FUNDAMENTA
MATHEMATICAE
188 (2005)

Virtual knot theory—unsolved problems

by

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Abstract. The present paper gives a quick survey of virtual and classical knot theory and presents a list of unsolved problems about virtual knots and links. These are all problems in low-dimensional topology with a special emphasis on virtual knots. In particular, we touch new approaches to knot invariants such as biquandles and Khovanov homology theory. Connections to other geometrical and combinatorial aspects are also discussed.

1. INTRODUCTION

The purpose of this paper is to give an introduction to virtual knot theory and to record a collection of research problems that the authors have found fascinating. The second section of the paper introduces the theory and discusses problems in that context. The third section is a list of specific problems.

We would like to take this opportunity to thank the many people who have, at the time of this writing, worked on the theory of virtual knots and links. The ones explicitly mentioned or referenced in this paper are: R. S. Avdeev, V. G. Bardakov, A. Bartholomew, D. Bar-Natan, S. Budden, S. Carter, H. Dye, R. Fenn, R. Furmaniak, M. Goussarov, J. Green, D. Hrencecin, D. Jelsovsky, M. Jordan, T. Kadokami, N. Kamada, S. Kamada, L. Kauffman, T. Kishino, G. Kuperberg, S. Lambropoulou, V. O. Manturov, S. Nelson, M. Polyak, D. E. Radford, S. Satoh, J. Sawollek, M. Saito, W. Schellhorn, D. Silver, V. Turaev, V. V. Vershinin, O. Viro, S. Williams, P. Zinn-Justin and J. B. Zuber. See [1–3, 6–12, 16–20, 22, 25, 27–29, 32, 34–35, 37–42, 49–53, 55, 57, 60, 61, 65, 66, 68–70, 73–77, 79–85, 87–89, 91, 94–107, 110]. We apologize to anyone who was left out of this list of participant researchers, and we hope that the problems described herein will stimulate people on and off this list to enjoy the beauty of virtual knot theory!

2000 *Mathematics Subject Classification*: Primary 57M25.

Key words and phrases: knot, link, virtual knot, invariant, Jones polynomial, fundamental group, quandle, biquandle, virtual quandle, long knot, Khovanov homology, atom, braid, virtual braid.

Acknowledgements. It gives the second author great pleasure to acknowledge support from NSF Grant DMS-0245588.

2. VIRTUAL KNOT THEORY

Knot theory studies the embeddings of curves in three-dimensional space. Virtual knot theory studies the embeddings of curves in thickened surfaces of arbitrary genus, up to the addition and removal of empty handles from the surface. Virtual knots have a special diagrammatic theory that makes handling them very similar to the handling of classical knot diagrams. In fact, this diagrammatic theory simply involves adding a new type of crossing to the knot diagrams, a *virtual crossing* that is neither under nor over. From a combinatorial point of view, the virtual crossings are artifacts of the representation of the virtual knot or link in the plane. The extension of the Reidemeister moves that takes care of them respects this viewpoint. A virtual crossing (see Figure 1) is represented by two crossing arcs with a small circle placed around the crossing point.

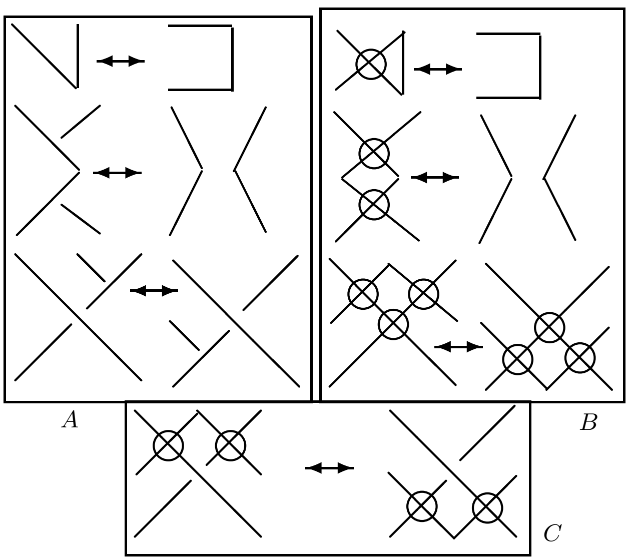


Fig. 1. Generalized Reidemeister moves for virtuals

Moves on virtual diagrams generalize the Reidemeister moves for classical knot and link diagrams. See Figure 1. One can summarize the moves on virtual diagrams by saying that the classical crossings interact with one another according to the usual Reidemeister moves. One adds the detour moves for consecutive sequences of virtual crossings and this completes the description of the moves on virtual diagrams. It is a consequence of moves

(B) and (C) in Figure 1 that an arc going through any consecutive sequence of virtual crossings can be moved anywhere in the diagram keeping the endpoints fixed; the places where the moved arc crosses the diagram become new virtual crossings. This replacement is the *detour move*. See Figure 1.1.

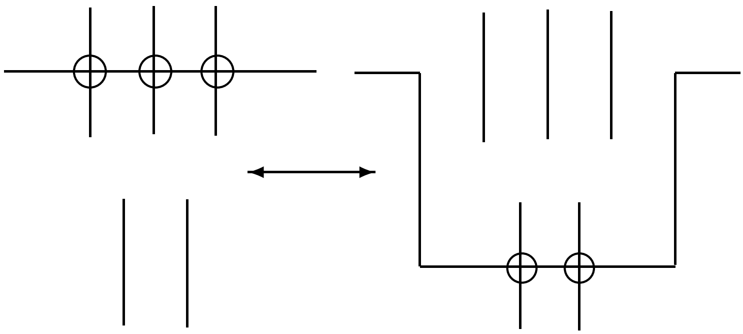


Fig. 1.1. Detour move

One can generalize many structures in classical knot theory to the virtual domain, and use the virtual knots to test the limits of classical problems such as the question whether the Jones polynomial detects knots, and the classical Poincaré conjecture. Counterexamples to these conjectures exist in the virtual domain, and it is an open problem whether any of these counterexamples are equivalent (by addition and subtraction of empty handles) to classical knots and links. Virtual knot theory is a significant domain to be investigated for its own sake and for a deeper understanding of classical knot theory.

Another way to understand the meaning of virtual diagrams is to regard them as representatives for oriented Gauss codes (Gauss diagrams) [49, 32]. Such codes do not always have planar realizations and an attempt to embed such a code in the plane leads to the production of virtual crossings. The detour move makes the particular choice of virtual crossings irrelevant. Virtual equivalence is the same as the equivalence relation generated on the collection of oriented Gauss codes modulo an abstract set of Reidemeister moves on the codes.

One intuition for virtual knot theory is the idea of a particle moving in three-dimensional space along a trajectory that occasionally disappears, and then reappears elsewhere. By connecting the disappearance points and the reappearance points with detour lines in the ambient space we get a picture of the motion, but the detours, being artificial, must be treated as subject to replacements.

2.1. Flat virtual knots and links. Every classical knot or link diagram can be regarded as a 4-regular plane graph with extra structure at the nodes. This extra structure is usually indicated by the over and under crossing conventions that give instructions for constructing an embedding of the link in three-dimensional space from the diagram. If we take the diagram without this extra structure, it is the shadow of some link in three-dimensional space, but the weaving of that link is not specified. It is well known that if one is allowed to apply the Reidemeister moves to such a shadow (without regard to the types of crossing since they are not specified) then the shadow can be reduced to a disjoint union of circles. This reduction is no longer true for virtual links. More precisely, let a *flat virtual diagram* be a diagram with virtual crossings as we have described them and *flat crossings* consisting in undecorated nodes of the 4-regular plane graph. Virtual crossings are flat crossings that have been decorated by a small circle. Two flat virtual diagrams are *equivalent* if there is a sequence of generalized flat Reidemeister moves (as illustrated in Figure 1) taking one to the other. A generalized flat Reidemeister move is any move as shown in Figure 1, but one can ignore the over or under crossing structure. Note that in studying flat virtuals the rules for changing virtual crossings among themselves and the rules for changing flat crossings among themselves are identical. However, detour moves as in Figure 1C are available for virtual crossings with respect to flat crossings and not the other way around.

We shall say that a virtual diagram *overlies* a flat diagram if the virtual diagram is obtained from the flat diagram by choosing a crossing type for each flat crossing in the virtual diagram. To each virtual diagram K there is associated a flat diagram $F(K)$ that is obtained by forgetting the extra structure at the classical crossings in K. Note that if K is equivalent to K' as virtual diagrams, then $F(K)$ is equivalent to $F(K')$ as flat virtual diagrams. Thus, if we can show that $F(K)$ is not reducible to a disjoint union of circles, then it will follow that K is a non-trivial virtual link.

Figure 2 illustrates an example of a flat virtual link H. This link cannot be undone in the flat category because it has an odd number of virtual crossings between its two components and each generalized Reidemeister move preserves the parity of the number of virtual crossings between components. Also illustrated in Figure 2 is a flat diagram D and a virtual knot K that overlies it. This example is given in [49]. The knot shown is undetectable by many invariants (fundamental group, Jones polynomial) but it is knotted and this can be seen either by using a generalization of the Alexander polynomial that we describe below, or by showing that the underlying diagram D is a non-trivial flat virtual knot using the filamentation invariant that is introduced in [34]. The filamentation invariant is a combinatorial method that is sometimes successful in identifying irreducible flat virtuals.

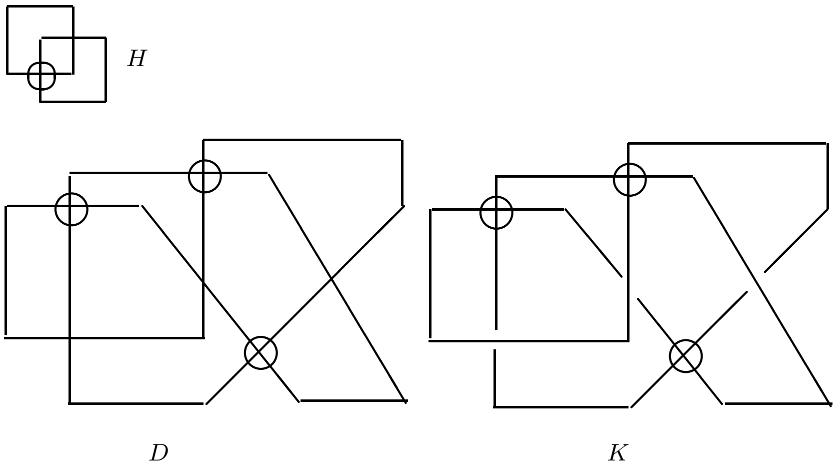


Fig. 2. Flats H and D, and the knot K

At this writing we know very few invariants of flat virtuals. The flat virtual diagrams present a strong challenge for the construction of new invariants. It is important to understand the structure of flat virtual knots and links. This structure lies at the heart of the comparison of classical and virtual links. We wish to be able to determine when a given virtual link is equivalent to a classcal link. The reducibility or irreducibility of the underlying flat diagram is the first obstruction to such an equivalence.

2.2. Interpretation of virtuals as stable classes of links in thickened surfaces. There is a useful topological interpretation for this virtual theory in terms of embeddings of links in thickened surfaces. See [49, 51, 71]. Regard each virtual crossing as a shorthand for a detour of one of the arcs in the crossing through a 1-handle that has been attached to the 2-sphere of the original diagram. By interpreting each virtual crossing in this way, we obtain an embedding of a collection of circles into a thickened surface $S_g \times R$ where g is the number of virtual crossings in the original diagram L, S_g is a compact oriented surface of genus g and R denotes the real line. We say that two such surface embeddings are *stably equivalent* if one can be obtained from the other by isotopy in the thickened surfaces, homeomorphisms of the surfaces and addition or subtraction of empty handles. Then we have

Theorem ([49, 54, 71]). *Two virtual link diagrams are equivalent if and only if the corresponding surface embeddings are stably equivalent.*

Virtual knots and links give rise to a host of problems. As we saw in the previous section, there are non-trivial virtual knots with unit Jones polynomial. Moreover, there are non-trivial virtual knots with integer fundamental group and trivial Jones polynomial. (Fundamental group is defined combinatorially by generalizing the Wirtinger presentation.) These phenomena underline the question of how planarity is involved in the way the Jones polynomial appears to detect classical knots, and that the relationship of the fundamental group (and peripheral system) is a much deeper one than the surface combinatorics for classical knots. It is possible to take the connected sum of two trivial virtual diagrams and obtain a non-trivial virtual knot (the Kishino knot).

Here long knots (or equivalently $1-1$ tangles) come into play. Having a knot, we can break it at some point and take its ends to infinity (say, in such a way that they coincide with the horizontal axis in the plane). One can study isotopy classes of such knots. A well known theorem says that in the classical case, knot theory coincides with long knot theory. However, this is not the case for virtual knots. By breaking the same virtual knot at different points, one can obtain non-isotopic long knots [25]. Furthermore, even if the initial knot is trivial, the resulting long knot may not be trivial. The "connected sum" of two trivial virtual diagrams may not be trivial in the compact case. The phenomenon occurs because these two knot diagrams may be non-trivial in the long category. It is sometimes more convenient to consider long virtual knots rather than compact virtual knots, since connected sum is well defined for long knots. It is important to construct long virtual knot invariants to see whether long knots are trivial and whether they are classical. One approach is to regard long knots as $1-1$ tangles and use extensions of standard invariants (fundamental group, quandle, biquandle, etc). Another approach is to distinguish two types of crossings: those having early undercrossing and those having later undercrossing with respect to the orientation of the long knot. The latter technique is described in [73].

Unlike classical knots, the connected sum of long knots is not commutative [91, 86]. Thus, if we show that two long knots K_1 and K_2 do not commute, then we see that they are different and both non-classical.

A typical example of such knots is the two parts of the Kishino knot, see Figure 2.1.

We have a natural map

$$\langle \text{Long virtual knots} \rangle \to \langle \text{Oriented compact virtual knots} \rangle,$$

obtained by taking two infinite ends of the long knots together to make a compact knot. This map is obviously well defined.

This map allows one to construct (weak) long virtual knot invariants from classical invariants, i.e., just to regard compact knot invariants as long knot invariants. There is no well defined inverse for this map. But, if we were able to construct the map from compact virtual knots to long virtual knots, we could apply the long techniques for the compact case. This map does have

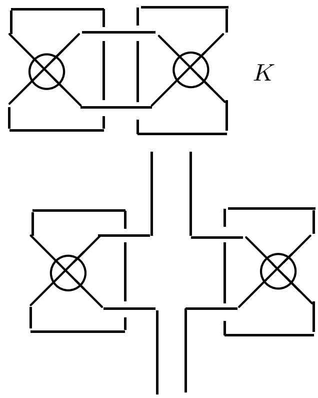


Fig. 2.1. Kishino and parts

an inverse for *classical* knots. Thus, the long techniques are applicable to classical (long) knots. It would be interesting to obtain new classical invariants from the representation of classical knots as closures of long knots using the approach described above. The long category can also be applied for the case of flat virtuals, where all problems formulated above occur as well.

There are examples of virtual knots that are very difficult to prove knotted, and there are infinitely many flat virtual diagrams that appear to be irreducible, but we have no techniques to prove it. How can one tell whether a virtual knot is classical? One can ask: Are there non-trivial virtual knots whose connected sum is trivial? The latter question cannot be handled by classical techniques, but it can be analyzed by using the surface interpretation for virtuals. See [90].

With respect to virtual knots, we are in the same position as the compilers of the original knot tables. We are, in fact, in the process of developing tables. At Sussex, tables of virtual knots are being constructed, and tables will appear in a book being written by Kauffman and Manturov. See also the website “Knotilus” [70] where there is a tabulation of virtual knots initiated by Ralph Furmaniak and Louis Kauffman, and the “Knot Atlas” of Dror Bar-Natan, containing a subatlas of virtual knots worked out in collaboration of Dror and Jeremy Green [6]. The theory of invariants of virtual knots needs more development. Flat virtuals (whose study is a generalization of the classification of immersions) are a nearly unknown territory (but see [34, 106]). The flat virtuals provide the deepest challenge since we have very few invariants to detect them. Curiously, there are many invariants of long flat virtual knots, due to the fact that the virtual (long) knot class of the descending virtual diagram associated with a long flat is an invariant of the long flat. (This observation is due to Turaev.)

3. JONES POLYNOMIAL OF VIRTUAL KNOTS

We use a generalization of the bracket state summation model for the Jones polynomial to extend it to virtual knots and links. We call a diagram in the plane *purely virtual* if the only crossings in the diagram are virtual crossings. Each purely virtual diagram is equivalent by the virtual moves to a disjoint collection of circles in the plane.

Given a link diagram K, a state S of this diagram is obtained by choosing a smoothing for each crossing in the diagram and labelling that smoothing with either A or A^{-1} according to the convention that a counterclockwise rotation of the overcrossing line sweeps two regions labelled A, and that a smoothing that connects the A regions is labelled by the letter A. Then, given a state S, one has the evaluation $\langle K|S\rangle$ equal to the product of the labels at the smoothings, and one has the evaluation $\|S\|$ equal to the number of loops in the state (the smoothings produce purely virtual diagrams). One then has the formula

$$\langle K\rangle = \sum_S \langle K|S\rangle d^{\|S\|-1}$$

where the summation runs over the states S of the diagram K, and $d = -A^2 - A^{-2}$. This state summation is invariant under all classical and virtual moves except the first Reidemeister move. The bracket polynomial is normalized to an invariant $f_K(A)$ of all the moves by the formula $f_K(A) = (-A^3)^{-w(K)}\langle K\rangle$ where $w(K)$ is the writhe of the (now) oriented diagram K. The writhe is the sum of the orientation signs (± 1) of the crossings of the diagram. The Jones polynomial, $V_K(t)$, is given in terms of this model by the formula

$$V_K(t) = f_K(t^{-1/4}).$$

The reader should note that this definition is a direct generalization to the virtual category of the state sum model for the original Jones polynomial. It is straightforward to verify the invariances stated above. In this way one has the Jones polynomial for virtual knots and links.

In terms of the interpretation of virtual knots as stabilized classes of embeddings of circles into thickened surfaces, our definition coincides with the simplest version of the Jones polynomial for links in thickened surfaces. In that version one counts all the loops in a state the same way, with no regard for their isotopy class in the surface. It is this equal treatment that makes the invariance under handle stabilization work. With this generalized version of the Jones polynomial, one has again the problem of finding a geometric/topological interpretation of this invariant. There is no fully satisfactory topological interpretation of the original Jones polynomial and the problem is inherited by this generalization.

In [51] we have

THEOREM. *To each non-trivial classical knot diagram K of one component there corresponds a non-trivial virtual knot diagram* $\mathrm{Virt}(K)$ *with unit Jones polynomial.*

This theorem is a key ingredient in the problems involving virtual knots. Here is a sketch of its proof. The proof uses two invariants of classical knots and links that generalize to arbitrary virtual knots and links. These invariants are the *Jones polynomial* and the *involutory quandle* denoted by $\mathrm{IQ}(K)$ for a knot or link K.

Given a crossing i in a link diagram, we define $s(i)$ to be the result of *switching* that crossing so that the undercrossing arc becomes an overcrossing arc and vice versa. We also define the *virtualization* $v(i)$ of the crossing by the local replacement indicated in Figure 3. In this figure we illustrate

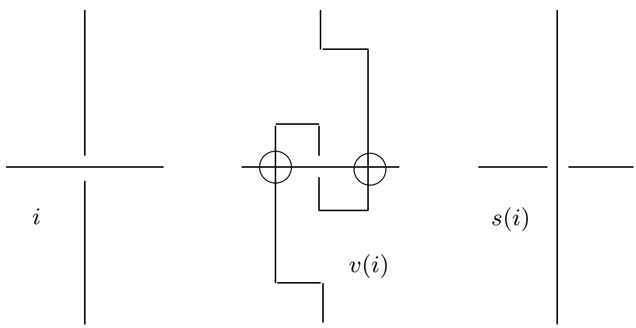


Fig. 3. Switching and virtualizing a crossing

how in the virtualization of the crossing the original crossing is replaced by a crossing that is flanked by two virtual crossings.

Suppose that K is a (virtual or classical) diagram with a classical crossing labelled i. Let $K^{v(i)}$ be the diagram obtained from K by virtualizing the crossing i while leaving the rest of the diagram just as before. Let $K^{s(i)}$ be the diagram obtained from K by switching the crossing i while leaving the rest of the diagram just as before. Then it follows directly from the definition of the Jones polynomial that

$$V_{K^{s(i)}}(t) = V_{K^{v(i)}}(t).$$

As far as the Jones polynomial is concerned, switching a crossing and virtualizing a crossing look the same.

The involutory quandle [45] is an algebraic invariant equivalent to the fundamental group of the double branched cover of a knot or link in the classical case. In this algebraic system one associates a generator of the algebra $\mathrm{IQ}(K)$ to each arc of the diagram K and there is a relation of the

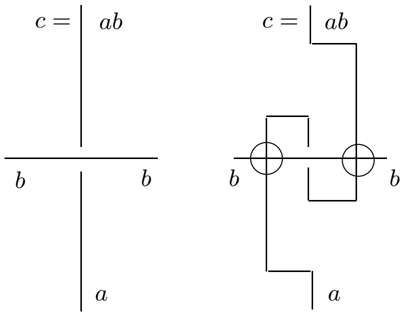


Fig. 4. $\mathrm{IQ}(\mathrm{Virt}(K)) = \mathrm{IQ}(K)$

form $c = ab$ at each crossing, where ab denotes the (non-associative) algebra product of a and b in $\mathrm{IQ}(K)$. See Figure 4. In this figure we have illustrated through the local relations the fact that

$$\mathrm{IQ}(K^{v(i)}) = \mathrm{IQ}(K).$$

As far as the involutory quandle is concerned, the original crossing and the virtualized crossing look the same.

If a classical knot is actually knotted, then its involutory quandle is non-trivial [108]. Hence if we start with a non-trivial classical knot, we can virtualize any subset of its crossings to obtain a virtual knot that is still non-trivial. There is a subset A of the crossings of a classical knot K such that the knot SK obtained by switching these crossings is an unknot. Let $\mathrm{Virt}(K)$ denote the virtual diagram obtained from A by virtualizing the crossings in the subset A. By the above discussion the Jones polynomial of $\mathrm{Virt}(K)$ is the same as the Jones polynomial of SK, and this is 1 since SK is unknotted. On the other hand, the IQ of $\mathrm{Virt}(K)$ is the same as the IQ of K, and hence if K is knotted, then so is $\mathrm{Virt}(K)$. We have shown that $\mathrm{Virt}(K)$ is a non-trivial virtual knot with unit Jones polynomial. This completes the proof of the theorem.

If there exists a classical knot with unit Jones polynomial, then one of the knots $\mathrm{Virt}(K)$ produced by this theorem may be equivalent to a classical knot. It is an intricate task to verify that specific examples of $\mathrm{Virt}(K)$ are not classical. This has led to an investigation of new invariants for virtual knots. In this investigation a number of issues appear. One can examine the combinatorial generalization of the fundamental group (or quandle) of the virtual knot and sometimes one can prove by pure algebra that the resulting group is not classical. This is related to observations by Silver and Williams [100], Manturov [80, 81] and by Satoh [96] showing that the fundamental group of a virtual knot can be interpreted as the fundamental group of the complement of a torus embedded in four-dimensional Euclidean space.

A very fruitful line of new invariants comes about by examining a generalization of the fundamental group or quandle that we call the *biquandle* of the virtual knot. The biquandle is discussed in the next section. Invariants of flat knots (when one has them) are useful in this regard. If we can verify that the flat knot $F(\mathrm{Virt}(K))$ is non-trivial, then $\mathrm{Virt}(K)$ is non-classical. In this way the search for classical knots with unit Jones polynomial expands to the exploration of the structure of the infinite collection of virtual knots with unit Jones polynomial.

Another way of putting this theorem is as follows: In the arena of knots in thickened surfaces there are many examples of knots with unit Jones polynomial. Might one of these be equivalent via handle stabilization to a classical knot? In [71] Kuperberg shows the uniqueness of the embedding of minimal genus in the stable class for a given virtual link. The minimal embedding genus can be strictly less than the number of virtual crossings in a diagram for the link. There are many problems associated with this phenomenon.

There is a generalization of the Jones polynomial that involves surface representation of virtual knots. See [16, 19, 82, 76]. These invariants essentially use the fact that the Jones polynomial can be extended to knots in thickened surfaces by keeping track of the isotopy classes of the loops in the state summation for this polynomial. In the approach of Dye and Kauffman, one uses this generalized polynomial directly. In the approach of Manturov, a polynomial invariant is defined using the stabilization description of the virtual knots.

3.1. Atoms. An *atom* is a pair (M^2, Γ) where M^2 is a closed 2-manifold and Γ is a 4-valent graph in M^2 dividing M^2 into cells which admit a checkerboard colouring (the colouring is also fixed). Γ is called the *frame* of the atom. See [91], [78].

Atoms are considered up to natural equivalence, that is, up to homeomorphisms of the underlying manifold M^2 mapping the frame to the frame and black cells to black cells. From this point of view, an atom can be recovered from the frame together with the following combinatorial structure:

1. A-structure: This indicates which edges for each vertex are *opposite edges*. That is, it indicates the cyclic structure at the vertex.
2. B-structure: This indicates pairs of "black angles". That is, one divides the four edges emanating from each vertex into two sets of adjacent (not opposite) edges such that the black cells are locally attached along these pairs of adjacent edges.

Given a virtual knot diagram, one can construct the corresponding atom as follows. Classical crossings correspond to the vertices of the atom, and generate both the A-structure and the B-structure at these vertices (the B-structure comes from over/under information). Thus, an atom is uniquely determined by a virtual knot diagram. It is easy to see that the inverse operation is well defined modulo virtualization. Thus the atom knows everything about the bracket polynomial (Jones polynomial) of the virtual link.

The crucial notions here are the minimal genus of the atom and the orientability of the atom. For instance, for each link diagram with a corresponding orientable atom (all classical link diagrams are in this class), all degrees of the bracket are congruent modulo 4 while in the non-orientable case they are congruent only modulo 2.

The orientability condition is crucial in the construction of the Khovanov homology theory for virtual links as in [91, 78].

3.2. Biquandles. In this section we give a sketch of some recent approaches to invariants of virtual knots and links.

A *biquandle* [12, 51, 25, 65, 7, 8] is an algebra with four binary operations written $a^b, a_b, a^{\bar{b}}, a_{\bar{b}}$ together with some relations which we will indicate below. The *fundamental* biquandle is associated with a link diagram and is invariant under the generalized Reidemeister moves for virtual knots and links. The operations in this algebra are motivated by the formation of labels for the edges of the diagram. View Figure 5. In this figure we have

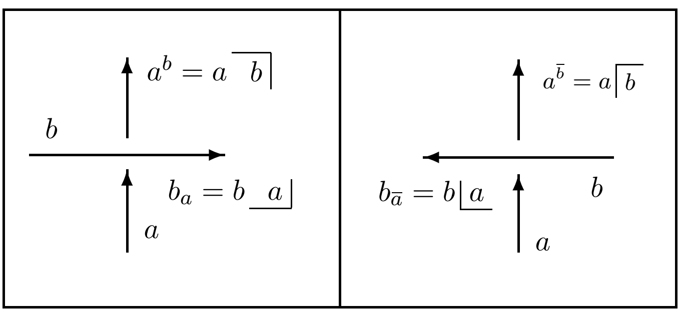


Fig. 5. Biquandle relations at a crossing

shown the format for the operations in a biquandle. The overcrossing arc has two labels, one on each side of the crossing. There is an algebra element labelling each *edge* of the diagram. An edge of the diagram corresponds to an edge of the underlying plane graph of that diagram.

Let the edges oriented toward a crossing in a diagram be called the *input* edges for the crossing, and the edges oriented away from the crossing be called the *output* edges for the crossing. Let a and b be the input edges for a positive crossing, with a the label of the undercrossing input and b the label on the overcrossing input. In the biquandle, we label the undercrossing output by

$$c = a^b,$$

while the overcrossing output is labelled

$$d = b_a.$$

The labelling for the negative crossing is similar using the other two operations.

To form the fundamental biquandle, $\mathrm{BQ}(K)$, we take one generator for each edge of the diagram and two relations at each crossing (as described above).

Another way to write this formalism for the biquandle is as follows:

$$a^b = a\,\overline{\;\;b\,}\big|, \quad a_b = a\,\underline{\;\;b\,}\big|, \quad a^{\bar{b}} = a\big|\overline{\,b\;}, \quad a_{\bar{b}} = a\big|\underline{\,b\;}.$$

We call this the *operator formalism* for the biquandle.

These considerations lead to the following definition.

Definition. A *biquandle* B is a set with four binary operations indicated above: $a^b, a^{\bar{b}}, a_b, a_{\bar{b}}$. We shall refer to the operations with barred variables as the *left* operations and the operations without barred variables as the *right* operations. The biquandle is closed under these operations and the following axioms are satisfied:

1. Given an element a in B, there exists an x in the biquandle such that $x = a_x$ and $a = x^a$. There also exists a y in the biquandle such that $y = a^{\bar{y}}$ and $a = y_{\bar{a}}$.
2. For any elements a and b in B we have
$$a = a^{b\overline{b_a}}, \quad b = b_{\overline{aa^b}}, \quad a = a^{\bar{b}\overline{a}}, \quad b = b_{\bar{a}a^{\bar{b}}}.$$
3. Given elements a and b in B, there exist elements x, y, z, t such that $x_b = a$, $y^{\bar{a}} = b$, $b^x = y$, $a_{\bar{y}} = x$ and $t^a = b$, $a_t = z$, $z_{\bar{b}} = a$, $b^{\bar{z}} = t$. The biquandle is called *strong* if x, y, z, t are uniquely defined and we then write $x = a_{b^{-1}}, y = b^{\bar{a}^{-1}}, t = b^{a^{-1}}, z = a_{\bar{b}^{-1}}$, reflecting the invertive nature of the elements.
4. For any a, b, c in B the following equations hold and the same equations hold when all right operations are replaced in these equations by left operations:
$$a^{bc} = a^{c_b b^c}, \quad c_{ba} = c_{a^b b_a}, \quad (b_a)^{c_{a^b}} = (b^c)_{a^{c_b}}.$$

These axioms are transcriptions of the Reidemeister moves. The first axiom transcribes the first Reidemeister move. The second axiom transcribes the directly oriented second Reidemeister move. The third axiom transcribes the reverse oriented Reidemeister move. The fourth axiom transcribes the third Reidemeister move. Much more work is needed in exploring these algebras and their applications to knot theory.

We may simplify the appearance of these conditions by defining

$$S(a,b) = (b_a, a^b), \qquad \overline{S}(a,b) = (b^{\overline{a}}, a_{\overline{b}})$$

and in the case of a strong biquandle,

$$S^+_-(a,b) = (b^{a_{b^{-1}}}, a_{b^{-1}}), \qquad S^-_+(a,b) = (b^{a^{-1}}, a_{b^{a^{-1}}})$$

and

$$\overline{S}^+_-(a,b) = (b_{\overline{a^{\overline{b}^{-1}}}}, a^{\overline{b}^{-1}}) = (b_{\overline{a^{b_{a^{-1}}}}}, a^{b_{a^{-1}}}),$$
$$\overline{S}^-_+(a,b) = (b_{\overline{a}^{-1}}, a^{\overline{b_{\overline{a}^{-1}}}}) = (b_{a^{b^{-1}}}, a^{\overline{b_{a^{b^{-1}}}}}),$$

which we call the *sideways* operators. The conditions then reduce to

$$S\overline{S} = \overline{S}S = 1,$$
$$(S \times 1)(1 \times S)(S \times 1) = (1 \times S)(S \times 1)(1 \times S),$$
$$\overline{S}^-_+ S^+_- = S^-_+ \overline{S}^+_- = 1,$$

and finally all the sideways operators leave the diagonal $\Delta = \{(a,a) \mid a \in X\}$ invariant.

3.3. The Alexander biquandle. It is not hard to see that the following equations in a module over $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$ give a biquandle structure:

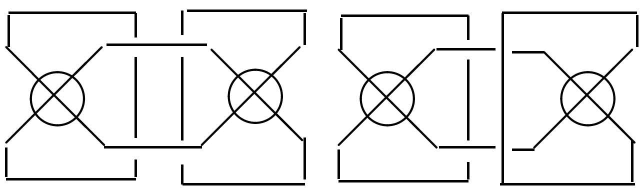
$$a^b = a\,\overline{\;\;b}\rvert = ta + (1 - st)b, \qquad a_b = a\,\underline{\;\;b}\rvert = sa,$$
$$a^{\overline{b}} = a\lvert\overline{b\;\;} = t^{-1}a + (1 - s^{-1}t^{-1})b, \qquad a_{\overline{b}} = a\lvert\underline{b\;\;} = s^{-1}a.$$

We shall refer to this structure, with the equations given above, as the *Alexander biquandle*.

Just as one can define the Alexander module of a classical knot, we have the Alexander biquandle of a virtual knot or link, obtained by taking one generator for each *edge* of the projected graph of the knot diagram and taking the module relations in the above linear form. Let $\mathrm{ABQ}(K)$ denote this module structure for an oriented link K. That is, $\mathrm{ABQ}(K)$ is the module generated by the edges of the diagram, factored by the submodule generated by the relations. This module then has a biquandle structure specified by the operations defined above for an Alexander biquandle.

The determinant of the matrix of relations obtained from the crossings of a diagram gives a polynomial invariant (up to multiplication by $\pm s^i t^j$ for integers i and j) of knots and links that we denote by $G_K(s,t)$ and call the *generalized Alexander polynomial. This polynomial vanishes on classical knots, but is remarkably successful at detecting virtual knots and links*. In fact $G_K(s,t)$ is the same as the polynomial invariant of virtuals of Sawollek [97] and defined by an alternative method by Silver and Williams [100] and by yet another method by Manturov [81]. It is a reformulation of the invariant for knots in surfaces due to Kauffman, Jaeger and Saleur [36, 66].

We end this discussion of the Alexander biquandle with two examples that show clearly its limitations. View Figure 6. In this figure we illustrate



K $\qquad\qquad$ KI

Fig. 6. The knot K and the Kishino diagram KI

two diagrams labelled K and KI. It is not hard to calculate that both $G_K(s,t)$ and $G_{KI}(s,t)$ are equal to zero. However, the Alexander biquandle of K is non-trivial—it is isomorphic to the free module over $\mathbb{Z}[s, s^{-1}, t, t^{-1}]$ generated by elements a and b subject to the relation $(s^{-1} - t - 1)(a - b) = 0$. Thus K represents a non-trivial virtual knot. This shows that it is possible for a non-trivial virtual diagram to be a connected sum of two trivial virtual diagrams. However, the diagram KI has a trivial Alexander biquandle. In fact the diagram KI, discovered by Kishino [13], is now known to be knotted and its general biquandle is non-trivial. The Kishino diagram has been shown non-trivial by a calculation of the three-strand Jones polynomial [69], by the surface bracket polynomial of Dye and Kauffman [16, 19], by the Ξ-polynomial (the surface generalization of the Jones polynomial of Manturov [82], and its biquandle has been shown to be non-trivial by a quaternionic biquandle representation [7] of Fenn and Bartholomew which we will now briefly describe.

The quaternionic biquandle is defined by the following operations where $i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$ in the associative, non-commutative algebra of the quaternions. The elements a, b are in a module over the ring of integer quaternions.

$$a\,\overline{\;b\,}\rvert = j \cdot a + (1+i) \cdot b, \qquad a\lvert\overline{\,b\;} = j \cdot a + (1-i) \cdot b,$$
$$a\,\underline{\;b\,}\rvert = -j \cdot a + (1+i) \cdot b, \qquad a\lvert\underline{\,b\;} = -j \cdot a + (1-i) \cdot b.$$

Amazingly, one can verify that these operations satisfy the axioms for the biquandle.

Equivalently, referring back to the previous section, define the linear biquandle by

$$S = \begin{pmatrix} 1+i & jt \\ -jt^{-1} & 1+i \end{pmatrix},$$

where i, j have their usual meanings as quaternions and t is a central variable. Let R denote the ring which they determine. Then as in the Alexander case considered above, for each diagram there is a square presentation of an R-module. We can take the (Study) determinant of the presentation matrix. In the case of the Kishino knot this is zero. However, the greatest common divisor of the codimension 1 determinants is $2 + 5t^2 + 2t^4$, showing that this knot is not classical.

3.3.1. *Virtual quandles.* There is another generalization of quandle [75] by means of which one can obtain the same polynomial as in [100, 97] from the other point of view [80, 81]. Namely, the formalism is the same as in the case of quandles at classical crossings but one adds a special structure at virtual crossings. The fact that these approaches give the same result in the linear case was proved recently by Roger Fenn and Andrew Bartholomew.

Virtual quandles (as well as biquandles) yield generalizations of the fundamental group and some other invariants. Also, virtual biquandles admit a generalization for multi-variable polynomials for the case of multicomponent links (see [80]). One can extend these definitions by bringing together the virtual quandle (at virtual crossings) and the biquandle (at classical crossings) to obtain what is called a *virtual biquandle*; this work is now in the process, [61].

3.4. A quantum model for $G_K(s,t)$, oriented and bi-oriented quantum algebras. We can understand the structure of the invariant $G_K(s,t)$ by rewriting it as a quantum invariant and then analysing its state summation. The quantum model for this invariant is obtained in a fashion analogous to the construction of a quantum model of the Alexander polynomial in [66, 36]. The strategy in those papers was to take the basic two-dimensional matrix of the Burau representation, view it as a linear transformation $T : V \to V$ on a two-dimensional module V, and then take the induced linear transformation $\widehat{T} : \Lambda^* V \to \Lambda^* V$ on the exterior algebra of V. This gives a transformation on a four-dimensional module that is a solution to the Yang–Baxter equation. This solution of the Yang–Baxter equation then becomes the building block for the corresponding quantum invariant. In the present instance, we have a generalization of the Burau representation, and this same procedure can be applied to it.

The normalized state summation $Z(K)$ obtained by the above process satisfies a skein relation that is just like that of the Conway polynomial:

$Z(K_+) - Z(K_-) = zZ(K_0)$. The basic result behind the correspondence of $G_K(s,t)$ and $Z(K)$ is

THEOREM ([65]). *For a (virtual) link K, the invariants $Z(K)$ ($\sigma = \sqrt{s}$, $\tau = 1/\sqrt{t}$) and $G_K(s,t)$ are equal up to a multiple of $\pm s^n t^m$ for integers n and m (this being the well definedness criterion for G).*

It is the purpose of this section to place our work with the generalized Alexander polynomial in a context of bi-oriented quantum algebras and to introduce the concept of an oriented quantum algebra. In [63, 64] Kauffman and Radford introduce the concept and show that *oriented quantum algebras encapsulate the notion of an oriented quantum link invariant.*

An *oriented quantum algebra* (A, ϱ, D, U) is an abstract model for an oriented quantum invariant of classical links [63, 64]. For the reader thinking about diagrams, the ϱ is an algebraic version of the Yang–Baxter equation and so corresponds to a classical crossing. The D and U are relatives of cups and caps in the diagrams. The definition of an oriented quantum algebra is as follows: We are given an algebra A over a base ring k, an invertible solution ϱ in $A \otimes A$ of the Yang–Baxter equation (in the algebraic formulation of this equation—differing from a braiding operator by a transposition), and commuting automorphisms $U, D : A \to A$ of the algebra, such that

$$(U \otimes U)\varrho = \varrho, \qquad (D \otimes D)\varrho = \varrho,$$
$$[(1_A \otimes U)\varrho][(D \otimes 1_{A^{\mathrm{op}}})\varrho^{-1}] = 1_{A \otimes A^{\mathrm{op}}},$$
$$[(D \otimes 1_{A^{\mathrm{op}}})\varrho^{-1}][(1_A \otimes U)\varrho] = 1_{A \otimes A^{\mathrm{op}}}.$$

The last two equations say that $[(1_A \otimes U)\varrho)]$ and $[(D \otimes 1_{A^{\mathrm{op}}})\varrho^{-1}]$ are inverses in the algebra $A \otimes A^{\mathrm{op}}$ where A^{op} denotes the opposite algebra.

If $U = D = T$, then A is said to be *balanced*. In the case where D is the identity mapping, we call the oriented quantum algebra *standard*. In [64] we show that *the invariants defined by Reshetikhin and Turaev (associated with a quasi-triangular Hopf algebra) arise from standard oriented quantum algebras.* It is an interesting structural feature of algebras that we have elsewhere [44] called *quantum algebras* (generalizations of quasi-triangular Hopf algebras) that they give rise to standard oriented quantum algebras. It would be of interest to see invariants such as the Links–Gould invariant [56] in this light.

We now extend the concept of oriented quantum algebra by adding a second solution to the Yang–Baxter equation γ that will take the role of the virtual crossing.

DEFINITION. A *bi-oriented quantum algebra* is a quintuple $(A, \varrho, \gamma, D, U)$ such that (A, ϱ, D, U) and (A, γ, D, U) are oriented quantum algebras and γ has the following properties:

1. $\gamma_{12}\gamma_{21} = 1_{A\otimes A}$. (This is the equivalent to the statement that the braiding operator corresponding to γ is its own inverse.)
2. Mixed identities involving ϱ and γ are satisfied. These correspond to the braiding versions of the virtual detour move of type three that involves two virtual crossings and one real crossing. See [65] for the details.

By extending the methods of [64], it is not hard to see that *a bi-oriented quantum algebra will always give rise to invariants of virtual links up to the type one moves* (*framing and virtual framing*).

In the case of the generalized Alexander polynomial, the state model $Z(K)$ translates directly into a specific example of a bi-oriented balanced quantum algebra (A, ϱ, γ, T). The main point about this bi-oriented quantum algebra is that the operator γ for the virtual crossing is *not* the identity operator; this non-triviality is crucial to the structure of the invariant. We will investigate bi-oriented quantum algebras and other examples of virtual invariants derived from them.

We have taken a path to explain not only the evolution of a theory of invariants of virtual knots and links, but also (in this subsection) a description of our oriented quantum algebra formulation of the whole class of quantum link invariants. Returning to the case of the original Jones polynomial, we want to understand its capabilities in terms of the oriented quantum algebra that generates the invariant.

4. INVARIANTS OF 3-MANIFOLDS

As is well known, invariants of 3-manifolds can be formulated in terms of Hopf algebras and quantum algebras and spin recoupling networks. In formulating such invariants it is useful to represent the 3-manifold via surgery on a framed link. Two framed links that are equivalent in the Kirby calculus of links represent the same 3-manifold and conversely. To obtain invariants of 3-manifolds one constructs invariants of framed links that are also invariant under the Kirby moves (handle sliding, blowing up and blowing down).

A classical 3-manifold is mathematically the same as a Kirby equivalence class of a framed link. The fundamental group of the 3-manifold associated with a link is equal to the fundamental group of the complement of the link modulo the subgroup generated by the framing longitudes for the link. We refer to the fundamental group of the 3-manifold as the 3-*manifold group*. If there is a counterexample to the classical Poincaré conjecture, then the counterexample would be represented by surgery on some link L whose 3-manifold group is trivial, but L is not trivial in Kirby calculus (i.e. it cannot be reduced to nothing).

Kirby calculus can be generalized to the class of virtual knots and links. We define a *virtual 3-manifold* to be a Kirby equivalence class of framed virtual links. The 3-manifold group generalizes via the combinatorial fundamental group associated to the virtual link (the framing longitudes still exist for virtual links). The *Virtual Poincaré Conjecture* for virtuals would say that a virtual 3-manifold with trivial fundamental group is trivial in Kirby calculus. However, **the virtual Poincaré conjecture is false** [20]. There exist virtual links whose 3-manifold group is trivial that are nevertheless not Kirby equivalent to nothing. The simplest example is the virtual knot in Figure 7. We detect the non-triviality of the Kirby class of this knot by

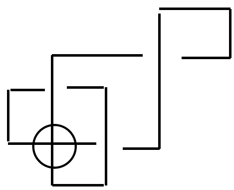


Fig. 7. A counterexample to the Poincaré conjecture for virtual 3-manifolds

computing that it has an $SU(2)$ Witten invariant that is different from the standard 3-sphere.

This counterexample to the Poincaré conjecture in the virtual domain shows how a classical counterexample might behave in the context of Kirby calculus. Virtual knot theory can be used to search for a counterexample to the classical Poincaré conjecture by searching for virtual counterexamples that are equivalent in Kirby calculus to classical knots and links. This is a new and exciting approach to the dark side of the classical Poincaré conjecture.

5. GAUSS DIAGRAMS AND VASSILIEV INVARIANTS

The reader should recall the notion of a *Gauss diagram* for a knot. If K is a knot diagram, then $G(K)$, its Gauss diagram, is a circle comprising the Gauss code of the knot by arranging the traverse of the diagram from crossing to crossing along the circle and putting an arrow (in the form of a chord of the circle) between the two appearances of the crossing. The arrow points from the overcrossing segment to the undercrossing segment in the order of the traverse of the diagram. (Note: Turaev uses another convention, [106].) Each chord is endowed with a sign that is equal to the sign of the corresponding crossing in the knot diagram. At the level of the Gauss diagrams, a virtual crossing is simply the absence of a chord. That is, if we wish to transcribe a virtual knot diagram to a Gauss diagram, we ignore the virtual crossings. Reidemeister moves on Gauss diagrams are defined by translation from the corresponding diagrams from planar representation. Virtual knot theory is precisely the theory of *arbitrary Gauss diagrams*, up to the Gauss diagram Reidemeister moves. Note that an arbitrary Gauss diagram is any pattern of directed, signed chords on an oriented circle. When transcribed back into a planar knot diagram, such a Gauss diagram may require virtual crossings for its depiction.

In [32] Goussarov, Polyak and Viro initiate a very important program for producing Vassiliev invariants of finite type of virtual and classical knots. The gist of their program is as follows. They define the notion of a semi-virtual crossing, conceived as a dotted, oriented, signed chord in a Gauss diagram for a knot. An *arrow diagram* is a Gauss diagram all of whose chords are dotted. Let $\mathcal{A}$ denote the collection of all linear combinations of arrow diagrams with integer coefficients. Let $\mathcal{G}$ denote the collection of all arbitrary Gauss diagrams (hence all representatives of virtual knots). Define a mapping

$$i : \mathcal{G} \to \mathcal{A}$$

by expanding at each chord of a Gauss diagram G into the sum of replacing the chord by a dotted chord and the removal of that chord. Thus

$$i(G) = \sum_{r \in R(G)} G^r$$

where $R(G)$ denotes all ways of replacing each chord in G either by a dotted chord, or by nothing; and G^r denotes that particular replacement applied to G.

Now let $\mathcal{P}$ denote the quotient of $\mathcal{A}$ by the subalgebra generated by the relations in $\mathcal{A}$ corresponding to the Reidemeister moves. Each Reidemeister move is of the form $X = Y$ for certain diagrams, and this translates to the relation $i(X) - i(Y) = 0$ in $\mathcal{P}$, where $i(X)$ and $i(Y)$ are individually certain linear combinations in $\mathcal{P}$. Let

$$I : \mathcal{G} \to \mathcal{P}$$

be the map induced by i. Then it is a formal fact that $I(G)$ is invariant under each of the Reidemeister moves, and hence that $I(G)$ is an invariant of the corresponding Gauss diagram (virtual knot) G. The algebra of relations that generate the image of the Reidemeister moves in $\mathcal{P}$ is called the *Polyak algebra.*

So far, we have only desribed a tautological and not a computable invariant. The key to obtaining computable invariants is to truncate. Let $\mathcal{P}_n$ denote $\mathcal{P}$ modulo all arrow diagrams with more than n dotted arrows.

Now $\mathcal{P}_n$ is a finitely generated module over the integers, and the composite map

$$I_n : \mathcal{G} \to \mathcal{P}_n$$

is also an invariant of virtual knots. Since we can choose a specific basis for $\mathcal{P}_n$, the invariant I_n is in principle computable, and it yields a large collection of Vassiliev invariants of virtual knots that are of finite type. The paper by Goussarov, Polyak and Viro investigates specific methods for finding and representing these invariants. They show that every Vassiliev invariant of finite type for classical knots can be written as a combinatorial state sum for long knots. They use the virtual knots as an intermediate stage in the construction.

By directly constructing Vassiliev invariants of virtual knots from known invariants of virtuals, we can construct invariants that are not of finite type in the above sense (see [49]). These invariants also deserve further investigation.

6. KHOVANOV AND OTHER INVARIANTS

The *Khovanov categorification of the Jones polynomial* [4] is important. This invariant is constructed by promoting the states in the bracket summation to tensor powers of a vector space V, where a single power of V corresponds to a single loop in the state. In this way a graded complex is constructed, whose graded Euler characteristic is equal to the original Jones polynomial, and the ranks of whose graded homology groups are themselves invariants of knots. It is now known that the information in the Khovanov construction exceeds that in the original Jones polynomial. It is an open problem whether a Khovanov type construction can generalize to virtual knots in the general case. The construction for the Khovanov polynomial for virtuals over $\mathbb{Z}_2$ was proposed in [77]. Khovanov homology (with arbitrary coefficient ring) can be defined for any link diagram for which the corresponding atom is orientable. This leads to two explicit geometric constructions of the virtual link Khovanov homology as in [78]. The main point here is that there is a well formulated theory of Khovanov invariants for virtual knots, and it needs more development.

Recent work by other authors related to knots in thickened surfaces [1] promises to shed light on this issue. More generally, the subject of uplevelling known polynomial-type invariants of knots and links to homology theories appears to be very fruitful, and new ways to accomplish this in the virtual category should shed light on the nature of these invariants.

One of the more promising directions for relating Vassiliev invariants to our present concerns is the theory of gropes [14], where one considers surfaces spanning a given knot, and then recursively the surfaces spanning curves embedded in the given surface. This hierarchical structure of curves and surfaces is likely to be a key to understanding the geometric underpinning of the original Jones polynomial. The same techniques in a new guise could elucidate invariants of virtual knots and links.

7. A LIST OF PROBLEMS

Below, we present a list of actual problems closely connected with virtual knot theory.

1. Recognising the Kishino knot. There have been invented at least five ways to recognize the Kishino virtual knot (from the unknot): The 3-strand Jones polynomial, i.e. the Jones polynomial of the 3-strand cabling of the knot) [69], the Ξ-polynomial [82], the quaternionic biquandle [7], and the surface bracket polynomial (Dye and Kauffman [19]). In [37] Kadokami proves the knot is non-trivial by examining the immersion class of a shadow curve in genus two.

Are we done with this knot? Perhaps not. Other proofs of its non-triviality may be illuminating. The fact that the Kishino diagram is non-trivial and yet a connected sum of trivial virtual knots suggests the question: *Classify when a non-trivial virtual knot can be the connected sum of two trivial virtual knots.* A key point here is that the connected sum of closed virtual knots is not well defined and hence the different choices give some interesting effects. With long virtual knots, the connected sum is ordered but well defined, and the last question is closely related to the question of classifying the different long virtual knots whose closures are equivalent to the unknot.

2. Flat virtuals. See Section 2.1. Flat virtual knots, also known as *virtual strings* [34, 106], are difficult to classify. Find new combinatorial invariants of flat virtuals.

We would like to know more about the flat biquandle algebra. This algebra is isomorphic to the Weyl algebra [8] and has no (non-trivial) finite-dimensional representations. One can make small examples of the flat biquandle algebra, that detect some flat linking beyond mod 2 linking numbers, but the absence of other finite-dimensional representations presents a problem.

3. The flat hierarchy. The flat hierarchy is constructed for any ordinal α. We label flat crossings with members of this ordinal. In a flat third Reidemeister move, a line with two a labels can slide across a crossing labelled b only if a is greater than b. This generalizes the usual theory of flat virtual diagrams to a system with arbitrarily many different types of flat crossings. Classify the diagrams in this hierarchy. This concept is due to L. H. Kauffman (unpublished). A first step in working with the flat hierarchy can be found in [89].

4. Virtuals and the theory of doodles. Compare flat theories of virtuals with theories of doodles. A *doodle link* has only one type of crossing, and that cannot slide over itself. See [31, 23, 67].

5. Virtual 3-manifolds. There is a theory of virtual 3-manifolds constructed as formal equivalence classes of virtual diagrams modulo generalized Kirby moves (see [20]). From this point of view, there are two equivalences for ordinary 3-manifolds: homeomorphisms and virtual equivalence. Do these equivalences coincide? That is, given two ordinary 3-manifolds, presented by surgery on framed links K and L, suppose that K and L are equivalent through the virtual Kirby calculus. Does this imply that they are equivalent through the classical Kirby calculus?

What is a virtual 3*-manifold?* That is, give an interpretation of these equivalence classes in the domain of geometric topology.

Construct *another* theory of virtual 3-manifolds by performing surgery on links in thickened surfaces $S_g \times \mathbb{R}$ considered up to stabilization. Will this theory coincide with that proposed by Kauffman and Dye [20]?

6. Welded knots. We would like to understand welded knots [96]. It is well known [94, 42] that if we admit forbidden moves to the virtual link diagrams, each virtual knot can be transformed to the unknot. If we allow only one forbidden move (e.g. the upper one), then there are lots of different equivalence classes of knots. In fact the fundamental group and the quandle of the virtual diagram are invariant under the upper forbidden move. The resulting equivalence classes are called *welded knots*. Similarly, *welded braids* were studied in [26], and every welded knot is the closure of a welded braid. The question is to construct good invariants of welded knots and, if possible, to classify them. In [96] a mapping is constructed from welded knots to ambient isotopy classes of embeddings of tori (ribbon tori to be exact) in four-dimensional space, and it is proved that this mapping is an isomorphism from the combinatorial fundamental group (in fact the quandle) of the welded knot to the fundamental group of the complement of the corresponding torus embedding in four-space. Is this Satoh mapping faithful from equivalence classes of welded knots (links) to ambient isotopy classes of ribbon torus embeddings in four-space?

7. Long knots and long flat knots. Enlarge the long knot invariant structure proposed in [73]. Can one get new classical knot invariants from the approach in this paper? Bring together the ideas from [73] with the biquandle construction from [65] to obtain more powerful invariants of long knots. Long flat virtuals can be studied via a powerful remark due to V. Turaev (in conversation) to the effect that one can associate to a given long flat virtual knot diagram F a *descending* diagram $D(F)$ (by always going over before going under in resolving the flat (non-virtual) crossings in the diagram). The long virtual knot type of $D(F)$ is an invariant of the long flat knot F. This means that one can apply any other invariant I of virtual knots that one likes to $D(F)$, and $I[D(F)]$ will be an invariant of the long flat F. It is quite interesting to do sample calculations of such invariants [55] and this situation underlines the deeper problem of finding a full classification of long flat knots.

8. Virtual biquandle. Construct presentations of the virtual biquandle with a linear (non-commutative) representation at classical crossings and some interesting structure at virtual crossings.

9. Virtual braids. Is there a birack such that its action on virtual braids is faithful? Is the invariant of virtual braids in [87] (see also [74, 3, 57]) faithful?

The action defined by *linear* biquandles is not faithful. This almost certainly means that the corresponding linear invariants of virtual knots and links are not faithful [24].

10. The fundamental biquandle. Does the fundamental biquandle (see [65]) classify virtual links up to mirror images? (We know that the biquandle has the same value on the orientation reversed mirror image where the mirror stands perpendicular to the plane (see [34, 35]).

Are there good examples of weak biquandles which are not strong?

We would like to know more about the algebra with two generators A, B and one relation $[B, (A-1)(A, B)] = 0$ (see [8]). It is associated to the linear case.

11. Virtualization and unit Jones polynomial. Suppose the knot K is classical and not trivial. Suppose that $\widetilde{K}$ (obtained from K by virtualizing a subset of its crossings) is not trivial and has a unit Jones polynomial, $V(\widetilde{K}) = 1$. Is it possible that $\widetilde{K}$ is classical (i.e. isotopic through virtual equivalence to a classical knot)?

Suppose K is a virtual knot diagram with unit Jones polynomial. Is K equivalent to a classical diagram via virtual equvalence plus crossing virtualization? (Recall that by crossing virtualization, we mean flanking a classical crossing by two virtual crossings. This operation does not affect the value of the Jones polynomial.)

Given two classical knots K and K', if K can be obtained from K' by a combination of crossing virtualization and virtual Reidemeister moves, is then K classically equivalent to K'?

If the above two questions have affirmative answers, then the only classical knot with unit Jones polynomial is the unknot.

12. Virtual quandle homology. Study virtual quandle homology in analogy to quandle homology [10, 33].

13. Khovanov homology. Construct a generalization of the Khovanov complex for the case of virtual knots that will work for arbitrary virtual diagrams. Investigate the Khovanov homology constructed in [91, 78]. The main construction in this approach uses an orientable atom condition to give a Khovanov homology over the integers for large classes of virtual links. The import of our question is to investigate this structure and to possibly find a way to do Khovanov homology for all virtual knots over the ring of integers. Similar questions can be raised for the presently evolving new classes of Khovanov homology theories related to other quantum invariants.

By a *K-full* virtual knot we mean a knot for which there exists a diagram such that the leading (the lowest, or both) term comes from the B-state. Analogously, one defines the *Kho-full* knot relative to the Khovanov invariant. Call such diagrams optimal diagrams. (It is easy to find knots which are neither K-full nor Kho-full.)

Classify all K-full (Kho-full) knots.

Are optimal diagrams always minimal with respect to the number of classical crossings?

Classify all diagram moves that preserve optimality.

Is it true that if a classical knot K has minimal classical diagram with n crossings then any virtual diagram of K has at least n classical crossings?

Can any virtual knot have torsion in the B-state of the Khovanov homology (the genuine leading term of some diagram)? Here we use the formulation of Khovanov homology given in [91], [78].

The behaviour of the lowest and leading terms of the Kauffman bracket for virtual knots was studied in [91], [2] and [38].

14. Brauer algebra. The appropriate domain for the virtual recoupling theory is to place the Jones–Wenzl projectors in the Brauer algebra. That is, when we add virtual crossings to the Temperley–Lieb algebra to obtain a “virtual Temperley–Lieb algebra” the result is the Brauer algebra of all connections from n points to n points. What is the structure of the projectors in this context? Can a useful algebraic generalization of the classical recoupling theory be formulated?

15. Virtual alternating knots. Define and classify alternating virtual knots.

Find an analogue of the Tait flyping conjecture and prove it. Compare [110].

Classify all alternating weaves on surfaces (without stabilization).

16. Crossing number problems. For each virtual link L, there are three crossing numbers: the minimal number C of classical crossings, the minimal number V of virtual crossings, and the minimal total number T of crossings for representatives of L. There are also a number of unknotting numbers: The classical unknotting number is the number of crossing switches needed to unknot the knot (using any diagram for the knot). The *virtual unknotting number* is the number of crossings one needs to convert from classical to virtual (by direct flattening) in order to unknot the knot (using any virtual diagram for the knot). Very little is known. Find out more about the virtual unknotting number.

What is the relationship between the least number of virtual crossings and the least genus in a surface representation of the virtual knot?

Is it true that $T = V + L$?

Is there any algorithm for finding V for some class of virtual knots? For T, this is partially done for two classes of links: quasialternating and some other, see [79]. For classical links and alternating diagrams see [93, 92].

Are there (non-trivial) upper and lower bounds for T, V, L coming from virtual knot polynomials?

17. Wild virtuals. Create the category of "wild virtual knots" and establish its axiomatics. In particular, one needs a theorem that states when a wild equivalence of tame virtual links implies a tame equivalence of these links.

18. Vassiliev invariants. Understand the connection between virtual knot polynomials and the Vassiliev knot invariants of virtual knots (in Kauffman's sense). Some of that was done in [49, 32, 98, 83].

The key question about this collection of invariants is this: *Does every Vassiliev invariant of finite type for classical knots extend to an invariant of finite type for long virtual knots?* Here we mean the problem in the sense of the formulation given in [32]. In [49] it was pointed out that there is a natural notion of Vassiliev invariants for virtual knots that has a different notion of finite type from that given in [32]. This alternative formulation needs further investigation.

19. Embeddings of surfaces. Given a non-trivial virtual knot K, prove that there exists a minimal realization of K in $N = S_g \times I$ and an *unknotted* embedding of $N \subset \mathbb{R}^3$ such that the resulting classical knot in $\mathbb{R}^3$ is not trivial. (This problem is partially solved by Heather Dye in [17].)

20. Non-commutativity and long knots. It is known that any classical long knot commutes with any long knot. Is it true that it is the only case of commutativity for virtual long knots? In other words, is it true that if K and K' are long knots and $K \# K'$ is isotopic to $K' \# K$ then there exists a virtual long knot L, classical long knots Q, Q', and non-negative integer numbers m, n such that

$$K = L^m \# Q, \qquad K' = L^n \# Q',$$

where by L^m we mean the connected sum of m copies of the same knot?

21. The rack space. The rack space was invented by Fenn, Rourke and Sanderson [22, 28, 29, 30]. The homology of the rack space has been considered by the above authors and Carter, Kamada and Saito [10]. For low dimensions, the homology has the following interesting interpretations. Two-dimensional cycles are represented by virtual link diagrams consistently coloured by the rack, and 3-dimensional cycles by the same but with the regions also coloured. See the thesis of Michale Greene [33]. So virtual links can give, in this way, information about classical knots! For the second homology of the dihedral rack, the results are given in Greene's thesis. Computer calculations suggest that for a prime p the third homology has a factor Z_p. Is this true in general?

Another line of enquiry is to look at properties of the birack space [29] and associated homology.

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Received 22 May 2004;
in revised form 17 June 2005