FUNDAMENTA
MATHEMATICAE
188 (2005)

On surface braids of index four with at most two crossings

by

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Dedicated to Professor Yukio Matsumoto for his sixtieth birthday

Abstract. Let Γ be a 4-chart with at most two crossings. We show that if the closure of the surface braid obtained from Γ is one 2-sphere, then the sphere is a ribbon surface.

1. Introduction. Kamada showed that any 3-chart can be modified by C-moves to a 3-chart without white vertices (see [3]). Nagase and Hirota showed that any 4-chart with at most one crossing can be modified by C-moves to a chart without white vertices (see [4]). Kamada showed that for any ribbon surface, for some positive integer n there exists an n-chart without white vertices such that the ribbon surface is ambient isotopic to the closure of the surface braid obtained from it (see [3, Proposition 20]). Conversely, if a chart can be modified by C-moves to a chart without white vertices, then the closure of the surface braid obtained from it is ambient isotopic to a ribbon surface. Hence the closure of the surface braid obtained from a 4-chart with at most one crossing is ambient isotopic to a ribbon surface.

Let Γ be a 4-chart with exactly two crossings. Aiba and Nagase showed that if Γ is a minimal 4-chart with exactly one special house, then each connected component of Γ is a free edge, a hoop, a 4-chart as shown in Figure 1, or its reflection. Here, a *house* means a complementary domain of the subgraph consisting of even labeled edges and their vertices, a *special house* means a house containing crossings, a *free edge* E means an edge such that each vertex of E has degree one, and a *hoop* means an edge without vertices.

2000 *Mathematics Subject Classification*: Primary 57Q45; Secondary 57Q35.
The second author is partially supported by Grant-in-Aid for Scientific Research (No. 16540082), Ministry of Education, Science and Culture, Japan.

The 4-chart in Figure 1 has eight black vertices, where a *black vertex* means a vertex of degree one. On the other hand, if the closure of the surface braid obtained from a 4-chart represents a 2-sphere, then the chart has exactly six black vertices. Therefore if Γ is a connected minimal 4-chart with exactly two crossings and one special house, then the closure of the surface braid obtained from Γ is either a ribbon surface, a connected closed surface of genus $g \geq 1$, or a disconnected closed surface.

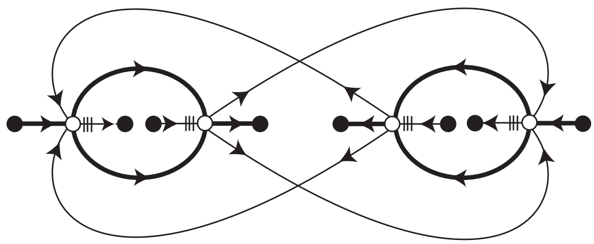

Fig. 1. The thick lines are edges of label 2. The other lines are edges of label 1 or 3.

The following is the main result of this paper.

THEOREM 1.1. *Let Γ be a 4-chart with at most two crossings. If the closure of the surface braid obtained from Γ is one 2-sphere, then Γ can be modified by C-moves to a 4-chart without white vertices. Hence the sphere is a ribbon surface.*

2. Definitions and preliminaries. In this section, we investigate distinguished arcs and terminal edges of label 2.

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices called *hoops*, satisfying the following four conditions:

(i) Every vertex has degree 1, 4, or 6.
(ii) The labels of edges are in $\{1, 2, \ldots, n-1\}$.
(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three outward, and those six arcs are labeled i and $i+1$ alternately for some i, where the orientation and label of each arc are inherited from the edge containing the arc.
(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j| > 1$.

A vertex of degree 1, 4, and 6 is called a *black vertex,* a *crossing*, and a *white vertex* respectively. *To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk.*

Among the six short arcs in a small neighborhood of a white vertex, the center arc of any three consecutive arcs oriented inward or outward is called a *middle* arc of the white vertex. There are two middle arcs in a small neighborhood of each white vertex. A middle arc of odd label is called a *distinguished arc*. We mark it as in Figure 2. If an edge contains a distinguished arc containing a white vertex w, then the edge is called a *distinguished edge at* w. An edge is called *free* if it has two black vertices; *terminal* if it contains one black vertex and one white vertex; and a *loop* if it contains only one vertex.

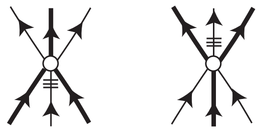

Fig. 2. The thick lines are edges of label 2. The other lines are edges of label 1 or 3.

Remark 2.1. Around each white vertex, there exists only one distinguished arc.

Let Γ be a connected 4-chart, and Γ_2 the subgraph of Γ consisting of edges of label 2 and their vertices. Then a complementary domain of Γ and Γ_2 are called a *room* and a *house*, respectively. Note that all rooms are open disks. In general, a house and a room of each component of a 4-chart are called a house and a room of the 4-chart, respectively.

Lemma 2.2 ([1]). *Let Γ be a connected* 4*-chart, $t(\Gamma)$ the number of terminal edges of label* 2*, and $w(\Gamma)$ the number of white vertices. For each natural number i, let n_i be the number of houses with exactly i boundary components each. Then*

$$w(\Gamma) - t(\Gamma) = 2(n_1 - 2) - 2\sum_{i\geq 3} n_i(i-2).$$

A *C-move* is a local modification of charts in a disk as shown in Lemma 16 of [3] (cf. Figures 19 and 20 in Chapter 3 of [2]). We show some C-moves in Figure 3. Two charts are *C-move equivalent* if there exists a finite sequence of C-moves which modify one of the charts to the other.

For each chart Γ, let $c(\Gamma)$, $w(\Gamma)$ and $f(\Gamma)$ be the number of crossings, of white vertices, and of free edges respectively. The triplet $(c(\Gamma), w(\Gamma), -f(\Gamma))$ is called the *complexity* of the chart. A chart is called *minimal* if its complexity is lexicographically minimal among the charts C-move equivalent to it.

Remark 2.3. Let Γ be a minimal 4-chart. Then

(i) no edge contains two distinguished arcs,

(ii) any terminal edge contains a middle arc, in particular, an odd labeled terminal edge is a distinguished edge (see the C-III-1 move in Figure 3).

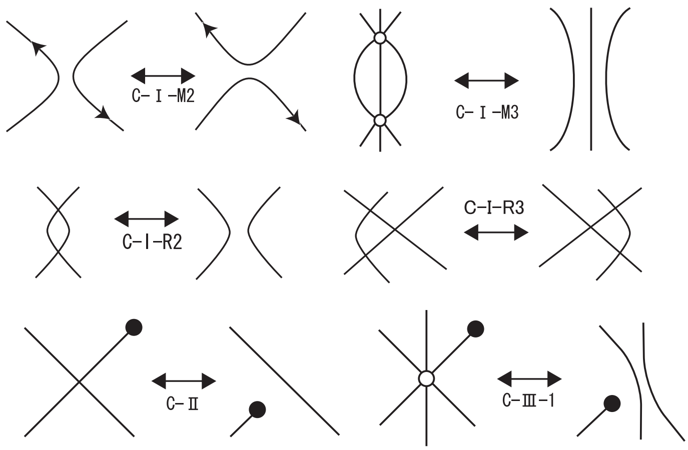

Fig. 3. In the left figure of a C-III-1 move, the edge containing the black vertex does not contain a middle arc.

It is easy to prove the following lemma.

LEMMA 2.4. *Each component of a minimal* 4*-chart is a minimal* 4*-chart.*

TERMINOLOGY. If a vertex or an edge is contained in the closure of a room or a house, then we say that the vertex or the edge *belongs* to the room or the house, or that the room or the house *possesses* the vertex or the edge.

Let Γ be a connected 4-chart, R a room of Γ, and X_R the closure of R. Let D be a disk, and $\overline{P}_1, \ldots, \overline{P}_n$ points on the boundary of D, ∂D, situated in this order. The points split ∂D into n arcs $\overline{A}_1, \ldots, \overline{A}_n$ where $\overline{P}_i$ and $\overline{P}_{i+1}$ are the end points of $\overline{A}_i$, with $\overline{P}_{n+1} = \overline{P}_1$. Let $g : D \to X_R$ be a continuous surjective map which:

(i) maps the interior of D homeomorphically onto R, and hence ∂D onto ∂R,
(ii) maps the interior of each arc $\overline{A}_i$ homeomorphically onto the interior of an edge belonging to the room R.

Then the set $\{g : D \to X_R; \overline{P}_1, \ldots, \overline{P}_n; \overline{A}_1, \ldots, \overline{A}_n\}$ is called an *associated set of the room* R. The set $\{g : D \to X_R; \overline{A}_1, \ldots, \overline{A}_n\}$ is also called an associated set of R. Similarly, we can define an associated set of a house with connected boundary.

Let Γ be a chart, and X_R the closure of a room R of Γ. Let A_1 and A_2 be different edges in X_R. The pair (A_1, A_2) is said to be *admissible with respect to a disk* E in X_R if:

(i) $E \cap \partial R$ consists of two disjoint arcs α_1, α_2 with $\alpha_i \subset A_i$ for $i = 1, 2$.
(ii) For any orientation of E, there exists some $i \in \{1, 2\}$ such that the orientation induced from the disk does not coincide with the orientation induced from A_i.

Let A, A' and A'' be edges which belong to a room R such that A is a terminal edge of label 2, and the labels of A' and A'' are odd (the case of $A' = A''$ is not excluded). Let $\{g : D \to X_R; \overline{P}_1, \ldots, \overline{P}_n; \overline{A}_1, \ldots, \overline{A}_n\}$ be an associated set of R with $A = g(\overline{A}_1)$. We can assume that $A' = g(\overline{A}_i)$, $A'' = g(\overline{A}_j)$, and $i < j$. The triplet (A', A, A'') is said to be *semi-reducible with respect to a disk* E if it satisfies condition (1) below, and *reducible with respect to* E if it satisfies (1) and (2).

(1) A splits E into two disks, say E_1 and E_2, so that the pair (A, A') is admissible with respect to one of them, and (A, A'') is admissible with respect to the other.
(2) If $g(\overline{A}_k \cap \overline{A}_{k+1})$ is a crossing for some k with $i \le k < j$, then the triplet $(g(\overline{A}_k), A, g(\overline{A}_{k+1}))$ is not semi-reducible.

LEMMA 2.5 ([4, Lemma 1]). *For any minimal* 4*-chart*, *there is no reducible triplet.*

A *special pair* is an admissible pair with a common crossing. A *special house* is a house containing a crossing.

LEMMA 2.6. *Let Γ be a connected minimal* 4*-chart. If a room R possesses exactly m special pairs*, *then it possesses at most m terminal edges of label* 2.

Proof. Let $\{g : D \to X_R; \overline{A}_1, \ldots, \overline{A}_n\}$ be an associated set of R. We prove the lemma by contradiction. Suppose that the room possesses more than $m + 1$ terminal edges of label 2. Then there exist two terminal edges $g(\overline{A}_i)$ and $g(\overline{A}_j)$ of label 2 with $i < j$ such that $g(\overline{A}_{i+1})$ and $g(\overline{A}_{j-1})$ are of odd label, and $(g(\overline{A}_k), g(\overline{A}_{k+1}))$ is a special pair for any $k = i + 1, i + 2, \ldots, j - 2$. Now $(g(\overline{A}_{i+1}), g(\overline{A}_{j-1}))$ is admissible. For, if not, we can increase the number of free edges by a C-I-M2 move between the terminal edges $g(\overline{A}_i)$ and $g(\overline{A}_j)$. Therefore the triplet $(g(\overline{A}_{i+1}), g(\overline{A}_i), g(\overline{A}_{j-1}))$ is reducible. This contradicts Lemma 2.5. ∎

The following lemma is a generalization of Proposition 1 in [4].

LEMMA 2.7. *Let Γ be a connected minimal* 4*-chart. If a house of Γ contains exactly n crossings*, *then the house possesses at most* $2n$ *terminal edges of label* 2. *In particular*, *no non-special house possesses a terminal edge of label* 2.

Proof. If a house is non-special, then it possesses no terminal edge of label 2 by Proposition 1 in [4]. If a house is special, then it contains crossings. For each crossing, there are two special pairs, so if the house contains exactly n crossings, then it contains exactly $2n$ special pairs. By Lemma 2.6, the house possesses at most $2n$ terminal edges of label 2. ∎

Let Γ be a connected 4-chart, and H a house of Γ. Let m_1 be the number of distinguished arcs in the closure of H, and m_2 the number of terminal edges of label 2 in that closure. Then $m_1 - m_2$ is denoted by $d(H)$. By Remark 2.1, the number of white vertices of Γ is equal to the number of distinguished arcs of Γ. Hence the sum of $d(H)$ for all houses is equal to $w(\Gamma) - t(\Gamma)$:

$$w(\Gamma) - t(\Gamma) = \sum_{H:\,\text{house}} d(H).$$

LEMMA 2.8. *Let Γ be a connected minimal 4-chart, and H a non-special house of Γ. If the boundary of H is connected, then H possesses an even number of distinguished arcs, at least two. Moreover, $d(H) \geq 2$.*

Proof. Let H be a non-special house with connected boundary. By Lemma 2.7, H possesses no terminal edge of label 2. Lemma 4 in [4] shows that H possesses an even number of distinguished arcs. By Proposition 4 in [4], H possesses at least two distinguished arcs. Since H possesses no terminal edge of label 2, it follows that $d(H) \geq 2$. ∎

In [4], Nagase and Hirota investigate edges near a terminal edge of label 2.

LEMMA 2.9 ([4, Lemma 3]). *Let Γ be a minimal 4-chart, and B, B' edges containing the same white vertex w such that B is terminal of label 2, and the label of B' is odd. If B' is not a distinguished edge at w, then it contains a crossing or is a distinguished edge at the other white vertex of itself.*

LEMMA 2.10 ([4, Lemma 6]). *In a minimal 4-chart, there exists no room whose boundary consists of exactly two edges with different parities and with the odd labeled edge containing no distinguished arc.*

LEMMA 2.11. *Let Γ be a connected minimal 4-chart, and H a special house of Γ. If the boundary of H is connected, and if H contains exactly one crossing, then $d(H) \geq 0$. Moreover, if H possesses distinguished arcs, then $d(H) \geq 1$.*

Proof. Let B_1, B_2, B_3, B_4 be the odd labeled edges containing the crossing which belong to H. They separate H into four components, say H_1, H_2, H_3, H_4. Each H_i possesses at most one special pair. By Lemma 2.7, each H_i possesses at most one terminal edge of label 2. Suppose that H_i possesses one terminal edge of label 2, say A. Let C_1, C_2 be the odd labeled edges which belong to H_i and contain the white vertex of A. Since H contains only one crossing, C_1 or C_2 does not contain a crossing. If neither of them does, then both contain distinguished arcs by Lemma 2.9. This is impossible. Suppose that only one of C_1 and C_2, say C_1, contains a crossing. Let R be the component of $H \setminus C_2$ with no crossing. Let $\{g : D \to X_R; \overline{P}_1, \ldots, \overline{P}_n; \overline{A}_1, \ldots, \overline{A}_n\}$ be an associated set of R such that $g(\overline{A}_1) = C_2$ and C_2 is the distinguished edge at $g(\overline{P}_1)$. Then R possesses two distinguished arcs. If not, then $(g(\overline{A}_k), g(\overline{A}_{k+1}))$ is not admissible for $2 \leq k \leq n-1$, and $(g(\overline{A}_2), g(\overline{A}_n))$ is not admissible. However, since $(C_2, g(\overline{A}_2))$ is not admissible, and $(C_2, g(\overline{A}_n))$ is, it follows that $(g(\overline{A}_2), g(\overline{A}_n))$ is admissible. This is a contradiction. Therefore R possesses at least two distinguished arcs, and if H_i possesses one terminal edge of label 2, then it contains at least two distinguished arcs. Hence $d(H) \geq 0$. Moreover, if H possesses distinguished arcs, then $d(H) \geq 1$. ■

3. Saturated 4-charts. In this section, we show that if Γ is a minimal 4-chart with at most two crossings, then Γ is a "saturated" 4-chart.

We say that a connected 4-chart Γ with white vertices is *saturated* if it satisfies the following two conditions:

(i) A house of Γ possesses distinguished arcs if and only if it is non-special with connected boundary.
(ii) If a house possesses distinguished arcs, then it possesses exactly two such arcs.

In general, we say that a 4-chart is *saturated* if each of its components is a free edge, a hoop, or a saturated 4-chart.

Nagase and Hirota showed that a minimal 4-chart with at most one crossing has no white vertices (see [4, Main Theorem]). Such a chart can be modified to a chart consisting of free edges and hoops by using C-II, C-I-R2 and C-I-R3 moves. This implies the following lemma.

LEMMA 3.1. *Any minimal* 4*-chart with at most one crossing is saturated.*

LEMMA 3.2 ([1, Main Theorem]). *Let Γ be a minimal* 4*-chart with exactly two crossings. If Γ has only one special house*, *then each component of Γ is a free edge*, *a hoop*, *the* 4*-chart of Figure* 1 *or its reflection.*

Let Γ be the 4-chart of Figure 1 or its reflection. Then Γ is saturated. By Lemma 3.2, we have the following lemma.

LEMMA 3.3. *Let Γ be a minimal* 4*-chart with exactly two crossings. If Γ has only one special house*, *then Γ is saturated.*

We wish to show that any minimal 4-chart with at most two crossings is saturated. To do this, it suffices to consider a connected minimal 4-chart with two special houses.

LEMMA 3.4. *Let Γ be a connected minimal 4-chart with exactly two crossings and with two special houses. Then*

$$w(\Gamma) - t(\Gamma) \geq 2(n_1 - 2)$$

where n_1 is the number of houses with connected boundary, $w(\Gamma)$ is the number of white vertices, and $t(\Gamma)$ is the number of terminal edges of label 2. Moreover:

(i) *If there exists a non-special house H_0 with connected boundary and with $d(H_0) \geq 4$, then $w(\Gamma) - t(\Gamma) \geq 2(n_1 - 2) + 2$.*
(ii) *If there exists a non-special house H_0 with disconnected boundary and with $d(H_0) \geq 1$, then $w(\Gamma) - t(\Gamma) \geq 2(n_1 - 2) + 1$.*

To prove Lemma 3.4, we need the following claim.

CLAIM. *Let Γ be a 4-chart as in Lemma 3.4. Let n be the number of non-special houses with connected boundary. Then $D(\Gamma) \geq 2n$. Moreover,*

$$D(\Gamma) \geq \begin{cases} 2n+2 & \text{if } \Gamma \text{ is as in Lemma 3.4(i)}, \\ 2n+1 & \text{if } \Gamma \text{ is as in Lemma 3.4(ii)}. \end{cases}$$

Here, $D(\Gamma)$ is the sum of $d(H)$ for all non-special houses H.

Proof of Claim. Let $H'_1, \dots, H'_n$ be all non-special houses with connected boundary. By Lemma 2.8, $d(H'_i) \geq 2$ for all i. By Lemma 2.7, no non-special house H possesses a terminal edge of label 2. Hence $d(H) \geq 0$. Therefore

$$D(\Gamma) \geq \sum_{i=1}^{n} d(H'_i) \geq 2n.$$

Suppose that Γ is as in Lemma 3.4(i). Then $d(H'_j) \geq 4$ for some j. Hence

$$D(\Gamma) \geq d(H'_j) + \sum_{1 \leq i \leq n,\, i \neq j} d(H'_i) \geq 4 + 2(n-1) = 2n+2.$$

Suppose now that Γ is as in Lemma 3.4(ii). Then there exists a non-special house H_0 such that ∂H_0 is disconnected and $d(H_0) \geq 1$. Hence $D(\Gamma) \geq d(H_0) + \sum_{i=1}^{n} d(H'_i) \geq 1 + 2n$. This completes the proof of Claim. ■

Let Γ be a 4-chart as in Lemma 3.4. Set

$$\varepsilon = \begin{cases} 2 & \text{if } \Gamma \text{ is as in Lemma 3.4(i)}, \\ 1 & \text{if } \Gamma \text{ is as in Lemma 3.4(ii)}, \\ 0 & \text{otherwise}. \end{cases}$$

By the above claim, we have $D(\Gamma) \geq 2n + \varepsilon$.

Proof of Lemma 3.4. Let H_1, H_2 be the two special houses of Γ. Since Γ has exactly two crossings, each H_i contains only one crossing. By the above claim,

$$w(\Gamma) - t(\Gamma) = d(H_1) + d(H_2) + D(\Gamma) \geq d(H_1) + d(H_2) + 2n + \varepsilon.$$

There are three possibilities:

(a) The boundaries of H_1 and H_2 are connected.
(b) One of ∂H_1 and ∂H_2 is connected, the other is disconnected.
(c) The boundaries of H_1 and H_2 are disconnected.

In case (a), by Lemma 2.11, $d(H_i) \geq 0$ for $i = 1, 2$. Since the number of special houses with connected boundary is two, we have $n_1 = n + 2$. Hence $w(\Gamma) - t(\Gamma) \geq 2(n_1 - 2) + \varepsilon$.

In case (b), we may assume that ∂H_1 is connected and ∂H_2 is not. By Lemma 2.11, $d(H_1) \geq 0$. Since H_2 contains only one crossing, it possesses at most two terminal edges by Lemma 2.7. Hence $d(H_2) \geq -2$. Since the number of special houses with connected boundary is 1, we have $n_1 = n+1$. Therefore

$$w(\Gamma) - t(\Gamma) \geq 0 - 2 + 2(n_1 - 1) + \varepsilon = 2(n_1 - 2) + \varepsilon.$$

In case (c), we have $d(H_i) \geq -2$ for $i = 1, 2$ in a similar way to the case above. Since there is no special house with connected boundary, we have $n_1 = n$. Therefore

$$w(\Gamma) - t(\Gamma) \geq -2 - 2 + 2n_1 + \varepsilon = 2(n_1 - 2) + \varepsilon.$$

This completes the proof of Lemma 3.4. ■

LEMMA 3.5. *Let Γ be a connected minimal* 4*-chart with exactly two crossings and with two special houses. Then the boundary of each house is connected or has two components.*

Proof. For each natural number i, let n_i be the number of houses with exactly i boundary components each. Suppose that there exists a house H such that ∂H has more than two components. Then $n_i \geq 1$ for some $i \geq 3$. By Lemma 2.2,

$$w(\Gamma) - t(\Gamma) \leq 2(n_1 - 2) - 2n_i(i - 2) \leq 2(n_1 - 2) - 2 = 2(n_1 - 3).$$

This contradicts Lemma 3.4. ■

LEMMA 3.6. *Let Γ be a connected minimal* 4*-chart with exactly two crossing and with two special houses. If there exists a special house which possesses distinguished arcs*, *then*

$$w(\Gamma) - t(\Gamma) \geq 2(n_1 - 2) + 1$$

where n_1 is the number of houses with connected boundary.

Proof. Let n be the number of non-special houses with connected boundary. Let H_1, H_2 be the special houses of Γ. Suppose that H_1 possesses distinguished arcs. By the Claim, $D(\Gamma) \geq 2n$.

Case 1: ∂H_1 *is connected.* Since H_1 possesses at least one distinguished arc, $d(H_1) \geq 1$ by Lemma 2.11. Suppose that ∂H_2 is connected. Then there are exactly two special houses with connected boundary, and $n_1 = n + 2$. By Lemma 2.11, $d(H_2) \geq 0$ and we have

$$w(\Gamma) - t(\Gamma) = d(H_1) + d(H_2) + D(\Gamma) \geq 1 + 0 + 2n = 2(n_1 - 2) + 1.$$

Suppose now that ∂H_2 is disconnected. Then $n_1 = n + 1$. Since H_2 contains exactly one crossing, by Lemma 2.7 it possesses at most two terminal edges of label 2. Hence $d(H_2) \geq -2$ and we have

$$w(\Gamma) - t(\Gamma) = d(H_1) + d(H_2) + D(\Gamma) \geq 1 - 2 + 2n = 2(n_1 - 2) + 1.$$

Case 2: ∂H_1 *is disconnected.* Since H_1 possesses a distinguished arc, we have $d(H_1) \geq -1$ in a similar way to the case above. Suppose that ∂H_2 is connected. Then $n_1 = n + 1$. By Lemma 2.11, $d(H_2) \geq 0$ and we have

$$w(\Gamma) - t(\Gamma) = d(H_1) + d(H_2) + D(\Gamma) \geq -1 + 0 + 2n = 2(n_1 - 2) + 1.$$

Suppose now that ∂H_2 is disconnected. Then $n_1 = n$. Similarly to the above, $d(H_2) \geq -2$ and we have

$$w(\Gamma) - t(\Gamma) = d(H_1) + d(H_2) + D(\Gamma) \geq -1 - 2 + 2n = 2(n_1 - 2) + 1. \blacksquare$$

Theorem 3.7. *Any minimal* 4*-chart with at most two crossings is saturated.*

Proof. Let Γ be a minimal 4-chart with at most two crossings. If each component of Γ has at most one crossing, then Γ is saturated by Lemmas 2.4 and 3.1. Hence we may assume that Γ is connected and has exactly two crossings. By Lemma 3.3, we may also assume that Γ has two special houses. By Lemma 3.5, the boundary of each house is connected or has two components. Lemma 2.2 yields

$$w(\Gamma) - t(\Gamma) = 2(n_1 - 2)$$

where n_1 is the number of houses with connected boundary. By Lemma 2.8, if a non-special house with connected boundary possesses distinguished arcs, then it possesses at least $2k$ such arcs ($k \geq 1$). Suppose that Γ is not saturated. Then there exists a house H_1 satisfying one of the following conditions:

(i) H_1 is non-special with connected boundary, and possesses at least four distinguished arcs.
(ii) H_1 is non-special with two boundary components, and possesses distinguished arcs.
(iii) H_1 is special and possesses distinguished arcs.

In case (i), Lemma 2.7 shows that H_1 possesses no terminal edge of label 2. Hence $d(H_1) \geq 4$. By Lemma 3.4(i), we then have $w(\Gamma) - t(\Gamma) \geq 2(n_1 - 2) + 2$. This is a contradiction.

In case (ii), Lemma 2.7 shows that H_1 possesses no terminal edge of label 2. Hence $d(H_1) \geq 1$, and then $w(\mathit{\Gamma}) - t(\mathit{\Gamma}) \geq 2(n_1 - 2) + 1$ by Lemma 3.4(ii). This is a contradiction.

In case (iii), $w(\mathit{\Gamma}) - t(\mathit{\Gamma}) \geq 2(n_1 - 2) + 1$ by Lemma 3.6. This is a contradiction. ■

LEMMA 3.8. *Let $\mathit{\Gamma}$ be a saturated connected minimal* 4*-chart. If a special house H of $\mathit{\Gamma}$ contains exactly one crossing, then H possesses no terminal edges of label* 2*. Moreover, $d(H) = 0$.*

Proof. Suppose that a terminal edge B of label 2 belongs to H. Let w be the white vertex of B. Since $\mathit{\Gamma}$ is saturated, H possesses no distinguished arcs. Hence no loop with vertex w has odd label. So, there exist two odd labeled edges B_1, B_2 such that B_i belongs to H and $w \in B_i$ for $i = 1, 2$. Since H possesses no distinguished arcs, B_i is a distinguished edge at neither of its vertices for $i = 1, 2$. By Lemma 2.9, B_1 and B_2 contain crossings. However, these crossings are not different. This contradicts the fact that H contains exactly one crossing. Therefore H possesses no terminal edges of label 2 and $d(H) = 0$. ■

LEMMA 3.9. *Let $\mathit{\Gamma}$ be a connected minimal* 4*-chart with exactly two crossings. If a special house H_1 of $\mathit{\Gamma}$ contains exactly one crossing, then the boundary of H_1 is connected.*

Proof. By Theorem 3.7, $\mathit{\Gamma}$ is saturated. Let H_1 be a special house containing exactly one crossing. Suppose that ∂H_1 is disconnected. By Lemmas 2.2 and 3.5,

$$w(\mathit{\Gamma}) - t(\mathit{\Gamma}) = 2(n_1 - 2)$$

where n_1 is the number of houses with connected boundary. Let H_2 be another special house of $\mathit{\Gamma}$. Since H_i contains exactly one crossing, $d(H_i) = 0$ for $i = 1, 2$ by Lemma 3.8. Moreover, since $\mathit{\Gamma}$ is saturated, $d(H) = 2$ for any non-special house H with connected boundary. Let n be the number of such houses. Since ∂H_1 is disconnected, $n = n_1 - 1$ or $n = n_1$. Hence $n \geq n_1 - 1$, and

$$w(\mathit{\Gamma}) - t(\mathit{\Gamma}) = 2n + d(H_1) + d(H_2) = 2n \geq 2(n_1 - 1).$$

This is a contradiction. ■

4. Rectangular rooms. In this section, we investigate a non-special house with connected boundary in a saturated minimal 4-chart.

Let $\mathit{\Gamma}$ be a chart, and w a white vertex of $\mathit{\Gamma}$. In a small neighborhood of w, there are six short arcs. Let $\alpha_1, \alpha_2, \alpha_3$ be three consecutive arcs in this neighborhood. For $i = 1, 2, 3$, let B_i be the edge containing α_i. Then B_1 and B_3 are called the *side edges of B_2 at the white vertex w*.

Lemma 4.1. *Let Γ be a connected* 4*-chart*, *and R a room of Γ. Let $\{g : D \to X_R; \overline{A}_1, \dots, \overline{A}_n\}$ be an associated set of R, and $A_i = g(\overline{A}_i)$ for each i $(1 \leq i \leq n)$. Suppose that A_1, A_3 are even labeled edges*, *and A_2 is odd labeled. If the room R satisfies one of the following conditions*, *then we can reduce the number of white vertices of Γ by C-moves.*

(i) *Neither (A_1, A_2) nor (A_2, A_3) is admissible.*
(ii) *A_2 does not contain distinguished arcs*, *and both (A_1, A_2) and (A_2, A_3) are admissible.*

Proof. Suppose that Γ satisfies (i). Then A_2 is not a loop. For $i = 1, 3$, let B_i be the side edge of A_2 at the white vertex $A_2 \cap A_i$ such that $B_i \neq A_i$. Then B_1, A_2, B_3 belong to the same room. Since (A_i, A_2) is not admissible for $i = 1, 3$, (B_i, A_2) is admissible. Hence (B_1, B_3) is not admissible. The pair (A_1, A_3) is not admissible. Apply a C-I-M2 move between A_1 and A_3, and a C-I-M2 move between B_1 and B_3. Then we can cancel the white vertices of A_2 by a C-I-M3 move. See Figure 4.

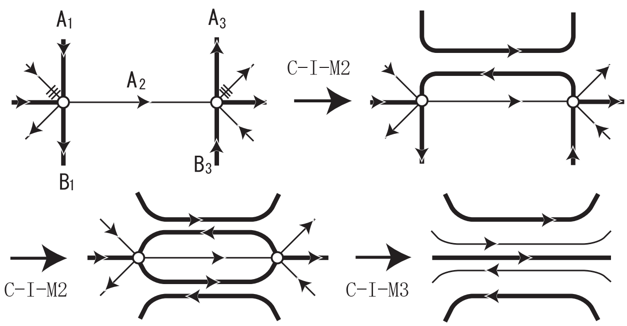

Fig. 4. The thick lines are edges of label 2. The other lines are edges of label 1 or 3.

Suppose now that Γ satisfies (ii). Since A_2 is not a distinguished edge, it is not a loop. For $i = 1, 3$, let B_i be the side edge of A_2 at the white vertex $A_2 \cap A_i$ such that $B_i \neq A_i$. Then B_1, A_2, B_3 belong to the same room. Since A_2 is not a distinguished edge and (A_i, A_2) is admissible, (B_i, A_2) is not admissible for $i = 1, 3$. Hence (B_1, B_3) is not admissible. The pair (A_1, A_3) is not admissible. Then we can cancel the white vertices of A_2 by C-I-M2 moves and a C-I-M3 move as above. ∎

Lemma 4.2. *Let Γ be a connected minimal* 4*-chart*, *H a house*, *and B an odd labeled edge of H. Suppose that H possesses no terminal edges of label* 2, *and B does not contain crossings and distinguished arcs. Let E be a component of $H \setminus B$. If $\partial E \setminus B$ is connected*, *then E possesses a distinguished arc or contains a crossing.*

Proof. Since B does not contain crossings, it connects the boundary points of X_H. Let $\{g : D \to X_E;\ \overline{A}_1, \dots, \overline{A}_n\}$ be an associated set of E such that $g(\overline{A}_1) = B$. Since ∂E contains only one odd labeled edge B, $g(\overline{A}_i)$ is an even labeled edge for each $i = 2, \dots, n$.

Suppose that E possesses neither distinguished arcs nor crossings. Take an outermost arc $\overline{B}'$ in D such that the odd labeled edge $g(\overline{B}') = B'$ belongs to E. Then one of the components of $E \setminus B'$ is a room with connected boundary, which possesses exactly one even labeled edge and one odd labeled edge, and which contains no distinguished arc. However, this contradicts Lemma 2.10. ∎

Let Γ be a connected 4-chart. A room R of Γ is *rectangular* if there exists an associated set of R, $\{g : D \to X_R;\ \overline{A}_1, \overline{A}_2, \overline{A}_3, \overline{A}_4\}$, such that $g(\overline{A}_1), g(\overline{A}_3)$ are edges of label 2, and $g(\overline{A}_2), g(\overline{A}_4)$ are edges of odd label.

Let Γ be a connected 4-chart, R a room of Γ, and $\{g : D \to X_R;\ \overline{P}_1, \dots, \overline{P}_n;\ \overline{A}_1, \dots, \overline{A}_n\}$ an associated set of R. If R satisfies one of the following conditions, then it is called an *end room*. See Figure 5.

(I) $n = 5$, $g(\overline{A}_1) = g(\overline{A}_5)$ is a distinguished edge, $g(\overline{A}_2), g(\overline{A}_4)$ are edges of label 2, and $g(\overline{A}_3)$ is an edge of odd label.

(II) $n = 2$, $g(\overline{A}_1)$ is an edge of label 2, and $g(\overline{A}_2)$ is a distinguished edge.

(III) $n = 6$, $g(\overline{A}_1) = g(\overline{A}_6)$, $g(\overline{A}_3) = g(\overline{A}_4)$ are distinguished edges, and $g(\overline{A}_2), g(\overline{A}_5)$ are edges of label 2.

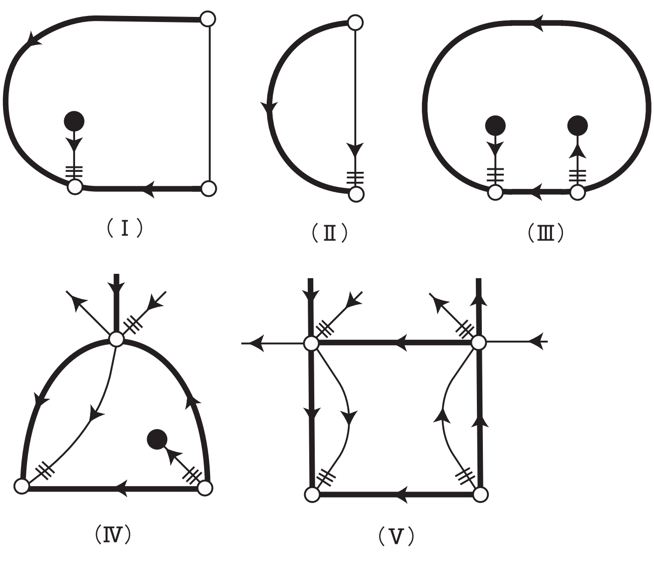

Fig. 5. The thick lines are edges of label 2. The other lines are edges of label 1 or 3.

The rooms satisfying (I), (II) and (III) are called *end rooms of type* (I), (II) *and* (III), respectively. Let H be a house. If H consists only of an end room of type (I) and an end room of type (II), then H is called a *house of type* (IV). The house in Figure 5(IV) is an example. If H consists only of two end rooms of type (II) and a rectangular room, then it is called a *house of type* (V). An example is shown in Figure 5(V).

LEMMA 4.3. *Let Γ be a saturated connected minimal 4-chart, and H a non-special house of Γ. If the boundary of H is connected, then H satisfies one of the following two conditions*:

(i) *H is an end room of type* (III).
(ii) *Two rooms in H, say R, R', are end rooms, and the other rooms, say $R_1, \dots, R_n$, are rectangular. Moreover, we can renumber the rectangular rooms so that for each $i = 0, 1, \dots, n$, R_i and R_{i+1} possess a common edge of odd label, where $R_0 = R$ and $R_{n+1} = R'$.*

Let A, A' be edges of label 2 which belong to H. If A, A' belong to the same rectangular room, then (A, A') is admissible.

Proof. Let $\{g : D \to X_H; \overline{P}_1, \dots, \overline{P}_n; \overline{A}_1, \dots, \overline{A}_n\}$ be an associated set of H. Since Γ is saturated, H possesses exactly two distinguished edges, say B, B'. Since Γ is minimal, $B \neq B'$ by Remark 2.3(i). Since H is non-special, it possesses no terminal edges of label 2 by Lemma 2.7. If H does not possess other edges of odd label, then it is of type (III), (IV) or (V). Therefore we may assume that there exist other edges of odd label belonging to H, say $B_1, \dots, B_k$. Let $\overline{B}_i$, $\overline{B}$, $\overline{B}'$ be the arcs in D with $g(\overline{B}) = B$, $g(\overline{B}') = B'$ and $g(\overline{B}_i) = B_i$ for all i $(1 \leq i \leq k)$. Since B_i is not a distinguished edge, it is not terminal by Remark 2.3(ii). Hence it connects boundary points of H.

Suppose that B or B' is terminal, say B is a terminal edge containing the white vertex $g(\overline{P}_1)$. Let $\overline{P}'$ be the point in ∂D such that B' is a distinguished edge at the vertex $g(\overline{P}')$. Let C_1, C_2 be the components of $\partial D \setminus \{\overline{P}_1, \overline{P}'\}$. By Lemma 4.2, each arc $\overline{B}_i$ $(1 \leq i \leq k)$ connects a point in C_1 and a point in C_2. If not, then there exists a component E of $H \setminus B_i$ possessing no distinguished arc and containing no crossing, and with $\partial E \setminus B_i$ connected. This contradicts Lemma 4.2. We may assume that the arc $\overline{B}_i$ connects $\overline{P}_{i+1}$ and $\overline{P}_{n-i-1}$ for $i = 1, \dots, k$. If B' is a terminal edge, then $n = 2k + 6$, and B' has the white vertex $g(\overline{P}_{k+2})$. See Figure 6(1). If B' is not terminal, then $n = 2k + 5$, and $\overline{B}'$ connects $\overline{P}_{k+2}$ and $\overline{P}_{k+3}$. See Figure 6(2).

Suppose that neither B nor B' is terminal. We may assume that $\overline{B}$ connects $\overline{P}_1$ and $\overline{P}_2$. As above, we may assume that $\overline{B}_i$ connects $\overline{P}_{i+2}$ and $\overline{P}_{n-i+1}$ for $i = 1, \dots, k$. Since B' is not a terminal edge, it follows that $n = 2k + 4$, and $\overline{B}'$ connects $\overline{P}_{k+3}$ and $\overline{P}_{k+4}$. See Figure 6(3).

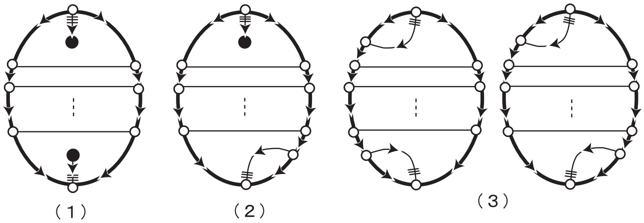

Fig. 6. The figures are preimages of associated sets. Hence two even labeled edges may be identified by the map g.

Let $\overline{P}, \overline{P}'$ be the points of ∂D such that B, B' are the distinguished edges at $g(\overline{P}), g(\overline{P}')$, respectively. Let C_1, C_2 be the components of $\partial D \setminus \{\overline{P}, \overline{P}'\}$. Since no odd labeled edges except B, B' contain distinguished arcs, even labeled edges are oriented as shown in Figure 6. That is, either both C_1 and C_2 are oriented from $\overline{P}$ to $\overline{P}'$, or both from $\overline{P}'$ to $\overline{P}$, where the orientations of C_1 and C_2 are induced from Γ. Hence we have the last statement in Lemma 4.3. ∎

LEMMA 4.4. *Let Γ be a saturated connected minimal 4-chart. Then any rectangular room of Γ satisfies one of the following two conditions*:

(i) *There exists only one edge of label 2 of the room.*
(ii) *There exist two edges of label 2 of the room such that this pair of edges is admissible.*

Proof. Let R be a rectangular room of Γ. Let $\{g : D \to X_R; \overline{A}_1, \overline{A}_2, \overline{A}_3, \overline{A}_4\}$ be an associated set of R such that $g(\overline{A}_1)$ is an edge of label 2. For $i = 1, 2, 3, 4$, set $A_i = g(\overline{A}_i)$. If $A_1 = A_3$, then R satisfies (i). Hence we may assume $A_1 \neq A_3$. Suppose that R possesses a distinguished arc. Since Γ is saturated, R is contained in a non-special house with connected boundary, say H. Since the two even labeled edges A_1 and A_3 belong to the rectangular room R, (A_1, A_3) is admissible by Lemma 4.3. We may assume that R possesses no distinguished arcs.

Suppose that (A_1, A_3) is not admissible. Then there are two possibilities:

(i) (A_1, A_2) or (A_1, A_4) is not admissible.
(ii) (A_1, A_2) and (A_1, A_4) are admissible, and A_2 and A_4 do not contain distinguished arcs.

In case (i), if (A_1, A_2) is not admissible, then neither is (A_2, A_3). By Lemma 4.1(i), we can reduce the number of white vertices of Γ by C-moves. This contradicts the minimality of Γ. Similarly, if (A_1, A_4) is not admissible, then we also have a contradiction.

In case (ii), by Lemma 4.1(ii) we can reduce the number of white vertices of Γ by C-moves. This contradicts the minimality of Γ. ∎

Lemma 4.5. *Let Γ be a saturated connected minimal* 4*-chart, and H a non-special house. If the boundary of H consists of two components, then H consists of rectangular rooms $R_1, \dots, R_n$ such that R_i and R_{i+1} possess a common edge of odd label for each $i = 1, \dots, n$, where $R_{n+1} = R_1$.*

Proof. Let C_1, C_2 be the components of ∂H. Since Γ is saturated, no odd labeled edge in H is a distinguished edge. Since H is a non-special house, it does not contain crossings. By Lemma 2.7, H possesses no terminal edges of label 2. By using Lemma 4.2, we can show that any odd labeled edge in H connects a point in C_1 and a point in C_2. Let H' be the complementary domain of C_k with $H \subset H'$ $(k = 1, 2)$. Let $\{g : \overline{D} \to X_{H'};\ A_1, \dots, A_n\}$ be an associated set of H'. Since H possesses no terminal edge of label 2, $g(\overline{A}_i) \neq g(\overline{A}_{i+1})$ for any i. Then the even labeled edges $g(\overline{A_i}), g(\overline{A}_{i+1})$ have a common white vertex. Since H possesses no distinguished arc, $(g(\overline{A}_i), g(\overline{A}_{i+1}))$ is not admissible. Hence $(g(\overline{A}_i), g(\overline{A}_j))$ is not admissible for all i, j $(1 \leq i \neq j \leq n)$, and $g(\overline{A}_i) \neq g(\overline{A}_j)$. Therefore each C_k is a simple closed curve $(k = 1, 2)$, each room in H is rectangular, and the closures of any two rooms have a common odd labeled edge or do not intersect. ∎

Lemma 4.6. *Let Γ be a connected minimal* 4*-chart with exactly two crossings and with two special houses. Then each room in a non-special house is an end room or a rectangular room.*

Proof. By Theorem 3.7, Γ is saturated. By Lemma 3.5, the boundary of any non-special house is connected or has two components. Applying Lemmas 4.3 and 4.5, we complete the proof. ∎

5. Towns. Let Γ be a connected 4-chart, T a union of houses and rooms of Γ, and X_T the closure of T. Let w be a white vertex in X_T. There are six short arcs in a small neighborhood of w, say $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. The white vertex w is *good* if it satisfies the following condition: If $N \cap \Gamma$ contains at most three arcs of α_i $(i = 1, \dots, 6)$, then no arc of $N \cap \Gamma$ is a distinguished arc. Here, N is a small neighborhood of w in X_T. We denote the number of arcs α_i lying in $N \cap \Gamma$ by $n(w, T)$.

We call T a *town* if:

(i) The closure X_T of T is a disjoint union of disks.
(ii) Either all even labeled edges in ∂X_T are oriented clockwise, or all are oriented anticlockwise.
(iii) Any white vertex in X_T is good.

A town T is *good* if the following holds: Let H be a house of type (IV) in X_T. Let B be the even labeled edge such that B belongs to H and both odd labeled edges in H are distinguished edges at the vertices of B. If all even labeled edges in ∂X_T are oriented clockwise (resp. anticlockwise), then B

is oriented anticlockwise in ∂X_H (resp. clockwise), where X_H is the closure of H.

Let Γ be a connected 4-chart, T a town, X_T the closure of T, B an even labeled edge in ∂X_T, and B_1, B_2 two odd labeled edges with $B_i \not\subset X_T$ for $i = 1, 2$. The triplet (B_1, B, B_2) is a *semi-repeat triplet with respect to* T if it satisfies the following three conditions:

(i) The three edges belong to the same room R.
(ii) There exists an associated set $\{g : D \to X_R; \overline{A}_1, \dots, \overline{A}_n\}$ of R such that $g(\overline{A}_1) = B_1$, $g(\overline{A}_2) = B$, and $g(\overline{A}_3) = B_2$.
(iii) The label of B_1 is different from that of B_2.

Moreover, if both B_1 and B_2 are distinguished edges at the vertices of B, then (B_1, B, B_2) is called a *repeat triplet with respect to* T.

Definition 5.1. Let Γ be a connected 4-chart, T a town, and (B_1, B, B_2) a semi-repeat triplet with respect to T. The triplet (B_1, B, B_2) is *good* if it satisfies one of the following two conditions:

(a) For $i = 1, 2$, if (B_i, B) is not admissible, then B_i is not a distinguished edge.
(b) If neither B_1 nor B_2 is a distinguished edge at a vertex of B, then B_1, B_2, B belong to the same rectangular room, and (B_i, B) is admissible for $i = 1, 2$.

Lemma 5.2. *Let Γ be a saturated connected minimal* 4*-chart*, T *a town*, *and* (B_1, B, B_2) *a repeat triplet with respect to* T. *Let* H *be the house which possesses* B_1 *and* B_2. *Then* $T \cup H$ *is a town*, *and* $X_T \cap X_H = B$, *where* X_T, X_H *are the closures of* T, H *respectively. Moreover*, *if* T *is good*, *then so is* $T \cup H$.

Proof. Since B_1 and B_2 each contain distinguished arcs, H possesses distinguished arcs. Since Γ is saturated, H is a non-special house with connected boundary. By Lemma 4.3,

(i) if B_1, B_2 are terminal edges, then H is of type (III),
(ii) if only one of B_1, B_2 is terminal, then H is of type (IV), and
(iii) if neither B_1 nor B_2 is terminal, then H is of type (V).

See Figure 5(III), (IV) and (V). Let B' be an edge of label 2 in ∂X_H different from B. Since T is a town, B' is not contained in X_T. Let w be a white vertex of X_H, and N a small neighborhood of w in the closure of $T \cup X$. We prove that N is a disk. There are two cases: (a) $w \in B$, (b) $w \notin B$.

In case (a), $n(w, H) = 3$ and $n(w, T \cup H) \geq 4$. If N is not a disk, then N is a cone over a disjoint union of two arcs, and then a neighborhood of w in X_T is a cone over a disjoint union of two arcs. This contradicts X_T being a disjoint union of disks. Hence N is a disk, and w is a good vertex with respect to $T \cup H$.

In case (b), if $w \notin \partial X_T$, then N is a disk. Suppose that $w \in \partial X_T$ and N is not a disk. Since $n(w, H) = 3$, we have $n(w, T) \leq 3$. Since a small neighborhood of w in X_T does not contain distinguished arcs, $n(w, T) = 2$. We see that there exists an even labeled edge C containing the white vertex w and contained in ∂X_T such that C is oriented clockwise (resp. anticlockwise) if the edge B in ∂X_T is oriented anticlockwise (resp. clockwise). However, this contradicts T being a town. Thus $w \notin \partial X_T$. We have $n(w, T \cup H) = 3$ and w is a good white vertex with respect to $T \cup H$, and N is a disk. Moreover, $X_T \cap X_H = B$, and the closure of $T \cup H$ is a disjoint union of disks. Therefore $T \cup H$ is a town.

Suppose that H is a house of type (IV). Let B be the even labeled edge such that B belongs to H and both odd labeled edges in H are distinguished edges at the vertices of B. If all even labeled edges in ∂X_T are oriented clockwise (resp. anticlockwise), then all even labeled edges in the boundary of the closure of $T \cup H$ are oriented clockwise (resp. anticlockwise), and then B is oriented anticlockwise (resp. clockwise) in ∂X_H. If T is good, then the other house of type (IV) satisfies the same condition. Therefore if T is good, then so is $T \cup H$. ∎

Lemma 5.3. *Let Γ be a saturated connected minimal 4-chart, T a town, and (B_1, B, B_2) a good semi-repeat triplet with respect to T such that B_1 is a distinguished edge at a vertex of B, but B_2 is not. Let R be the union of all rooms which possess B_1. Then $T \cup R$ is a town, and $X_T \cap X_R = B$, where X_T, X_R are the closures of T, R respectively. Moreover, if T is good, then so is $T \cup R$.*

Proof. Since Γ is minimal, B_2 is not a terminal edge. Since B_1 contains a distinguished arc, R possesses a distinguished arc. Since Γ is saturated, R is contained in a non-special house with connected boundary. By Lemma 4.3,

(i) if B_1 is a terminal edge, then R is an end room of type (I),
(ii) if neither B_1 nor B_2 is terminal, then there exist an end room R_1 of type (II) and a rectangular room R_2 with $R = R_1 \cup R_2$.

In case (i), let B' be another even labeled edge in R. In case (ii), let B' be another even labeled edge in R_2. If (B, B_2) is admissible, then (B', B_2) is not by Lemma 4.3. Hence the arcs in a small neighborhood of B_2 are oriented as shown in Figure 7(a). If (B, B_2) is not admissible, then B_2 is not a distinguished edge, because (B_1, B, B_2) is good. Hence the arcs in a small neighborhood of B_2 are oriented as shown in Figure 7(b).

Let B'' be an edge of label 2 in ∂X_R different from B. Since T is a town, B'' is not contained in X_T. Let w be a white vertex in X_R, and N a small

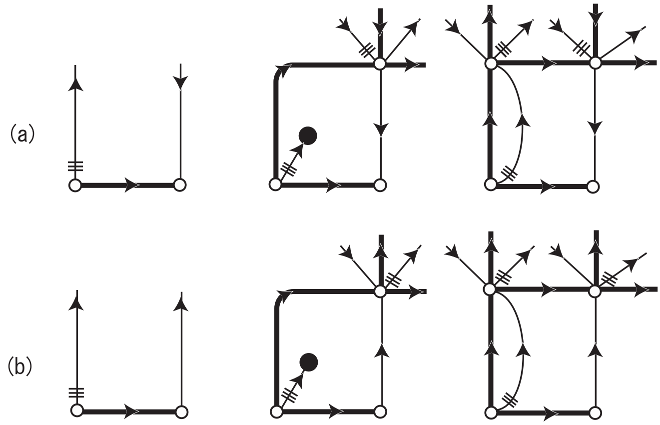

Fig. 7. The thick lines are edges of label 2. The other lines are edges of label 1 or 3.

neighborhood of w in the closure of $T \cup R$. We prove that N is a disk. For $n(w, R) = 3$ or $w \in B$, this can be shown in a similar way to Lemma 5.2. Hence we may assume that $n(w, R) = 2$, $w \notin B$, and N is not a disk. If $w \notin \partial X_T$, then N is a disk, so assume $w \in \partial X_T$.

If $n(w, T) = 4$, then there exists an even labeled edge C containing the white vertex w and contained in ∂X_T such that C is oriented clockwise (resp. anticlockwise) if the edge B in ∂X_T is oriented anticlockwise (resp. clockwise). However, this contradicts T being town.

If $n(w, T) = 2$ or 3, then N contains no distinguished arcs, because w is a good vertex with respect to T. We see that there exists an even labeled edge C containing the white vertex w and contained in ∂X_T such that C is oriented clockwise (resp. anticlockwise) if the edge B in ∂X_T is oriented anticlockwise (resp. clockwise). However, this contradicts T being town.

Hence $w \notin \partial X_T$, $n(w, T \cup R) = n(w, R) = 2$, N is a disk, and w is a good white vertex with respect to $T \cup R$. Moreover, $X_T \cap X_R = B$, and the closure of $T \cup R$ is a disjoint union of disks. Therefore $T \cup R$ is a town. ■

Lemma 5.4. *Let Γ be a saturated connected minimal 4-chart, T a town, and (B_1, B, B_2) a good semi-repeat triplet with respect to T such that neither B_1 nor B_2 is a distinguished edge at a vertex of B. Let R be the rectangular room which possesses B_1 and B_2. Then $T \cup R$ is a town, $X_T \cap X_R = B$, where X_T, X_R are the closures of T, R respectively, and there exists a repeat triple with respect to $T \cup R$. Moreover, if T is good, then so is $T \cup R$.*

Proof. Let B' be another labeled edge which belongs to R. Then (B, B') is admissible by Lemma 4.4. The proof is now similar to that of Lemma 5.3. ■

LEMMA 5.5 ([3, Remarks 8(2)]). *Let Γ be an n-chart, m the number of black vertices of Γ, and F the closure of the surface braid obtained from Γ. Then $\chi(F) = 2n - m$.*

To show Theorem 1.1, it suffices to prove that any saturated minimal 4-chart with exactly two crossings has at least eight black vertices. Let T be a town. Denote by $b(T)$ the number of black vertices in X_T, where X_T is the closure of T. If a black vertex w is contained in a house of type (IV), then the *weight* of w is defined to be 1. Otherwise it is 1/2. Denote by $b'(T)$ the sum of the weights of all black vertices in X_T.

LEMMA 5.6. *Let T_1, T_2 be good towns such that all even labeled edges in ∂X_{T_1} (resp. ∂X_{T_2}) are oriented clockwise (resp. anticlockwise). Then $b(T_1 \cup T_2) \geq b'(T_1) + b'(T_2)$.*

Proof. Let H be a house of type (IV) with $H \subset T_1 \cup T_2$. Let B be an even labeled edge in H such that both odd labeled edges in H are distinguished edges at the vertices of B. Since T_1 and T_2 are good towns, if B is oriented clockwise (resp. anticlockwise) in ∂H, then H is contained in T_2 (resp. T_1). Hence no house of type (IV) in T_i intersects T_j, $\{i, j\} = \{1, 2\}$. Thus no black vertex of weight 1 in T_i belongs to T_j. Therefore $b(T_1 \cup T_2) \geq b'(T_1) + b'(T_2)$. ∎

LEMMA 5.7. *Let Γ be a saturated connected minimal 4-chart, T a town, and (B_1, B, B_2) a repeat triplet with respect to T. Then there exists a town T' such that $b(T') \geq b(T) + 1$, $b'(T') = b'(T) + 1$, $T' \supset T$, and $X_T \cap X = B$, where X_T, X are the closures of $T, T' \setminus T$ respectively. Moreover, if T is good, then so is T'.*

Proof. Let H_1 be the house which possesses B_1 and B_2, and X_{H_1} its closure. Then we have the three possibilities (i), (ii) and (iii) as in the proof of Lemma 5.2. By Lemma 5.2, $T \cup H_1$ is a town, and $X_T \cap X_{H_1} = B$. In cases (i) and (ii), $T \cup H_1$ has more black vertices than T, and $b'(T \cup H_1) = b'(T) + 1$. In case (iii), the orientations of all edges in a neighborhood of X_{H_1} are as in Figure 5(V). Hence there exists a repeat triplet with respect to $T \cup H_1$. Let $T_0 = T$, $T_1 = T \cup H_1$, and $(B_{1,1}, A_1, B_{1,2}) = (B_1, B, B_2)$. To repeat this argument, there exists a repeat triplet $(B_{i,1}, A_i, B_{i,2})$ with respect to T_{i-1} $(i = 2, \ldots, n)$ and there exists a town T_i $(i = 2, \ldots, n)$ such that

- (a) T_i is the union of T_{i-1} and a house H_i which possesses $B_{i,1}$ and $B_{i,2}$ for $i = 1, \ldots, n$,
- (b) $B_{n,1}$ or $B_{n,2}$ is a terminal edge,
- (c) $X_{T_{i-1}} \cap X_{H_i} = A_i$ for $i = 1, \ldots, n$,

where X_{T_i} and X_{H_i} are the closures of T_i and H_i respectively. See Figure 8. By Lemma 5.2, T_n is the desired town. ∎

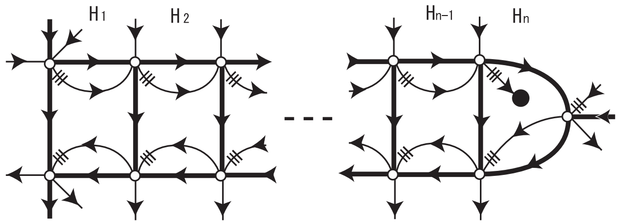

Fig. 8. The thick lines are edges of label 2. The other lines are edges of label 1 or 3.

The set $H_1 \cup \cdots \cup H_n$ is called the *tower* with respect to (B_1, B, B_2), and the house H_n is called the *top house* of this tower.

LEMMA 5.8. *Let Γ be a saturated connected minimal* 4*-chart, and T a town. Let (B_1, B, B_2) be a good semi-repeat triplet with respect to T such that neither B_1 nor B_2 is a distinguished edge at a vertex of B. Then there exists a town T' such that $b(T') \geq b(T) + 1$, $b'(T') = b'(T) + 1$, $T' \supset T$, and $X_T \cap X = B$, where X_T, X are the closures of $T, T' \setminus T$ respectively. Moreover, if T is good, then so is T'.*

Proof. By Definition 5.1(b), (B_1, B) and (B_2, B) are admissible. Let R be the rectangular room which possesses B_1 and B_2, B' the even labeled edge in R with $B \neq B'$, and C_i the side edge of B' with $C_i \neq B_i$ for $i = 1, 2$. By Lemma 4.4, (B, B') is admissible. The pair (C_i, B') is not admissible, and C_i is a distinguished edge at the vertex of B' for $i = 1, 2$. By Lemma 5.4, $T \cup R$ is a town. The triplet (C_1, B', C_2) is a repeat triplet with respect to the town $T \cup R$. Hence we have the desired town by Lemma 5.7. ∎

6. A special house with connected boundary. In this section, we investigate a special house with connected boundary.

LEMMA 6.1. *Let Γ be a saturated connected minimal* 4*-chart, and H a special house with connected boundary. If H contains exactly one crossing, then there exist only four even labeled edges A_1, A_2, A_3, A_4 and only four odd labeled edges B_1, B_2, B_3, B_4 such that*

(i) *$B_i \cap B_j$ is a crossing for $i \neq j$,*
(ii) *$A_i \cap A_{i+1}$ is a white vertex w_i for $i = 1, 2, 3, 4$,*
(iii) *$A_i \cap B_i = \{w_i\}$ for $i = 1, 2, 3, 4$,*

where $A_5 = A_1$. That is, H is as illustrated in Figure 9.

Proof. Let B be an edge containing a crossing which belongs to H. Since H contains exactly one crossing, the other vertex of B is white. There exist four such edges of odd label. By Lemma 3.8, H possesses no terminal edges of label 2.

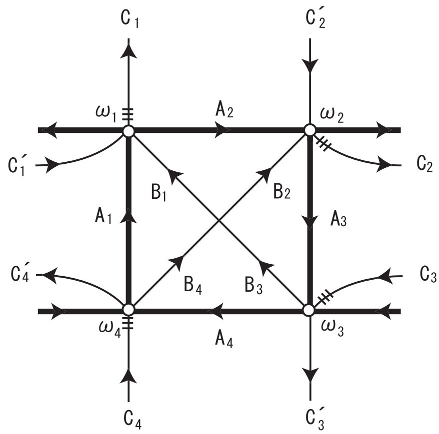

Fig. 9

Suppose that an odd labeled edge B' belongs to H and contains no crossings. Since Γ is saturated, H possesses no distinguished edges and B' is not a distinguished edge. By Lemma 4.2, each component of $H\backslash B'$ contains a crossing. However, this contradicts the assumption that H contains exactly one crossing. Hence no such odd labeled edge B' exists. Therefore there exist only four edges of label 2 and only four edges of odd label satisfying the conditions in Lemma 6.1. ∎

Let Γ, H, A_i, B_i, w_i be as above $(i = 1, 2, 3, 4)$. We may assume that (B_1, B_2), (A_1, B_1) are admissible, For $i = 1, 2, 3, 4$, let C_i, C_i' be the odd labeled edges containing the white vertex w_i such that C_i is a distinguished edge at w_i and $C_i' \neq B_i$. Then C_1, A_2, C_2' belong to the same room, and so do C_2, A_3, C_3 and C_3', A_4, C_4 and C_1', A_1, C_4'.

LEMMA 6.2. *Let $\Gamma, H, A_i, B_i, C_i, C_i', w_i$ be as above $(i = 1, 2, 3, 4)$. Then*:

(i) *H is a good town.*
(ii) *(C_2, A_3, C_3) is a repeat triplet with respect to H.*
(iii) *(C_1, A_2, C_2') and (C_3', A_4, C_4) are good semi-repeat triplets with respect to H.*
(iv) *If Γ has exactly two crossings, then (C_1', A_1, C_4') is a good semi-repeat triplet with respect to H.*
(v) *C_1 or C_4 is not a terminal edge.*

Proof. We first show (iv). Suppose that Γ has exactly two crossings. Since H contains only one crossing, Γ has two special houses. If C_1' and C_4' belong to a non-special house, then C_1', A_1, C_4' belong to a rectangular room by Lemma 4.6. The pairs (C_1', A_1) and (C_4', A_1) are admissible. Hence (C_1', A_1, C_4') is a good semi-repeat triplet with respect to H. If C_1' and C_4' belong to a special house, say H', then H' contains exactly one crossing, and then $\partial H'$ is connected by Lemma 3.9. By Lemma 6.1, H and H' are houses as shown in Figure 10.

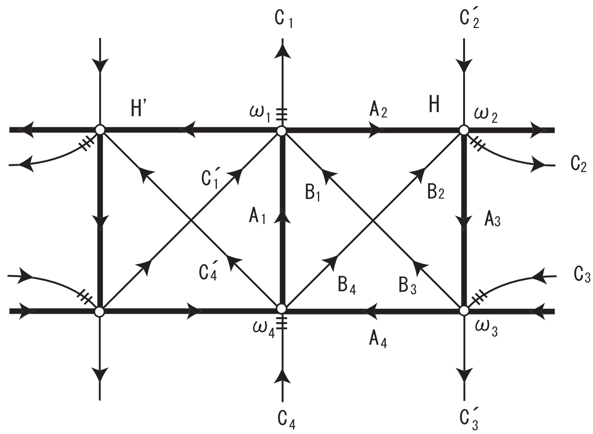

Fig. 10

Let A be the even labeled edge containing the white vertex w_4 different from A_1 and A_4, and w the white vertex of A with $w \neq w_4$. Let B be the odd labeled edge containing w such that B is not a distinguished edge at w, and $H' \cap B = \emptyset$. Since Γ is saturated, B, C_4, C_3' belong to the same non-special house with connected boundary. By Lemma 4.3, C_4 is a terminal edge or $C_4 = B$ or $C_4 = C_3'$. Since the label of C_4 is different from those of C_3' and B, C_4 is a terminal edge and $B = C_3'$. However, B is oriented from w to the other white vertex, and C_3' is oriented from w_3 to the other white vertex. This is a contradiction. Therefore C_1' and C_4' belong to a non-special house, and (C_1', A_1, C_4') is a good semi-repeat triplet with respect to H.

We now show (v) by means of Figure 11. Let W be the tower with respect to (C_2, A_3, C_3), and H' the top house of W. Suppose that C_1 and C_4 are both terminal edges. In general, for any two charts, if there exists a disk E such that ∂E intersects both charts transversally or intersects neither, and if the charts do not have black vertices on E and coincide in the complement of E, then one chart is C-move equivalent to the other (see [3, Lemma 16]). Let E be a neighborhood of the closure of $(H \cup W) \setminus H'$. The second figure in Figure 11 can be obtained from the first by applying this move to the disk E. The third figure can be obtained from the second by C-II moves. The fourth figure can be obtained from third by a C-III-1 move. The final figure can be obtained from the fourth by a C-II move. Hence one white vertex can be cancelled and the number of crossings does not change. This contradicts the minimality of Γ. ∎

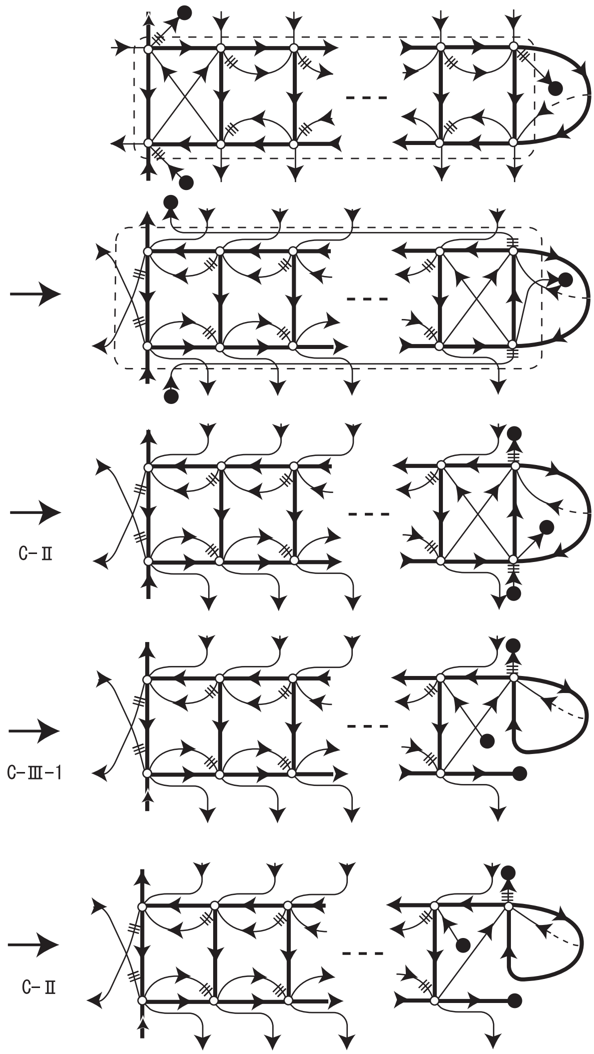

Fig. 11

Lemma 6.3. *Let $\Gamma, H, A_i, B_i, C_i, C_i', w_i$ be as above* ($i = 1, 2, 3, 4$), *and T a town such that $X_T \cap X_H = \emptyset$ and $T \cup H$ is a good town. Suppose that Γ has exactly two crossings. If neither C_1 nor C_4 is a terminal edge, then there exists a good town T' with $H \subset T'$, $X_T \cap X_{T'} = \emptyset$, $b(T') \geq 4$, and $b'(T') = 4$. Here, $X_T, X_H, X_{T'}$ are the closures of T, H, T'.*

Proof. By Lemma 6.2(ii), (C_2, A_3, C_3) is a repeat triplet with respect to H. Let R_i be the union of all rooms which possess C_i for $i = 1, 4$. By Lemma 6.2(iii), (C_1, A_2, C_2') and (C_3', A_4, C_4) are good semi-repeat triplets with respect to H. By Lemma 5.3, $H \cup R_1 \cup R_4$ is a good town as shown

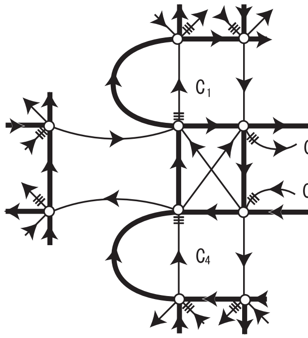

Fig. 12

in Figure 12. Let A'_1, A'_4 be the even labeled edges such that C_i, A'_i belong to the same rectangular room for $i = 1, 4$, $A'_1 \neq A_2$ and $A'_4 \neq A_4$. For $i = 1, 4$, let v_i, v'_i be the white vertices of A'_i, and E_i, E'_i the distinguished edges at v_i, v'_i respectively. Then (E_i, A'_i, E'_i) is a repeat triplet with respect to $H \cup R_1 \cup R_4$ for $i = 1, 4$. Hence there are three repeat triplets with respect to $H \cup R_1 \cup R_4$. Let W_1, W_2, W_3 be the towers with respect to $(C_2, A_3, C_3), (E_1, A'_1, E'_1), (E_4, A'_4, E'_4)$ respectively. Let

$$T_1 = H \cup R_1 \cup R_4 \cup W_1 \cup W_2 \cup W_3.$$

By Lemma 5.7, T_1 is a good town, $b(T_1) \geq 3$, $b'(T_1) = 3$, $X_T \cap X_{T_1} = \emptyset$, and $T \cup T_1$ is a good town, where X_{T_1} is the closure of T_1. Since Γ has exactly two crossings, by Lemma 6.2(iv), (C'_1, A_1, C'_4) is a good semi-repeat triplet for H. We see that (C'_1, A_1, C'_4) is a good semi-repeat triplet for T_1. By Lemma 5.8, there exists a good town T_2 such that $H \subset T_1 \subset T_2$, $b(T_2) \geq 4$, $b'(T_2) = 4$, $X_T \cap X_{T_2} = \emptyset$ and $T \cup T_2$ is a good town, where X_{T_2} is the closure of T_2. ■

LEMMA 6.4. *Let $\Gamma, H, A_i, B_i, C_i, C'_i, w_i$ be as above $(i = 1, 2, 3, 4)$, and T a good town such that $X_T \cap X_H = \emptyset$ and $T \cup H$ is a good town. Suppose that Γ has exactly two crossings. If C_1 or C_4 is a terminal edge, then there exists a good town T' with $H \subset T'$, $X_T \cap X_{T'} = \emptyset$, $b(T') \geq 4$, and $b'(T') = 3 + 1/2$. Here, $X_T, X_H, X_{T'}$ are the closures of T, H, T'.*

Proof. By Lemma 6.2(v), we may assume that only one of C_1 and C_4, say C_4, is a terminal edge. Since Γ is saturated, C_4 belongs to a non-special house with connected boundary. By Lemma 4.3, C_4 belongs to an end room of type (I), and the weight of the black vertex of C_4 is $1/2$. For $i = 1, 4$ let R_i be the union of all rooms which possess C_i. By Lemma 5.3, $H \cup R_1 \cup R_4$ is a good town as shown in Figure 13 and there are two repeat triplets with respect to $H \cup R_1 \cup R_4$. Since Γ has exactly two crossings, (C'_1, A_1, C'_4) is a good semi-repeat triplet for H by Lemma 6.2(iv). We conclude that

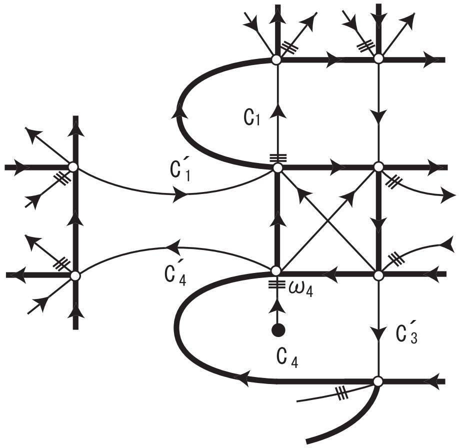

Fig. 13

(C'_1, A_1, C'_4) is a good semi-repeat triplet for $H \cup R_1 \cup R_4$. In a similar way to Lemma 6.3, there exists a good town T_1 such that $H \cup R_1 \cup R_4 \subset T_1$, $b(T_1) \geq 4$, $b'(T_1) = 3 + 1/2$, $X_T \cap X_{T_1} = \emptyset$ and $T \cup T_1$ is a good town, where X_{T_1} is the closure of T_1. ∎

7. Proof of the main theorem

THEOREM 7.1. *Let Γ be a connected minimal 4-chart with exactly two crossings and two special houses H_1, H_2. Then Γ contains at least eight black vertices.*

Proof. By Theorem 3.7, Γ is saturated. Suppose that ∂H_1 and ∂H_2 are both oriented clockwise or both anticlockwise. Then $H_1 \cup H_2, H_1, H_2$ are good towns. By Lemma 6.2, each H_i satisfies one of the conditions in Lemmas 6.3 and 6.4. Hence for each i there exists a good town T_i with $H_i \subset T_i$ and $b(T_i) \geq 4$. Moreover $T_1 \cap T_2 = \emptyset$. Hence $b(T_1 \cup T_2) \geq 8$.

Suppose that the orientation of one of H_1 and H_2 is clockwise, and of the other anticlockwise. By Lemmas 6.3 and 6.4, for each $i = 1, 2$ there exists a good town T_i with $H_i \subset T_i$ and $b'(T_i) = 3 + 1/2$. Lemma 5.6 yields $b(T_1 \cup T_2) \geq b'(T_1) + b'(T_2) = 7$, which completes the proof since the number of black vertices of Γ is even. ∎

Proof of Theorem 1.1. We may assume that Γ is a connected minimal saturated 4-chart with exactly two crossings by the Main Theorem in [4]. Suppose that the closure of the surface braid obtained from Γ is one 2-sphere. Since Γ is a 4-chart, it has exactly six black vertices by Lemma 5.5. If Γ has only one special house, then it contains at least eight black vertices by Lemma 3.2. If Γ has two special houses, then it contains at least eight black vertices by Theorem 7.1. This contradicts Γ having exactly six black vertices. ∎

Finally, in Figure 14 we exhibit a saturated 4-chart with eight black vertices.

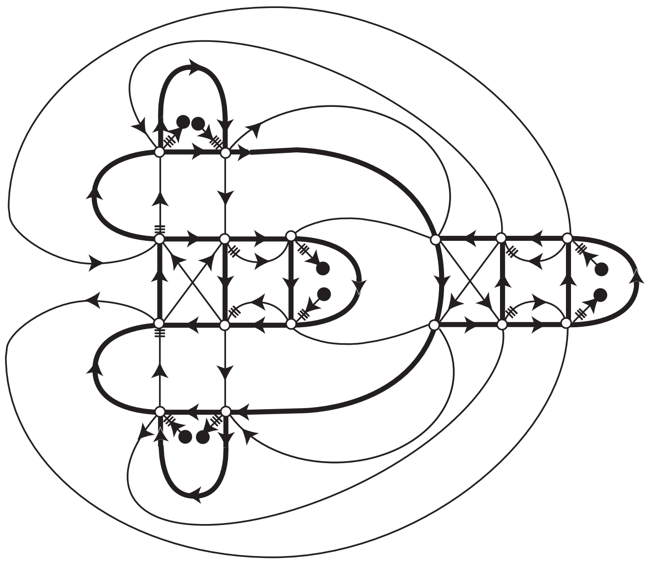

Fig. 14

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Received 11 April 2004;
in revised form 22 September 2005