FUNDAMENTA
MATHEMATICAE
187 (2005)

Forcing relation on interval patterns

by

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Abstract. We consider–without restriction to the piecewise monotone case—a forcing relation on interval (transitive, roof, bottom) patterns. We prove some basic properties of this type of forcing and explain when it is a partial ordering. Finally, we show how our approach relates to the results known from the literature.

1. Introduction. A (line) *system* is a pair $\langle T, g\rangle$, where $T \subset \mathbb{R}$ is compact and $g\colon T \to T$ is a continuous map. Two systems $\langle T_1, g_1\rangle$, $\langle T_2, g_2\rangle$ are equal if $T_1 = T_2$ and $g_1 = g_2$. For a nonempty compact set $T \subset \mathbb{R}$ we denote by $C(T)$ the set of all continuous functions that map T into itself. In particular, if $I \subset \mathbb{R}$ is a closed interval, any element of $C(I)$ will be called an *interval map.* A function $f \in C(\widetilde{T})$ *has a system* $\langle T, g\rangle$ if $T \subset \widetilde{T}$ and $f|T = g$.

It is quite easy to see that any continuous interval map has infinitely many distinct systems. One can ask the following question: if it is known that a continuous interval map has a given system, what can be said about other systems of that map? Some interesting results concerning this question are known—they are included in the theory of the forcing relation on interval patterns (a pattern is an equivalence class of systems). For periodic patterns, the systematic theory was summarized in [1], in [8] the case of finite patterns (given by finite sets) has been deeply studied, and recently the case of piecewise monotone minimal patterns (also infinite) has been examined [5]. The main aim of our paper is to extend the notion of forcing on minimal (periodic, finite) patterns to more general (not only piecewise monotone) cases.

In Section 2 we define a natural *equivalence of transitive systems.* Then a *transitive pattern* is a corresponding equivalence class. Using the usual definition of the forcing relation we characterize when a transitive pattern forces another one. The main statement of this part is Theorem 2.4.

2000 *Mathematics Subject Classification*: Primary 37E05; Secondary 37B20.
Key words and phrases: forcing relation, interval pattern.
Author supported by MYES of the Czech Republic via contract MSM 6840770010.

In Section 3 we introduce a special type of transitive pattern—we call it a *roof pattern*—that arose from maximal ω-limit sets. Defining a *nonfractal structure* of a transitive pattern, in Theorem 3.5 we show that the forcing relation restricted to the set of nonfractal roof patterns is a partial ordering.

In Section 4 we explain how our results relate to the ones known from the literature [1], [5]. For this purpose we define a (*nonfractal*) *bottom system* as a system (minimal with respect to inclusion) coding a (nonfractal) roof system. In particular we show that any minimal (periodic) system is a nonfractal bottom system. Saying that two bottom systems are equivalent when they code equivalent roof systems we introduce a *bottom pattern*. Again using the usual definition of forcing, in Lemma 4.5 we prove that our forcing relation on bottom patterns extends the one used on minimal (periodic) patterns. In Theorem 4.8 we prove that the forcing relation on nonfractal bottom patterns is a partial ordering. Finally, Theorem 4.9 says that our result generalizes the ones known from the literature—see Theorem 6.4.

Section 5 is mainly devoted to the technical statements needed to prove our results.

In Section 6 we present some important notions and results known from the literature. Mainly we recall Blokh's classification of maximal ω-limit sets [4] that plays a central role in our paper.

By $\mathbb{R}$, $\mathbb{N}$, $\mathbb{N}_0$ we denote the sets of real, positive integer and nonnegative integer numbers respectively. For $g \in C(T)$ we define g^n inductively by $g^0 = \mathrm{id}$ and (for $n \geq 1$) $g^n = g \circ g^{n-1}$. Let $g \in C(T)$. If J is a nonempty subset (maybe one point) of T, then the *orbit* of J under g is $\mathrm{orb}(g, J) = \{g^n(J)\}_{n=0}^{\infty}$. We often write $\mathrm{orb}(g, J)$ instead of $\bigcup \mathrm{orb}(g, J)$. We say that J is *g-periodic*, resp. *weakly g-periodic* of period $n \in \mathbb{N}$ if $J, \dots, g^{n-1}(J)$ are pairwise disjoint and $g^n(J) = J$, resp. $g^n(J) \subset J$. A fixed point is a periodic point of period 1 and $\mathrm{Per}(g)$, resp. $\mathrm{Fix}(g)$ is the set of all periodic, resp. fixed points of g. The *ω-limit set* $\omega(g, x)$ of x consists of all the limit points of $\mathrm{orb}(g, x)$.

If a function $f \in C(\widetilde{T})$ has a system $\langle T, g\rangle$ then we often write $\langle T, f\rangle$ instead of $\langle T, g = f|T\rangle$.

2. Transitive patterns

2.1. *Classification of transitive systems.* A system $\langle T, g\rangle$ is said to be *transitive* if $\omega(g, x) = T$ for some $x \in T$. Such a point will be called *transitive* and we denote by $\mathrm{Tran}\langle T, g\rangle$ the set of all transitive points in T. We will use the known classification of possible types of transitive (line) systems (see for example [3]): Any transitive system $\langle T, g\rangle$ satisfies either (i), (ii) or (iii), where:

(i) T is finite, there is a least $n \in \mathbb{N}$ such that $T = \{x, g(x), \dots, g^{n-1}(x)\}$ for any $x \in T$. In this case $\langle T, g\rangle$ is called a *cycle*. The set of all cycles is denoted by $\mathcal{P}$.

(ii) T is a Cantor set and all points of T have an orbit dense in T. In this case $\langle T, g\rangle$ is called *minimal*. The set of all minimal systems is denoted by $\mathcal{M}$.

(iii) T is either a Cantor set ($\langle T, g\rangle \in \mathcal{NM}_C$) or a finite union of closed intervals ($\langle T, g\rangle \in \mathcal{NM}_I$) and not all points of T have an orbit dense in T. The set of all such systems is denoted by $\mathcal{NM} = \mathcal{NM}_C \cup \mathcal{NM}_I$.

We put $\mathfrak{T} = \mathcal{P} \cup \mathcal{M} \cup \mathcal{NM}$.

2.2. *Transitive pattern, forcing relation.* Lemma 5.2 shows one can define an equivalence relation for transitive systems in the following natural manner.

2.1. Definition. Transitive systems $\langle T, g\rangle, \langle S, f\rangle$ are *equivalent* if there are points $x_T \in \operatorname{Tran}\langle T, g\rangle$, $y_S \in \operatorname{Tran}\langle S, f\rangle$ such that for any $i, j \in \mathbb{N}_0$,

$$g^i(x_T) < g^j(x_T) \;\Leftrightarrow\; f^i(y_S) < f^j(y_S). \tag{2.1}$$

In that case we write $\langle T, g\rangle \sim \langle S, f\rangle$ and $x_T \leftrightarrow y_S$.

A *transitive pattern* is a corresponding equivalence class in $\mathfrak{T}_\sim$. We denote by $[\langle T, g\rangle]_\sim$ a transitive pattern from $\mathfrak{T}_\sim$ with representative $\langle T, g\rangle$. The cardinality of a transitive pattern A is equal to $\operatorname{card} T$, where $\langle T, g\rangle \in A$ (it does not depend on the choice of a representative). If a map $f \in C(\widetilde{T})$ has a transitive system $\langle T, g\rangle$ then we also say that f *exhibits a transitive pattern* $[\langle T, g\rangle]_\sim$.

Put $\mathfrak{C} = \{f : I \to I : I \subset \mathbb{R}$ is a compact interval and f is continuous$\}$.

2.2. Definition. A transitive pattern A *forces* a transitive pattern B we write $A \hookrightarrow B$ if all maps in $\mathfrak{C}$ exhibiting A also exhibit B.

A cycle $\langle T, g\rangle$ (resp. a pattern $[\langle T, g\rangle]_\sim$) is a 2-*extension* of a cycle $\langle S, f\rangle$ (resp. of a pattern $[\langle S, f\rangle]_\sim$) with $S = \{s_1 < \dots < s_k\}$ if there are T-blocks $B_i = \{a_i, b_i\} \subset T$, $i \in \{1, \dots, k\}$, such that $a_i < b_i < a_{i+1}$ for $i \in \{1, \dots, k-1\}$, $T = \bigcup_{i=1}^k B_i$, and $g(B_i) = B_j$ if and only if $f(s_i) = s_j$.

For a system $\langle R, p\rangle$, $p_R \in C(\operatorname{conv} R)$ denotes a map such that $p_R|R = p$ and p_R is affine on each component of $\operatorname{conv} R \setminus R$ (such a component called an R-*contiguous interval*).

2.3. Definition. For a system $\langle R, p\rangle \in \mathfrak{T}$ we say that $\langle T, g\rangle \in \mathcal{P}$ is a *reducible system* (*cycle*) of p_R if

- $T \subset \operatorname{conv} R$ and $p_R|T = g$,
- $\langle T, g\rangle$ is a 2-extension,
- each T-block is a subset of a closed R-contiguous interval.

We say that the map p_R *exhibits a pattern* B *irreducibly* if p_R has a system $\langle T, g\rangle \in B$ which is not a reducible system of p_R.

2.4. THEOREM. *Let* $A \neq B$ *be transitive patterns. The following conditions are equivalent*:

(i) A *forces* B.
(ii) *For every* $\langle R, p\rangle \in A$, p_R *exhibits the pattern* B *irreducibly.*
(iii) *For some* $\langle R, p\rangle \in A$, p_R *exhibits the pattern* B *irreducibly.*

Proof. The implication (ii)⇒(iii) is clear. Thus it is sufficient to prove (i)⇒(ii) and (iii)⇒(i).

(i)⇒(ii). Assume to the contrary that there is a system $\langle R, p\rangle \in A$ such that the map p_R has only representatives of B which are reducible systems of p_R. By Definition 2.3 and Lemma 5.16(iv) this means that

- $\langle R, p\rangle \in \mathcal{NM}$ and B is a 2-extension (a periodic pattern),
- whenever p_R has a cycle $\langle T, g\rangle \in B$ with T-blocks $B_i \subset [a_i, b_i]$, where each $[a_i, b_i]$ is a closed R-contiguous interval, then $\langle T^\star = \bigcup_i \{a_i, b_i\}, p\rangle$ is also a reducible cycle of p_R.

The cycle $\langle T^\star, p\rangle$ satisfies $T^\star \subset R$; such a system will be called a *maximal reducible cycle* of p_R. Let $\{\langle T^j, p\rangle\}_j \subset B$ contain all maximal reducible cycles of p_R with T^j-blocks $B_i^j = \{a_i^j, b_i^j\}$. Define a continuous surjective nondecreasing map $\alpha\colon \operatorname{conv} R \to \operatorname{conv} R$ by

$$\alpha|J \text{ is constant } \Leftrightarrow \exists i, j, k\colon p_R^k(J) \subset [a_i^j, b_i^j].$$

Such a map exists since $\operatorname{Tran}\langle R, p\rangle \cap \bigcup [a_i^j, b_i^j] = \emptyset$ and the set

$$\operatorname{conv} R \setminus \bigcup_{i,j} \bigcup_{k \in \mathbb{N}_0} p_R^{-k}((a_i^j, b_i^j))$$

is perfect. Using α, we can find (see [1, Lemma 4.6]) a map $\varrho \in C(\operatorname{conv} R)$ satisfying

$$\alpha \circ p_R = \varrho \circ \alpha \quad \text{ on } \operatorname{conv} R.$$

Clearly, $\langle \alpha(R), \varrho\rangle \in \mathcal{NM}$ and since α is increasing on $\operatorname{Tran}\langle R, p\rangle$, we also have $\langle R, p\rangle \sim \langle \alpha(R), \varrho\rangle$, i.e., ϱ exhibits the pattern A. At the same time, since α "kills" all representatives of B, ϱ does not exhibit B, which contradicts our assumption (i). Summarizing, we have shown that if (i) is true then the map p_R has to exhibit the pattern B irreducibly.

(iii)⇒(i). Assume that for some $\langle R, p\rangle \in A$, p_R exhibits the pattern B irreducibly, and fix a map $f \in \mathfrak{C}$, $f\colon I \to I$ exhibiting A. We need to prove that f also exhibits B. Let p_R have a representative $\langle T, p_R\rangle$ of B which is not a reducible system of p_R, and assume that $S \subset I$ is a closed f-invariant set satisfying $\langle S, f\rangle \in A$. We will distinguish two possibilities.

Case I: $T \subset R$. In this case Lemma 5.5(i) guarantees the existence of a closed f-invariant set $T^\star \subset \operatorname{conv} S$ such that $\langle T^\star, f\rangle \sim \langle T, p_R\rangle$, hence $\langle T^\star, f\rangle \in B$, i.e., the map f exhibits the pattern B.

Case II: $\operatorname{Tran}\langle T, p_R\rangle \cap R = \emptyset$ and T is infinite. Then we can apply Lemma 5.10 putting $\langle A, \alpha\rangle = \langle R, p\rangle$, $\langle T, r\rangle = \langle T, p_R\rangle$, $\langle S, q\rangle = \langle S, f|S\rangle$, $\widetilde{q} = f|\operatorname{conv} S$ and $B = S$. By that lemma there exists a set $T^\star \subset [\min S, \max S]$ for which $\langle T^\star, f\rangle \sim \langle T, p_R\rangle$, i.e., f exhibits the pattern B.

If $\operatorname{Tran}\langle T, p_R\rangle \cap R = \emptyset$ and T is finite we will apply Lemma 5.11 for $\langle R, p\rangle$, $\langle S, q = f|S\rangle$, $\widetilde{q} = f|\operatorname{conv} S$ and $\langle T, p_R\rangle$. This is possible since $\langle T, p_R\rangle$ is not a reducible system of p_R. By Lemma 5.11 there exists a set $T^\star \subset [\min S, \max S]$ for which $\langle T^\star, f\rangle \sim \langle T, p_R\rangle$, i.e., f exhibits the pattern B. ∎

3. Roof patterns. By Blokh [4], if $\omega \subset I$ is a maximal ω-limit set of an interval map $f\colon I \to I$ then $\langle \omega, f\rangle$ is a transitive system. In this part we use the equivalence relation $\sim$ only on a set of transitive systems that arose from maximal ω-limit sets—we call them *roof systems*.

3.1. Definition. A transitive system $\langle T, g\rangle$ is a *roof system* if for any closed set S such that $T \subset S \subset \operatorname{conv} T$ and the system $\langle S, g_T\rangle$ is transitive we necessarily have $S = T$. The set of all roof systems will be denoted by $\mathfrak{R}\mathfrak{S}$.

A *roof pattern* is a corresponding equivalence class in $\mathfrak{R}\mathfrak{S}_\sim$.

By the definition, if $\langle T, g\rangle$ is a roof system then T is a maximal ω-limit set of a map g_T. Thus, to distinguish all possible types of roof systems we can use the properties of maximal ω-limit sets described in [4] and recalled in Section 6. Here we recall two definitions used below.

Solenoidal system. Let $\langle S, f\rangle$ be a system and let $K_0 \supset K_1 \supset \cdots$ be f_S-periodic intervals containing S with periods $n_0, n_1, \ldots$. Obviously n_{i+1} is a multiple of n_i for all i. If $n_i \to \infty$ then the intervals $\{K_i\}_{i\in\mathbb{N}_0}$ are said to be *Q-generating*, where

$$S \subset Q = \bigcap_{i\in\mathbb{N}_0} \operatorname{orb}(f_S, K_i).$$

If $\omega(f_S, x) = S$ for any $x \in Q$, the system $\langle S, f\rangle$ is minimal and it is called a *solenoidal system*.

Basic system. For a system $\langle B, f\rangle$ let K be an f_B-periodic interval with a period n, and $L = \operatorname{orb}(f_B, K)$. The system $\langle B, f\rangle$ is called a *basic system* provided that the set B is infinite and if $J(x)$ denotes a neighbourhood of $x \in L$ (in L) then

$$B = B(L, f_B) = \{x \in L\colon \overline{\operatorname{orb}}(f_B, J(x)) = L \text{ for each } J(x)\}.$$

Using the results from Section 6, we show in Lemmas 5.12, 5.13 and 5.15 that for a roof system $\langle T, g\rangle$ exactly one of the following three possibilities holds:

(i) T is finite and either $\operatorname{card} T = 1$ (a trivial roof system) or $\langle T, g\rangle$ is a 2-extension;
(ii) $\langle T, g\rangle$ is a solenoidal system;
(iii) $\langle T, g\rangle$ is a basic system.

First, let us emphasize that our definition of a roof pattern is fully compatible with the equivalence relation $\sim$ on $\mathfrak{T}$.

3.2. Lemma. *If A is a roof pattern and $\langle S, q\rangle \in A$ then $A = [\langle S, q\rangle]_{\sim}$. Moreover, if $\langle S, q\rangle$ is basic (resp. solenoidal, a 2-extension, a trivial roof system) then any element of A is basic (resp. solenoidal, a 2-extension, a trivial roof system).*

Proof. This is an immediate consequence of Lemmas 5.18, 5.12, 5.13 and 5.15. ∎

In accordance with the previous lemma we can say about a roof pattern that it is trivial, a 2-extension, solenoidal or basic. In what follows we introduce another notion useful for our purpose: fractal and nonfractal transitive systems.

3.3. Definition. A transitive system $\langle S, f\rangle$ is said to be *fractal* if there is a set $\widetilde{S} \subsetneq S$ such that $\langle \widetilde{S}, f\rangle$ is transitive and $\langle \widetilde{S}, f\rangle \sim \langle S, f\rangle$. A transitive system which is not fractal is called *nonfractal.*

3.4. Lemma. *Two equivalent transitive systems $\langle T, g\rangle$, $\langle R, p\rangle$ are simultaneously fractal, resp. nonfractal.*

Proof. Let $\langle T, g\rangle$ be fractal, i.e., $\langle S, f = g|S\rangle \sim \langle T, g\rangle$ for some $S \subsetneq T$. Then $\langle T, g\rangle \in \mathcal{NM}$. Fix $u \in \operatorname{Tran}\langle S, f\rangle$ and consider $v \in \mathcal{B}_{T,R}(\{u\})$ as in Lemma 5.2. By properties (i), (ii) of that lemma, the orbits $\operatorname{orb}(g, u)$, $\operatorname{orb}(p, v)$ have the same order. Putting $K_i = p^i(v)$ in Lemma 5.4, we infer that there is a p-recurrent point $r^\star \in R$ such that $p^{m(n)}(v) \searrow r^\star$ and for $R^* = \omega(p, r^*)$ we have $\langle R^*, p\rangle \sim \langle S, f\rangle \sim \langle T, g\rangle \sim \langle R, p\rangle$. Since by Lemma 5.2(iii), $v \notin \operatorname{Tran}\langle R, p\rangle$, also $r^\star \notin \operatorname{Tran}\langle R, p\rangle$, i.e., the system $\langle R, p\rangle$ is fractal. This proves the lemma. ∎

Thus we can also talk about fractal and nonfractal transitive patterns. The main result of this section follows. We say that a system $\langle R, p\rangle$ is *piecewise monotone* if the map $p_R \in C(\operatorname{conv} R)$ is piecewise monotone.

3.5. Theorem. *The forcing relation on nonfractal roof patterns is a partial ordering.*

Proof. In the proof we say briefly "pattern" instead of "nonfractal roof pattern".

Clearly, if A is a pattern, then $A \hookrightarrow A$ (reflexivity); if A, B, C are patterns such that $A \hookrightarrow B$ and $B \hookrightarrow C$, then $A \hookrightarrow C$ (transitivity). Thus it remains to prove the weak antisymmetry of the forcing relation. It holds trivially when $\min\{\operatorname{card} A, \operatorname{card} B\} \leq 2$. Therefore we will assume that $\min\{\operatorname{card} A, \operatorname{card} B\} > 2$.

Suppose that for patterns A, B, $A \hookrightarrow B$ and $B \hookrightarrow A$. Using Lemma 5.16(i) we see that A is piecewise monotone if and only if B is. We need to show that $A = B$. Let us distinguish several possibilities.

Case I: *A, B not piecewise monotone, A solenoidal.* Fix $\langle R, p\rangle \in A$. From Lemma 3.2 we know that $\langle R, p\rangle \in \mathcal{M}$; by our assumption the map p_R exhibits B, i.e., $\langle S, p_R\rangle \in B$ for some $S \subset \operatorname{conv} R$. Since $\langle S, p_R\rangle$ is not piecewise monotone, $S \cap J$ is nonempty for infinitely many R-contiguous intervals J, hence from minimality of $\langle R, p\rangle$ we get $R \subset S$. But $\langle R, p\rangle$ is a roof system. Then Definition 3.1 gives $R = S$, hence also $A = B$.

Case II: *A, B basic, not piecewise monotone.* Using Theorem 6.2 we can fix $\langle R, p\rangle \in A \cap \mathcal{NM}_I$. As before the map p_R exhibits B, i.e., $\langle S, p_R\rangle \in B$ for some $S \subset \operatorname{conv} R$. We assume that $\langle S, p_R\rangle$ is not piecewise monotone and $A \neq B$. Using the fact that the set R has finitely many R-contiguous intervals we get $S \subsetneq R$. Similarly we can take a system $\langle S', q\rangle \in B \cap \mathcal{NM}_I$ to show that $\langle R', q\rangle \in A$ for some $R' \subsetneq S'$. Since $\langle R', q\rangle$ is not a reducible system of $q_{S'}$, Lemma 5.5(i) gives a set $R'' \subset S \subsetneq R$ such that $\langle R'', p\rangle \in A$ and $\langle R'', p\rangle \sim \langle R, p\rangle$. This contradicts our assumption that A is nonfractal. Thus, $A = B$.

Case III: *A, B piecewise monotone, A solenoidal or a 2-extension.* Suppose $A \neq B$ and $\langle T, g\rangle \in A$ (with $\operatorname{card} T > 2$). Since A forces B, the map g_T exhibits the pattern B. Fix $\langle S, f = g_T|S\rangle \in B$. If $T \cap S \neq \emptyset$ then since $\langle T, g\rangle \in \mathcal{P} \cup \mathcal{M}$ we would have $T \subsetneq S$, which is impossible for the roof system $\langle T, g\rangle$. Thus, $T \cap S = \emptyset$. In particular, $\min T < \min S$ and $\max S < \max T$. Define the map $h \in C(\operatorname{conv} T)$ by

$$h(x) = \begin{cases} g_T(\min S), & x \in [\min T, \min S], \\ g_T(x) & \text{for } x \in [\min S, \max S], \\ g_T(\max S), & x \in [\max S, \max T]. \end{cases}$$

Then since B forces A and the map h exhibits B it has to exhibit also A, i.e., we can consider some $\widehat{T} \subset \operatorname{conv} S$ such that $\langle \widehat{T}, h\rangle = \langle \widehat{T}, g_T\rangle \in A$. Since $T \neq \widehat{T}$, this is impossible by Lemma 5.16(ii). This implies $A = B$.

Case IV: *A, B basic, piecewise monotone.* Using Theorem 6.2 we can fix $\langle T = [a_1, b_1] \cup \cdots \cup [a_k, b_k], g\rangle \in A \cap \mathcal{NM}_I$ that has a block structure over a cycle $\langle S = \{s_1 < \cdots < s_k\}, f\rangle$ where $s_i \in [a_i, b_i]$ for each $i \in \{1, \ldots, k\}$

and $f = g|S$. The map g_T exhibits B, i.e., $\langle R, g_T\rangle \in B$ for some $R \subset \operatorname{conv} T$. Without loss of generality we can assume that $\operatorname{Tran}\langle R, g_T\rangle \cap T = \emptyset$ (otherwise we could start from B and use the fact that both patterns A, B are nonfractal).

Now we define a continuous nondecreasing map $\tau\colon \operatorname{conv} T \to \operatorname{conv} S$ such that

$$\tau([a_i, b_i]) = s_i, \quad (\tau|J \text{ is constant } \Leftrightarrow \exists n \in \mathbb{N}_0\colon g_T^n(J) \subset T).$$

Such a map exists since $\operatorname{Tran}\langle R, g_T\rangle \cap T = \emptyset$ and $\operatorname{conv} T \setminus \bigcup_{k\in\mathbb{N}_0} g_T^{-k}(T^\circ)$ is perfect. Using τ, we can find (see [1, Lemma 4.6]) a map $\varrho \in C(\operatorname{conv} S)$ satisfying

$$\tau \circ g_T = \varrho \circ \tau \quad \text{on } \operatorname{conv} T.$$

Clearly, $\langle \tau(T) = S,\ \varrho|\tau(T) = f\rangle \in \mathcal{P}$ and $\varrho \in C\langle S, f\rangle$. Since $R \subset \operatorname{conv} T \setminus \bigcup_{k\in\mathbb{N}_0} g_T^{-k}(T^\circ)$ and $\tau|R$ is strictly monotone, by Lemma 5.4 there is a set $R^\star \subset \operatorname{conv} S$ for which $\langle R, g_T\rangle \sim \langle R^\star, \rho\rangle \in B$. Then since $S \cap \operatorname{Tran}\langle R^\star, \rho\rangle = \emptyset$, from Lemma 5.10 we deduce that f_S exhibits B irreducibly. By Theorem 2.4, our assumption $A \hookrightarrow B$, $B \hookrightarrow A$, and Lemma 5.16(iii), this implies that $[\langle S, f\rangle]_\sim \hookrightarrow C$ and $C \hookrightarrow [\langle S, f\rangle]_\sim$ for $C \in \{A, B\}$.

Now, from Theorem 2.4 we know that f_S exhibits the patterns A, B; let $\langle U, u = f_S|U\rangle$, $\langle V, v = f_S|V\rangle$ be representatives of A, B respectively. If $\min S = \min U = \min V$ then since $f_S = u_U = v_V$, Lemma 5.17 implies $A = B$. Without loss of generality assume that $\min S < \min V$. Then by the above, v_V exhibits the pattern $[\langle S, f\rangle]_\sim$ and by Lemma 5.6 we can consider a set $S^\star \subset \operatorname{conv} V$ such that $\langle S^\star, f_S\rangle \sim \langle S, f\rangle$, which contradicts Lemma 5.16(ii).

Thus, $A = B$. ∎

4. Bottom patterns. In this section we explain how the forcing relation on roof patterns relates to the results on the forcing relation on periodic and minimal patterns known from the literature [1], [5].

For a system $\langle S, f\rangle$ we can consider the set

$$T = \overline{\bigcup\{\widetilde{T}\colon S \subset \widetilde{T} \subset \operatorname{conv} S \text{ and } \langle \widetilde{T}, f_S\rangle \text{ is a transitive system}\}}. \tag{4.1}$$

For example, if $\langle S, f\rangle$ itself is transitive then $T \neq \emptyset$. We will say that $\langle S, f\rangle$ is *supporting* if the set T defined in (4.1) is nonempty. Then $\langle T, f_S\rangle$ is a roof system (see Lemma 5.17) and we will denote it by $\uparrow\langle S, f\rangle$ (we put $\uparrow\langle S, f\rangle = \emptyset$ if $\langle S, f\rangle$ is not supporting). As usual, a *proper subsystem* of a system $\langle S, f\rangle$ is a system $\langle S_1, f_1\rangle$ such that $S_1 \subsetneq S$ and $f_1 = f|S_1$. We start with the following definition.

4.1. Definition. A supporting system $\langle S, f\rangle$ is a *bottom system* if there is no proper subsystem $\langle S_1, f_1\rangle$ of $\langle S, f\rangle$ such that $\uparrow\langle S_1, f_1\rangle \sim \uparrow\langle S, f\rangle$. The set of all bottom systems will be denoted by $\mathfrak{B}\mathfrak{S}$.

Bottom systems $\langle S_1, f_1\rangle, \langle S_2, f_2\rangle$ are *equivalent* (we write $\langle S_1, f_1\rangle \bowtie \langle S_2, f_2\rangle$) if $\uparrow\langle S_1, f_1\rangle \sim \uparrow\langle S_2, f_2\rangle$.

A *bottom pattern* is a corresponding equivalence class in $\mathfrak{BS}_{\bowtie}$. We denote by $[\langle S, f\rangle]_{\bowtie}$ a bottom pattern from $\mathfrak{BS}_{\bowtie}$ with representative $\langle S, f\rangle$. If a map $f \in C(\widetilde{S})$ has a bottom system $\langle S, f\rangle$ then we also say that f exhibits a bottom pattern $[\langle S, f\rangle]_{\bowtie}$.

The set $\mathfrak{BS}$ of bottom systems seems to be of independent interest. In the next lemma we state some of its basic properties.

4.2. Lemma.

(i) $\mathcal{P} \cup \mathcal{M} \subset \mathfrak{BS}$.
(ii) $\mathfrak{BS} \cap \mathcal{NM} \neq \emptyset$, $\mathcal{NM} \setminus \mathfrak{BS} \neq \emptyset$.
(iii) $\mathfrak{BS} \setminus (\mathcal{P} \cup \mathcal{M} \cup \mathcal{NM}) \neq \emptyset$.

Proof. (i) follows directly from Definition 4.1.

(ii) To see $\mathfrak{BS} \cap \mathcal{NM} \neq \emptyset$, one can consider a transitive interval map $f\colon [0,1] \to [0,1]$ such that for some $c \in (0,1)$, $f(c) = 1$, $f|[0,c]$, resp. $f|[c,1]$ is increasing, resp. decreasing (f is unimodal), and 0 is a transitive point. Then $\langle S = [0,1],\ f\rangle$ is a supporting (transitive) system and any proper subsystem $\langle S_1, f_1\rangle$ of $\langle S, f\rangle$ has to satisfy $0 < \min S_1 \leq \max S_1 < 1$, hence $\uparrow\langle S_1, f_1\rangle \nsim \uparrow\langle S, f\rangle$ (see [7]). The set $\mathcal{NM}\setminus\mathfrak{BS}$ is nonempty since it contains the system $\langle S = [0,1],\ f = 1 - |1 - 2\,\mathrm{id}|\rangle$ (the transitive full tent map on the unit interval). Indeed, the system $\langle S_1, f_1\rangle$ defined by $S_1 = \{0, 1/2, 1\}$, $f_1(0) = 0$, $f_1(1/2) = 1$, $f_1(1) = 0$ is a proper subsystem and $\uparrow\langle S_1, f_1\rangle = \langle S, f\rangle = \uparrow\langle S, f\rangle$.

In order to prove (iii), consider the systems $\langle S, f\rangle$, $\langle S_1, f_1\rangle$ as above. Then $\uparrow\langle S_1, f_1\rangle = \langle S, f\rangle$, hence the system $\langle S_1, f_1\rangle$ is supporting. Since there is only one proper supporting subsystem $\langle \{0\}, f|\{0\}\rangle$ of $\langle S_1, f_1\rangle$ and $\uparrow\langle S_1, f_1\rangle \nsim \uparrow\langle \{0\}, f|\{0\}\rangle = \langle \{0\}, f|\{0\}\rangle$, we conclude that $\langle S_1, f_1\rangle \notin \mathcal{P} \cup \mathcal{M} \cup \mathcal{NM}$ is a bottom system according to Definition 4.1. ∎

By the previous lemma any periodic or minimal system is a bottom system. Now we show that our approach preserves "classical" periodic and minimal patterns corresponding to the relation $\sim$ (see [1], [5]).

4.3. Lemma. *If* $\langle S, q\rangle \in \mathcal{P} \cup \mathcal{M}$ *then* $[\langle S, q\rangle]_{\bowtie} = [\langle S, q\rangle]_{\sim}$.

Proof. Let $\langle S, q\rangle$ be a roof system, i.e., $\langle S, q\rangle = \uparrow\langle S, q\rangle$. Fix any $\langle R, p\rangle \in [\langle S, q\rangle]_{\bowtie}$. Then $\langle S, q\rangle \sim \uparrow\langle R, p\rangle$, hence the roof system $\uparrow\langle R, p\rangle$ is in $\mathcal{P} \cup \mathcal{M}$. But then $\langle R, p\rangle = \uparrow\langle R, p\rangle$ and $\langle R, p\rangle \in [\langle S, q\rangle]_{\sim}$.

If $\langle R, p\rangle \in [\langle S, q\rangle]_{\sim}$ then Lemma 5.3 implies that $\langle R, p\rangle \in \mathcal{P} \cup \mathcal{M}$. Moreover, our assumption $\langle S, q\rangle = \uparrow\langle S, q\rangle$ and Lemmas 5.12, 5.13 and 5.18 give $\langle R, p\rangle \sim \uparrow\langle R, p\rangle$, hence $\langle R, p\rangle \in [\langle S, q\rangle]_{\bowtie}$. Thus, $[\langle S, q\rangle]_{\sim} = [\langle S, q\rangle]_{\bowtie}$ in this case.

Suppose $\langle S,q\rangle$ is not a roof system, i.e., $\langle S,q\rangle \neq \uparrow\langle S,q\rangle$. If $\langle R,p\rangle \in [\langle S,q\rangle]_\sim$ then by Lemma 5.3, $\langle R,p\rangle \in \mathcal{P}\cup\mathcal{M}$ and from Lemma 3.2 we know that also $\langle R,p\rangle \neq \uparrow\langle R,p\rangle$. Using Lemma 5.15 we deduce that $\uparrow\langle R,p\rangle = \langle T,p_R\rangle \in \mathcal{NM}$ is a basic system. Since $R \subsetneq T$, by Lemma 5.10 there is a set $T^\star \subset \operatorname{conv} S$ such that $\langle T,p_R\rangle \sim \langle T^\star,q_S\rangle$. Note that by Lemma 5.2(iv)–(vi) the point $t=\min T^\star$ is a strongly q_S-recurrent point and $\langle\omega(q_S,t^\star),q_S\rangle \sim \langle R,p\rangle \sim \langle S,q\rangle$.

Let us show that $\langle\omega(q_S,t^\star),q_S\rangle = \langle S,q\rangle$. If $\langle S,q\rangle$ is piecewise monotone then this fact follows directly from Lemma 5.16(ii). If $\langle S,q\rangle$ is not piecewise monotone then $\langle T,p_R\rangle$, $\langle T^\star,q_S\rangle$ are not piecewise monotone either. In particular, $T^\star$ is contained in infinitely many S-contiguous intervals. This means that the distance between the compact sets S and $T^\star$ is zero and $S \subset T^\star$. Since by Lemma 3.2, the system $\langle T^\star,q_S\rangle$ is a roof system, we obtain $\uparrow\langle R,p\rangle = \langle T,p_R\rangle \sim \langle T^\star,q_S\rangle = \uparrow\langle S,q\rangle$, i.e., $\langle R,p\rangle \in [\langle S,q\rangle]_\bowtie$.

Assume $\langle R,p\rangle \in [\langle S,q\rangle]_\bowtie$; we will show that $\langle R,p\rangle \in [\langle S,q\rangle]_\sim$. From Lemma 5.2(iv)–(vi) we get $\langle\omega(p_R,\min R),p_R\rangle \sim \langle S,q\rangle$. We have proved above that then also $\langle\omega(p_R,\min R),p_R\rangle \bowtie \langle S,q\rangle$. Since $\omega(p_R,\min R)\subset R$ and $\langle R,p\rangle$ is a bottom system, we have $\omega(p_R,\min R)=R$ and $\langle R,p\rangle \sim \langle S,q\rangle$.

This proves the lemma. ∎

In order to define the forcing relation on bottom patterns we use an analogous definition to that for transitive patterns.

4.4. Definition. A bottom pattern A *forces* a bottom pattern B (we write $A \rightharpoondown B$) if all maps in $\mathfrak{C}$ exhibiting A also exhibit B.

As a consequence of Lemma 4.3 we obtain

4.5. Lemma. *Let $\langle R,p\rangle,\langle S,q\rangle \in \mathcal{P}\cup\mathcal{M}$. The following statements are equivalent.*

(i) $[\langle R,p\rangle]_\bowtie \rightharpoondown [\langle S,q\rangle]_\bowtie$.
(ii) $[\langle R,p\rangle]_\sim \hookrightarrow [\langle S,q\rangle]_\sim$.

Proof. By Lemma 4.3, $[\langle R,p\rangle]_\bowtie = [\langle R,p\rangle]_\sim$ and $[\langle S,q\rangle]_\bowtie = [\langle S,q\rangle]_\sim$. Since Definitions 2.2 and 4.4 coincide, the equivalence (i)⇔(ii) follows. ∎

4.6. Lemma. *Let $\langle A,\alpha\rangle$ be a bottom system and $f\in\mathfrak{C}$. If f exhibits $[\langle A,\alpha\rangle]_\bowtie$ then it also exhibits $[\uparrow\langle A,\alpha\rangle]_\sim$.*

Proof. Suppose f has a system $\langle B,q\rangle \in [\langle A,\alpha\rangle]_\bowtie$, and set $\langle S,q_B\rangle = \uparrow\langle B,q\rangle$. We know that $B\subset S$ and $\langle S,q_B\rangle \in [\uparrow\langle A,\alpha\rangle]_\sim$. The conclusion is clear when $B=S$. Assume that $B\subsetneq S$. Since $\mathcal{B}_{S,S}(B)=B$, Lemma 5.10 for $\widetilde{q}=f|[\min B,\max B]$ yields a set $T^\star\subset[\min B,\max B]$ such that $\langle T^\star,f\rangle \sim \langle S,q_B\rangle$. Then from Lemma 3.2 we get $\langle T^\star,f\rangle \in [\uparrow\langle A,\alpha\rangle]_\sim$, i.e., f exhibits $[\uparrow\langle A,\alpha\rangle]_\sim$. ∎

4.7. Definition. A bottom system $\langle S, f\rangle$ is said to be *fractal*, resp. *nonfractal* if the system $\uparrow\langle S, f\rangle$ is fractal, resp. nonfractal.

It follows from Lemma 3.4 that we can also talk about fractal, resp. nonfractal bottom patterns. If A is a bottom pattern then using Lemma 3.2 we put

$$\uparrow A = \{\uparrow\langle S, q\rangle : \langle S, q\rangle \in A\}.$$

As a consequence of Theorem 3.5 and Lemma 4.6 one can prove

4.8. Theorem. *The forcing relation on nonfractal bottom patterns is a partial ordering.*

Proof. In this proof we say briefly "pattern" instead of "nonfractal bottom pattern".

Clearly, if A is a pattern, then $A \rightharpoondown A$ (reflexivity); if A, B, C are patterns such that $A \rightharpoondown B$ and $B \rightharpoondown C$, then $A \rightharpoondown C$ (transitivity). Thus it remains to prove the weak antisymmetry of the forcing relation.

Suppose that $A \rightharpoondown B$ and $B \rightharpoondown A$. We will show that also $\uparrow A \hookrightarrow \uparrow B$ and $\uparrow B \hookrightarrow \uparrow A$. Then Theorem 3.5 yields $\uparrow A = \uparrow B$, hence also $A = B$.

By our assumption on forcing of A, B and by Lemma 4.6, for any representative $\langle T, g\rangle \in A$, the map g_T exhibits B (in addition to A, of course) and the roof patterns $\uparrow A$, $\uparrow B$. Moreover, from Definition 4.1 and Lemma 5.17 it follows that there exists a set $T^\star$ with $T \subset T^\star \subset \operatorname{conv} T$ such that $\langle T^\star, r = g_T | T^\star\rangle \in \uparrow A$. Since $g_T = r_{T^\star}$, we can use Theorem 2.4(iii). It states that if $r_{T^\star}$ exhibits the pattern $\uparrow B$ irreducibly (for example, when B is not a 2-extension) then $\uparrow A \hookrightarrow \uparrow B$. We will use this argument to show that $\uparrow A \hookrightarrow \uparrow B$, resp. $\uparrow B \hookrightarrow \uparrow A$. To simplify the writing define the set of periodic patterns $\mathrm{EX}_2 = \{A : A \text{ is a 2-extension}\}$ (see Lemma 4.3).

Case I: $A \notin \mathrm{EX}_2$, $B \notin \mathrm{EX}_2$. By the above, the assumption $A \rightharpoondown B$ and $B \rightharpoondown A$ gives $\uparrow A \hookrightarrow \uparrow B$ and $\uparrow B \hookrightarrow \uparrow A$. Then Theorem 3.5 implies $\uparrow A = \uparrow B$ and hence also $A = B$.

Case II: $A, B \in \mathrm{EX}_2$. The conclusion follows directly from Lemmas 5.12, 4.5 and Theorem 6.4.

Case III: $A = [\langle T, g\rangle]_{\bowtie} \in \mathrm{EX}_2$, $B = [\langle U, h\rangle]_{\bowtie} \notin \mathrm{EX}_2$. By the above, $\uparrow A \hookrightarrow \uparrow B$. We will show that also $\uparrow B \hookrightarrow \uparrow A$. Let $\langle T = \{t_1 < \cdots < t_{2k}\}, g\rangle$ be a 2-extension over a cycle $\langle R, p\rangle$. As explained above, the map g_T exhibits, resp. irreducibly exhibits, the pattern B, resp. the roof pattern $\uparrow B$, and $\uparrow A \hookrightarrow \uparrow B$. Let $\langle S, q = g_T | S\rangle \in \uparrow B$ for some infinite $S \subset \operatorname{conv} T$, and let $U^\star \subset \operatorname{conv} U$ be such that $U \subset U^\star \subset \operatorname{conv} U$ and $\langle U^\star, i = h_U | U^\star\rangle \in \uparrow B$. As above, if $i_{U^\star}$ exhibits $\uparrow A$ irreducibly, then by Theorem 2.4, $\uparrow B \hookrightarrow \uparrow A$ and we are done. So assume that $\langle V, i_{U^\star}\rangle \in \uparrow A$ is a reducible system of $i_{U^\star}$. Using Lemma 5.16(iv) we can assume that $V \subset U^\star$. Then by Lemma 5.5(ii) there is a set $S_0 \subset S$ such that exactly one of the following two possibilities holds: either $\langle S_0, q\rangle \sim \langle T, g\rangle$ or $\langle S_0, q\rangle \sim \langle R, p\rangle$. Since

$$S \cap \bigcup_{i=1}^{k} [t_{2i-1}, t_{2i}] = S_0 \cap \bigcup_{i=1}^{k} [t_{2i-1}, t_{2i}] = \emptyset, \tag{4.2}$$

the first possibility contradicts Lemma 5.16(ii). Again by (4.2), the second one is impossible because of Lemma 5.16(v).

Thus, also in this case from $A \rightharpoondown B$ and $B \rightharpoondown A$ we get $\uparrow A \hookrightarrow \uparrow B$ and $\uparrow B \hookrightarrow \uparrow A$. By Theorem 3.5, $\uparrow A = \uparrow B$, which implies $A = B$. ∎

In order to show that our Theorem 4.8 generalizes Theorem 6.4 we need to prove

4.9. Theorem. *For any system $\langle S, f\rangle \in \mathcal{P} \cup \mathcal{M}$, the roof pattern $[\uparrow\langle S, f\rangle]_\sim$ is nonfractal.*

Proof. By Lemma 3.4 it is sufficient to show that the roof system $\uparrow\langle S, f\rangle$ is nonfractal. This holds trivially when $\uparrow\langle S, f\rangle = \langle S, f\rangle$.

Suppose $\uparrow\langle S, f\rangle = \langle T, g\rangle \neq \langle S, f\rangle$. Then $\langle T, g\rangle \in \mathcal{NM}$ is a basic roof system (see Theorem 6.1). Let $\widetilde{T} \subsetneq T$ be such that $\langle \widetilde{T}, g\rangle$ is transitive and $\langle \widetilde{T}, g\rangle \sim \langle T, g\rangle$. By Lemmas 5.18 and 5.15, $\langle \widetilde{T}, g\rangle$ is also a basic roof system.

We show that $\widetilde{T} \cap S = \emptyset$. Indeed, otherwise $f_S = g_T = g_{\widetilde{T}}$ and by Lemma 5.17, $T = \widetilde{T}$, a contradiction. Notice that by Lemma 5.2(iv)–(vi), the point $t = \min \widetilde{T}$ is strongly g_T-recurrent and

$$\langle R = \omega(g_T, t), g_T\rangle \sim \langle S, f\rangle, \qquad S \cap R = \emptyset. \tag{4.3}$$

The last property (4.3) is impossible for piecewise monotone $\langle S, f\rangle$ by Lemma 5.16(ii). If $\langle S, f\rangle$ is not piecewise monotone, then $\langle R, g_T\rangle$ is not piecewise monotone either and the set R has to be contained in infinitely many S-contiguous intervals. Then the distance of the closed sets S, R is zero, which contradicts $S \cap R = \emptyset$ again.

Thus, the system $\uparrow\langle S, f\rangle = \langle T, g\rangle$ is nonfractal. ∎

4.10. Remark. It would be of interest to describe in detail the properties of bottom systems. We conjecture that any bottom system according to our Definition 4.1 is in fact nonfractal.

5. Technical results. For two closed sets $K, L \subset \mathbb{R}$ we write

$$K < L \ \Leftrightarrow\ \max K < \min L \tag{5.1}$$

(and analogously $K \leq L$ iff $\max K \leq \min L$).

We will need a generalized version of (2.1). Recall that $C(T)$ denotes the set of all continuous functions that map a nonempty compact set T into itself. If T has empty interior then any closed subinterval of T consists of a single point.

5.1. DEFINITION. Let $f_j \in C(T_j)$, $j \in \{1, 2\}$. Assume there are closed (maybe one-point) intervals $K^j \subset T_j$ such that if we set $K_i^j = f_j^i(K^j)$, $i \in \mathbb{N}_0$, then

(i) K_i^j is a point or a closed interval,
(ii) for $i(1) \neq i(2)$ either $K_{i(1)}^j \cap K_{i(2)}^j = \emptyset$ or $K_{i(1)}^j = K_{i(2)}^j$.

We say that the orbits $\mathrm{orb}(f_1, K^1)$, $\mathrm{orb}(f_2, K^2)$ *have the same order* if for any $i(1), i(2) \in \mathbb{N}_0$,

$$K_{i(1)}^1 < K_{i(2)}^1 \;\Leftrightarrow\; K_{i(1)}^2 < K_{i(2)}^2.$$

We denote by $\mathrm{Exp}\, X$ the set of all subsets of a set X. For two equivalent systems $\langle T, g\rangle, \langle S, f\rangle \in \mathfrak{T}$ with $\mathrm{Tran}\langle T, g\rangle \ni x_T \leftrightarrow y_S \in \mathrm{Tran}\langle S, f\rangle$ we define a set operator $\mathcal{B}_{T,S}\colon \mathrm{Exp}\, T \to \mathrm{Exp}\, S$ by (we write $u_n \rightsquigarrow u$ if $\lim_n u_n = u$ and $\{u_n\}_n$ is monotone)

$$\mathcal{B}_{T,S}(R) = \{f^{m(n)}(y_S)\colon g^{m(n)}(x_T) \rightsquigarrow x \in R\}, \qquad R \in \mathrm{Exp}\, T.$$

For a map $f \in C(T)$, a point $x \in T$ is called *f-recurrent*, resp. *strongly f-recurrent* if $x \in \omega(f, x)$, resp. x is f-recurrent and $\langle \omega(f, x), f\rangle$ is minimal. The set of all strongly f-recurrent points will be denoted by $\mathrm{Min}(f)$. The following lemma can be left to the reader as an exercise.

5.2. LEMMA. *Let $\langle T, g\rangle \sim \langle S, f\rangle$, $u \in T$ and $v \in \mathcal{B}_{T,S}(\{u\})$.*

(i) $\mathrm{card}\{g^n(u)\colon n \in \mathbb{N}_0\} = \infty$ *iff* $\mathrm{card}\{f^n(v)\colon n \in \mathbb{N}_0\} = \infty$.
(ii) *If* $\mathrm{card}\{g^n(u)\colon n \in \mathbb{N}_0\} = \infty$ *then the orbits* $\mathrm{orb}(g, u)$, $\mathrm{orb}(f, v)$ *have the same order.*
(iii) $u \in \mathrm{Tran}\langle T, g\rangle$ *iff* $v \in \mathrm{Tran}\langle S, f\rangle$.
(iv) $u = \min T$ *iff* $v = \min S$.
(v) *If* $u = \min T \in \mathrm{Per}(g)$ *then* $v \in \mathrm{Per}(f)$ *and*

$$\langle \mathrm{orb}(g, u), g\rangle \sim \langle \mathrm{orb}(f, v), f\rangle.$$

(vi) *If* $u = \min T \in \mathrm{Min}(g)$ *then* $v \in \mathrm{Min}(f)$ *and*

$$\langle \omega(g, u), g\rangle \sim \langle \omega(f, v), f\rangle.$$

From Lemma 5.2 we obtain

5.3. LEMMA. *Let $\langle T, g\rangle \sim \langle S, f\rangle$. Then $\langle T, g\rangle$, $\langle S, f\rangle$ belong to the same element of $\{\mathcal{P}, \mathcal{M}, \mathcal{NM}\}$.*

In order to study transitive systems we need a method to recognize that a fixed map $f \in C(I)$ has such a system of prescribed order. The following lemmas will be helpful.

5.4. Lemma. *Let* $f \in C(\widetilde{T})$ *and* $\langle T, g\rangle \in \mathfrak{T}$. *Assume there is a* $K_0 \subset \widetilde{T}$ *such that* (i) $K_0 \subset \widetilde{T}$ *is a closed interval* (*maybe degenerate*), (ii) $K_i = f^i(K_0)$ *for each* $i \in \mathbb{N}_0$ *and for some* $t \in \operatorname{Tran}\langle T, g\rangle$ *the orbits* $\operatorname{orb}(f, K_0)$, $\operatorname{orb}(g, t)$ *have the same order. Then there is an* f*-recurrent point* $t^\star \in \widetilde{T}$ *such that for* $T^* = \omega(f, t^*)$ *we have* $\langle T^*, f\rangle \sim \langle T, g\rangle$. *Moreover*, *if* $\operatorname{orb}(g, t)$ *is infinite and a sequence* $g^{m(n)}(t)$ *decreases to* t *then we can put* $t^\star = \inf \bigcup_n K_{m(n)}$, *hence* $\max K_0 \le t^\star$.

Proof. The conclusion is well known when $\langle T, g\rangle \in \mathcal{P}$ (see [1]). The case when $\widetilde{T}$ is an interval and $\langle T, g\rangle \in \mathcal{M}$ was proven in [5, Lemma 2.2]. All other possibilities can be handled in the same manner. ∎

We write $u_n \rightsquigarrow u$, $u_n \nearrow u$, $u_n \searrow u$ if $\lim_n u_n = u$ and $\{u_n\}_n$ is monotone, increasing, decreasing respectively.

5.5. Lemma. *Let* $\langle T, g\rangle \sim \langle S, f\rangle$.

(i) *For any set* $T_0 \subset T$ *such that* $\langle T_0, g\rangle \in \mathfrak{T}$ *is not a reducible system of* g_T *there is a set* $S_0 \subset \operatorname{conv} S$ *for which* $\langle S_0, f_S\rangle \in \mathfrak{T}$ *and* $\langle T_0, g\rangle \sim \langle S_0, f_S\rangle$.

(ii) *Let* $T_0 \subset T$ *satisfy*

— $\langle T_0, g\rangle$ *is a* 2*-extension of a cycle* $\langle R, p\rangle$,
— $\langle T_0, g\rangle$ *is a reducible system of* g_T.

There is a set $S_0 \subset S$ *for which either* $\langle S_0, f\rangle \sim \langle T_0, g\rangle$ *or* $\langle S_0, f\rangle \sim \langle R, p\rangle$.

Proof. (i) Fix $u \in \operatorname{Tran}\langle T_0, g\rangle$ and $v \in \mathcal{B}_{T,S}(\{u\})$.

Let $\langle T_0, g\rangle \in \mathcal{M} \cup \mathcal{NM}$. By Lemma 5.2(ii) the orbits $\operatorname{orb}(g, u)$ and $\operatorname{orb}(f, v)$ have the same order. Now the conclusion follows from Lemma 5.4.

Assume that $\langle T_0, g\rangle \in \mathcal{P}$ and $T_0 \subsetneq T$ (the case when $T_0 = T$ is trivial). Then $\langle T, g\rangle \in \mathcal{NM}$ and $u \in T_0$ is a periodic point of period $k \in \mathbb{N}$. Obviously u is a limit point of T. Let $\operatorname{Tran}\langle T, g\rangle \ni x_T \leftrightarrow y_S \in \operatorname{Tran}\langle S, f\rangle$. Without loss of generality we can assume that for an increasing sequence $\{m(n)\}_n$, $g^{m(n)}(x_T) \nearrow u$, $g^{m(n)+k}(x_T) \rightsquigarrow u$ and $f^{m(n)}(y_S) \nearrow v \in S$.

Put $v_i = \lim_n f^{m(n)+i}(y_S)$, $i \in \{1, \ldots, k-1\}$. Suppose that $i_0 \in \{1, \ldots, k-1\}$ is the least for which $v = v_{i_0}$. Then the cycle $\langle T_0, g\rangle$ has a block structure with the block $T_0 \cap \operatorname{conv}\{u, u_{i_0}\}$, and any T_0-block is a subset of a T-contiguous interval. By Lemma 5.16(iv), the cycle $\langle T_0, g\rangle$ is a reducible system of g_T, a contradiction. Thus, $v \ne v_i = \lim_n f^{m(n)+i}(y_S)$ for any $i \in \{1, \ldots, k-1\}$.

If $\lim_n f^{m(n)+k}(y_S) = v$ then v is a periodic point of period k and

$$\langle S_0 = \operatorname{orb}(f, v), f), f\rangle \sim \langle T_0, g\rangle.$$

In the case when $\lim_n f^{m(n)+k}(y_S) = w \ne v$, from $f^{m(n)}(y_S) \nearrow v$ it follows that $v < w$, the interval $[v, w]$ is an S-contiguous interval and the orbits $\mathrm{orb}(g, u)$ and $\mathrm{orb}(f_S, [v, w])$ have the same order. Now the existence of $\langle S_0, f_S\rangle \in \mathfrak{T}$ satisfying $\langle S_0, f_S\rangle \sim \langle T_0, g\rangle$ follows from Lemma 5.4.

(ii) Let $\operatorname{card} T_0 = 2k$ and let $\{t_0 < t_1\}$ be the leftmost block of $\langle T_0, g\rangle$. By our assumption, $[t_0, t_1]$ is a T-contiguous interval and $\mathcal{B}_{T,S}(\{t_0\}) = \{s_0\}$, $\mathcal{B}_{T,S}(\{t_1\}) = \{s_1\}$. If $s_0 < s_1$ then $[s_0, s_1]$ is an S-contiguous interval and

$$\Big\langle S_0 = \bigcup_{i=0}^{k-1} f^i(\{s_0, s_1\}), f\Big\rangle \sim \langle T_0, g\rangle.$$

If $s_0 = s_1$, we get $\langle S_0 = \bigcup_{i=0}^{k-1} f^i(\{s_0\}), f\rangle \sim \langle R, p\rangle$. ∎

5.6. Lemma. *Let $f \in C(I)$, $S \subset I$ be closed such that $f(S) \subset S$, and put $q = f|S$. Then for any $t' \in \mathrm{Per}(q_S)$ there is a $t^\star \in \mathrm{Per}(f) \cap \operatorname{conv} S$ such that $\langle \mathrm{orb}(q_S, t'), q_S\rangle \sim \langle \mathrm{orb}(f, t^\star), f\rangle$.*

Proof. See [6, Th. 3.12]. ∎

As before, I denotes a compact real subinterval of $\mathbb{R}$.

5.7. Definition. Let $f\colon I \to \mathbb{R}$ be a continuous map and $[x, y] \subset I$. We define

$$\mathrm{sign}_f([x, y]) = \begin{cases} +1, & f(x) < f(y), \\ -1, & f(x) > f(y). \end{cases}$$

5.8. Lemma. *Let $f\colon I \to \mathbb{R}$ be a continuous map, $[a, b] \subset I$, $[c, d] \subset \mathbb{R}$, $f(a) \neq f(b)$ and*

$$\operatorname{conv}\{f(a), f(b)\} \supset [c, d].$$

There are $a^*, b^* \in [a, b]$ such that $f([a^*, b^*]) = [c, d]$, $f(\{a^*, b^*\}) = \{c, d\}$ and $\mathrm{sign}_f([a^*, b^*]) = \mathrm{sign}_f([a, b])$.

Proof. If $f(a) > f(b)$ put

$$a^* = \sup\{x \in [a, b] : f(x) = d\}, \qquad b^* = \inf\{x \in [a^*, b] : f(x) = c\}.$$

The second case is similar. ∎

5.9. Remark. For $\langle T, g\rangle \in \mathfrak{T}$, the set $\mathrm{Tran}\langle T, g\rangle$ is a dense G_δ set in the compact metric space T equipped by the Euclidean metric. Using this fact and the classification of Section 2.1 we infer that for $\langle T, g\rangle \in \mathcal{M} \cup \mathcal{NM}$ and $U \subset T$ countable we can consider a point $t \in \mathrm{Tran}\langle T, g\rangle$ such that $\mathrm{orb}(g, t) \cap U = \emptyset$.

For a system $\langle R, p\rangle$, a map $r \in C(\operatorname{conv} R)$ is said to be $\langle R, p\rangle$-*monotone* if $r|R = p$ and $r|J$ is monotone for any interval $J \subset \operatorname{conv} R$ such that $J \cap R = \emptyset$. We write $C\langle R, p\rangle$ for the set of all $\langle R, p\rangle$-monotone maps. In particular, $p_R \in C\langle R, p\rangle$.

As before, a *subsystem* of a system $\langle R, p\rangle$ is a system $\langle A, \alpha\rangle$ such that $A \subset R$ and $\alpha = f|A$.

5.10. Lemma. *Let $\langle A,\alpha\rangle$ be a system. Assume that for some $r\in C\langle A,\alpha\rangle$ and a set $T\subset \operatorname{conv} A$,*

- $\langle T,r\rangle \in \mathcal{M}\cup\mathcal{NM}$,
- *there is a point $t\in \operatorname{Tran}\langle T,r\rangle$ satisfying $\operatorname{orb}(r,t)\cap A=\emptyset$.*

Assume that $\langle A,\alpha\rangle$ is a subsystem of a transitive system $\langle R,p\rangle \sim \langle S,q\rangle$, and let $\langle B,q\rangle$ be a subsystem of $\langle S,q\rangle$ such that $\mathcal{B}_{S,R}(B)=A$. Then for any continuous map $\widetilde{q}\colon [\min B,\max B]\to\mathbb{R}$ satisfying $\widetilde{q}|B=q$ there exists a set $T^\star\subset[\min B,\max B]$ such that $\langle T^\star,\widetilde{q}\rangle\sim\langle T,r\rangle$ and $T^\star\setminus B\neq\emptyset$.

Proof. An A-contiguous interval L (in $\operatorname{conv} A$) will be called *active* if $r^j(t)\in L^\circ$ for some $j\in\mathbb{N}_0$. Obviously for any active interval L, the map $r|L$ is not constant and there is an $n\in\mathbb{N}$ for which

$$r^n(L^\circ)\cap L^\circ\neq\emptyset. \tag{5.2}$$

Let $\{L_i^A\}_{i\in\mathbb{N}}$ consist of all active closed A-contiguous intervals and define $\{L_i^B\}_{i\in\mathbb{N}}$ as follows: if $L_i^A=[u_A,v_A]$ then $L_i^B=[u_B,v_B]$ satisfies

$$u_B,v_B\in B,\qquad (u_B,v_B)\cap B=\emptyset,\qquad \mathcal{B}_{S,R}(\{u_B,v_B\})=\{u_A,v_A\}. \tag{5.3}$$

Note that L_i^B is well defined since $\mathcal{B}_{S,R}(B)=A$. Moreover, $u_B<v_B$. Indeed, otherwise by (5.2), (5.3), the point u_B would be periodic (of period n, say), the intervals $L_i^A,\dots,r^{n-1}(L_i^A)$ would be pairwise disjoint closed A-contiguous intervals, $r^n(L_i^A)=L_i^A$ and $r^n|L_i^A$ would be monotone and by our assumption also $\langle L_i^A\cap T,r^n\rangle\in\mathcal{M}\cup\mathcal{NM}$, a contradiction.

Since $\langle A,\alpha\rangle$ is a system, we have the implication

(i) $r(L_{i(1)}^A)\cap[L_{i(2)}^A]^\circ\neq\emptyset \;\Rightarrow\; r(L_{i(1)}^A)\supset L_{i(2)}^A$.

Our choice of $\{L_i^A\}_{i\in\mathbb{N}}$, $\{L_i^B\}_{i\in\mathbb{N}}$ implies, for each i and $i(1)\neq i(2)$,

(ii) $L_i^A\subset\operatorname{conv} A$ and $L_i^B\subset\operatorname{conv} B$,
(iii) $L_{i(1)}^A\le L_{i(2)}^A$ iff $L_{i(1)}^B\le L_{i(2)}^B$,
(iv) $r(L_{i(1)}^A)\supset L_{i(2)}^A\Rightarrow\widetilde{q}(L_{i(1)}^B)\supset L_{i(2)}^B$ (in particular when $\widetilde{q}=q_B$).

We have shown above that each L_i^B is nondegenerate. Using this fact and (iv) one can see that $q_B|L_i^B$ is not constant and

(v) $\operatorname{sign}_r(L_i^A)=\operatorname{sign}_{\widetilde{q}}(L_i^B)$.

We assume that $\operatorname{orb}(r,t)\cap A=\emptyset$. Define the map $\widetilde{\pi}\colon \operatorname{orb}(r,t)\times\mathbb{N}_0\to\mathbb{N}$ and $\pi=\widetilde{\pi}|(\{t\}\times\mathbb{N}_0)$ by

$$\widetilde{\pi}(s,i)=j\quad\text{ if } r^i(s)\in L_j^A,\qquad \pi(i)=\widetilde{\pi}(t,i).$$

Set $I_i^1=L_{\pi(i)}^A$ for $i\in\mathbb{N}_0$. We define closed intervals I_i^j, $(i,j)\in\mathbb{N}_0\times\mathbb{N}$, by the conditions $I_i^j\subset I_i^{j-1}$ and $r(I_i^j)=I_{i+1}^{j-1}$ (clearly from (i) we have $r(I_i^{j-1})\supset I_{i+1}^{j-1}$). Put $\mathcal{I}_i=\bigcap_{j\in\mathbb{N}}I_i^j$. We have $r^i(t)\in\mathcal{I}_i$ for each $i\in\mathbb{N}_0$; by our definition of the intervals I_i^j we even get $r^i(\mathcal{I}_0) = \mathcal{I}_i$, i.e. the itineraries of t and $\mathcal{I}_0$ with respect to $\{L_1^A, \dots, L_k^A, \dots\}$ are the same. Obviously each $\mathcal{I}_i$ is a point or a closed interval.

Without loss of generality we can assume that $t \neq \max T$. Using Remark 5.9 the transitive point t can be taken to satisfy

$$\text{(5.4)} \qquad \forall s \in \operatorname{orb}(r,t)\colon s \text{ is a two-sided limit point of} \operatorname{orb}(r,t).$$

The map π would be periodic if there were a positive integer n such that $\pi(i) = \pi(i+n)$ for each $i \in \mathbb{N}_0$. Let us show that π is not periodic for $\langle T, r\rangle \in \mathcal{M} \cup \mathcal{NM}$. We know that $r^i(t) \in [L_{\pi(i)}^A]^\circ$. If such an n did exist, then the closed interval

$$J = \overline{\operatorname{conv}\{s \in \operatorname{orb}(r,t)\colon \widetilde{\pi}(s,i) = \pi(i) \text{ for each } i \in \mathbb{N}_0\}}$$

would be r-periodic (not weakly) with period n and $\langle J, r^n\rangle \in \mathcal{M} \cup \mathcal{NM}$ for the monotone map $r^n|J$, a contradiction.

Now we show that $\mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)} = \emptyset$ for $i(1) \neq i(2)$. If $\mathcal{I}_{i(1)} \cap \mathcal{I}_{i(2)} \neq \emptyset$, from (5.4) we get some $i(3) \in \mathbb{N}$ greater than $i(1), i(2)$ for which $r^{i(3)}(t) \in \operatorname{conv}\{r^{i(1)}(t), r^{i(2)}(t)\}$. Since $r^i(t) \in \mathcal{I}_i$, we necessarily have either $r^{i(3)}(t) \in \mathcal{I}_{i(1)}$ or $r^{i(3)}(t) \in \mathcal{I}_{i(2)}$, which is impossible for the nonperiodic function π. Using (5.4) again, for $E^A = \bigcup_{i\in\mathbb{N}_0}\{\min L_i^A, \max L_i^A\}$ we can show similarly that $\mathcal{I}_i \cap E^A = \emptyset$ for each $i \in \mathbb{N}_0$. Summarizing, $r^j(\mathcal{I}_0) \subset [L_i^A]^\circ$ if and only if $r^j(t) \subset [L_i^A]^\circ$ and the orbits $\operatorname{orb}(r, \mathcal{I}_0)$, $\operatorname{orb}(r,t)$ have the same order.

As above (for A-contiguous intervals), let $K_i^1 = L_{\pi(i)}^B$ for $i \in \mathbb{N}_0$. Since any interval L_i^B is nondegenerate, using properties (ii)–(v) and Lemma 5.8 we can choose closed intervals $K_i^j = [a_i^j, b_i^j]$, $(i,j) \in \mathbb{N}_0 \times \mathbb{N}$, such that

(a) $K_i^j \subset K_i^{j-1}$,
(b) $\widetilde{q}(K_i^j) = K_{i+1}^{j-1}$ and $\operatorname{conv}\{\widetilde{q}(a_i^j), \widetilde{q}(b_i^j)\}) = K_{i+1}^{j-1}$,
(c) $\operatorname{sign}_r(I_i^j) = \operatorname{sign}_{\widetilde{q}}(K_i^j)$,
(d) for each $j \in \mathbb{N}$ (see (5.1) and use (c)),

$$K_{i(1)}^j \leq K_{i(2)}^j \;\Leftrightarrow\; I_{i(1)}^j \leq I_{i(2)}^j,\; i(1), i(2) \in \mathbb{N}_0.$$

Put $\mathfrak{K}_i = \bigcap_{j\in\mathbb{N}} K_i^j$ and $E^B = \bigcup_{i\in\mathbb{N}_0}\{\min L_i^B, \max L_i^B\}$. Clearly $\mathfrak{K}_i$ is a point or a closed interval in $\operatorname{conv} B$. Using (a)–(d) and the property analogous to (5.4) formulated with the help of (d), we can show as for $\mathcal{I}_i$ the following properties for each $i, j, i(1), i(2) \in \mathbb{N}_0$, $i(1) \neq i(2)$:

(e) $\mathfrak{K}_{i(1)} \cap \mathfrak{K}_{i(2)} = \emptyset$, $\mathfrak{K}_i \cap E^B = \emptyset$ and $\widetilde{q}^i(\mathfrak{K}_0) = \mathfrak{K}_i \subset L_{\pi(i)}^B$,
(f) $\widetilde{q}^j(\mathfrak{K}_0) \subset [L_i^B]^\circ \Leftrightarrow r^j(\mathcal{I}_0) \subset [L_i^A]^\circ \Leftrightarrow r^j(t) \subset [L_i^A]^\circ$,
(g) the orbits $\operatorname{orb}(\widetilde{q}, \mathfrak{K}_0)$, $\operatorname{orb}(r, \mathcal{I}_0)$, $\operatorname{orb}(r,t)$ have the same order.

Now, Lemma 5.4 and property (g) yield a $\widetilde{q}$-recurrent point $t^\star \in [L^B_{\pi(0)}]^\circ$ such that for $T^* = \omega(\widetilde{q}, t^*) \subset [\min B, \max B]$ we have $\langle T^*, \widetilde{q}\rangle \sim \langle T, r\rangle$ and $t^\star \in T^\star \setminus B \neq \emptyset$.

This proves the lemma. ∎

5.11. Lemma. *Let* $\langle R, p\rangle \sim \langle S, f\rangle$ *and* $\widetilde{q}\colon [\min S, \max S] \to \mathbb{R}$ *be a continuous map satisfying* $\widetilde{q}|S = q$. *Moreover, assume that for some set* $T \subset \operatorname{conv} R$,

- $\langle T, p_R\rangle \in \mathcal{P}$,
- $T \cap R = \emptyset$,
- $\langle T, p_R\rangle$ *is not a reducible system of* p_R.

Then there exists a set $T^\star \subset [\min S, \max S]$ *for which* $\langle T^\star, \widetilde{q}\rangle \sim \langle T, p_R\rangle$.

Proof. Assume that $\langle T, p_R\rangle$ is not equivalent to any subsystem of $\langle S, q\rangle$ (otherwise we are done). Moreover, our assumption that $\langle T, p_R\rangle$ is not a reducible system of p_R together with Lemma 5.16(iv) shows that $\langle T, p_R\rangle$ does not have a block structure with blocks in R-contiguous intervals.

Set $I = \operatorname{conv} S$ and define a map $f \in C(I)$ by

$$f(x) = \begin{cases} \widetilde{q}(x) & \text{if } \widetilde{q}(x) \in I, \\ \max S & \text{for } \widetilde{q}(x) > \max S, \\ \min S & \text{for } \widetilde{q}(x) < \min S. \end{cases}$$

Notice that if there is a set $S^\star \subset I$ for which $\langle S^\star, q_S\rangle \sim \langle T, p_R\rangle$, then by Lemma 5.6 there is a set $T^\star \subset I$ for which $\langle T^\star, f\rangle = \langle T^\star, \widetilde{q}\rangle \sim \langle T, p_R\rangle$. Thus, it is sufficient to show the existence of $S^\star$.

An R-contiguous interval L (in $\operatorname{conv} R$) will be called *active* if $T \cap L^\circ \neq \emptyset$.

Obviously for any active interval L, the map $p_R|L$ is not constant and for some $n > 0$,

$$p_R^n(L^\circ) \cap L^\circ \neq \emptyset. \tag{5.5}$$

Let $\{L_i^R\}_{i=0}^{k-1}$ consist of all active closed R-contiguous intervals and define $\{L_i^S\}_{i=0}^{k-1}$ as follows: if $L_i^R = [u_R, v_R]$ then $L_i^S = [u_S, v_S]$ satisfies

$$u_S, v_S \in S, \qquad (u_S, v_S) \cap S = \emptyset, \qquad \mathcal{B}_{S,R}(\{u_S, v_S\}) = \{u_R, v_R\}.$$

Note that L_i^S is well defined since $\mathcal{B}_{S,R}(S) = R$. Moreover, $u_S < v_S$. Indeed, otherwise by (5.5) the point u_S would be periodic of a period k; analogously, the intervals $L_i^R, p_R(L_i^R), \ldots, p_R^{k-1}(L_i^R)$ would be pairwise disjoint closed R-contiguous intervals satisfying $p_R^k(L_i^R) = L_i^R$. But then the set $T \cap L_i^R$ would be a block of $\langle T, p_R\rangle$. Since we assume that $\langle T, p_R\rangle$ is not equivalent to any subsystem of $\langle S, q\rangle$, we would have $\operatorname{card} T \cap L_i^R \geq 2$. This is impossible for $\langle T, p_R\rangle$ that does not have a block structure with blocks in R-contiguous intervals.

Note that by our choice of L_i^R and L_i^S, we have

$$\operatorname{sign}_{p_R}(L_i^R) = \operatorname{sign}_{q_S}(L_i^S) \quad \text{for each } i. \tag{5.6}$$

Put $t = \min T$, $n = \operatorname{card} T$ and define $\pi\colon \{0, \dots, n-1\} \to \mathbb{N}$ by $\pi(i) = j$ if $p_R^i(t) \in L_j^R$. Since $\langle T, p_R\rangle$ does not have a block structure with blocks in R-contiguous intervals, the finite sequence $\pi(0), \dots, \pi(n-1)$ is not repetitive. By the above, for $Q \in \{R, S\}$ and $h \in \{p_R, q_S\}$,

$$L^Q_{\pi(0)} \xrightarrow{h} \cdots \xrightarrow{h} L^Q_{\pi(n-1)} \xrightarrow{h} L^Q_{\pi(0)},$$

where $K \xrightarrow{h} L$ denotes the fact that $h(K) \supset L$. Since the finite sequence $\pi(0), \dots, \pi(n-1)$ is not repetitive, there is a periodic point $s \in L^S_{\pi(0)}$ of period n such that $q_S^i(s) \in L^S_{\pi(i)}$. From (5.6) it follows that $\langle S^\star = \operatorname{orb}(q_s, s), q_S\rangle \sim \langle T, p_R\rangle$.

This proves the lemma. ∎

5.12. Lemma. *Let $\langle T, g\rangle \in \mathcal{P}$. The system $\langle T, g\rangle$ is a roof system if and only if either* $\operatorname{card} T = 1$ *or $\langle T, g\rangle$ is a 2-extension.*

Proof. Let $\langle T, g\rangle \in \mathcal{P}$ be a 2-extension with T-blocks B_i, and assume that $x \in T$ and $g(x) = \max T$. Then for a sufficiently small neighbourhood $U(x)$ of x, $g(U(x)) \subset \operatorname{conv} B_{(\operatorname{card} T)/2} \subset \operatorname{Per}(g_T)$. This implies that for any closed set S such that $T \subset S \subset \operatorname{conv} T$ and the system $\langle S, g_T\rangle$ is transitive we necessarily have $S = T$.

Conversely, assume that a roof system $\langle T, g\rangle \in \mathcal{P}$ is not a 2-extension. Let $\langle T, g\rangle$ have a block structure over a cycle $\langle S = \{s_i\}_{i=1}^k, f\rangle$ with maximal number of points $k = \operatorname{card} S$ and T-blocks $B_i \subset T$. Obviously, $k \ge 2$ and $(\operatorname{card} T)/k = \operatorname{card} B_i > 2$. Since $\langle T, g\rangle$ is a roof system, no system $\langle R = \operatorname{conv} B_i, h = g_T^k | R\rangle$ is transitive; by Theorem 6.3 the system $\langle B_i, g^k\rangle$ has a nontrivial ($l \ge 2$) block structure with B_i-blocks C_j, $\operatorname{card} C_j \ge 2$, $j = 1, \dots, l$, and for any $m \in \{0, \dots, k-1\}$,

$$\{g_T^{k+m}(\operatorname{conv} C_j), g_T^{2k+m}(\operatorname{conv} C_j), \dots, g_T^{lk+m}(\operatorname{conv} C_j)\}$$

is an orbit (formed by disjoint intervals) of a periodic interval $g_T^m(\operatorname{conv} C_j)$ in $g_T^m(\operatorname{conv} B_i) = \operatorname{conv} B_p$ (if $f^m(s_i) = s_p$). Hence $\langle T, g\rangle$ has a block structure over a cycle $\langle S', f'\rangle$ with $S' = \{s'_1 < \cdots < s'_{kl}\}$, which contradicts the maximality of k. ∎

5.13. Lemma. *Let $\langle S, f\rangle \in \mathcal{M}$. The system $\langle S, f\rangle$ is a roof system if and only if it is a solenoidal system.*

Proof. By the definition, if $\langle S, f\rangle$ is a roof system then S is a maximal ω-limit set of a map f_S. Thus, a roof system $\langle S, f\rangle \in \mathcal{M}$ is solenoidal by

Theorem 6.1. Let $\{K_i\}_{i\in\mathbb{N}_0}$ be a sequence of Q-generating intervals, where

$$S \subset Q = \bigcap_{i\in\mathbb{N}_0} \operatorname{orb}(f_S, K_i)$$

and $\omega(f_S, x) = S$ for any $x \in Q$. If there were a set T for which $S \subsetneq T \subset \operatorname{conv} S$ and $\langle T, f_S\rangle \in \mathfrak{T}$, then there would exist a point $y \in \operatorname{Tran}\langle T, f_S\rangle \setminus \operatorname{orb}(f_S, K_i)$ for some i. Without loss of generality we can assume that $\min S = \min K_i$ for each i. Then $y \notin \omega(f_S, z)$ for any $z \in \operatorname{Tran}\langle T, f_S\rangle \cap K_i$, a contradiction. Thus $S = T$. ∎

5.14. Lemma. *Let $\langle T, g\rangle$ be a system and suppose that for some $[\alpha,\beta] \subset \operatorname{conv}(T)$ and $m \in \mathbb{N}$, $g_T^m([\alpha,\beta]) \cap [\alpha,\beta] \neq \emptyset$. Then there exist an $n \in \mathbb{N}$ and a weakly g_T-periodic closed interval $J \subset \operatorname{conv}(T)$ of period n such that*

$$\overline{\operatorname{orb}}(g_T, [\alpha,\beta]) = \operatorname{orb}(g_T, J).$$

Proof. Since $g_T^m([\alpha,\beta]) \cap [\alpha,\beta] \neq \emptyset$, the set $\widetilde{J} = \operatorname{orb}(g_T^m, [\alpha,\beta])$ is a g_T^m-invariant interval, i.e. $g_T^m(\widetilde{J}) \subset \widetilde{J}$. Take pairwise disjoint components $J_1, \dots, J_n$ of the set $\bigcup_{i=0}^{m-1} g_T^i(\overline{\widetilde{J}})$ and if $[\alpha,\beta] \subset J_i$, put $J = J_i$. Clearly, $J, g_T(J), \dots, g_T^{n-1}(J)$ are pairwise disjoint, $g_T^n(J) \subset J$ and $\overline{\operatorname{orb}}(g_T, [\alpha,\beta]) = \operatorname{orb}(g_T, J)$. ∎

5.15. Lemma. *Let $\langle B, f\rangle \in \mathcal{NM}$. The system $\langle B, f\rangle$ is a roof system if and only if it is a basic system.*

Proof. By definition, if $\langle B, f\rangle$ is a roof system then B is a maximal ω-limit set of a map f_B. Thus, a roof system $\langle B, f\rangle \in \mathcal{NM}$ is basic by Theorem 6.1. By our assumption the set B is infinite. Let K be an f_B-periodic set with a period n, $L = \operatorname{orb}(f_B, K)$ and (see Section 3)

$$B = \{x \in L \colon \overline{\operatorname{orb}}(f_B, J(x)) = L \text{ for each neighbourhood } J(x)\}.$$

If there were a set T for which $B \subsetneq T \subset \operatorname{conv} B$ and $\langle T, f_B\rangle \in \mathfrak{T}$, then we would also have $T \subsetneq L$ ($\langle L, f_B\rangle$ is not transitive). Let $J(x)$ be a neighbourhood of $x \in T$ (in L). Since $\langle T, f_B\rangle$ is transitive, by Lemma 5.14 we can consider a weakly f_B-periodic closed interval $J \subset L$ such that

$$\overline{\operatorname{orb}}(f_B, J(x)) = \operatorname{orb}(f_B, J). \tag{5.7}$$

By our assumption $B \subset T$, we have $B \subset \overline{\operatorname{orb}}(f_B, J(x))$, hence also $B \subset \operatorname{orb}(f_B, J)$. Then from (5.7) we get $L = \operatorname{orb}(f_B, J) = \overline{\operatorname{orb}}(f_B, J(x))$, i.e., $x \in B$. Thus $T \subset B$, hence $B = T$. ∎

A system $\langle T, g\rangle$ *has a block structure over a cycle* $\langle S, f\rangle$ with $S = \{s_1 < \dots < s_k\}$ if there are T-blocks $B_i = [a_i, b_i] \cap T$, $i \in \{1, \dots, k\}$, such that $a_i \le b_i$, $b_i < a_{i+1}$ for $i \in \{1, \dots, k-1\}$, $\bigcup_{i=1}^k \{a_i, b_i\} \subset T$, $T = \bigcup_{i=1}^k B_i$ and $g(B_i) = B_j$ if and only if $f(s_i) = s_j$. In this case we sometimes briefly write that $\langle T, g\rangle$ *has a block structure* (with blocks $B_i = [a_i, b_i] \cap T$, $i \in \{1, \dots, k\}$).

5.16. LEMMA.

(i) *If A, B are transitive patterns, $A \hookrightarrow B$ and A is piecewise monotone then B is also piecewise monotone.*

(ii) *Let $\langle T, g\rangle \in \mathcal{P} \cup \mathcal{M}$ with $\operatorname{card} T > 2$ and piecewise monotone map g_T. If $\langle T, g\rangle \sim \langle S, g_T\rangle$ for some $S \subset \operatorname{conv} T$, then $S = T$.*

(iii) *If $\langle R, p\rangle \in \mathfrak{T}$ has a block structure over a cycle $\langle S, f\rangle$ then the pattern $[\langle R, p\rangle]_\sim$ forces the pattern $[\langle S, f\rangle]_\sim$.*

(iv) *Let $\langle R, p\rangle \in \mathfrak{T}$ and suppose that for some $S \subset \operatorname{conv} R$,*

— *$\langle S, p_R\rangle$ has a block structure with blocks D_i and $\operatorname{card} D_i \geq 2$ for each $i = 0, \ldots, k-1$,*

— *each block is a subset of an R-contiguous interval.*

Then the system $\langle S, p_R\rangle$ is a 2-extension, different blocks D_i, D_j are contained in different R-contiguous intervals $[c_i, d_i]$, $[c_j, d_j]$, and $\langle \bigcup_{i=0}^{k-1}\{c_i, d_i\}, p\rangle \in \mathcal{P}$ is a 2-extension equivalent to $\langle S, p_R\rangle$. In particular, the cycle $\langle S, p_R\rangle$ is a reducible system of p_R.

(v) *If $\langle T = \{t_1 < t_2 < \cdots < t_{2k-1} < t_{2k}\}, g\rangle$ is a 2-extension of a cycle $\langle S, f\rangle$ then g_T has a unique representative*

$$\langle\{(t_1 + t_2)/2 < \cdots < (t_{2k-1} + t_{2k})/2\}, g_T\rangle$$

of the pattern $[\langle S, f\rangle]_\sim$.

Proof. Property (i) is clear. (ii) is well known for $\langle T, g\rangle$ periodic [1]. For the case of piecewise monotone minimal $\langle T, g\rangle$, see the proof of Theorem 3.2 in [5]. Property (iii) follows from Lemma 5.4 applied to the map p_R.

(iv) If $D_0 = \{a_1 < \cdots < a_l\} \subset [c_0, d_0]$ then for each $i \in \{0, \ldots, k-1\}$, $p_R^i([a_1, a_l])$ is a subset of an R-contiguous interval $[c_i, d_i]$. Since the map p_R is affine on each R-contiguous interval, it follows that $l = 2$, $p_R^k(a_1) = a_2$, $p_R^k(a_2) = a_1$ and $p_R^k|[a_1, a_2]$, resp. $p_R^{2k}|[a_1, a_2]$ is an affine map with slope -1, resp. 1. Since for any two R-contiguous intervals L_1, L_2 we have

$$p_R(L_1) \cap [L_2]^\circ \neq \emptyset \ \Rightarrow\ p_R(L_1) \supset L_2,$$

we can consider a closed interval J such that $[a_1, a_2] \subset J \subset [c_0, d_0]$, $p_R^i(J)$ is a subset of $[c_i, d_i]$ and $p_R^k(J) = [c_0, d_0]$. By the above, $p_R^k|J$ has slope -1. It follows that $J = [c_0, d_0]$ and $p_R^i(J) \cap [c_0, d_0] = \emptyset$ for each $i \in \{1, \ldots, k-1\}$. Starting from $[c_i, d_i]$ instead of $[c_0, d_0]$ we obtain $p_R^{k-i}([c_i, d_i]) \subset [c_0, d_0]$. This implies $p_R^i[c_0, d_0] = [c_i, d_i]$ since otherwise $p_R^k([c_0, d_0]) \subsetneq [c_0, d_0]$, a contradiction. Thus $\langle \bigcup_{i=0}^{k-1}\{c_i, d_i\}, p\rangle \in \mathcal{P}$ is a 2-extension equivalent to $\langle S, p_R\rangle$. All other properties follow immediately.

For property (v) see [8]. ∎

Let us recall that a roof system was defined in Definition 3.1. In the proof of Theorem 3.5 we need the following description of those systems.

5.17. LEMMA. *The following statements are equivalent*:

(i) $\langle T, g\rangle$ *is a roof system.*

(ii) $\langle T, g\rangle$ *is a system and there is a closed* $S \subset T$ *such that for* $f = g|S$, $f_S = g_T$ *and* $T = \overline{\bigcup\{\widetilde{T}\colon \widetilde{T} \supset S \text{ and } \langle \widetilde{T}, g_T\rangle \text{ is transitive}\}}$.

Proof. Put

$$T^\star = \overline{\bigcup\{\widetilde{T}\colon \widetilde{T} \supset S \text{ and } \langle \widetilde{T}, g_T\rangle \text{ is transitive}\}}. \tag{5.8}$$

We will show that $\langle T^\star, g_T\rangle$ is transitive if $T^\star$ is nonempty. To show (i)⇒(ii) we can put $S = T$. The opposite implication (ii)⇒(i) follows from $T \subset T^\star$ and Definition 3.1.

Since $f_S = g_T$ and the set S is closed,

$$\min T = \min T^\star = \min S \quad \text{and} \quad \max T = \max T^\star = \max S.$$

In what follows we will work with closed intervals $[\alpha,\beta] \subset \operatorname{conv} T$ satisfying $[\alpha,\beta]^\circ \cap S \neq \emptyset$. In particular, this holds when for a sufficiently small $\varepsilon > 0$ either $[\alpha = \min S,\ \beta = \varepsilon + \min S]$ or $[\alpha = -\varepsilon + \max S,\ \beta = \max S]$ (we use the relative topology of $\operatorname{conv} T$). Since S is contained in T and $\langle T, g_T\rangle$ is transitive, there is an $m \in \mathbb{N}$ satisfying $g_T^m([\alpha,\beta]) \cap [\alpha,\beta] \neq \emptyset$. By Lemma 5.14 and (5.8) we get a weakly g_T-periodic closed interval $J \subset \operatorname{conv} T$ with a period $n \in \mathbb{N}$ such that

$$\overline{\operatorname{orb}}(g_T, [\alpha,\beta]) = \operatorname{orb}(g_T, J) \quad \text{and} \quad \operatorname{orb}(g_T, J) \supset T^\star;$$

then the interval $K = \bigcap_{l\in\mathbb{N}_0} g_T^{ln}(J)$ is g_T-periodic of period n and with $\operatorname{orb}(g_T, K) \supset T^\star$.

I. The conclusion of our lemma holds true when $\operatorname{card} T^\star \in \mathbb{N}$. Then Lemma 5.12 implies that $\langle T^\star, g_T\rangle$ is a cycle.

Let $T^\star$ be infinite. Using the classification from Section 2.1 we can verify that $T^\star$ is a perfect set.

II. There exist an increasing sequence $\{n_i\}_{i\in\mathbb{N}}$ of positive integers and a decreasing sequence $\{K_i\}_{i\in\mathbb{N}}$ of closed intervals such that K_i is g_T-periodic with a period n_i and $\operatorname{orb}(g_T, K_i) \supset T^\star$ for each $i \in \mathbb{N}$. Then due to Section 3 there exists a unique infinite set $T_0 \subset Q = \bigcap_{i\in\mathbb{N}_0} \operatorname{orb}(g_T, K_i)$ such that $\omega(g_T, x) = T_0$ for any $x \in Q \supset T^\star$ and $\langle T_0, g_T\rangle$ is minimal. It follows that $T_0 = T^\star$ and $\langle T^\star, g_T\rangle$ is minimal.

III. There exists an $n \in \mathbb{N}$ and a closed interval K which is g_T-periodic with a period n, $L = \operatorname{orb}(g_T, K) \supset T^\star$ and

$$\forall [\alpha,\beta] \subset L\colon [\alpha,\beta]^\circ \cap S \neq \emptyset \ \Rightarrow\ \overline{\operatorname{orb}}(g_T, [\alpha,\beta]) = L. \tag{5.9}$$

Consider the set $B = B(L, g_T)$ defined in Section 3. Immediately from (5.9), it follows that $S \subset B$.

We need to show that $T^\star \subset B$. Consider an interval $[\alpha,\beta] \subset L$ satisfying $[\alpha,\beta]^\circ \cap T^\star \neq \emptyset$. Repeating the procedure from Lemma 5.14 for this interval we obtain a weakly g_T-periodic closed interval $\widetilde{L}$ of period k, hence $M = \bigcap_{l\in\mathbb{N}_0} g_T^{lk}(\widetilde{L})$ is a g_T-periodic closed interval of period k such that

$$\overline{\mathrm{orb}}(g_T,[\alpha,\beta]) = \mathrm{orb}(g_T,M), \qquad S\subset \mathrm{orb}(g_T,M)\subset L.$$

Without loss of generality we can assume that $\min M = \min S$. Then from (5.9) applied to M we obtain $\overline{\mathrm{orb}}(g_T,[\alpha,\beta]) = \mathrm{orb}(g_T,M) = L$. Therefore, $T^\star \subset B$. We have argued that $T^\star$ is a perfect set. It follows that B is infinite and by Section 3 the system $\langle B, g_T\rangle$ is transitive. But then $B = T^\star$, i.e., $\langle T^\star, g_T\rangle$ is a transitive system. ■

The following lemma is a direct consequence of Definition 2.1 and the ones of solenoidal and basic systems from Section 3. We leave its proof to the reader.

5.18. Lemma. *Two equivalent transitive systems $\langle T, g\rangle$, $\langle S, f\rangle$ are simultaneously solenoidal*, *resp. basic.*

6. The most important known needed notions and results. For $f \in C(I)$ and $x\in I$, the ω-limit set $\omega(f,x)$ is a *maximal ω-limit set* of f if for any $y\in I$ and $\omega(f,y)\supset\omega(f,x)$ we have $\omega(f,y)=\omega(f,x)$. The most important properties of maximal ω-limit sets are presented in Theorem 6.1. This result uses the notions of solenoidal and basic systems, recalled in Section 3. We present a simplified version using only piecewise affine extensions of systems.

6.1. Theorem ([4]). *If $\omega\subset I$ is a maximal ω-limit set of an interval map $f\in C(I)$ then the system $\langle\omega, f\rangle$ is transitive. Moreover*, *when* $\mathrm{card}\,\omega = \infty$ *then $\langle\omega, f\rangle$ is either a solenoidal system* ($\in\mathcal{M}$) *or a basic system* ($\in\mathcal{NM}$).

The sets $\mathcal{NM}_C$, $\mathcal{NM}_I$, $\mathcal{NM}$ have been defined in Section 2.

6.2. Theorem. *Let $\langle R,p\rangle\in\mathcal{NM}$ be a roof system. There is a system $\langle T=[a_1,b_1]\cup\cdots\cup[a_k,b_k], g\rangle \in \mathcal{NM}_I$ ($k\in\mathbb{N}$) such that $\langle R,p\rangle\sim\langle T,g\rangle$. Moreover*, *if $\langle T,g\rangle$ has a block structure over a cycle $\langle S=\{s_1<\cdots<s_k\}, f\rangle$ then we can suppose that $s_i\in[a_i,b_i]$ for each $i\in\{1,\ldots,k\}$ and $f=g|S$.*

Proof. See [4]. ■

An interval map $f\in C(I)$ is said to be *mixing* if $\langle I, f^n\rangle\in\mathcal{NM}$ for each $n\in\mathbb{N}$. A pattern $[\langle T,g\rangle]_\sim$ is said to be mixing if its adjusted map $g_T\in C(\mathrm{conv}\,T)$ is mixing. A periodic pattern $[\langle S,f\rangle]_\sim$ *has a division* if the system $\langle S,f\rangle$ has a block structure over a 2-cycle.

6.3. Theorem ([8]). *If A is a periodic pattern then it has either a division or a block structure over a mixing pattern.*

In accordance with the classification given in Section 2 we consider separately periodic, resp. minimal piecewise monotone patterns.

6.4. Theorem ([2], [5]). *The forcing relation on periodic and on minimal piecewise monotone patterns is a partial ordering.*

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Received 26 October 2004;
in revised form 27 July 2005