result for torus knots will be given in Section 4.

PROPOSITION 3 ([6]). *The two-bridge knot having Conway parameters* $[2a_1, 2b_1, \ldots, 2a_n, 2b_n]$ *is the* $(1,1)$*-knot* K_ψ *with*
$$\psi = t_\beta t_\alpha t_\beta \tau_m^{-b_n} t_\varepsilon^{a_n} \cdots \tau_m^{-b_1} t_\varepsilon^{a_1},$$
where $t_\varepsilon = \tau_l^{-1} \tau_m \tau_l \tau_m^{-1}$.

Observe that the representations $\Theta_{p,q}$ are also not injective, since K_ψ and $K_{\psi\tau_m^c}$ are equivalent knots for all $c \in \mathbb{Z}$.

Another way to obtain a "more injective" representation seems to be the characterization of the subgroup $\mathcal{E}$ of $\mathrm{PMCG}_2(T)$, consisting of the isotopy classes of the homeomorphisms admitting an extension to a homeomorphism of H which fixes A. For each $\varepsilon \in \mathcal{E}$, the knot K_ε is the trivial knot in $\mathbf{S}^1 \times \mathbf{S}^2$. Moreover, ψ and $\psi\varepsilon$ produce equivalent $(1,1)$-knots for every $\psi \in \mathrm{PMCG}_2(T)$ and $\varepsilon \in \mathcal{E}$. Therefore, there exists an induced surjective map
$$\Theta'' : \mathrm{PMCG}_2(T)/\mathcal{E} \to \mathcal{K},$$
where $\mathrm{PMCG}_2(T)/\mathcal{E}$ is the set of left cosets of $\mathcal{E}$ in $\mathrm{PMCG}_2(T)$.

The following proposition gives a characterization of the elements of $\mathcal{E}$ in terms of their action on the fundamental groups of $T - \{P_1, P_2\}$ and $H - A$. Let $* \in T$ be a base point of $T - \{P_1, P_2\}$. We define the loops $\overline{\alpha} = \xi \cdot \alpha \cdot \xi^{-1}$, $\overline{\beta} = \xi_1 \cdot \beta \cdot \xi_1^{-1}$ and $\overline{\gamma} = \xi_2 \cdot \gamma \cdot \xi_2^{-1}$, where ξ, ξ_1, ξ_2 are paths connecting $*$ to α, β and γ respectively. Obviously, $\pi_1(T - \{P_1, P_2\}, *)$ is freely generated by the set $\{\overline{\alpha}, \overline{\beta}, \overline{\gamma}\}$, and $\pi_1(H - A, *)$ is freely generated by the set $\{\overline{\alpha}, \overline{\gamma}\}$.

PROPOSITION 4. *Let* $\psi \in \mathrm{PMCG}_2(T)$*. Then* ψ *belongs to* $\mathcal{E}$ *if and only if* $i_\#(\psi_\#(\overline{\beta})) = 1$, *where* $i_\# : \pi_1(T - \{P_1, P_2\}, *) \to \pi_1(H - A, *)$ *is induced by inclusion.*

Proof. $\Rightarrow$ Trivial.

$\Leftarrow$ By the proof of [11, Theorem 10.1], ψ extends to a homeomorphism $\widetilde{\psi}$ of H. Moreover, $\psi(\beta)$ bounds a disk D such that $D \cap A = D \cap \widetilde{\psi}(A) = \emptyset$, and cutting H along D produces a 3-ball. Therefore, up to isotopy we can suppose that $\widetilde{\psi}(A) = A$. ∎

It is easy to verify that t_β, t_γ and $(t_\beta t_\alpha t_\beta)^2$ belong to $\mathcal{E}$, while t_α does not, but the problem of finding a (possibly finite) presentation for $\mathcal{E}$ is still open.

3. Parametric representation of $(1,1)$-knots. As proved in [7], a $(1,1)$-knot K_ψ is completely determined by the curve $\psi(\beta)$ on $T - \{P_1, P_2\}$. Moreover, in the open Heegaard diagram obtained by cutting T along β, the curve $\psi(\beta)$ is, up to Singer moves [13] fixing the set $\{P_1, P_2\}$, one of the three types depicted in Figure 6 (see proof of [7, Theorem 3]). In all the cases the circles C' and C'' represent the curve β.

In case (1), the parameters a, b and c denote a, b and c parallel arcs respectively, which are $\psi(\beta)$ after the cutting. In this case, we have $d = 2a + b + c > 0$. The parameter r gives the gluing rule between the circles C' and C''. Obviously, r can be taken mod d. The corresponding $(1,1)$-knot is denoted by $K(a,b,c,r)$.

In case (2), the corresponding $(1,1)$-knot is the trivial knot in $\mathbf{S}^1\times\mathbf{S}^2$, denoted by $K(0,0,0,0)$.

In case (3), the corresponding $(1,1)$-knot is the core knot $\mathbf{S}^1\times\{P\}\subset\mathbf{S}^1\times\mathbf{S}^2$, which admits no parametrization, as will be explained in the following.

In this way we obtain a parametrization of $(1,1)$-knots by 4-tuples of integers (a,b,c,r), with $a,b,c\geq 0$ and either $r\in\mathbb{Z}_d$, when $d>0$, or $r=0$, when $d=0$.

An interesting property of this parametrization is its connection with Dunwoody manifolds, which are closed orientable 3-manifolds introduced in [9] using a class of trivalent regular planar graphs (called *Dunwoody diagrams*), depending on six integer parameters a,b,c,n,r,s such that $n>0$ and $a,b,c\geq 0$.

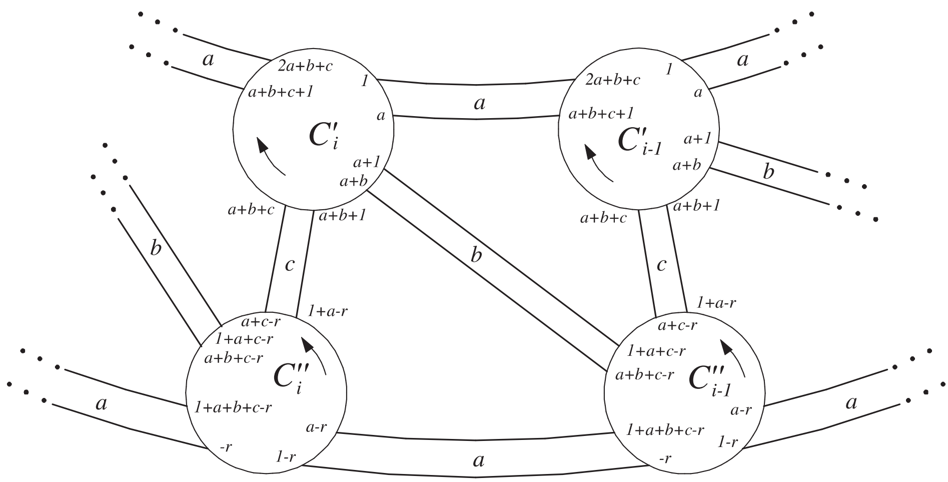

Fig. 5. Heegaard diagram of Dunwoody type

More precisely, for particular values of the parameters, called *admissible*, a Dunwoody diagram is an (open) Heegaard diagram of genus n (see Figure 5), which contains n internal circles $C'_1,\ldots,C'_n$, and n external circles $C''_1,\ldots,C''_n$, each having $d=2a+b+c$ vertices. For every $i=1,\ldots,n$, the circle C'_i (resp. C''_i) is connected to the circle C'_{i+1} (resp. C''_{i+1}) by a parallel arcs, to the circle C''_i by c parallel arcs and to the circle C''_{i-1} by b parallel arcs (subscripts mod n). The cycle C'_i is glued to the cycle C''_{i+s} (subscripts mod n) so that equally labelled vertices are identified.

Observe that the parameters r and s can be considered mod d and n, respectively. Since the identification rule and the diagram are invariant with respect to an obvious cyclic action of order n, the Dunwoody manifold $D(a,b,c,r,n,s)$ admits a cyclic symmetry of order n.

Theorem 5.

(i) ([10]) *The Dunwoody manifold* $D(a, b, c, n, r, s)$ *is the* n*-fold strongly-cyclic covering of the lens space* $D(a, b, c, 1, r, 0)$ *(possibly* $\mathbf{S}^3$*), branched over* $K(a, b, c, r)$.

(ii) ([7]) *If* M^3 *is an* n*-fold strongly-cyclic branched covering of* $K(a, b, c, r)$*, then there exists* $s \in \mathbb{Z}_n$ *such that* M^3 *is homeomorphic to the Dunwoody manifold* $D(a, b, c, n, r, s)$.

Therefore, the class of Dunwoody manifolds coincides with the class of strongly-cyclic branched coverings of $(1, 1)$*-knots.*

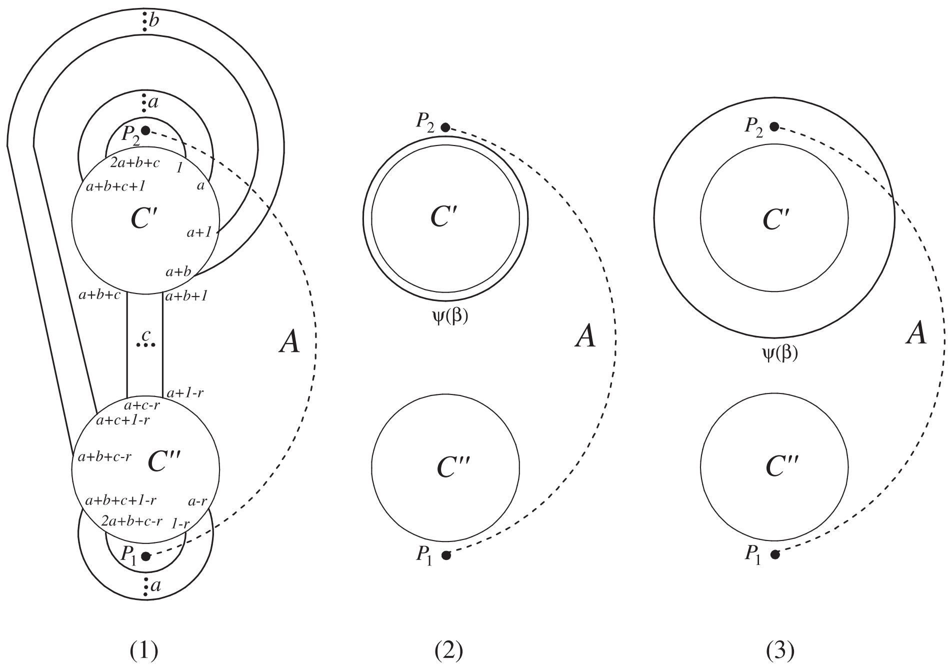

Fig. 6

The core knot cannot be parametrized as $K(a, b, c, r)$, since it admits no strongly-cyclic branched coverings (see [5]).

Observe that not every 4-tuple of non-negative integers (a, b, c, r) determines a $(1, 1)$-knot $K(a, b, c, r)$, since the corresponding diagram could fail to be a Heegaard diagram. For example, the 4-tuples $(a, 0, a, a)$, with $a > 1$, and $(1, 0, c, 2)$, with c even, do not determine any $(1, 1)$-knot (see [10]).

Example 2.

(a) The trivial knot in $L(p, q)$ (including $L(1, 0) \cong \mathbf{S}^3$) is $K(0, 0, p, q)$.
(b) The two-bridge knot of type $(2a + 1, 2r)$ is $K(a, 0, 1, r)$ (see [10]).

(c) The (1, 1)-knot $K(1,1,1,2) \subset \mathbf{S}^1 \times \mathbf{S}^2$ admits three 3-fold strongly-cyclic branched coverings. One of them is the 3-torus $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$, which is homeomorphic to the Dunwoody manifold $D(1,1,1,3,2,1)$. It is well known that this manifold cannot be a cyclic branched covering of any knot in $\mathbf{S}^3$.

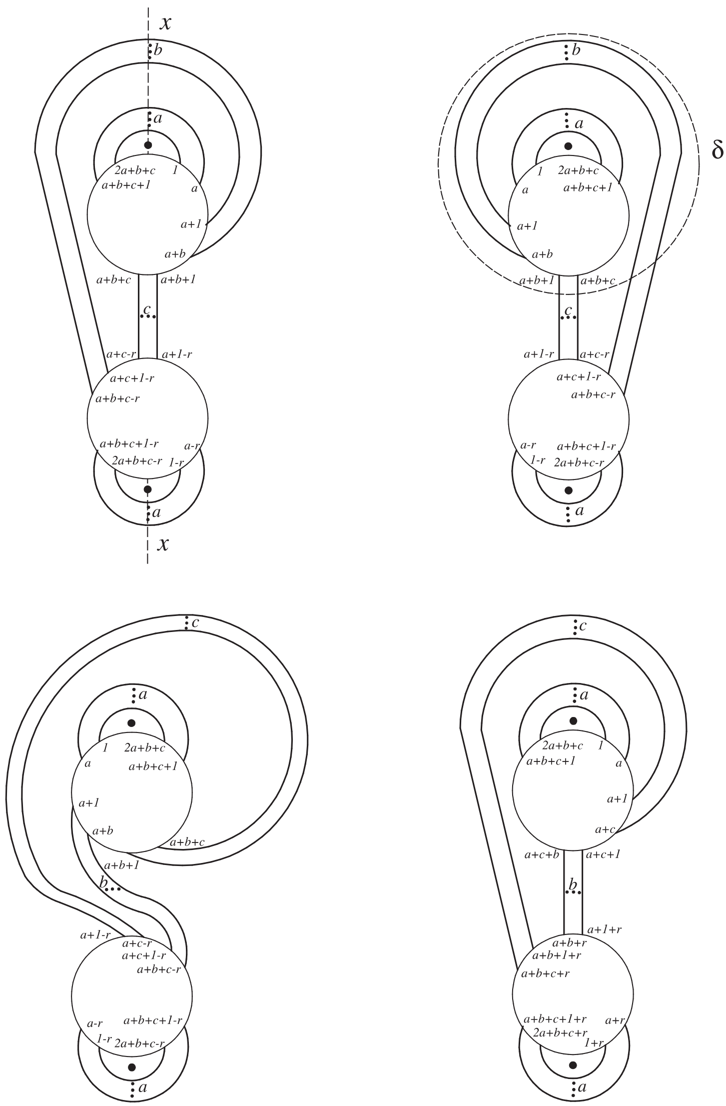

Fig. 7. From $K(a,b,c,r)$ to $K(a,c,b,-r)$

Just as the algebraic representation, the parametric representation of a (1, 1)-knot is not unique, as proved by the following lemma.

Lemma 6.

(a) $K(a,b,c,r)$ *and* $K(a,c,b,-r)$ *are equivalent*;
(b) $K(a,0,c,r)$ *and* $K(a,c,0,r)$ *are equivalent.*

Proof. (a) Looking at Figure 7, we pass from the first diagram, representing $K(a,b,c,r)$, to the second by a reflection along an axis passing through the punctures (denoted by x-x in the figure). Operating a Singer move of type IIB along δ, and relabelling the vertices, we obtain $K(a,c,b,-r)$.

(b) The application of a Singer move of type IIB along δ (see Figure 7) on $K(a,b,0,r)$ gives $K(a,0,b,r)$. ∎

A different parametrization of $(1,1)$-knots, involving four parameters for the knot and two additional parameters for the ambient space, can be found in [8].

4. The case of torus knots. As previously remarked, a very important class of $(1,1)$-knots in $\mathbf{S}^3$ are torus knots. Without loss of generality, we can consider torus knots $\mathbf{t}(k,h)$ with $0 < k < h$. The next result gives the algebraic representation for torus knots. In the following, $\lfloor x \rfloor$ denotes the integral part of x.

Proposition 7 ([6]). *The torus knot* $\mathbf{t}(k,h)$ *is the* $(1,1)$*-knot* K_ψ *with*

$$\psi = \prod_{j=0}^{h-1} (\tau_l^{-1} \tau_m^{\varepsilon_{h-j}}) t_\beta t_\alpha t_\beta, \tag{1}$$

where $\varepsilon_{h-j} = \lfloor (j+1)k/h \rfloor - \lfloor (j+2)k/h \rfloor$, $\tau_m = t_\beta t_\gamma^{-1}$ *and* $\tau_l = \tau_m^{-1} t_\alpha \tau_m t_\alpha^{-1}$.

Moreover, the following proposition tells us how to pass from the algebraic to the parametric representation of a torus knot.

Proposition 8 ([7]). *Let* $\mathbf{t}(k,h) \subset \mathbf{S}^3$ *be a torus knot and* ψ *be as in* (1). *Then* $\mathbf{t}(k,h) = K(a,b,c,r)$, *where* $(a,b,c,r) = (a_h, b_h, c_h, r_h)$ *is the final step of the following algorithm, applied for* $i = h - j = 1, \ldots, h$:

- $(a_0, b_0, c_0, r_0) = (0,0,1,0)$ *and* $z_0 = 0$;
- *for* $i = 1, \ldots, h$:

$$\begin{cases} a_i = a_{i-1} + v, \\ b_i = r_{i-1} - 2w - ud, \\ c_i = d - b_i, \\ r_i = a_{i-1} + v + w, \\ z_i = u - \varepsilon_i, \end{cases}$$

where

$$w = \begin{cases} a_{i-1} + b_{i-1} + c_{i-1} & \text{if } z_{i-1} < -1 - \varepsilon_i, \\ a_{i-1} + c_{i-1} & \text{if } z_{i-1} = -1 - \varepsilon_i, \\ a_{i-1} & \text{if } z_{i-1} > -1 - \varepsilon_i, \end{cases}$$

$$v = \begin{cases} -(b_{i-1} + c_{i-1})(z_{i-1} + 1 + \varepsilon_i) - b_{i-1} & \text{if } z_{i-1} < -1 - \varepsilon_i, \\ 0 & \text{if } z_{i-1} = -1 - \varepsilon_i, \\ (b_{i-1} + c_{i-1})(z_{i-1} + 1 + \varepsilon_i) - c_{i-1} & \text{if } z_{i-1} > -1 - \varepsilon_i, \end{cases}$$

and $u = \lfloor (r_{i-1} - 2w)/d \rfloor$, *with* $d = 2a_{i-1} + b_{i-1} + c_{i-1}$.

Explicit formulae for torus knots of type $\mathbf{t}(k, ck \pm 1)$ have been obtained in [1, 7] (see Appendix).

The next result gives an explicit parametric representation of another family of torus knots, which contains all the torus knots with bridge number at most three.

PROPOSITION 9. *The torus knot* $\mathbf{t}(sq' + 1, (sq' + 1)q + s)$ *is*

$$K(q', q'(2qq'(s-1) + 2q + s - 2), 1 + (s-2)q', 2q'^2(s-1) + sq' + 1)$$

for every $q, q' > 0$ *and* $s > 1$.

Proof. From Proposition 7, we see that $\mathbf{t}(sq' + 1, (sq' + 1)q + s)$ is represented by

$$\psi = ((\tau_l^{-q}\tau_m^{-1})^{q'}\tau_l^{-1})^{s-1}(\tau_l^{-q}\tau_m^{-1})^{q'+1}\tau_l^{-1}t_\beta t_\alpha t_\beta.$$

By [7, Corollary 8], the application of $(\tau_l^{-q}\tau_m^{-1})^{q'+1}\tau_l^{-1}$ to $K(0,0,1,0)$ gives $K(1, q'-1, q'(2q-1), q'+1)$ and $z = 0$. Applying τ_l^{-1}, we get $K(q', q'-1, 2 + q'(2q-1), q'+1)$ and $z = 0$. Then we have to apply q' times $\tau_l^{-q}\tau_m^{-1}$. If $q' > 1$, applying the first $\tau_l^{-q}\tau_m^{-1}$, we get $K(q', q'-2, q'(4q-1)+3, 3q'+2)$ and $z = 0$. So, each time we apply $\tau_l^{-q}\tau_m^{-1}$, as long as it is not the final step, the a and z terms remain unchanged, the b term decreases by one, the c term increases by $2qq' + 1$ and the r term increases by $2q' + 1$. So, after $q' - 1$ steps, we get $K(q', 0, 2qq'^2 + 1, 2q'^2)$ and $z = 0$. Now applying $\tau_l^{-q}\tau_m^{-1}$ for the last time, we get $K(q', 2qq'^2 + 2qq', 1, 2q'^2 + 2q' + 1)$ and $z = -1$, which is as asserted for $s = 2$. If $s > 2$ we have to apply $(\tau_l^{-q}\tau_m^{-1})^{q'}\tau_l^{-1}$ to $K(q', 2qq'^2 + 2qq', 1, 2q'^2 + 2q' + 1)$, with $z = -1$, another time. Proceeding as before, we obtain $K(q', 4qq'^2 + 2qq' + q', q' + 1, 4q'^2 + 3q' + 1)$ and $z = -1$. So each time we apply $(\tau_l^{-q}\tau_m^{-1})^{q'}\tau_l^{-1}$, the a and z terms remain unchanged, the b term increases by $2qq'^2 + q'$, the c term increases by q', and the r term increases by $2q'^2 + q'$. So, after $s - 1$ steps, we get $K(q', (2qq'^2 + q')(s-2) + 2qq'^2 + 2qq', q'(s-2) + 1, (2q'^2 + q')(s-2) + 2q'^2 + 2q' + 1)$, as stated. ∎

The algorithm of Proposition 8 can easily be implemented. The table in the Appendix is obtained by computer and contains the parametrizations of all torus knots $\mathbf{t}(k,h)$ with $k,h \le 25$, not included in the previous cases.

5. Appendix: $(1,1)$-parametrization of torus knots

- $\mathbf{t}(k,qk+1)$ is $K(1,k-2,(k-1)(2q-1),k)$ for all $k>1$ and $q>0$ (see [1, 7]).
- $\mathbf{t}(k,qk-1)$ is $K(1,k-2,(k-1)(2q-1)-2,(k-1)(2q-3))$ for all $k,q>1$ (see [1]).
- For all $q,q'>0$ and $s>1$, $\mathbf{t}(sq'+1,(sq'+1)q+s)$ is

$$K(q',q'(2qq'(s-1)+2q+s-2),1+(s-2)q',2q'^2(s-1)+sq'+1).$$

The following table gives the parametrization of the torus knots $\mathbf{t}(k,h)$ with $k,h \le 25$, not included in the previous formulae.

Knot	Parametrization
$\mathbf{t}(5,8)$	$K(2,1,14,11)$
$\mathbf{t}(5,13)$	$K(2,1,26,11)$
$\mathbf{t}(5,18)$	$K(2,1,38,11)$
$\mathbf{t}(5,23)$	$K(2,1,50,11)$
$\mathbf{t}(7,11)$	$K(2,3,24,17)$
$\mathbf{t}(7,12)$	$K(3,1,34,23)$
$\mathbf{t}(7,18)$	$K(2,3,44,17)$
$\mathbf{t}(7,19)$	$K(3,1,58,23)$
$\mathbf{t}(7,25)$	$K(2,3,64,17)$
$\mathbf{t}(8,11)$	$K(3,2,33,28)$
$\mathbf{t}(8,13)$	$K(3,41,2,32)$
$\mathbf{t}(8,19)$	$K(3,2,63,28)$
$\mathbf{t}(8,21)$	$K(3,71,2,32)$
$\mathbf{t}(9,14)$	$K(2,5,34,23)$
$\mathbf{t}(9,16)$	$K(4,1,62,39)$
$\mathbf{t}(9,23)$	$K(2,5,62,23)$
$\mathbf{t}(9,25)$	$K(4,1,102,39)$
$\mathbf{t}(10,17)$	$K(3,4,61,38)$

Knot	Parametrization
$\mathbf{t}(11,14)$	$K(4,3,60,53)$
$\mathbf{t}(11,15)$	$K(3,5,54,43)$
$\mathbf{t}(11,17)$	$K(2,7,44,29)$
$\mathbf{t}(11,18)$	$K(3,68,5,53)$
$\mathbf{t}(11,19)$	$K(4,86,3,59)$
$\mathbf{t}(11,20)$	$K(5,1,98,59)$
$\mathbf{t}(11,25)$	$K(4,3,116,53)$
$\mathbf{t}(12,17)$	$K(5,2,87,68)$
$\mathbf{t}(12,19)$	$K(5,99,2,72)$
$\mathbf{t}(13,18)$	$K(5,98,3,83)$
$\mathbf{t}(13,20)$	$K(2,9,54,35)$
$\mathbf{t}(14,17)$	$K(5,4,95,86)$
$\mathbf{t}(14,19)$	$K(3,8,75,58)$
$\mathbf{t}(15,19)$	$K(4,7,96,81)$
$\mathbf{t}(17,20)$	$K(6,5,138,127)$
$\mathbf{t}(18,23)$	$K(7,179,4,158)$
$\mathbf{t}(18,25)$	$K(5,163,8,138)$
$\mathbf{t}(19,23)$	$K(5,9,150,131)$
$\mathbf{t}(19,24)$	$K(4,11,132,109)$
$\mathbf{t}(20,23)$	$K(7,6,189,176)$

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