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Cohomology of the boundary of Siegel modular varieties of degree two, with applications

by

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To our friend and colleague Ronnie Lee

Abstract. Let $\mathcal{A}_2(n) = \Gamma_2(n)\backslash\mathfrak{S}_2$ be the quotient of Siegel's space of degree 2 by the principal congruence subgroup of level n in $\mathbf{Sp}(4,\mathbb{Z})$. This is the moduli space of principally polarized abelian surfaces with a level n structure. Let $\mathcal{A}_2(n)^*$ denote the Igusa compactification of this space, and $\partial\mathcal{A}_2(n)^* = \mathcal{A}_2(n)^* - \mathcal{A}_2(n)$ its "boundary". This is a divisor with normal crossings. The main result of this paper is the determination of $\mathrm{H}^*(\partial\mathcal{A}_2(n)^*)$ as a module over the finite group $\Gamma_2(1)/\Gamma_2(n)$. As an application we compute the cohomology of the arithmetic group $\Gamma_2(3)$.

1. Introduction. This paper is a continuation of the authors' previous investigations of the topology of moduli spaces of abelian surfaces, [20]–[22], [26]–[29]. In this paper, we will study the contribution to the cohomology of Siegel modular threefolds that arises, in a manner to be explained below, from the boundary of those threefolds. The term "boundary" can be interpreted in several ways depending on which compactification one is using. In our study, we will work with the toroidal compactification, due to Igusa and later generalized by Mumford and his coworkers. In this introduction we will motivate the calculations done in this paper by discussing their relation with the primary concern—that of computing the cohomology of congruence subgroups of $\mathbf{Sp}(4,\mathbb{Z})$. Much of the general discussion that follows in this introduction is valid more generally for arithmetic subgroups of reductive algebraic groups defined over $\mathbb{Q}$. Two excellent introductions to these topics can be found in [4] and [37].

Let $\mathfrak{S}_d$ be the Siegel space of degree d:

$$X = \mathfrak{S}_d = \{\tau \in \mathbf{M}_d(\mathbb{C}) : {}^t\tau = \tau,\ \mathrm{Im}(\tau) \text{ is positive definite}\}.$$

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The group $\mathbf{Sp}(2d, \mathbb{R})$ operates on $\mathfrak{S}_d$ in the usual way:

$$\gamma.\tau = (A\tau + B)(C\tau + D)^{-1}, \qquad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and $\mathfrak{S}_d$ is the corresponding symmetric space $\mathbf{Sp}(2d, \mathbb{R})/K$, where the maximal compact K is isomorphic to the unitary group $\mathbf{U}(d)$. Let

$$\Gamma \subset \mathbf{Sp}(2d, \mathbb{Q})$$

be a subgroup commensurable with $\mathbf{Sp}(2d, \mathbb{Z})$. Then Baily and Borel have shown that the quotient $X_\Gamma = \Gamma \backslash \mathfrak{S}_d$ admits the structure of a quasi-projective algebraic variety of dimension $d(d + 1)/2$. If Γ is torsion-free, this quotient is a complex manifold of this dimension. These varieties are moduli spaces of abelian varieties with a principal polarization and a level structure. Set

$$\Gamma_d(n) = \{\gamma \in \mathbf{Sp}(2d, \mathbb{Z}) : \gamma \equiv \mathbf{1} \bmod n\},$$

the principal congruence subgroup of degree d and level n. For this case we use the notation

$$\mathcal{A}_d(n) = \Gamma_d(n) \backslash \mathfrak{S}_d = X_{\Gamma_d(n)}.$$

Let

$$\varrho : \mathbf{Sp}(2d, \mathbb{R}) \to \mathrm{GL}(V)$$

be a rational representation on a finite-dimensional complex vector space. By restricting this representation to the subgroup Γ we obtain in a well known way a sheaf $\mathbb{V}$ on X_Γ, which is a local system if Γ is torsion-free, and it is also well known that

$$\mathrm{H}^*(X_\Gamma; \mathbb{V}) \cong \mathrm{H}^*(\Gamma; V).$$

One goal is to compute the dimensions of these spaces. Aside from the case of modular curves, $d = 1$, even in the case of trivial coefficients $V = \mathbb{C}$, very few actual results are known. Already for $d = 2$ it is an open problem to determine these dimensions. These cohomology spaces carry additional structures that are significant. Especially relevant to us here are the parts of the cohomology related to boundary components of various compactifications. In all cases, the combinatorics of the boundary components is described by a finite geometry—the quotient by Γ of the Tits building of rational parabolic subgroups of $\mathbf{Sp}_{2d}$. This is a geometric reflection of Harish-Chandra's philosophy of cusp forms and of Langlands' theory of Eisenstein series. There are several compactifications:

1. The Satake, or Baily–Borel, compactification X_Γ^{bb}. This is a generally singular projective algebraic variety.

2. The Borel–Serre compactification X_Γ^{bs}. This is a manifold with corners with the same homotopy type as X_Γ (see [6]). The sheaf $\mathbb{V}$ extends to the

compactification and one has

$$\mathrm{H}^*(X_\Gamma;\mathbb{V}) \cong \mathrm{H}^*(X_\Gamma^{\mathrm{bs}};\mathbb{V}).$$

There are maps $X_\Gamma^{\mathrm{bs}} \to X_\Gamma^{\mathrm{rbs}} \to X_\Gamma^{\mathrm{bb}}$ where X_Γ^{rbs} is the reductive Borel–Serre compactification (see [15]).

3. The toroidal compactifications $X_{\Gamma,\Sigma}$ (see [2], [33], [23]). When $d=2$, this was first constructed by Igusa. These compactifications depend on the choice of a Γ-admissible rational polyhedral subdivision Σ of the cone of positive $d\times d$ symmetric real matrices. When $d=1$ this is trivial; when $d=2$ there is a canonical choice; for $d\geq 3$ one has existence but not uniqueness. In case $d=2$ with the canonical choice of subdivision we will call it the *Igusa compactification* and denote it by X_Γ^*. When Γ is torsion-free it is a nonsingular projective algebraic variety and the boundary

$$\partial X_{\Gamma,\Sigma} = X_{\Gamma,\Sigma} - X_\Gamma = \bigcup_{i\in I} D_i$$

is a divisor with normal crossings. There is a morphism $X_{\Gamma,\Sigma} \to X_\Gamma^{\mathrm{bb}}$ which extends the identity $X_\Gamma \to X_\Gamma$; otherwise said, $X_{\Gamma,\Sigma}$ is a desingularization of X_Γ^{bb} along $\partial X_\Gamma^{\mathrm{bb}} = X_\Gamma^{\mathrm{bb}} - X_\Gamma$.

The main results of this paper concern the cohomology of the boundary of the Igusa compactification $\mathrm{H}^i(\mathcal{A}_2(n)^*;\mathbb{C})$. Actually, we often work with homology rather than cohomology, which allows us to work with explicit topological cycles. Also, wherever possible, we work with integer coefficients, enabling us to obtain information about torsion.

The space $\mathcal{A}_2(n)$ has a natural action of the finite group

$$\Gamma_2(1)/\Gamma_2(n) = \mathbf{Sp}(4,\mathbb{Z}/n),$$

and in fact this action factors through the projective group

$$G = \mathbf{PSp}(4,\mathbb{Z}/n) = \mathbf{Sp}(4,\mathbb{Z}/n)/\pm 1.$$

For n an odd prime this is a simple group. The group G acts on both $\mathcal{A}_2(n)^*$ and $\partial\mathcal{A}_2(n)^*$, making the (co)homology groups of these spaces into representation spaces of G. On the one hand, we are most interested in these spaces as G-representations, and on the other hand, their structure as G-representations is important for our argument. Thus our main result, Theorem 2.13, is stated in these terms. We also call the reader's attention to the applications in Section 2.5.

These arguments are for general $n\geq 3$. In Section 3 we investigate further the case $n=3$, using our results in [21], using a complementary approach and refining somewhat the conclusions there.

Before turning to the proofs of our main theorems, we comment on previous work on the cohomology of the boundary of Siegel modular varieties.

There is an exact sequence:

$$(1) \qquad \cdots \to \mathrm{H}^i_{\mathrm{c}}(X_\Gamma;\mathbb{V}) \to \mathrm{H}^i(X_\Gamma^{\mathrm{bs}};\mathbb{V}) \xrightarrow{r} \mathrm{H}^i(\partial X_\Gamma^{\mathrm{bs}};\mathbb{V}) \to \dots$$

We denote $\mathrm{Ker}(r)$, which is the image of the cohomology with compact supports, by $\mathrm{H}^i_!(X_\Gamma;\mathbb{V})$. The theory of Eisenstein cohomology attempts to determine $\mathrm{Im}(r)$, as well as a natural section $\mathrm{H}^i_{\mathrm{Eis}}(X_\Gamma;\mathbb{V})$, so that

$$\mathrm{H}^i(X_\Gamma;\mathbb{V}) = \mathrm{H}^i_!(X_\Gamma;\mathbb{V}) \oplus \mathrm{H}^i_{\mathrm{Eis}}(X_\Gamma;\mathbb{V}).$$

The general theory of Eisenstein cohomology can be found in [35]. Results specific to Siegel modular varieties appear in [34], [36] and [38]. Although $\partial X_\Gamma^{\mathrm{bs}}$ is not an algebraic variety, Harris and Zucker have shown that $\mathrm{H}^i(\partial X_\Gamma^{\mathrm{bs}};\mathbb{V})$ carries a canonical mixed Hodge structure so that (1) is an exact sequence of Hodge structures. See [17], [18], [45]. A key point here is the identification of the cohomology of the Borel–Serre boundary with deleted neighborhood cohomology. Let N_Γ^{bs} be a tubular neighborhood of $\partial X_\Gamma^{\mathrm{bs}}$. Then since X_Γ^{bs} is a manifold with corners, it is easy to see that there is a homotopy equivalence

$$N_\Gamma^{\mathrm{bs}} - \partial X_\Gamma^{\mathrm{bs}} \cong \partial X_\Gamma^{\mathrm{bs}}.$$

Thus,

$$\mathrm{H}^i(N_\Gamma^{\mathrm{bs}} - \partial X_\Gamma^{\mathrm{bs}};\mathbb{V}) = \mathrm{H}^i(\partial X_\Gamma^{\mathrm{bs}};\mathbb{V}).$$

Harris and Zucker construct a system of compatible homotopy equivalences

$$N_\Gamma^{\mathrm{bs}} - \partial X_\Gamma^{\mathrm{bs}} \simeq N_{\Gamma,\Sigma} - \partial X_{\Gamma,\Sigma},$$

where $N_{\Gamma,\Sigma}$ is a system of neighborhoods of $\partial X_{\Gamma,\Sigma}$ in a projective and smooth toroidal compactification. Thus, the deleted neighborhoods of the boundary $N_{\Gamma,\Sigma} - \partial X_{\Gamma,\Sigma}$, have the same homotopy type as the Borel–Serre boundary. Using the excision

$$(N_{\Gamma,\Sigma}, N_{\Gamma,\Sigma} - \partial X_{\Gamma,\Sigma}) \sim (X_{\Gamma,\Sigma}, X_{\Gamma,\Sigma} - \partial X_{\Gamma,\Sigma})$$

one can often relate information obtained from these two different compactifications.

Last we want to comment on an alternative viewpoint already alluded to, namely the connection with automorphic forms. As is true for any arithmetic subgroup Γ of a reductive algebraic group G over $\mathbb{Q}$, the cohomology is entirely expressible in terms of automorphic forms. The main assertion here, a concatenation of results of Borel, Wallach, Casselman, Garland and Franke, is that

$$\mathrm{H}^*(\Gamma;V) = \mathrm{H}^*(\mathfrak{g},K;A(G/\Gamma)\otimes V),$$

where the right-hand side is relative Lie algebra cohomology, $\mathfrak{g}$ is the complexified Lie algebra of the derived semisimple group G^{der}, and $A(G/\Gamma)$ is the space of automorphic forms for Γ. For definitions and precise statements, see [7], [12], [43]. These results are generalized in the context of weighted cohomology in [32]. Related to this is a decomposition, originally due to Langlands (unpublished),

$$\mathrm{H}^*(\mathit{\Gamma};V) = \bigoplus_{\{P\}\in\mathcal{C}} \mathrm{H}^*_{\{P\}}(\mathit{\Gamma};V),$$

where the direct sum is over the associativity equivalence classes of parabolic subgroups of G. The summand corresponding to G is called *cuspidal cohomology*, $\mathrm{H}^i_{\mathrm{cusp}}$. See [5] for a proof. The summands corresponding to each P are further decomposed in [13], the pieces of which are related to Eisenstein series. One has $\mathrm{H}^i_{\mathrm{cusp}} \subset \mathrm{H}^i_! \subset \mathrm{H}^i_{(2)}$, where the last one is the subset of cohomology representable by square-integrable differential forms. For $G = \mathbf{Sp}(4)$, one has (see [36], [34], [38])

$$\mathrm{H}^2 = \mathrm{H}^2_{(2)}, \quad \mathrm{H}^3_{\mathrm{cusp}} = \mathrm{H}^3_! = \mathrm{H}^3_{(2)}, \quad \mathrm{H}^4_! = \mathrm{H}^4_{(2)}.$$

2. The case of general $n \geq 3$. We now compute the homology of the "boundary" $\partial\mathcal{A}_2(n)^*$ for $n \geq 3$. In addition to its intrinsic interest, this provides us with a second proof of [21, Thm. 1.1b]. Although the proof is longer, it yields more information. It allows us to control the torsion, and enables us to more easily identify the various (co)homology groups as representation spaces of $G = \mathbf{PSp}(4, \mathbb{Z}/n) = P\mathit{\Gamma}_2(1)/P\mathit{\Gamma}_2(n)$, where $\mathit{\Gamma}_2(k)$ denotes the principal congruence subgroup of level k in $\mathbf{Sp}(4,\mathbb{Z})$.

As a matter of notation, we let 1 denote the 1-dimensional trivial representation (of whatever group is operating). Also, since all our representations are defined over $\mathbb{Q}$, they are self-dual, so we do not distinguish between a representation and its dual.

Recall that $\partial\mathcal{A}_2(n)^*$ is a union of corank 1 boundary components $D(l)$ glued along a set of disjoint corank 2 boundary components $C(h)$. Their descriptions will be discussed below. The combinatorics of this gluing is governed by the Tits building

$$\mathfrak{T}(\mathbb{Z}/n) = (\mathfrak{P}_1(\mathbb{Z}/n), \mathfrak{P}_2(\mathbb{Z}/n), \mathfrak{P}_{1,2}(\mathbb{Z}/n)).$$

We will describe this finite geometry. Assume that $n \geq 3$. We consider the module $(\mathbb{Z}/n)^4$ with the standard alternating form

$$\langle x, y\rangle = (x_1y_3 + x_2y_4) - (x_3y_1 + x_4y_2).$$

A submodule $M \subset (\mathbb{Z}/n)^4$ is *isotropic* if $\langle x, y\rangle = 0$ for all $x, y \in M$. Consider the following finite sets:

$$\mathfrak{P}_1(\mathbb{Z}/n) = \left\{ \begin{array}{l} \text{nonzero vectors } l \in (\mathbb{Z}/n)^4 \text{ modulo } \pm 1 \\ \text{such that the submodule generated by } l \\ \text{is a free direct factor} \end{array} \right\},$$

$$\mathfrak{P}_2(\mathbb{Z}/n) = \left\{ \begin{array}{l} \text{nonzero decomposable vectors} \\ h = v_1 \wedge v_2 \in \bigwedge^2(\mathbb{Z}/n)^4 \text{modulo } \pm 1 \\ \text{such that the submodule } \widetilde{h} \text{ generated by } v_1, v_2 \\ \text{is an isotropic free direct factor} \end{array} \right\},$$

$$\mathfrak{P}_{1,2}(\mathbb{Z}/n) = \left\{ \begin{array}{l} \text{pairs } (l, h) \text{ with } l,\ h \text{ as above} \\ \text{such that } l \text{ is a direct factor of } \widetilde{h} \end{array} \right\}.$$

A vector $l = (l_1, l_2, l_3, l_4)$ will generate a free direct factor precisely when (l_1, l_2, l_3, l_4) is the unit ideal in $\mathbb{Z}/n$. The submodule $\widetilde{h}$ will be a free direct factor precisely when the six Plücker coordinates of h generate the unit ideal in $\mathbb{Z}/n$. Note that when n has two or more prime factors, there are direct factors of $\mathbb{Z}/n$ that are not free. In the cases $n = 3$ and $n = 4$, the units of the ring $\mathbb{Z}/n$ consist only of ± 1, and in these cases, we may identify the above objects with subsets of the projective space $\mathbf{P}^3(\mathbb{Z}/n)$. Indeed,

$$\mathbf{P}^3(\mathbb{Z}/n) = \mathfrak{P}_1(\mathbb{Z}/n)$$

for $n = 3, 4$, and the elements of $\mathfrak{P}_2(\mathbb{Z}/n)$ are certain kinds of lines in that projective 3-space.

These sets have the cardinalities:

$$\#\mathfrak{P}_1(\mathbb{Z}/n) = \frac{n^4}{2} \prod_{p|n} (1 - p^{-4}), \qquad \#\mathfrak{P}_2(\mathbb{Z}/n) = \frac{n^4}{2} \prod_{p|n} (1 - p^{-4}),$$

$$\#\mathfrak{P}_{1,2}(\mathbb{Z}/n) = \frac{n^6}{4} \prod_{p|n} (1 - p^{-2})(1 - p^{-4}).$$

There is an obvious notion of incidence among the configurations introduced here, which coincides with the usual notion of incidence in projective space when $n = 3, 4$. The number of l's on each h is the same as the number of h's on each l, which is

$$\frac{n^2}{2} \prod_{p|n} (1 - p^{-2}).$$

The intrinsic definition of the sets $\mathfrak{P}(\mathbb{Z}/n)$ is that they index the $\Gamma_2(n)$-equivalence classes of nontrivial $\mathbb{Q}$-parabolic subgroups of $\mathbf{Sp}(4)$. Recall that apart from $\mathbf{Sp}(4)$ itself there are three classes of such—two maximal ones, and the Borel subgroups.

Finally we note that the group $\mathbf{Sp}(4, \mathbb{Z}/n)$ acts transitively on each of the four sets here.

2.1. *The surfaces $D(l)$.* We begin by considering a single boundary component $D = D(l)$. Recall that D is an elliptic modular surface over $M = M(l)$, a curve of genus g, with $\pi : D \to M$ having t exceptional fibers.

Here

$$t = \frac{n^2}{2}\prod_{p|n}(1-p^{-2}), \qquad g = 1 + \frac{(n-6)t}{12}.$$

We let F denote the union of the exceptional fibers and $D^\circ = D - F$. Then D° is an elliptic curve bundle over $M^\circ = M - \pi(F)$ and each fiber in F is an "n-gon" (a fiber of type I_n in Kodaira's classification [25]).

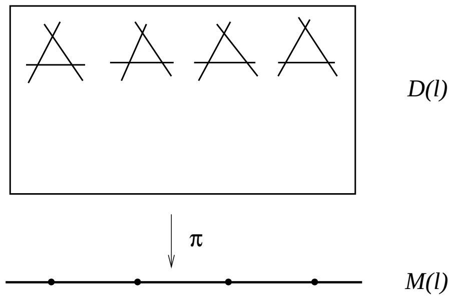

Fig. 1. Corank 1 boundary component for $n = 3$

LEMMA 2.1. *The projection $\pi : D \to M$ induces an isomorphism of fundamental groups $\pi_1(D) \to \pi_1(M)$.*

Proof. This is well known; see for example [9, Prop. 1.31]. ∎

COROLLARY 2.2. $\mathrm{H}_i(D;\mathbb{Z})$ *and* $\mathrm{H}^i(D;\mathbb{Z})$ *are torsion-free for all i. The ranks are*

$$1, \quad 2g, \quad 4g + nt - 2, \quad 2g, \quad 1$$

for $i = 0, 1, 2, 3, 4$.

Proof. The previous lemma shows that H_1 is free of rank $2g$, since it is the H_1 of a compact Riemann surface of genus g. Application of the universal coefficient theorem ([41, Cor. 4, p. 244]) shows then that H^1, H^2 and H_3 are free. Poincaré duality ([41, Thm. 18, p. 297]) implies that H_2 and H^3 are free. The second Betti number follows from an easy Euler characteristic argument. ∎

PROPOSITION 2.3. *For each i,* $\mathrm{H}_i(D^\circ;\mathbb{Z})$ *has no torsion of order prime to n. The ranks are given by*

$$1, \quad 2g + t - 1, \quad 4g + 2t - 3, \quad 2g + t - 1$$

for $i = 0, 1, 2, 3$. *In fact, these are torsion-free for* $i = 0, 2, 3$, *and*

$$\mathrm{H}_1(D^\circ;\mathbb{Z})_{\mathrm{tor}} = \mathbb{Z}/n \oplus \mathbb{Z}/n.$$

Proof. We are going to consider both the homology and cohomology spectral sequence of the fibration $D^\circ \to M^\circ$. The fiber is topologically a torus T^2. The homology version of this is

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(M^\circ; \mathrm{H}_q(T^2; A)),$$

where we have a nontrivial local coefficient system, converging to $\mathrm{H}_*(D^\circ; A)$. Here A is a coefficient ring. We have $\mathrm{E}^2_{p,q} = 0$ for $p \geq 2$, so these spectral sequences collapse for dimensional reasons.

The data of a local system on M° is equivalent to that of a representation of $\pi = \pi_1(M^\circ)$. Since $n \geq 3$, the principal congruence subgroup of level n in $\mathbf{SL}(2, \mathbb{Z})$, denoted $\Gamma_1(n) = \Gamma$, acts fixed-point free on the upper half-plane, so this fundamental group is isomorphic to $\Gamma_1(n)$. Note also that as M° is a connected open surface, it has the homotopy type of a wedge of circles, so that $\pi_1(M^\circ)$ is a free group of finite rank. This rank s is the dimension of the first homology of M°, which is $2g + t - 1$, since it is a genus g Riemann surface with t punctures. Let $\gamma_1, \ldots, \gamma_s$ be a set of free generators for this group. Let R be the group ring $\mathbb{Z}[\Gamma]$. Since M° is a $K(\pi, 1)$-space, we have canonical isomorphisms

$$\mathrm{H}_i(M^\circ; V) = \mathrm{H}_i(\Gamma; V)$$

for any R-module V, where on the right-hand side above we identify V with its local system over M°, and the left-hand side is Eilenberg–MacLane group homology. We have a free resolution $K_* \to \mathbb{Z}$:

$$0 \to R^s \xrightarrow{\partial} R \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

of the trivial R-module $\mathbb{Z}$, where the maps are defined as follows:

$$\partial(r_1, \ldots, r_s) = \sum_{j=1}^{s} (\gamma_j - 1) r_j, \qquad \varepsilon\Big(\sum_{\gamma \in \Gamma} n_\gamma \, \gamma\Big) = \sum_{\gamma \in \Gamma} n_\gamma.$$

Then $\mathrm{H}_i(\Gamma; V) = \mathrm{H}_i(K_* \otimes_R V)$, $\mathrm{H}^i(\Gamma; V) = \mathrm{H}^i(\mathrm{Hom}_R(K_*, V))$. Similar formulas hold for V taken to be an A-module with a Γ-action, with R replaced by $A[\Gamma]$.

We apply this to V either $\mathrm{H}^i(T^2; A)$ or $\mathrm{H}_i(T^2; A)$. First, $\mathrm{H}^0(T^2; A) = A$ and $\mathrm{H}^2(T^2; A) = A$ are acted on trivially, but $\mathrm{H}^1(T^2; A)$ is not. Indeed, as a representation space of $\Gamma_1(n)$, $\mathrm{H}^1(T^2; A)$ is $\mathbb{Z}^2 \otimes A$ where $\Gamma_1(n)$ acts on $\mathbb{Z}^2$ by the dual of its natural action on $\mathbb{Z}^2$, the natural action being matrix multiplication of $\mathbf{SL}(2, \mathbb{Z})$ on $\mathbb{Z}^2$ (we have the dual action here as we are dealing with cohomology rather than homology). Also, $\mathrm{H}^p(M^\circ; A) = 0$ for $p > 1$ from the above complex.

Hence we have, for any ring A, $\mathrm{E}_2^{0,0} = \mathrm{E}_2^{0,2} = A$, and $\mathrm{E}_2^{1,0} = \mathrm{E}_2^{1,2} = \mathrm{H}^1(M^\circ; A)$, free of rank s independent of A. Furthermore, from the above complex,

$$\mathrm{E}_2^{0,1} = \mathrm{H}^0(M^\circ; \mathrm{H}^1(T^2; A))$$

is isomorphic to the invariant elements in $\mathrm{H}^1(T^2;A) = A^2$ under the action of $\Gamma_1(n)$. Note this is zero for $A = \mathbb{Z}$ or $\mathbb{Z}/p$, p prime to n. It is $\mathbb{Z}/n \oplus \mathbb{Z}/n$ for $A = \mathbb{Z}/n^k$, for any $k \geq 1$, because the fixed vectors in $\mathbb{Z}/n^k \oplus \mathbb{Z}/n^k$ under the dual of the natural action of $\Gamma_1(n)$ are the vectors (a, b) with na and nb both zero in $\mathbb{Z}/n^k \oplus \mathbb{Z}/n^k$, and this subgroup is isomorphic to $\mathbb{Z}/n \oplus \mathbb{Z}/n$. We see thus that $\mathrm{H}^1(D^\circ; A)$ is a free A-module of rank s, for $A = \mathbb{Z}$ or $\mathbb{Z}/p$, p prime to n, but that $\mathrm{H}^1(D^\circ; \mathbb{Z}/n)$ is free of rank $s + 2$. Since

$$\mathrm{H}^1(D^\circ; A) = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_1(D^\circ; \mathbb{Z}), A)$$

we see that $\mathrm{H}_1(D^\circ; \mathbb{Z})$ has no torsion prime to n, and that

$$\mathrm{Hom}(\mathrm{H}_1(D^\circ; \mathbb{Z})_{\mathrm{tor}}, \mathbb{Z}/n) = \mathbb{Z}/n \oplus \mathbb{Z}/n.$$

For the assertions about $i = 2, 3$, we argue from the homology spectral sequence. Note that, from the above complex, $\mathrm{H}_1(\Gamma; V)$ is a submodule of V^s, so that if V is a free $\mathbb{Z}$-module of finite rank, so is $\mathrm{H}_1(\Gamma; V)$. Applied to $V = \mathrm{H}_1(T^2; \mathbb{Z})$, this shows that $\mathrm{E}^2_{1,1}$ is a free $\mathbb{Z}$-module. Its rank is easy to determine from an Euler characteristic argument. First, the universal coefficient theorem ([41, Theorem 3, p. 243]) shows that $\mathrm{E}^2_{1,1} \otimes \mathbb{Q}$ and $\mathrm{E}_2^{1,1} \otimes \mathbb{Q}$ have the same dimension. Since $\mathrm{E}_2^{0,1} \otimes \mathbb{Q}$ is 0, as we have argued above, we get

$$-\dim \mathrm{E}_2^{1,1} \otimes \mathbb{Q} = \chi(\mathrm{E}_2^{*,1}) = (1 - s) \dim \mathrm{H}_1(T^2; \mathbb{Q}) = 2 - 2s,$$

which implies that the rank of $\mathrm{E}^2_{1,1}$ is $2s - 2$. Since $\mathrm{E}^2_{0,2}$ is clearly free of rank 1, we see that $\mathrm{H}_2(D^\circ; \mathbb{Z})$ is free of rank $4g + 2t - 3$.

The argument for $i = 3$ is easier. The only contribution is $\mathrm{E}^2_{1,2} = \mathrm{H}_1(M^\circ; \mathbb{Z})$, which is visibly free of rank $2g + t - 1$. ∎

COROLLARY 2.4. *Let E be the representation of $\Gamma_1(n)$ on $\mathbb{Q}^2$ given by $g(v) = g \cdot v$, $g \in \Gamma_1(n)$, $v \in \mathbb{Q}^2$, where the right-hand side is matrix multiplication. Then*

$$\dim \mathrm{H}_1(\Gamma_1(n); E) = 2t + 4g - 4, \qquad \mathrm{H}_1(D^\circ; \mathbb{Q}) = 1 + \mathrm{H}_1(\Gamma_1(n); E).$$

Proof. The spectral sequence in the proof collapses at E_2 for dimensional reasons. Also, $\Gamma_1(n)$ acts freely on $\{\mathrm{Im}(z) > 0\}$ with quotient M°, so we may identify the cohomology of M° with the cohomology of the group $\Gamma_1(n)$. We then dualize to homology to obtain the corollary. ∎

To compute representations we now look at specific matrix groups. Henceforth $\varepsilon = \pm 1$.

DEFINITION 2.1. Let

$$Q_1(k) = \left\{ \begin{pmatrix} 1 & m & m' \\ 0 & a & b \\ 0 & c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(k),\ m, m' \equiv 0 \bmod k \right\}.$$

Then $Q_1(k)$ acts on $\mathbb{C} \times \{\mathrm{Im}(z) > 0\}$, covering the action of $\Gamma_1(k)$ on $\{\mathrm{Im}(z) > 0\}$, by

$$\begin{pmatrix} 1 & m & m' \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \begin{pmatrix} z \\ \tau \end{pmatrix} = \begin{pmatrix} (z + m\tau + m')/(c\tau + d) \\ (a\tau + b)/(c\tau + d) \end{pmatrix}.$$

Letting $k = n$, we obtain D° as quotient of this action, and furthermore the projection $(z, \tau) \mapsto \tau$ descends to the natural $\pi : D^\circ \to M^\circ$ ([23, Sec. I.2B]).

The group $Q_1(n)$ is a normal subgroup of $Q_1(1)$ and so $Q_1(1)/Q_1(n)$ acts as a group of automorphisms of D°. Letting

$$\pi : Q_1(1) \to \mathbf{SL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$

be the obvious map suggested by the notation, this action covers the action of $\Gamma_1(1)/\Gamma_1(n)$ on $\{\mathrm{Im}(z) > 0\}$ by Möbius transformations. We also denote by π the map $Q_1(1)/Q_1(n) \to \Gamma_1(1)/\Gamma_1(n) = \mathbf{SL}(2, \mathbb{Z}/n)$. Henceforth we set $Q = Q_1(1)/Q_1(n)$ and $H = \Gamma_1(1)/\Gamma_1(n)$. We have a split exact sequence

$$0 \to \mathbb{Z}/n \oplus \mathbb{Z}/n \to Q \xrightarrow{\pi} H \to 0.$$

If $\varphi : G_1 \to G_2$ is a homomorphism of groups and $\sigma : G_2 \to \mathbf{GL}(V)$ is a representation, we denote by $\varphi^*(\sigma)$ the representation of G_1 given by $\sigma \circ \varphi$.

Corollary 2.5. *As a representation of Q, $\mathrm{H}_i(D^\circ; \mathbb{Q})$ is given by*

$$\mathrm{H}_i(D^\circ; \mathbb{Q}) = \begin{cases} 1, & i = 0, \\ \pi^*(\mathrm{H}_1(\Gamma_1(n); \mathbb{Q})), & i = 1, 3, \\ 1 + \pi^*(\mathrm{H}_i(\Gamma_1(n); E)), & i = 2, \end{cases}$$

where H acts by conjugation on $\Gamma_1(n)$, and E is as in 2.4. *These representations have dimensions*

$$1, \quad 2g + (t - 1), \quad 4g + 2t - 3, \quad 2g + (t - 1)$$

respectively.

Proof. For $i = 0$, there is nothing to prove.

For $i = 1, 3$ this is immediate from the identifications

$$\mathrm{H}_i(D^\circ; \mathbb{Q}) = \mathrm{H}_i(Q_1(n); \mathbb{Q}) = \mathrm{H}_1(\Gamma_1(n); \mathrm{H}_{i-1}(T^2; \mathbb{Q})) = \mathrm{H}_1(\Gamma_1(n); \mathbb{Q})$$

in the proof of 2.3.

For $i = 2$ this also follows from the identifications there, together with the observation that

$$\pi^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} 1 & m & m' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

operates trivially on $\mathrm{H}_1(T^2; \mathbb{Q})$.

As for the dimensions, $\mathrm{H}_1(\Gamma_1(n);\mathbb{Q}) = \mathrm{H}_1(M^\circ;\mathbb{Q})$ and M° is an open Riemann surface of genus g with t punctures. ∎

While the corollary identifies these representations in purely algebraic terms, it leaves us with difficult computations. We now present a geometric description which is much easier to compute, assuming we understand $\mathrm{H}_1(M;\mathbb{Q})$ as a representation of H.

Let

$$Q'' = \left\{ \begin{pmatrix} 1 & m \\ 0 & \varepsilon \end{pmatrix} : m \in \mathbb{Z}/n,\, \varepsilon = \pm 1 \right\}.$$

Then Q'' is isomorphic to the dihedral group D_{2m}. We let R_1 be the 1-dimensional representation of Q'' where $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ acts trivially and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ acts by multiplication by -1, and we let R_n be the n-dimensional representation of Q'' with basis $e_0, \dots, e_{n-1}$ where $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ takes e_i to e_{i+1}, and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right)$ takes e_i to e_{-i} (indices mod n).

Observe that the stabilizer of a single exceptional fiber in the action of Q on D is isomorphic to the subgroup

$$Q' = \left\{ \begin{pmatrix} 1 & m & m' \\ 0 & \varepsilon & b \\ 0 & 0 & \varepsilon \end{pmatrix} \right\}.$$

Let $\varrho : Q' \to Q''$ be the obvious map suggested by the notation. Also, let P be the subgroup of $H = \Gamma_1(1)/\Gamma_1(n)$ given by

$$P = \left\{ \begin{pmatrix} \varepsilon & b \\ 0 & \varepsilon \end{pmatrix} \right\}.$$

Corollary 2.6. *As a representation of Q, $\mathrm{H}_i(D^\circ;\mathbb{Q})$ is given by*

$$\mathrm{H}_i(D^\circ;\mathbb{Q}) = \begin{cases} 1, & i = 0, \\ \pi^*(\mathrm{H}_1(M;\mathbb{Q})) + \pi^*(\mathrm{Ind}_P^H(1)) - 1, & i = 1,3, \\ \mathrm{H}_2(D;\mathbb{Q}) - \mathrm{Ind}_{Q'}^Q \varrho^*(R_n) + \mathrm{Ind}_{Q'}^Q(1) - 1 & \\ \qquad + \mathrm{Ind}_{Q'}^Q \varrho^*(R_1), & i = 2. \end{cases}$$

Proof. We have the exact sequence

$$0 \to \mathrm{H}_2(M;\mathbb{Q}) \to \mathrm{H}_2(M, M^\circ;\mathbb{Q}) \to \mathrm{H}_1(M^\circ;\mathbb{Q}) \to \mathrm{H}_1(M;\mathbb{Q}) \to 0.$$

Now $\mathrm{H}_2(M;\mathbb{Q})$ is the trivial representation of $\Gamma_1(1)/\Gamma_1(n)$, and

$$\mathrm{H}_2(M, M^\circ;\mathbb{Q}) = \mathrm{H}^0(M - M^\circ;\mathbb{Q})$$

is the permutation representation on the t cusps, hence so is the representation induced from the trivial representation of the stabilizer of any one of them, and P stabilizes the cusp ∞. Now

$$\mathrm{H}_1(D^\circ;\mathbb{Q}) \simeq \mathrm{H}_3(D^\circ;\mathbb{Q}) \simeq \mathrm{H}_1(M^\circ;\mathbb{Q}),$$

establishing the claim for $i = 1, 3$.

To compute $\mathrm{H}_2(D^\circ;\mathbb{Q})$ we consider the exact sequence of the pair (D, F), from which we see

$$\begin{aligned}\mathrm{H}^2(D^\circ;\mathbb{Q}) &= \mathrm{H}_2(D,F;\mathbb{Q}) \\ &= \mathrm{Coker}(\mathrm{H}_2(F;\mathbb{Q}) \to \mathrm{H}_2(D;\mathbb{Q})) \oplus \mathrm{Ker}(\mathrm{H}_1(F;\mathbb{Q}) \to \mathrm{H}_1(D;\mathbb{Q})).\end{aligned}$$

Let f denote a single exceptional fiber, the fiber over ∞. Then the stabilizer of f is $\pi^{-1}(P) = Q'$.

Let Σ be the union of the cusps in M. We have a commutative diagram

$$\begin{tikzcd} \mathrm{H}_1(F) \arrow[r] \arrow[d] & \mathrm{H}_1(D) \arrow[d] \\ 0 = \mathrm{H}_1(\Sigma) \arrow[r] & \mathrm{H}_1(M) \end{tikzcd}$$

and by Lemma 2.1 the map $\mathrm{H}_1(D) \to \mathrm{H}_1(M)$ is an isomorphism, so

$$\mathrm{H}_1(F;\mathbb{Q}) \to \mathrm{H}_1(D;\mathbb{Q})$$

is the zero map. Thus the second summand above is $\mathrm{Ind}_{Q'}^{Q}\mathrm{H}_1(f;\mathbb{Q})$.

As for the first summand, we have seen that for each exceptional fiber f, $\mathrm{H}_2(f;\mathbb{Q}) \to \mathrm{H}_2(D;\mathbb{Q})$ is an injection, and that in fact the only relation between the images of these homology groups in $\mathrm{H}_2(D;\mathbb{Q})$ is that the sum of the homology classes of the $\mathbf{P}^1$'s in each n-gon is the homology class of a general fiber. Note that the sum of the homology classes of the $\mathbf{P}^1$'s in $\mathrm{H}_2(f;\mathbb{Q})$ is acted on trivially by Q', and the homology class of a general fiber is acted on trivially by the whole group Q. Hence the image of the map $\mathrm{H}_2(F;\mathbb{Q}) \to \mathrm{H}_2(D;\mathbb{Q})$ is isomorphic to

$$\mathrm{Ind}_{Q'}^{Q}(\mathrm{H}_2(f;\mathbb{Q}) - 1) + 1.$$

Thus to complete the proof it remains to determine $\mathrm{Ind}_{Q'}^{Q}\mathrm{H}_i(f;\mathbb{Q})$ for $i = 1, 2$.

Now as f is an n-gon of $\mathbf{P}^1$'s, the elements

$$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

of Q' act trivially on f, while the element

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}$$

acts by "rotating" the n-gon, taking the ith $\mathbf{P}^1$ to the $(i+1)$st $\mathbf{P}^1$, mod n, and the element

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

acts by "reflecting" the n-gon, taking the ith $\mathbf{P}^1$ to the $(-i)$th $\mathbf{P}^1$, mod n. (For all this see the construction in [23, Section I.2B]. In other words, the action of Q' factors through Q'', and Q'' acts as R_1 on $\mathrm{H}_1(f;\mathbb{Q})$ and as R_n on $\mathrm{H}^2(f;\mathbb{Q})$.)

Assembling these terms gives $\mathrm{H}_2(D,F;\mathbb{Q})$ and thus $\mathrm{H}_2(D^\circ;\mathbb{Q})$. ∎

Finally, let $P(l)$ denote the stabilizer of l in G. We wish to find the action of $P(l)$ on $D(l)$. To do so, we choose $l = (0,0,0,1)$. Then

$$P(0,0,0,1) = \left\{ \begin{pmatrix} a & 0 & b & * \\ m & 1 & m' & s \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

where the entries marked $*$ are determined by the condition that the matrix be symplectic. (Compare [23, I.3.100].) Note that although the entries in σ are only determined up to a common sign, letting the element in the lower right-hand corner be $+1$ eliminates any ambiguity. The element

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

acts trivially on $D(0,0,0,1)$. We thus see:

PROPOSITION 2.7. *The action of* $P(0,0,0,1)$ *on* $\mathrm{H}_*(D(0,0,0,1);\mathbb{Z})$ *factors through the map* $\mathrm{pr}_l : P(l) \to Q$ *given by*

$$\mathrm{pr}_l \begin{pmatrix} a & 0 & b & * \\ m & 1 & m' & s \\ c & 0 & d & * \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m & m' \\ 0 & a & b \\ 0 & c & d \end{pmatrix}.$$

We can recapitulate these calculations in the language of sheaf cohomology. For a treatment of this for an arbitrary fibration of elliptic curves, see [9]. The Leray spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(M^\circ; R^q\pi_*\mathbb{Q}) \Rightarrow \mathrm{H}^*(D^\circ;\mathbb{Q})$$

is degenerate at E_2 by a theorem of Deligne [10, Section 1]. One has

$$\pi_*\mathbb{Q} = \mathbb{Q}, \qquad R^2\pi_*\mathbb{Q} = \mathbb{Q}$$

(ignoring Tate twists). The local system $\mathcal{E} = R^1\pi_*\mathbb{Q}$ is equivalent to the dual E^* representation of $\pi_1(M^\circ) = \Gamma_1(n)$ described in Corollary 2.4. Therefore, $\mathrm{H}^i(M^\circ; \mathcal{E})$ is 0 for $i = 0, 2$, being the $\pi_1(M^\circ)$-invariants in E^* and coinvariants in E, respectively. The rank for $i = 1$, namely $4g + 2t - 4$, follows easily from an Euler characteristic calculation. We recover the calculation of $\mathrm{H}^i(D^\circ)$ this way.

The spectral sequence for $\pi : D \to M$ has $\pi_*\mathbb{Q} = \mathbb{Q}$ and nontrivial sheaves $\mathcal{E} = R^1\pi_*\mathbb{Q}$ and $\mathcal{F} = R^2\pi_*\mathbb{Q}$. We claim that the sequence is E_2-degenerate, and we will evaluate the terms. Let $\Sigma \subset M$ be the set of t cusps, and $j : M^\circ \to M$ the inclusion of their open complement.

The constructible sheaf $\mathcal{E}$ is given as the local system $j^*\mathcal{E}$ on M°, equivalent to the representation E^* of $\pi_1(M^\circ)$. On Σ it is

$$\bigoplus_{s\in\Sigma} (R^1\pi_*\mathbb{Q})_s = \bigoplus_{s\in\Sigma} \mathbb{Q}_s,$$

since over each cusp sits an n-gon of $\mathbf{P}^1$'s. The monodromy of E^* around the cusp s is equivalent to

$$\gamma_s = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

(see the proof of Theorem 2.13) and at each cusp the gluing map

$$\mathcal{E}_s \cong \mathbb{Q} \to (j_*j^*\mathcal{E})_s = \mathrm{Ker}(\gamma_s - 1) \cong \mathbb{Q}$$

is an isomorphism. Thus, $\mathcal{E} = j_*j^*\mathcal{E}$. We have the local cohomology exact sequence

$$\cdots \to \mathrm{H}^i_\Sigma(M; \mathcal{E}) \to \mathrm{H}^i(M; \mathcal{E}) \to \mathrm{H}^i(M^\circ; j^*\mathcal{E}) \to \ldots$$

and the local-to-global spectral sequence

$$\mathrm{E}_2^{a,b} = \mathrm{H}^a(M; \mathcal{H}^b_\Sigma(\mathcal{E})) \Rightarrow \mathrm{H}^*_\Sigma(M; \mathcal{E})$$

(see [16]). The exact sequence

$$0 \to \mathcal{H}^0_\Sigma(\mathcal{E}) \to \mathcal{E} \to j_*j^*\mathcal{E} \to \mathcal{H}^1_\Sigma(\mathcal{E}) \to 0$$

shows that $\mathcal{H}^i_\Sigma(\mathcal{E}) = 0$ for $i = 0, 1$. These are 0 for $i \geq 3$ and

$$\mathcal{H}^2_\Sigma(\mathcal{E})_s = R^1j_*(j^*\mathcal{E})_s = \mathrm{Coker}(\gamma_s - 1) \cong \mathbb{Q}.$$

Thus, the only nonzero local cohomology group is

$$\mathrm{H}^2_\Sigma(M; \mathcal{E}) = \mathrm{H}^0(M; \mathcal{H}^2_\Sigma(\mathcal{E})) = \mathbb{Q}^t.$$

The exact sequence

$$0 \to j_!j^*\mathcal{E} \to j_*j^*\mathcal{E} = \mathcal{E} \to \bigoplus_{s\in\Sigma} \mathbb{Q}_s \to 0$$

shows that

$$\mathrm{H}^2(M;\mathcal{E}) \cong \mathrm{H}^2(M; j_! j^*\mathcal{E}) = \mathrm{H}^2_{\mathrm{c}}(M^\circ; j^*\mathcal{E}).$$

This last one is Verdier dual ([42]) to $\mathrm{H}^0(M^\circ; (j^*\mathcal{E})^*)$, which vanishes, as mentioned before. Therefore, $\mathrm{H}^i(M;\mathcal{E}) = 0$ for $i \neq 1$, and we have an exact sequence

$$0 \to \mathrm{H}^1(M;\mathcal{E}) \to \mathrm{H}^1(M^\circ; j^*\mathcal{E}) \to \mathbb{Q}^t \to 0,$$

which gives the rank of the left-hand term as $4g + t - 4$.

The sheaf $j^*\mathcal{F}$ is the constant sheaf $\mathbb{Q}$ on M°, hence $j_*j^*\mathcal{F} = \mathbb{Q}$. For each $s \in \Sigma$ we have $\mathcal{F}_s \cong \mathbb{Q}^n$, generated by the fundamental classes of the $\mathbf{P}^1$'s in the n-gon lying over s. The gluing map

$$\mathbb{Q}^n \cong \mathcal{F}_s \to (j_*j^*\mathcal{F})_s = \mathbb{Q}$$

is $(n_i) \mapsto \sum n_i$. We get the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^0_\Sigma(\mathcal{F}) \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] & j_*j^*\mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{H}^1_\Sigma(\mathcal{F}) \arrow[r] \arrow[d, equal] & 0 \\ & \bigoplus_{s\in\Sigma}(\mathbb{Q}^{n-1})_s & & \mathbb{Q} & 0 & \end{tikzcd}$$

From the long exact cohomology sequence of this sheaf sequence we see that $\mathrm{H}^i(M;\mathcal{F})$ has dimension $t(n-1)+1$, $2g$, 1 for $i = 0,1,2$ and is 0 for $i \geq 3$.

It is now clear that the spectral sequence degenerates as claimed, and that it gives the dimensions of $\mathrm{H}^i(D)$ as calculated. Actually, the degeneration of this spectral sequence is a general phenomenon [44, §15].

There is a precision of these results in Hodge theory. Namely,

$$\mathrm{H}^1(D) \cong \mathrm{H}^1(M) \quad \text{and} \quad \mathrm{H}^3(D) \cong \mathrm{H}^1(M)(-1)$$

as Hodge structures (over $\mathbb{Q}$). Also, there is a canonical Hodge structure on

$$\mathrm{H}^1(M; j_*j^*\mathcal{E}) = \mathrm{H}^1(M;\mathcal{E})$$

(see [44]) of pure weight 2. This contributes the $(2,0)$ and $(0,2)$ part of $\mathrm{H}^2(D)$. Shioda [40] has identified the $(2,0)$ part as the space of $\Gamma_1(n)$-cusp forms of weight 3, and the image of

$$\mathrm{H}^1(M; j_*j^*R^1\pi_*\mathbb{Z}) \to \mathrm{H}^1(M; j_*j^*R^1\pi_*\mathbb{C})$$

is the lattice of Eichler periods. It is more elementary, and classical, that the $(1,0)$ part of $\mathrm{H}^1(M)$ is identified with the weight 2 cusp forms. As to $\mathrm{H}^{1,1}(D)$, it is entirely generated by algebraic cycles. This is now clear from the spectral sequence: $\mathrm{E}_2^{0,2}$ contributes $n-1$ of the $\mathbf{P}^1$'s over each of the cusps, plus a general fiber of the map π. In turn, $\mathrm{E}_2^{2,0}$ contributes the class of any one of the sections of π. Finally, $\mathrm{H}^1(M^\circ; j^*\mathcal{E})$ carries a mixed Hodge structure. All these various Hodge structures are compatible with the maps arising from the Leray sequence (for details on this see [9], [39] and [44]).

REMARK 2.1. The referee has pointed out that the above calculations can be summarized neatly using the Decomposition Theorem for perverse sheaves ([3, Theorem 6.2.5]). The simple sheaves over an algebraic curve M are of the form (up to dimension shift) $j_*\mathcal{L}$ for a local system $\mathcal{L}$ on a Zariski open subset $j : U \to M$, or $i_*\mathbb{Q}$ for the inclusion of a point $i : p \to M$. Then the above calculations show that

$$R\pi_*\mathbb{Q}_D = \mathbb{Q}_M \oplus j_*j^*\mathcal{E}[-1] \oplus \mathbb{Q}_M[-2] \oplus \bigoplus_{s\in\Sigma} \mathbb{Q}_s^{n-1}[-2]$$

in $D^b_c(M(\mathbb{C}), \mathbb{Q})$.

2.2. *The curves* $C(h)$. We now consider a corank 2 boundary component $C(h)$. Note that $C(h)$ is a union of n-gons, one for each l with $l \subset h$. The $\mathbf{P}^1$'s in $C(h)$ are $D(l_1, l_2)$ where $h = l_1 \wedge l_2$, so we see that each $\mathbf{P}^1$ is contained in two n-gons. Each point which is the intersection of two $\mathbf{P}^1$'s is in fact a triple point $D(l_1, l_2) \cap D(l_1, l_3) \cap D(l_2, l_3)$ with $h = l_1 \wedge l_2 = l_1 \wedge l_3 = l_2 \wedge l_3$; necessarily, $l_3 = l_1 \pm l_2$. Recall that $D(l) = D(-l)$. We set

$$r_1 = \dim \mathrm{H}_1(C(h); \mathbb{Q}), \qquad r_2 = \dim \mathrm{H}_2(C(h); \mathbb{Q}),$$

and note that $\mathrm{H}_i(C(h); \mathbb{Q}) = 0$ for $i \geq 3$.

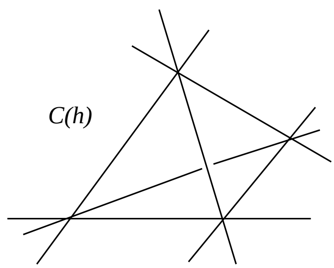

Fig. 2. Corank 2 boundary component for $n = 3$

PROPOSITION 2.8. *We have*

$$r_2 = nt/2, \qquad r_1 = 1 + nt/6,$$

where n is the level, and t the number of cusps of the modular curve (*see the beginning of Section* 2.1).

Proof. The integer r_2 is simply the number of $\mathbf{P}^1$'s. Now an h-vertex in $\mathfrak{T}(\mathbb{Z}/n)$ has valence t, an edge leading out from h corresponds to an n-gon of $\mathbf{P}^1$'s. This counts nt $\mathbf{P}^1$'s, but each is counted twice, yielding $r_2 = nt/2$.

To compute r_1, we observe that the Euler characteristic of $C(h)$ is $r_2 - r_1 + 1$. On the other hand, as each $\mathbf{P}^1$ contains two triple points, and the Euler characteristic of $\mathbf{P}^1 - 2$ points is zero, the Euler characteristic of $C(h)$ is equal to the number of triple points, which by the same logic is $nt/3$. ■

In fact, it is easy to see, say by Mayer–Vietoris, that $\mathrm{H}_2(C(h);\mathbb{Z})$ is free on the fundamental classes of the $D(l_1, l_2)$.

We further observe that, as is well known and can be verified by an Euler characteristic computation, if each $D(l_1, l_2) = \mathbf{P}^1 = S^2$ in $C(h)$ is replaced by the interval $[0, 1]$, then the 1-complex so obtained is the 1-skeleton $C'(h)$ of a tessellation of a Riemann surface $N = N(h)$ of genus g by t n-gons. To determine $\mathrm{H}_1(C(h))$, we replace the complex $C(h)$ by the 1-skeleton $C'(h)$, which has the same homology in dimensions $i = 0, 1$. Here we want to be careful about orientations, as an element of the stabilizer subgroup $P(h)$ of $C(h)$ may reverse the orientation of a path. To this end, if $(l_1,\ l_2)$ is an ordered pair of elements of $(\mathbb{Z}/n)^4$ with $h = l_1 \wedge l_2$, we let $d(l_1, l_2)$ denote an oriented path in $D(l_1, l_2)$ connecting the vertices $D(l_1, l_2, l_1 + l_2)$ and $D(l_1, l_2, l_1 - l_2)$. We define a chain complex

$$\begin{aligned} C_0(h) &= \text{the free } \mathbb{Z}\text{-module on } d(l_1, l_2, l_3) \\ &\qquad \text{with } l_1 \wedge l_2 = l_1 \wedge l_3 = l_2 \wedge l_2 = \pm h, \\ C_1(h) &= \text{the free } \mathbb{Z}\text{-module on } d(l_1, l_2) \text{ with } l_1 \wedge l_2 = \pm h. \end{aligned}$$

The boundary is defined by

$$\partial d(l_1, l_2) = d(l_1, l_2, l_1 + l_2) - d(l_1, l_2, l_1 - l_2).$$

This provides an orientation of these 1-cells. These symbols cannot depend essentially on the order or sign changes of their arguments. Consistency demands that the symbol $d(l_1, l_2, l_3)$ is invariant under $l_i \to \pm l_i$ but that

$$d(l_1, -l_2) = d(-l_1, l_2) = -d(l_1, l_2).$$

Note that this is in contrast to the symbol $D(l_1, l_2)$ which is insensitive to sign changes of l_1 and l_2. The symbols $d(l_1, l_2)$ and $d(l_1, l_2, l_3)$ are symmetrical under permutations of their respective arguments. The reader can check that the complex is well defined. Then $\mathrm{H}_1(C(h)) = \mathrm{H}_1(C'(h))$, and $\mathrm{H}_1(C'(h))$ is generated by the cycles given by the n-gons. The n-gons are the faces of the polyhedron $C'(h)$. These are indexed by the flags $l \subset h$. For a given such flag we have $l \wedge l' = \pm h$ for some other l'. The edges of the face are labeled by pairs $(l' + il, l)$ for $i \in \mathbb{Z}/n$. Then

$$c(h, l) = d(l, l') + d(l, l' + l) + \ldots + d(l, l' + (n - 1)l)$$

is a generating cycle on this face (one checks that $\partial c(h, l) = 0$). It is well defined up to a sign. Let $P(h)$ be the stabilizer of h in G. We can now determine the action of $P(h)$ on $\mathrm{H}_*(C(h))$. To do so, we choose

$$h = (0, 0, 1, 0) \wedge (0, 0, 0, 1).$$

Then

$$P(h)=\left\{\begin{pmatrix} a & c & * & * \\ b & d & * & * \\ 0 & 0 & \varepsilon d & -\varepsilon b \\ 0 & 0 & -\varepsilon c & \varepsilon a \end{pmatrix}\right\},$$

where $\varepsilon = ad - bc = \pm 1$ and the entries marked $*$ are determined by the condition that the matrix be symplectic.

If l is a line with $l \subset h$ then l is of the form $(0,0,x,y)$, and the image of the line under the action of $g \in P(h)$ is $lg^{-1} = (0,0,x',y')$, where

$$(x',y') = (x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The action of $P(h)$ on the generators $D(l_1,l_2)$ is induced from the obvious action on the symbols l_i. The same goes for its action on the symbols $d(l_1,l_2)$ and $d(l_1,l_2,l_3)$. Notice that this action does factor, as it should, across the center ± 1 (a nontrivial assertion only in the case of the $d(l_1,l_2)$). We thus see:

Proposition 2.9. *For $h = (0,0,1,0) \wedge (0,0,0,1)$, the action of $P(h)$ on $\mathrm{H}_*(C(h))$ factors through $\mathrm{pr}_h : P(h) \to H$ where*

$$\mathrm{pr}_h \begin{pmatrix} a & c & * & * \\ b & d & * & * \\ 0 & 0 & \varepsilon d & -\varepsilon b \\ 0 & 0 & -\varepsilon c & \varepsilon a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{GL}(2, \mathbb{Z}/n).$$

For future purposes we record (for notations, see 2.6):

Proposition 2.10. *Let $l = (0,0,0,1)$ and $h = (0,0,1,0) \wedge (0,0,0,1)$, and $f(h,l)$ the corresponding exceptional fiber. The stabilizer $P(h,l) = P(h) \cap P(l)$ of $f(h,l)$ is $\mathrm{pr}_l^{-1}(Q') \subset P(l)$. Its action on $\mathrm{H}_*(f(h,l))$ factors through the projections*

$$\mathrm{pr}_l^{-1}(Q') \to Q' \to Q''.$$

2.3. *Homology of the boundary.* We now turn to the boundary $\partial\mathcal{A}_2(n)^*$, $n \geq 3$, as a whole. We let q be the number of corank 1 boundary components, which is the same as the number of corank 2 boundary components; we have computed

$$q = \frac{n^4}{2}\prod_{p|n}(1-p^{-4}).$$

The corank 1 and corank 2 boundary components each have valence t, which is the integer so denoted in the previous section.

The Tits building $\mathfrak{T} = \mathfrak{T}(\mathbb{Z}/n)$ is a connected 1-complex with $2q$ vertices and qt edges, so $\mathrm{H}_1(\mathfrak{T};\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank $q(t-2)+1$.

LEMMA 2.11. *As a representation of* G,

$$\mathrm{H}_1(\mathfrak{T};\mathbb{Q}) = \mathrm{Ind}_{P(l,h)}^G(1) - \mathrm{Ind}_{P(l)}^G(1) - \mathrm{Ind}_{P(h)}^G(1) + 1.$$

Proof. We know that the Euler characteristic of $\mathfrak{T}$ is equal to the Euler characteristic of $\mathrm{H}_*(\mathfrak{T};\mathbb{Z})$. Now the vertices of $\mathfrak{T}$ are indexed by $\{l\}$ and $\{h\}$ and the edges by $\{(l,h) : l \subset h\}$, so $\mathfrak{T}$ has Euler characteristic

$$-\mathrm{Ind}_{P(l,h)}^G(1) + \mathrm{Ind}_{P(l)}^G(1) + \mathrm{Ind}_{P(h)}^G(1),$$

while $\mathrm{H}_*(\mathfrak{T};\mathbb{Z})$ has Euler characteristic

$$-\mathrm{H}_1(\mathfrak{T};\mathbb{Z}) + \mathrm{H}_0(\mathfrak{T};\mathbb{Z}).$$

But $\mathfrak{T}$ is connected, so $\mathrm{H}_0(\mathfrak{T};\mathbb{Z}) = 1$, the trivial 1-dimensional representation of G, and the result follows. ■

LEMMA 2.12. *The representation of* G *on* $\mathrm{H}_1(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$ *contains* $\mathrm{H}_1(\mathfrak{T};\mathbb{Q})$.

Proof. Let $D^\circ(l)$ be $D(l)$ with the exceptional fibers removed, and let $C^\bullet(h)$ be a regular neighborhood of $C(h)$ in $\partial\mathcal{A}_2(n)^*$. Let $\mathfrak{I} = \mathfrak{P}_1(\mathbb{Z}/n) \coprod \mathfrak{P}_2(\mathbb{Z}/n)$. Then $\{U_i\}_{i\in\mathfrak{I}} = \{D^\circ(l), C^\bullet(h)\}$ is an open cover of $\partial\mathcal{A}_2(n)^*$ with nerve $\mathfrak{T}(\mathbb{Z}/n)$, and furthermore each $D^\circ(l), C^\bullet(h)$, $D^\circ(l) \cap C^\bullet(h)$ for $l \subset h$ is connected. Then the sheaf cohomology spectral sequence for this open covering,

$$\mathrm{E}_1^{a,b} = \bigoplus_{\#I=a+1} \mathrm{H}^b(U_I;\mathbb{Q}) \Rightarrow \mathrm{H}^*(\partial\mathcal{A}_2(n)^*;\mathbb{Q}),$$

where $U_I = U_{i_0} \cap \ldots \cap U_{i_a}$ for $I = \{i_0,\ldots,i_a\}$, has

$$\mathrm{E}_2^{1,0} = \mathrm{Coker}(\mathrm{E}_1^{0,0} \to \mathrm{E}_1^{1,0}),$$

which is clearly $\mathrm{H}^1(\mathfrak{T};\mathbb{Q})$. The spectral sequence collapses at E_2 for dimensional reasons, showing $\mathrm{H}^1(\mathfrak{T};\mathbb{Q})$ does indeed exist inside $\mathrm{H}^1(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$ and hence by duality $\mathrm{H}_1(\mathfrak{T};\mathbb{Q})$ is in $\mathrm{H}_1(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$ (see [14, Theorem 5.4.1]). ■

REMARK 2.2. This conclusion holds for the boundary of the Borel–Serre compactification, as observed in [36], and for the boundary of the Satake compactification.

In case $l \subset h$, we let $C^\bullet(h,l)$ be the intersection $C^\bullet(h) \cap D(l)$, and $C^\circ(h,l)$ be the intersection $C^\bullet(h) \cap D^\circ(h,l)$, where $D^\circ(h,l)$ denotes D with the singular fiber $f(h,l)$ associated with (l,h) removed. Note that the inclusion $f(h,l) \hookrightarrow C^\bullet(h,l)$ is a homotopy equivalence, and hence this has the homotopy type of an n-gon of $\mathbf{P}^1$'s. Also $C^\circ(h,l)$ has the homotopy type of

$\partial C^\bullet(h,l)$, which is that of a T^2-bundle over S^1 with monodromy $\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$. We let

$$C^\circ(h) = \bigcup_{l\subset h} C^\circ(h,l) \quad \text{for fixed } h, \qquad C^\circ(l) = \bigcup_{l\subset h} C^\circ(h,l) \quad \text{for fixed } l.$$

These are both disjoint unions. We set

$$C^\bullet(l) = \bigcup_{l\subset h} C^\bullet(h,l).$$

Then we have decompositions

$$\begin{aligned} D(l) &= C^\bullet(l) \cup D^\circ(l), & \partial\mathcal{A}_2(n)^* &= C^\bullet \cup D^\circ, \\ C^\circ(l) &= C^\bullet(l) \cap D^\circ(l), & C^\circ &= C^\bullet \cap D^\circ \end{aligned}$$

with

$$C^\bullet = \bigcup C^\bullet(l), \qquad D^\circ = \bigcup D^\circ(l), \qquad C^\circ = \bigcup C^\circ(l) = \bigcup C^\circ(h),$$

where the last three unions are disjoint. We shall use these decompositions to calculate homology.

THEOREM 2.13. (a) *For* $i = 0,1,2,3,4$, $\dim \mathrm{H}_i(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$ *is*

$$1, \quad q(r_1+2g-1)+1, \quad q(r_2+4g-1), \quad 2gq, \quad q$$

respectively. (*The definition of* q *is at the beginning of Section* 2.3; *that of* r_1, r_2 *before Proposition* 2.8; *that of* g *at the beginning of Section* 2.1.)

(b) $\mathrm{H}_i(\partial\mathcal{A}_2(n)^*;\mathbb{Z})$ *is torsion-free for* $i = 0,3,4$. *It has no torsion of order prime to* n *for* $i=1$, *and no torsion of order prime to* $2n$ *for* $i=2$.

(c) *As representations of* $G = P\Gamma_2(1)/P\Gamma_2(n)$, $\mathrm{H}_i(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$ *is as follows*:

$$\begin{cases} 1, & i=0,\\ \mathrm{Ind}_{P(h)}^G \mathrm{H}_1(N(h);\mathbb{Q}) + \mathrm{Ind}_{P(l)}^G \mathrm{H}_1(M(l);\mathbb{Q}) + \mathrm{H}_1(\mathfrak{T};\mathbb{Q}), & i=1,\\ \mathrm{Ind}_{P(h)}^G \mathrm{H}_2(C(h);\mathbb{Q}) + \mathrm{Ind}_{P(l)}^G \mathrm{H}_2(D^\circ(l);\mathbb{Q}) & \\ \qquad - \mathrm{Ind}_{P(h,l)}^G \mathrm{H}_1(f(h,l);\mathbb{Q}) - \mathrm{H}_1(\mathfrak{T};\mathbb{Q}) + 1, & i=2,\\ \mathrm{Ind}_{P(l)}^G \mathrm{H}_1(M(l);\mathbb{Q}), & i=3,\\ \mathrm{Ind}_{P(l)}^G(1), & i=4. \end{cases}$$

The definition of the Riemann surface $N(h)$ *is in the second paragraph after the proof of Proposition* 2.8.

REMARK 2.3. Note that the representations of $P(h)$ on $\mathrm{H}_*(C(h);\mathbb{Q})$ are identified in Section 2.2 and the representations of $P(l)$ on $\mathrm{H}_*(D^\circ(l);\mathbb{Q})$ are identified in Section 2.1 (see especially Corollary 2.5 and Propositions 2.7 and 2.9).

REMARK 2.4. If $n = p$ is an odd prime, the dimensions in (a) are

$$\begin{aligned}&1,\\&1 + ((p^4 - 1)/2)(2 + (p - 3)(p^2 - 1)/6),\\&((p^4 - 1)/2)(3 + (5p - 12)(p^2 - 1)/12),\\&(p^4 - 1)(1 + (p - 6)(p^2 - 1)/24),\\&(p^4 - 1)/2.\end{aligned}$$

For $p = 3$ these integers are 1, 81, 200, 0, and 40. For $p = 5$ they are 1, 3121, 9048, 0, and 312.

Proof of Theorem 2.13. In what follows, we will sometimes omit reference to the coefficient ring in our homology groups, but we will be careful to specify it when necessary. In all cases where group characters are involved, the coefficient ring will be $\mathbb{Q}$.

We apply Mayer–Vietoris to the decomposition $\partial\mathcal{A}_2(n)^* = C^\bullet \cup D^\circ$. We shall write $\mathrm{H}_*(C)$ rather than $\mathrm{H}_*(C^\bullet)$ and $\mathrm{H}_*(\partial C^\bullet)$ rather than $\mathrm{H}_*(C^\circ)$, using the fact that $C \to C^\bullet$ and $\partial C^\bullet \to C^\circ$ are homotopy equivalences. Then Mayer–Vietoris reads:

$$\ldots \to \mathrm{H}_i(\partial C^\bullet) \to \mathrm{H}_i(C) \oplus \mathrm{H}_i(D^\circ) \xrightarrow{\varphi_i} \mathrm{H}_i(\partial\mathcal{A}_2(n)^*) \to \ldots \tag{2}$$

A segment of this implies an exact sequence

$$0 \to \mathrm{Im}(\varphi_3) \to \mathrm{H}_3(\partial\mathcal{A}_2(n)^*) \to \mathrm{H}_2(\partial C^\bullet) \xrightarrow{i_2} \mathrm{H}_2(C) \oplus \mathrm{H}_2(D^\circ).$$

We analyze the image. For any fixed l we have an exact sequence coming from the Mayer–Vietoris applied to $D(l) = C^\bullet(l) \cup D^\circ(l)$:

$$0 \to \mathrm{H}_4(D(l)) \to \mathrm{H}_3(C^\circ(l)) \to \mathrm{H}_3(D^\circ(l)) \xrightarrow{m_l} \mathrm{H}_3(D(l)) \to \mathrm{H}_2(C^\circ(l)) \tag{3}$$

(as the term $\mathrm{H}_3(C^\bullet(l))$ vanishes), and where the first three terms have dimensions 1, t, and $2g + (t - 1)$ respectively, by Corollary 2.5. We may replace the last group above by 0, in other words, the map m_l is onto. With $\mathbb{Q}$-coefficients this follows because the dimension of the term

$$\mathrm{H}_3(D(l)) \simeq \mathrm{H}_1(D(l)) \simeq \mathrm{H}_1(M(l))$$

is $2g$. With $\mathbb{Z}$-coefficients now, if m_l were not onto, its cokernel would be finite by the previous argument. But this would inject into the last term. But this is a free $\mathbb{Z}$-module, because C° is a disjoint union of spaces whose homology is shown to have a free H_2 in the spectral sequence argument that comes next. Thus the cokernel of m_l is 0. In fact, with $\mathbb{Z}$-coefficients all terms in the above exact sequence are free $\mathbb{Z}$-modules of the indicated ranks.

Since $\partial\mathcal{A}_2(n)^*$ is a union of complex surfaces $D(l)$, which intersect in complex curves, it is easily seen by repeated application of Mayer–Vietoris that $\mathrm{H}_4(\partial\mathcal{A}_2(n)^*; \mathbb{Z})$ is a free $\mathbb{Z}$-module on the fundamental classes of the surfaces $D(l)$. The same argument shows that $\mathrm{H}^4(\partial\mathcal{A}_2(n)^*;\mathbb{Z})$ is a free $\mathbb{Z}$-module on the duals of these fundamental classes. The universal coefficient theorem ([41, Cor. 4, p. 244]) then shows that $\mathrm{H}_3(\partial\mathcal{A}_2(n)^*;\mathbb{Z})$ is free.

As G acts transitively on the $D(l)$ with stabilizer $P(l)$ it is clear that

$$\mathrm{H}_4(\partial\mathcal{A}_2(n)^*) = \mathrm{Ind}_{P(l)}^G \mathrm{H}_4(D(l)).$$

For essentially the same reason, easier here because the unions $\partial C^\bullet \simeq C^\circ = \bigcup C^\circ(l)$ and $D^\circ = \bigcup D^\circ(l)$ are disjoint, we have

$$\mathrm{H}_3(\partial C^\bullet) = \mathrm{Ind}_{P(l)}^G \mathrm{H}_4(C^\circ(l)), \qquad \mathrm{H}_3(D^\circ) = \mathrm{Ind}_{P(l)}^G \mathrm{H}_4(D^\circ(l)).$$

Applying the functor $\mathrm{Ind}_{P(l)}^G$ to the first three terms of the exact sequence (3) we thus get the segment of Mayer–Vietoris

$$0 \to \mathrm{H}_4(\partial\mathcal{A}_2(n)^*) \to \mathrm{H}_3(\partial C^\bullet) \to \mathrm{H}_3(D^\circ) \xrightarrow{\varphi_3} \dots$$

Therefore, $\mathrm{Ind}_{P(l)}^G$ applied to the entire exact sequence (3) shows that

$$\mathrm{Im}(\varphi_3) = \mathrm{Ind}_{P(l)}^G \mathrm{H}_3(D(l)) = \mathrm{Ind}_{P(l)}^G \mathrm{H}_1(M(l)).$$

Now for any fixed (h,l), $\partial C^\bullet(h,l)$ is an $S^1 \times S^1$-bundle over S^1 with monodromy $\gamma = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. The homology of this is computed from the spectral sequence of a fibration

$$\mathrm{E}^2_{p,q} = \mathrm{H}_p(S^1; \mathrm{H}_q(S^1 \times S^1; R)) \Rightarrow \mathrm{H}_*(\partial C^\bullet(h,l); R),$$

which for dimensional reasons degenerates at E^2. Local systems on S^1 are identified canonically with $R[\gamma]$-modules V and their homology is computed from the complex

$$V \xrightarrow{\gamma - 1} V,$$

so in the case at hand we see that

$$\mathrm{H}_i(\partial C^\bullet(h,l); R) = \begin{cases} \mathrm{E}^2_{0,0} = R & \text{for } i = 0, \\ \mathrm{E}^2_{0,1} \oplus \mathrm{E}^2_{1,0} = (R \oplus R/n) \oplus R & \text{for } i = 1, \\ \mathrm{E}^2_{0,2} \oplus \mathrm{E}^2_{1,1} = R \oplus R & \text{for } i = 2, \\ \mathrm{E}^2_{1,2} = R & \text{for } i = 3. \end{cases}$$

Lemma 2.14. *$i_2 : \mathrm{H}_2(\partial C^\bullet) \to \mathrm{H}_2(C) \oplus \mathrm{H}_2(D^\circ)$ is injective for the coefficient rings $R = \mathbb{Q}, \mathbb{Z}, \mathbb{Z}/p$ for p odd. The kernel of i_2 is nontrivial for $p = 2$.*

Proof. The above argument shows that $\mathrm{H}_2(\partial C^\bullet(h,l))$ is free of rank 2 over any ring. We can decompose the cohomology spaces as

$$\bigoplus_{h,l} \mathrm{H}_2(\partial C^\bullet(h,l)) \xrightarrow{i_2} \Big(\bigoplus_h \mathrm{H}_2(C(h))\Big) \oplus \Big(\bigoplus_l \mathrm{H}_2(D^\circ(l))\Big)$$

corresponding to disjoint unions of the various spaces involved. We are going to show that the map i_2 can be represented in an appropriate basis in the shape

$$\begin{pmatrix} \alpha_2 & 0 \\ * & \beta_2 \end{pmatrix}$$

and we will show that β_2 is injective and that $\mathrm{Ker}(\alpha_2)$ is 2-torsion, which suffices. From the spectral sequence argument above, we can write

$$\mathrm{H}_2(\partial C^\bullet(h,l)) = A(h,l) \oplus B(h,l),$$

where the first summand is represented by the class of a fiber of this bundle, and that class includes into $\mathrm{H}_2(C^\bullet(h,l))$ with image the sum of the fundamental classes of the $\mathbf{P}^1$'s in an n-gon, a class that is a direct summand of $\mathrm{H}_2(C^\bullet(h,l))$. The second generator is represented by a (trivial) S^1-bundle over S^1. We can give an explicit geometric representative for this as follows: In any fiber T^2 of the projection $\partial C^\bullet(h,l) \to S^1$ we let σ_1 be an $S^1 \subset T^2$ invariant under the monodromy γ and let σ_2 be the image of the base S^1 under any section s of this fibration (recall that there are n^2 global sections of the map $D(l) \to M(l)$ for any l). Then $\sigma_1 \times \sigma_2$ represents a generator of the summand $B(h,l)$. Since the base $S^1 \subset M(l)$ bounds a disk Δ in $M(l)$, we see that $\sigma_1 \times \sigma_2$ bounds $\sigma_1 \times s(\Delta)$ in $C^\bullet(h,l)$. In other words, the summand $B(h,l)$ maps to zero under

$$\mathrm{H}_2(\partial C^\bullet(h,l)) \to \mathrm{H}_2(C^\bullet(h,l)) \to \mathrm{H}_2(C^\bullet(h)) = \mathrm{H}_2(C(h)).$$

Thus, the map i_2 does have the matrix form as claimed relative to this splitting.

The map β_2: A segment of the Mayer–Vietoris sequence for a single boundary component $D(l) = C^\bullet(l) \cup D^\circ(l)$ is

$$\begin{aligned} 0 \to \mathrm{H}_4(D(l)) &\to \mathrm{H}_3(\partial C^\bullet(l)) \to \mathrm{H}_3(D^\circ(l)) \\ &\to \mathrm{H}_3(D(l)) \to \mathrm{H}_2(\partial C^\bullet(l)) \xrightarrow{j_2(l)} \mathrm{H}_2(C^\bullet(l)) \oplus \mathrm{H}_2(D^\circ(l)) \end{aligned}$$

(as $\mathrm{H}_3(C^\bullet(l)) = 0$). The ranks of the first 4 of these are respectively $1, t$, $2g + (t-1), 2g$, independently of the coefficient ring R we are considering (which in all cases we are considering is a field), showing that the map $j_2(l)$ is an injection. Note that

$$\mathrm{H}_2(C^\bullet(l)) = \bigoplus_{h \supset l} \mathrm{H}_2(\partial C^\bullet(h,l)).$$

Restricted to the subspace $\bigoplus_{h \supset l} B(h,l)$, this is precisely $0 \oplus \beta_2$, so β_2 is an injection.

The map α_2: This is a sum over h of components

$$\bigoplus_{l \subset h} A(h,l) \to \mathrm{H}_2(C(h)).$$

$C(h)$ is a polyhedron of $\mathbf{P}^1$'s, and $\mathrm{H}_2(C(h))$ is free on the fundamental classes of the $\mathbf{P}^1$'s in it. The image of each $A(h,l) = \mathbb{Q} \cdot a(h,l)$ is generated by the cycle of $\mathbf{P}^1$'s in one face of this polyhedron. That the above map is injective means that these face-cycles $a(h,l)$ are linearly independent. This can be seen as follows. Suppose that there were a relation

$$\sum_{l \subset h} m_l \cdot a(h,l) = 0.$$

Consider any vertex of $C(h)$. This is a triple l_1, l_2, l_3 with $h = l_1 \wedge l_2$ and $l_3 = l_1 \pm l_2$ (recall that all of these are well defined up to multiplication by ± 1.) The given relation implies that the coefficients m_l cancel along each of the three edges emanating from this vertex, in other words that

$$m_{l_1} + m_{l_2} = 0, \qquad m_{l_1} + m_{l_3} = 0, \qquad m_{l_2} + m_{l_3} = 0,$$

which implies that all three are zero provided we are not in characteristic 2, where there visibly is a kernel. As this is true at all vertices, the m_l are all 0 in characteristic not 2. Thus the map α_2 is an injection in these characteristics as well. ∎

Since i_2 is an injection we see that

$$\mathrm{Im}(\varphi_3) = \mathrm{H}_3(\partial \mathcal{A}_2(n)^*) = \mathrm{Ind}_{P(l)}^{G} \mathrm{H}_1(M(l))$$

as claimed in the theorem for $i = 3$, where the first equality holds with $\mathbb{Z}$-coefficients, and $\mathbb{Q}$-coefficients are understood in the second equality.

Examining this further, we see that the subspace of $\mathrm{H}_2(\partial C^\bullet; \mathbb{Q})$ spanned by the first generators is

$$\bigoplus_{h,l} A(h,l) = \mathrm{Ind}_{P(h,l)}^{G}(1),$$

since as we have noted, the first generator is invariant under the automorphisms of $\partial C^\bullet(h,l)$. Also, $\mathrm{H}_2(C) = \mathrm{Ind}_{P(h)}^{G} \mathrm{H}_2(C(h))$.

Now we must identify the subspace spanned by the second generators of $\mathrm{H}_2(\partial C^\bullet; \mathbb{Q})$. As we have just observed, for any fixed pair $(h,\ l)$ this generator is represented by an S^1-bundle over S^1. Under the inclusion of $\partial C^\bullet$ into $C^\bullet$, the base includes trivially, but the fiber includes to a path representing a generator of $\mathrm{H}_1(f(h,l))$. Put another way, the exact sequence of the pair $(C^\bullet, \partial C^\bullet)$ and the Alexander duality theorem gives an equivariant (for $P(h,l)$) identification

$$\begin{aligned} B(h,l) = \mathrm{H}_3(C^\bullet(h,l); \partial C^\bullet(h,l)) &= \mathrm{H}^1_{\mathrm{c}}(C^\bullet(h,l) - \partial C^\bullet(h,l)) \\ &\cong \mathrm{H}^1(f(h,l)) \cong \mathrm{H}_1(f(h,l)). \end{aligned}$$

Therefore the space of second generators is

$$\bigoplus_{h,l} B(h,l) = \mathrm{Ind}_{P(h,l)}^{G} \mathrm{H}_1(f(h,l)),$$

and we have described how $P(h,l)$ acts on $\mathrm{H}_1(f(h,l))$ in the previous section (see the proof of Corollary 2.6).

Also, $\mathrm{H}_2(D^\circ) = \mathrm{Ind}_{P(l)}^{G} \mathrm{H}_2(D^\circ(l))$ and we have described the action of $P(l)$ on $\mathrm{H}_2(D^\circ(l))$ in the previous section (see Corollaries 2.5, 2.6, and Proposition 2.7).

If we examine the continuation of the Meyer–Vietoris starting at the term $\mathrm{H}_2(\partial C^\bullet)$, the above results show that, as representations,

$$\begin{aligned}\mathrm{Coker}(i_2) &= \mathrm{H}_2(C) + \mathrm{H}_2(D^\circ) - \mathrm{H}_2(\partial C^\bullet)\\ &= \mathrm{Ind}_{P(h)}^{G} \mathrm{H}_2(C(h)) + \mathrm{Ind}_{P(l)}^{G} \mathrm{H}_2(D^\circ(l))\\ &\qquad - \mathrm{Ind}_{P(h,l)}^{G}(1) - \mathrm{Ind}_{P(h,l)}^{G} \mathrm{H}_1(f(h,l)).\end{aligned}$$

Now we consider $\mathbb{Z}$-coefficients again. We have an exact sequence:

$$0 \to \mathrm{Coker}(i_2) \to \mathrm{H}_2(\partial\mathcal{A}_2(n)^*) \to \mathrm{H}_1(\partial C^\bullet) \xrightarrow{i_1} \mathrm{H}_1(C) \oplus \mathrm{H}_1(D^\circ).$$

This sequence shows that the torsion of $\mathrm{H}_2(\partial\mathcal{A}_2(n)^*)$ will be contained in that of $\mathrm{Coker}(i_2)$ and $\mathrm{H}_1(\partial\mathcal{A}_2(n)^*)$. The latter has no torsion prime to n as we have seen in the spectral sequence argument preceding Lemma 2.14. As to the former, the following lemma shows that it has only 2-power torsion. Thus, $\mathrm{H}_1(\partial C^\bullet)$ has no torsion prime to $2n$.

LEMMA 2.15. $\mathrm{Coker}(i_2)$ *has no torsion prime to* 2*. Its* 2*-power torsion is nontrivial.*

Proof. We may consider i_2 as defining the differential in a 2-term complex of free $\mathbb{Z}$-modules of finite type, $K = [K_1 \to K_0]$. Lemma 2.14 shows that $\mathrm{H}_1(K \otimes \mathbb{Z}/p) = 0$ for all odd primes, but that it is nonzero for $p = 2$. Since $\mathrm{H}_1(K) = 0$, the universal coefficient theorem [41, Thm. 8, p. 222] shows that

$$\mathrm{H}_1(K \otimes \mathbb{Z}/p) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_0(K), \mathbb{Z}/p),$$

which is nonzero if and only if $p = 2$. Of course, $\mathrm{H}_0(K) = \mathrm{Coker}(i_2)$. ∎

We now study $\mathrm{H}_1(\partial\mathcal{A}_2(n)^*)$. We must analyze the last map i_1 above. In what follows, our coefficient ring is such that n is invertible. Again, for each (h,l), $\mathrm{H}_1(\partial C^\bullet(h,l))$ is two-dimensional, and we split this into two subspaces according to the spectral sequence of a fibration. The first generator is represented by an S^1 in a fiber, and the second is represented by a section (the fiber bundle has a section as it is the restriction of the $S^1 \times S^1$-bundle over $D^\circ(l)$, which we have seen has sections). Now $\mathrm{H}_1(C^\bullet(h,l))$ is one-dimensional (recall that $C^\bullet(h,l)$ is an n-gon) and the inclusion $\partial C^\bullet(h,l) \hookrightarrow C^\bullet(h,l)$

sends the first generator to the generator of $\mathrm{H}_1(C^\bullet(h,l);\mathbb{Q})$ and the second generator to zero, as the section over $D^\circ(l)$ extends to a section over $D(l)$, so this second generator bounds a disk in $C^\bullet(h,l)$. On the other hand, in the inclusion $\partial C^\bullet(h,l) \hookrightarrow D^\circ(l)$ the first generator goes to zero, as we see from the commutative diagram

$$\begin{tikzcd} \mathrm{H}_1(S^1;\mathbb{Q}) \arrow[r] \arrow[d] & \mathrm{H}_1(D^\circ;\mathbb{Q}) \arrow[d, "\simeq"] \\ \mathrm{H}_1(\mathrm{pt};\mathbb{Q}) \arrow[r] & \mathrm{H}_1(M^\circ;\mathbb{Q}) \end{tikzcd}$$

whereas we may compute the image of the second generator by considering its image in M°, from the diagram

$$\begin{tikzcd} & \mathrm{H}_1(D^\circ;\mathbb{Q}) \arrow[dd, "\simeq"] \\ \mathrm{H}_1(S^1;\mathbb{Q}) \arrow[ur] \arrow[dr] & \\ & \mathrm{H}_1(M^\circ;\mathbb{Q}) \end{tikzcd}$$

In other words, if we let A (resp. B) be the subspace of $\mathrm{H}_1(\partial C^\bullet)$ spanned by the first (resp. second) generators, then the map from $\mathrm{H}_1(\partial C^\bullet)$,

$$\bigoplus_{h,l} \mathrm{H}_1(\partial C^\bullet(h,l)) \xrightarrow{i_1} \Big(\bigoplus_h \mathrm{H}_1(C(h))\Big) \oplus \Big(\bigoplus_l \mathrm{H}_1(D^\circ(l))\Big),$$

can be represented as a block matrix

$$i_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{pmatrix}$$

Thus it remains to calculate α_1 and β_1.

The map α_1: Fix h and consider $C(h)$. As we have observed in Section 2.2 there is an isomorphism $\mathrm{H}_1(C(h)) \simeq \mathrm{H}_1(C'(h))$. Each first generator is a path around an n-gon and is represented by a cycle

$$c(h,l) = d(l,l') + d(l,l'+l) + \ldots + d(l,l'+(n-1)l)$$

in $C'(h)$, well defined up to sign. But $C'(h)$ is the 1-skeleton of a cell decomposition of the oriented surface $N(h)$, and each of these cycles is the boundary of a 2-cell in $N(h)$. Hence there is a single relation between them: The sum of all these cycles, with signs chosen suitably, is zero. Thus the map α_1 has kernel a 1-dimensional trivial representation of $P(h)$ on each $C(h)$. We also see that, for fixed h, the map α_1 is exactly the boundary map on 2-cells of $N(h)$ in a complex for the cellular homology of $N(h)$, so its cokernel is isomorphic to $\mathrm{H}_1(N(h))$. Hence we see that

$$\mathrm{Ker}(\alpha_1) = \mathrm{Ind}_{P(h)}^G(1), \quad \mathrm{Coker}(\alpha_1) = \mathrm{Ind}_{P(h)}^G \mathrm{H}_1(N(h)).$$

The map β_1: Fix l and consider $D(l)$. As we observed earlier in this proof, we may calculate β_1 by looking at it on $\mathrm{H}_1(M^\circ(l)) \simeq \mathrm{H}_1(D^\circ(l))$. Here it is the inclusion of the loops around the t punctures of the oriented surface into this surface, so there is again a single relation between them: The sum of all these cycles is zero. Thus the map β_1 has kernel a 1-dimensional trivial representation of $P(l)$ on each $D^\circ(l)$. We also see that, for fixed l, the cokernel of the inclusion of the classes of the loops around the punctures of $M^\circ(l)$ into $\mathrm{H}_1(M^\circ(l))$ is just $\mathrm{H}_1(M^\circ)$. Hence we see that

$$\mathrm{Ker}(\beta_1) = \mathrm{Ind}_{P(l)}^G(1), \quad \mathrm{Coker}(\beta_1) = \mathrm{Ind}_{P(l)}^G \mathrm{H}_1(M(l)).$$

Applied to the coefficient rings $\mathbb{Z}/p$ for primes p not dividing n, these arguments show that $\mathrm{Ker}(i_1)$ has a rank independent of p and equal to the rank of $\mathrm{Ker}(i_1)$ when computed with coefficients $\mathbb{Z}[1/n]$. This is free, since it is a submodule of a free $\mathbb{Z}[1/n]$-module.

We have an exact sequence:

$$0 \to \mathrm{Coker}(i_1) \to \mathrm{H}_1(\partial \mathcal{A}_2(n)^*) \to \mathrm{H}_0(\partial C^\bullet) \xrightarrow{i_0} \mathrm{H}_0(C) \oplus \mathrm{H}_1(D^\circ).$$

Since $\mathrm{H}_0(\partial C^\bullet)$ is torsion-free, the only possible torsion in $\mathrm{H}_1(\partial \mathcal{A}_2(n)^*)$ comes from $\mathrm{Coker}(i_1)$. The next lemma shows that this is contained in the divisors of n:

LEMMA 2.16. $\mathrm{Coker}(i_1)$ *has no torsion prime to* n.

Proof. We may consider i_1 as defining the differential in a 2-term complex of free $\mathbb{Z}[1/n]$-modules of finite type, $K = [K_1 \to K_0]$. According to [41, Thm. 8, p. 222] we have an exact sequence

$$0 \to \mathrm{H}_1(K) \otimes R \xrightarrow{\mu} \mathrm{H}_1(K \otimes R) \to \mathrm{Tor}_1^{\mathbb{Z}[1/n]}(\mathrm{H}_0(K), R) \to 0$$

for any $\mathbb{Z}[1/n]$-module R. The arguments in the previous paragraph show that μ is an isomorphism for $R = \mathbb{Z}/p$, where p is a prime not dividing n. Thus the Tor term is zero for these. Therefore, $\mathrm{H}_0(K) = \mathrm{Coker}(i_1)$ has no torsion prime to n. ∎

To complete the analysis of $\mathrm{H}_1(\partial \mathcal{A}_2(n)^*)$ we must look at the terms that follow it in the Mayer–Vietoris sequence:

$$\mathrm{H}_0(\partial C^\bullet) \xrightarrow{i_0} \mathrm{H}_0(C^\bullet) \oplus \mathrm{H}_0(D^\circ) \to \mathrm{H}_0(\partial \mathcal{A}_2(n)^*) \to 0.$$

As representations of G, this is the sequence

$$\mathrm{Ind}_{P(h,l)}^G(1) \to \mathrm{Ind}_{P(h)}^G(1) \oplus \mathrm{Ind}_{P(l)}^G(1) \to 1 \to 0,$$

so

$$\mathrm{Ker}(i_0) = \mathrm{Ind}_{P(h,l)}^G(1) - \mathrm{Ind}_{P(h)}^G(1) - \mathrm{Ind}_{P(l)}^G(1) + 1,$$

and by Lemma 2.11 this is simply $\mathrm{H}_1(\mathfrak{T})$. Note that this part of the sequence is exact over $\mathbb{Z}$, though the representations do not split.

We have

$$\begin{aligned}\mathrm{H}_2(\partial\mathcal{A}_2(n)^*) &= \mathrm{Coker}(i_2)\oplus\mathrm{Ker}(i_1) = \mathrm{Coker}(i_2)\oplus\mathrm{Ker}(\alpha_1)\oplus\mathrm{Ker}(\beta_1),\\ \mathrm{H}_1(\partial\mathcal{A}_2(n)^*) &= \mathrm{Coker}(i_1)\oplus\mathrm{Ker}(i_0) = \mathrm{Coker}(\alpha_1)\oplus\mathrm{Coker}(\beta_1)\oplus\mathrm{Ker}(i_0).\end{aligned}$$

Then, assembling and identifying all the pieces, Theorem 2.13 follows. ∎

Remark 2.5. We may alternatively consider the projectivization $P\mathfrak{T}$ of $\mathfrak{T}$. For $n \geq 3$ the map $P\mathfrak{T} \to \mathfrak{T}$ is $\varphi(n)/2$-to-1. (It is 1-1 for $n = 1, 2$.) The group $\mathbf{PSp}(4, \mathbb{Z}/n)$ acts on $\mathrm{H}_1(P\mathfrak{T};\mathbb{Q})$, and the representation so obtained is known as the *Steinberg representation.* We denote it by St_n. For $n = p$ prime, St_p is known to be the unique irreducible representation of $\mathbf{PSp}(4, \mathbb{Z}/p)$ of degree p^4. In particular, $\mathrm{H}_1(\mathfrak{T}(\mathbb{Z}/n);\mathbb{Q}) = \mathrm{St}_n$ for $n = 3$, but these differ for $n > 3$.

2.4. *The mixed Hodge structure.* We will determine the weight filtrations on the mixed Hodge structures of the cohomology of the boundary. Let D_i, $i \in I$, be the components of the boundary. The indexing set I is a set of vertices in the building $\mathfrak{T}(\mathbb{Z}/n)$, and $\#I = q$. For each $J \subset I$ we let

$$D_J = \bigcap_{j\in J} D_j$$

and for each $a \geq 0$ the disjoint union

$$D^{[a]} = \coprod_{\#J=a+1} D_J.$$

These are smooth projective varieties, which as a varies, form a simplicial scheme in a natural way which "resolves" $\partial\mathcal{A}_2(n)^*$. In this case, $D^{[0]}$ is the disjoint union of q copies of the boundary component D, $D^{[1]}$ is the disjoint union of qr_2 copies of $\mathbf{P}^1$, $D^{[2]}$ is the disjoint union of $2q(r_1 - 1) = qnt/3$ copies of a point, and all other $D^{[a]}$ are empty.

We have a spectral sequence

$$\mathrm{E}_1^{a,b} = \mathrm{H}^b(D^{[a]};\mathbb{Q}) \Rightarrow \mathrm{H}^{a+b}(\partial\mathcal{A}_2(n)^*;\mathbb{Q}).$$

It is a theorem of Deligne [11] that $\mathrm{E}_2 = \mathrm{E}_\infty$, and the weight filtration

$$W_0 \subset \ldots \subset W_i = \mathrm{H}^i(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$$

has

$$\mathrm{Gr}_a^W = W_a/W_{a-1} = \mathrm{E}_\infty^{i-a,a}.$$

The dimensions of the E_1 terms are known and the only nontrivial differentials to determine are

$$\mathrm{E}_1^{*,0} : \mathrm{H}^0(D^{[0]};\mathbb{Q}) \xrightarrow{d_1} \mathrm{H}^0(D^{[1]};\mathbb{Q}) \to \mathrm{H}^0(D^{[2]};\mathbb{Q}), \tag{4}$$

the dimensions of whose terms are q, qr_2, $2q(r_1-1)$, respectively, and

$$\mathrm{E}_1^{*,2} : \mathrm{H}^2(D^{[0]};\mathbb{Q}) \xrightarrow{d_2} \mathrm{H}^2(D^{[1]};\mathbb{Q}) \to \mathrm{H}^2(D^{[2]};\mathbb{Q}) = 0, \tag{5}$$

whose dimensions are $q(4g+nt-2)$, qr_2 and 0 (see Corollary 2.2). It is known that $\partial\mathcal{A}_2(n)^*$ is connected, so that

$$\mathrm{Ker}(d_1) = \mathrm{E}_2^{0,0} = \mathrm{H}^0(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$$

is one-dimensional, spanned by the function that is 1 on all components D_i. For the right-hand side of the sequence (4) we can break it up into a sum of complexes indexed by the corank 2 boundary components h. Namely, fix an h, and only consider those boundary components $D(l)$ such that $l \subset h$. The resulting subcomplex

$$\bigoplus \mathrm{H}^0(D(l_1,l_2);\mathbb{Q}) \xrightarrow{d_3} \bigoplus \mathrm{H}^0(D(l_1,l_2,l_3);\mathbb{Q}) \tag{6}$$

of complex (4) has $\mathrm{Coker}(d_3)$ equal to the homology in degree 2 of (4) because the $C(h)$ for various h are disjoint. In the above sequence the

$$D(l_1,l_2) = D(l_1) \cap D(l_2) \simeq \mathbf{P}^1$$

are the various pairwise intersections of boundary components $D(l)$, $l \subset h$. Similarly, the $D(l_1,l_2,l_3)$, which are points, are the various threefold intersections of boundary components. There are r_2 $\mathbf{P}^1$'s in $C(h)$, and $2(r_1-1)$ such triple points (see the proof of Proposition 2.8). It is known that the nerve of $C(h)$ is that of the 1-skeleton of a tessellation by n-gons of the compact oriented surface $N(h)$ of genus equal to $g = (r_1+1-t)/2$, the genus of the modular curve $\mathcal{A}_1(n)^*$. Thus, $\mathrm{Coker}(d_3)$ is the same as the second homology of that surface, whose dimension is 1. Since this holds for each of the q h's, we have $\dim \mathrm{E}_2^{2,0} = q$. By taking Euler characteristics, we see that $\dim \mathrm{E}_2^{1,0} = q(r_2 - 2r_1 + 2) + 1$.

We now claim that d_2 is surjective, which gives $\mathrm{E}_2^{1,2} = 0$ and

$$\begin{aligned}\dim \mathrm{E}_2^{0,2} &= q \dim \mathrm{H}^2(D;\mathbb{Q}) - qr_2 \dim \mathrm{H}^2(\mathbf{P}^1;\mathbb{Q}) = q((4g+nt-2)-r_2)\\ &= q(2r_1 - r_2 + (n-2)t).\end{aligned}$$

To prove the surjectivity of d_2, fix any one $L = D(l_1,l_2) \simeq \mathbf{P}^1$. We will show the existence of a class γ in $D^{[0]}$ restricting to a generator of $\mathrm{H}^2(L)$ and zero on any other $D(l_1',l_2')$. The various $D(l_1,l_2)$, call them simply $\mathbf{P}^1$'s, belong to the disjoint configurations $C(h)$, so in this argument we need only consider those $\mathbf{P}^1$'s belonging to the same $C(h)$ as L, namely $h = l_1 \wedge l_2$. In fact, we will only need three boundary components, $D(l_1)$, $D(l_2)$, $D(l_3)$, where l_3 is chosen so that $D(l_1,l_2,l_3)$ is a triple point of $C(h)$, i.e., l_3 is in the span of l_1 and l_2. We claim that there is a class $c_1 \in \mathrm{H}^2(D(l_1))$ such that $\varphi|D(l_1,l_2)$ is the generator (12) of $\mathrm{H}^2(D(l_1,l_2))$ and $\varphi|D(l_1,l_3)$ is $-(13)$ in $\mathrm{H}^2(D(l_1,l_3))$ and restricts to 0 on any other $\mathbf{P}^1$ contained in $D(l_1)$. Similarly

$c_2 \in \mathrm{H}^2(D(l_2))$ restricts to $(12)-(23)$, and c_3 to $(23)-(13)$. Accept this for the moment. We let $\gamma_1 \in \mathrm{H}^2(D^{[0]})$ stand for the class that is c_1 in $\mathrm{H}^2(D(l_1))$ and 0 in all other $\mathrm{H}^2(D(l))$, and similarly for γ_2, γ_3. In defining the spectral sequence, we have chosen an ordering of the l's. Without loss of generality, assume that $l_1 < l_2 < l_3$. Then, for $\varphi \in \mathrm{H}^2(D^{[0]}) = \bigoplus \mathrm{H}^2(D(l))$,

$$d_2\varphi | D(l_i, l_j) = \mathrm{Res}^{D(l_i)}_{D(l_i,l_j)}(\varphi) - \mathrm{Res}^{D(l_j)}_{D(l_i,l_j)}(\varphi),$$

where $l_i < l_j$ is understood. Define $\gamma = (\gamma_1 - \gamma_2 + \gamma_3)/2$. One checks that $d_2\gamma$ is the generator (12) on $D(l_1, l_2) = L$, and is zero on $D(l_1, l_3)$ and $D(l_2, l_3)$, and on any other $\mathbf{P}^1$, and is thus the required class. For instance,

$$\begin{aligned} d_2\gamma | D(l_1, l_2) &= \mathrm{Res}^{D(l_1)}_{D(l_1,l_2)}(\gamma) - \mathrm{Res}^{D(l_2)}_{D(l_1,l_2)}(\gamma) \\ &= (\mathrm{Res}^{D(l_1)}_{D(l_1,l_2)}(c_1) - \mathrm{Res}^{D(l_2)}_{D(l_1,l_2)}(-c_2))/2 \\ &= ((12) + (12))/2 = (12). \end{aligned}$$

To complete the proof we must show that a class such as c_1 exists. Recall that the various $D(l_1, l_2)$ in D are projective lines appearing in the n-gons lying over the t cusps of the modular curve. The class c_1 lives on components $D(l_1, l_2)$, $D(l_1, l_3)$ of an n-gon of $\mathbf{P}^1$'s lying over a cusp, namely the n-gon corresponding to $h = l_1 \wedge l_2 \supset l_1$. The total degree of this class on this n-gon is 0, while it vanishes identically on any other n-gon. Thus the claim follows from the following lemma:

Lemma 2.17. *Let D be the elliptic modular surface of level n, and fix any cusp of the modular curve M of level n. Recall the morphism $\pi : D \to M$ whose fiber over any cusp is an n-gon of $\mathbf{P}^1$'s. Let k_i be a collection of integers, $i = 1, \ldots, n$. There exists an element $c \in \mathrm{H}^2(D; \mathbb{Q})$ such that*

(1) *$\deg(c|L_i) = k_i$ for each i, where L_i are the $\mathbf{P}^1$'s lying over this cusp,*
(2) *$c|L = 0$ for any $\mathbf{P}^1$ lying over a different cusp,*

if and only if $\sum k_i = 0$.

Proof. Recall that there is a canonical isomorphism

$$\deg : \mathrm{H}^2(\mathbf{P}^1; \mathbb{Q}) \simeq \mathbb{Q}.$$

The condition is necessary: Because H^2 of the projective line is purely of Hodge type $(1,1)$, the only classes in $\mathrm{H}^2(D)$ that can restrict nontrivially to a projective line L lying in D must themselves have Hodge type $(1,1)$. Such a class c is known to be the class of an algebraic cycle C on D (see [40]). But in that case, $\deg(c|L)$ is the intersection number $C \cdot L$. The sum of these intersection numbers for the L's in a cuspidal fiber is the same for all cusps because, being fibers of π, they are all homologically equivalent. Any c satisfying condition (2) in the lemma has total degree 0 over every cusp except the chosen one, so it has to have total degree 0 on the chosen one as well.

To see sufficiency, we will construct c as a linear combination of the cycle classes of the L_i. These clearly have intersection number 0 with the L's above any other cusp. Now each L_i has self-intersection -2 and intersection number 1 with its two adjacent L_j's and intersection number 0 for all the rest. This leads to an n by n intersection matrix as follows:

$$\begin{pmatrix} -2 & 1 & 0 & 0 & \dots & 1 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 1 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}.$$

Each row (and column) sums to 0, reflecting the fact that each element in the span of the cycles L_i has total degree 0 in the n-gon. An inductive computation shows that the determinant of the upper $(n-1)$ by $(n-1)$ minor is $(-1)^{n-1}n$. Hence the rank of the above matrix is $n-1$, so that any vector (k_i) summing to 0 is in the column span, proving the lemma. ∎

We have now proved:

PROPOSITION 2.18. *Let* $h^i_j = \dim \mathrm{Gr}^W_j(\mathrm{H}^i(\partial\mathcal{A}_2(n)^*;\mathbb{Q}))$. *Then*

$$\begin{aligned} &h^0_0 = 1, \\ &h^1_0 = 2qg, \quad h^1_1 = q(r_2 - 2r_1 + 2) + 1, \\ &h^2_0 = q, \qquad h^2_1 = 0, \quad h^2_2 = q(4g + nt - r_2 - 2), \\ &h^3_3 = 2qg, \\ &h^4_4 = q, \end{aligned}$$

all others being 0.

The reader can check that these numbers give the same dimensions as Theorem 2.13(a).

2.5. *Further results.* We have already observed that, by the work of Kazhdan [24], for any $d \geq 2$ and any n, $\mathrm{H}^1(\mathit{\Gamma}_d(n);\mathbb{Q}) = 0$, or equivalently, $\mathrm{H}_1(\mathit{\Gamma}_d(n);\mathbb{Z})$ is finite. We refine this.

PROPOSITION 2.19. *For any* $d \geq 2$ *and any* n, $\mathrm{H}_1(\mathit{\Gamma}_d(n);\mathbb{Z})$ *is a finite group of exponent dividing* $2n$.

Proof. For ease of notation we prove this for $d = 2$; the general case is immediate.

Mennicke [31] has shown that, as a subgroup of $\Gamma_2(1)$, $\Gamma_2(n)$ is normally generated by the single element

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & n \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and similarly for $\Gamma_d(n),\ d \geq 2$. Let

$$a = \begin{pmatrix} 1 & 0 & 0 & 0 \\ n & 1 & 0 & 0 \\ 0 & 0 & 1 & -n \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 & 0 & n \\ 0 & 1 & n & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that $a, b \in \Gamma_2(n)$. Then direct calculation shows that $a^{-1}b^{-1}ab = g^{2n}$. This proves the claim since $\mathrm{H}_1(\Gamma_d(n);\mathbb{Z})$ is the commutator quotient $\Gamma_d(n)/[\Gamma_d(n),\Gamma_d(n)]$. ∎

Proposition 2.20. $\mathrm{H}^4(\Gamma_2(n);\mathbb{Q})$ *contains* $\mathrm{H}_1(\mathfrak{T}(\mathbb{Z}/n);\mathbb{Q})$.

Proof. We have $\mathrm{H}^4(\Gamma_2(n);\mathbb{Q}) = \mathrm{H}_2(\mathcal{A}_2(n)^*, \partial\mathcal{A}_2(n)^*;\mathbb{Q})$ and the exact sequence

$$\mathrm{H}_2(\mathcal{A}_2(n)^*, \partial\mathcal{A}_2(n)^*;\mathbb{Q}) \to \mathrm{H}_1(\partial\mathcal{A}_2(n)^*;\mathbb{Q}) \to \mathrm{H}_1(\mathcal{A}_2(n)^*;\mathbb{Q}).$$

But $\mathrm{H}_1(\partial\mathcal{A}_2(n)^*;\mathbb{Q})$ contains $\mathrm{H}_1(\mathfrak{T}(\mathbb{Z}/n);\mathbb{Q})$ by Lemma 2.12, while we have $\mathrm{H}_1(\mathcal{A}_2(n)^*;\mathbb{Q}) = 0$ as $\mathcal{A}_2(n)^*$ is simply connected [19]. ∎

Using our detailed computations in Theorem 2.13, we may refine this.

Proposition 2.21. $\dim \mathrm{H}^4(\Gamma_2(n);\mathbb{Q}) \geq q(r_1 + 2g - 1) + 1$.

Proof. Same as that of the above proposition, noting that

$$\dim \mathrm{H}_1(\partial\mathcal{A}_2(n)^*;\mathbb{Q}) = q(r_1 + 2g - 1) + 1$$

by Theorem 2.13. ∎

Remark 2.6. For $n = p$ an odd prime, the lower bound in Proposition 2.21 is

$$q(r_1 + 2g - 1) + 1 = \frac{p^4 - 1}{2}\,\frac{p^3 - 3p^2 - p + 15}{6} + 1.$$

For $p = 3$ this lower bound is 81, and this is indeed the dimension of $\dim \mathrm{H}^4(\Gamma_2(n);\mathbb{Q})$, as shown in Theorem 3.2 below.

Adem in [1] has also investigated this question. His methods and interests in [1] are different from ours, but he obtains a lower bound for $\dim \mathrm{H}^4(\Gamma_2(p);\mathbb{Q})$ which is asymptotic to ours. Actually, his bound is slightly sharper, but there is unfortunately an error in his calculations, as his lower bound in case $p = 3$ is 161, which is too high.

REMARK 2.7. For $n = p$ an odd prime, $\dim \mathrm{H}_1(\mathfrak{T}(\mathbb{Z}/p); \mathbb{Q})$ is asymptotic to $p^6/4$, while the lower bound in Proposition 2.21 is asymptotic to $p^7/12$.

PROPOSITION 2.22. $\dim \mathrm{H}_3(\mathcal{A}_2(n)^*; \mathbb{Q}) \geq q(r_2 + 4g - 1)$.

Proof. Oda and Schwermer [34] have shown that the map

$$\mathrm{H}^2(\mathcal{A}_2(n)^*; \mathbb{Q}) \to \mathrm{H}^2(\mathcal{A}_2(n); \mathbb{Q})$$

is an epimorphism. Dualizing, we see that

$$\mathrm{H}_4(\mathcal{A}_2(n)^*; \mathbb{Q}) \to \mathrm{H}_4(\mathcal{A}_2(n)^*, \partial\mathcal{A}_2(n)^*; \mathbb{Q})$$

is an epimorphism, so the exact sequence of the pair $(\mathcal{A}_2(n)^*, \partial\mathcal{A}_2(n)^*)$ shows that the map

$$\mathrm{H}_3(\partial\mathcal{A}_2(n)^*; \mathbb{Q}) \to \mathrm{H}_3(\mathcal{A}_2(n)^*; \mathbb{Q})$$

is a monomorphism, and the conclusion is immediate from Theorem 2.13. ∎

REMARK 2.8. In particular, this shows that $\mathrm{H}_3(\mathcal{A}_2(n)^*; \mathbb{Q})$ is nonzero for $n \geq 6$. Our lower bound is asymptotic to $p^7/24$ for $n = p$ an odd prime.

3. The case $n = 3$

THEOREM 3.1. *The cohomology of the principal congruence subgroup of level* 3 *in* $\mathbf{Sp}(4, \mathbb{Z})$ *is given by*:

$$\begin{aligned}
\mathrm{H}^0(\Gamma_2(3); \mathbb{Z}) &= \mathbb{Z}, \\
\mathrm{H}^1(\Gamma_2(3); \mathbb{Z}) &= 0, \\
\mathrm{H}^2(\Gamma_2(3); \mathbb{Z}) &= \mathbb{Z}^{21} \oplus (\mathbb{Z}/3)^{10} \oplus \mathbb{Z}/2, \\
\mathrm{H}^3(\Gamma_2(3); \mathbb{Z}[1/6]) &= \mathbb{Z}[1/6]^{139}, \\
\mathrm{H}^4(\Gamma_2(3); \mathbb{Z}[1/3]) &= \mathbb{Z}[1/3]^{81}, \\
\mathrm{H}^i(\Gamma_2(3); \mathbb{Z}) &= 0 \quad \textit{for } i > 4.
\end{aligned}$$

REMARK 3.1. These calculations immediately translate into results for homology:

$$\begin{aligned}
\mathrm{H}_0(\Gamma_2(3); \mathbb{Z}) &= \mathbb{Z}, \\
\mathrm{H}_1(\Gamma_2(3); \mathbb{Z}) &= (\mathbb{Z}/3)^{10} \oplus \mathbb{Z}/2, \\
\mathrm{H}_2(\Gamma_2(3); \mathbb{Z}[1/6]) &= \mathbb{Z}[1/6]^{21}, \\
\mathrm{H}_3(\Gamma_2(3); \mathbb{Z}[1/3]) &= \mathbb{Z}[1/3]^{139}, \\
\mathrm{H}_4(\Gamma_2(3); \mathbb{Z}) &= \mathbb{Z}^{81}, \\
\mathrm{H}_i(\Gamma_2(3); \mathbb{Z}) &= 0 \quad \text{for } i > 4.
\end{aligned}$$

For $i = 1, 4$ these confirm the results of MacPherson–McConnell [30, Section 10], which were obtained by massive computer calculation.

Proof of Theorem 3.1. We consider the exact sequence of the pair $(\mathcal{A}_2(3)^*, \partial\mathcal{A}_2(3)^*)$. In [21, Thm. 1.1] we computed that $\mathrm{H}_i(\mathcal{A}_2(3)^*) = \mathbb{Z}, 0, \mathbb{Z}^{61}, 0, \mathbb{Z}^{61}, 0, \mathbb{Z}$ for $i = 0, \dots, 6$. Theorem 2.13 shows that

$$\mathrm{H}_0(\partial\mathcal{A}_2(3)^*; \mathbb{Z}) = \mathbb{Z}, \quad \mathrm{H}_1(\partial\mathcal{A}_2(3)^*; \mathbb{Z}[1/3]) = \mathbb{Z}[1/3]^{81},$$
$$\mathrm{H}_2(\partial\mathcal{A}_2(3)^*; \mathbb{Z}[1/6]) = \mathbb{Z}[1/6]^{200}, \quad \mathrm{H}_3(\partial\mathcal{A}_2(3)^*; \mathbb{Z}) = 0,$$

and

$$\mathrm{H}_4(\partial\mathcal{A}_2(3)^*; \mathbb{Z}) = \mathbb{Z}^{40}.$$

It is an easy corollary of [21, Prop. 4.3] that the map

$$\mathrm{H}_4(\partial\mathcal{A}_2(3)^*) \to \mathrm{H}_4(\mathcal{A}_2(3)^*)$$

is a monomorphism: The left-hand side above is free on the fundamental classes $[D(l)]$. Thus, the image of $\mathrm{H}_4(\partial\mathcal{A}_2(3)^*)$ in $\mathrm{H}_4(\mathcal{A}_2(3)^*)$ is a subspace of rank 40. A computer calculation as in the proof of [20, Theorem 4.9] (computing the intersection numbers of the classes $D(l)$ with generators of $\mathrm{H}_2(\mathcal{A}_2(3)^*)$) shows that this subspace has elementary divisors 1 (multiplicity 30), 3 (multiplicity 9), and 6, i.e., that

$$\mathrm{H}_4(\mathcal{A}_2(3)^*)/\mathrm{Im}\mathrm{H}_4(\partial\mathcal{A}_2(3)^*) = (\mathbb{Z}/3)^9 \oplus \mathbb{Z}/6.$$

In view of the vanishing of $\mathrm{H}_3(\partial\mathcal{A}_2(3)^*; \mathbb{Z})$, this computes $\mathrm{H}_4(\mathcal{A}_2(3)^*, \partial\mathcal{A}_2(3)^*; \mathbb{Z})$, hence $\mathrm{H}^2(\mathcal{A}_2(3)^* - \partial\mathcal{A}_2(3)^*; \mathbb{Z})$, by Alexander duality:

$$\mathrm{H}_{6-i}(\mathcal{A}_2(3)^*, \partial\mathcal{A}_2(3)^*) = \mathrm{H}^i(\mathcal{A}_2(3)) = \mathrm{H}^i(\Gamma_2(3)\backslash\mathfrak{S}_2) = \mathrm{H}^i(\Gamma_2(3)).$$

The last isomorphism follows from the fact that $\Gamma_2(3)$ acts freely on the contractible space $\mathfrak{S}_2$.

Also,

$$\mathrm{H}_2(\partial\mathcal{A}_2(3)^*; \mathbb{Z}) \to \mathrm{H}_2(\mathcal{A}_2(3)^*; \mathbb{Z})$$

is onto: By [21, Thm. 4.9], $\mathrm{H}_2(\mathcal{A}_2(3)^*; \mathbb{Z})$ is generated by the 130 classes denoted $h_1(\Delta)$, $h_1(\Delta)$, $4d(l)$ there. But all of these classes are supported in $\partial\mathcal{A}_2(3)^*$ by their very construction.

The homology sequence shows that $\mathrm{H}_1(\mathcal{A}_2(3)^*, \partial\mathcal{A}_2(3)^*; \mathbb{Z}) = 0$, being a quotient of $\mathrm{H}_1(\mathcal{A}_2(3)^*; \mathbb{Z}) = 0$. Thus by duality, $\mathrm{H}^5(\Gamma_2(3); \mathbb{Z}) = 0$. The rest of the argument is chasing through the homology sequence of the pair, using Alexander duality. ∎

We now decompose $\mathrm{H}^*(\Gamma_2(3); \mathbb{Q})$ into irreducible representations of $\mathbf{PSp}(4, \mathbf{F}_3)$. Henceforth we use the notations of [8], which we refer to as the Atlas, except that we continue to use 1 to denote the trivial representation of the relevant group.

THEOREM 3.2. *As representation spaces of* $\mathbf{PSp}(4, \mathbf{F}_3)$,

$$\begin{aligned}
\mathrm{H}^0(\mathit{\Gamma}_2(3); \mathbb{Q}) &= 1, \\
\mathrm{H}^1(\mathit{\Gamma}_2(3); \mathbb{Q}) &= 0, \\
\mathrm{H}^2(\mathit{\Gamma}_2(3); \mathbb{Q}) &= 1 + 20a, \\
\mathrm{H}^3(\mathit{\Gamma}_2(3); \mathbb{Q}) &= 5a + 5b + 15a + 24a + 30a + 60a, \\
\mathrm{H}^4(\mathit{\Gamma}_2(3); \mathbb{Q}) &= 81a.
\end{aligned}$$

Proof. For $i = 0, 1$ this is trivial. In [21, Cor. 6] we determined

$$\mathrm{H}_2(\mathcal{A}_2(3)^*; \mathbb{Q}) = \mathrm{H}_4(\mathcal{A}_2(3)^*; \mathbb{Q}) = 1 + 1 + 15a + 20a + 24a.$$

(Actually, this uses results from the Atlas which we will derive in Lemma 4.1 independently below.) Also, by Lemma 4.1,

$$\mathrm{H}_4(\mathcal{A}_2(3)^*; \mathbb{Q}) = \mathrm{Ind}_{P(l)}^G(1) = 1 + 15a + 24a.$$

This yields the case $i = 2$ and reduces the case $i = 3$ to the computation of $\mathrm{H}_2(\partial\mathcal{A}_2(3)^*; \mathbb{Q})$.

To identify this representation we return to the proof of Theorem 2.13. Recall that we showed there that

$$i_2 : \mathrm{H}_2(\partial C^\bullet; \mathbb{Q}) \;\to\; \mathrm{H}_2(C; \mathbb{Q}) \oplus \mathrm{H}_2(D^\circ; \mathbb{Q})$$

is an injection and that

$$i_1 : \mathrm{H}_1(\partial C^\bullet; \mathbb{Q}) \;\to\; \mathrm{H}_1(C; \mathbb{Q}) \oplus \mathrm{H}_1(D^\circ; \mathbb{Q})$$

has cokernel

$$\mathrm{Ind}_{P(h)}^G \mathrm{H}_1(N(h); \mathbb{Q}) \oplus \mathrm{Ind}_{P(l)}^G \mathrm{H}_1(M(l); \mathbb{Q}),$$

where $N(h)$ and $M(l)$ are both Riemann surfaces of genus g. But $g = 0$ for $n = 3$, so i_1 is a surjection. Hence,

$$\begin{aligned}
\mathrm{H}_2(\partial\mathcal{A}_2(3)^*; \mathbb{Q}) &= \mathrm{Coker}(i_2) \oplus \mathrm{Ker}(i_1) \\
&= (\mathrm{H}_2(C; \mathbb{Q}) + \mathrm{H}_2(D^\circ; \mathbb{Q})) - \mathrm{H}_2(\partial C^\bullet; \mathbb{Q}) \\
&\qquad + \mathrm{H}_1(\partial C^\bullet; \mathbb{Q}) - (\mathrm{H}_1(C; \mathbb{Q}) + \mathrm{H}_1(D^\circ; \mathbb{Q})) \\
&= \mathrm{H}_2(C; \mathbb{Q}) + \mathrm{H}_2(D^\circ; \mathbb{Q}) - \mathrm{H}_1(C; \mathbb{Q}) - \mathrm{H}_1(D^\circ; \mathbb{Q}) \\
&\qquad \text{as } \mathrm{H}_2(\partial C^\bullet; \mathbb{Q}) \text{ and } \mathrm{H}_1(\partial C^\bullet; \mathbb{Q}) \\
&\qquad \text{are isomorphic representations,} \\
&= \mathrm{Ind}_{P(h)}^G \mathrm{H}_2(C(h); \mathbb{Q}) + \mathrm{Ind}_{P(l)}^G \mathrm{H}_2(D^\circ(l); \mathbb{Q}) \\
&\qquad - \mathrm{Ind}_{P(h)}^G \mathrm{H}_1(C(h); \mathbb{Q}) - \mathrm{Ind}_{P(l)}^G \mathrm{H}_1(D^\circ l; \mathbb{Q}).
\end{aligned}$$

We determine these four representations.

$\mathrm{Ind}_{P(h)}^G \mathrm{H}_2(C(h); \mathbb{Q})$: A single $\mathbf{P}^1$ in $C(h)$ is specified by a set $\{l_1, l_2\}$ of two lines with $l_i \subset h$ and $h = l_1 \wedge l_2$ (this last condition is automatic), and the stabilizer of this set acts trivially on the second homology group of this $\mathbf{P}^1$ (as it preserves its orientation). Thus by Lemma 4.4 below, this representation is

$$1 + 15a + 15b + 20a + 2 \cdot 24a + 60a + 81a.$$

$\mathrm{Ind}_{P(l)}^{G} \mathrm{H}_2(D^\circ(l); \mathbb{Q})$: While we have described $\mathrm{H}_2(D^\circ(l); \mathbb{Q})$ in Theorem 2.13, here, because $n = 3$, we have a slightly simpler description. If F denotes the union of the 4 singular fibers in $D(l)$, then we have, as in Corollary 2.6, the exact sequence

$$\mathrm{H}_2(F) \xrightarrow{j} \mathrm{H}_2(D(l)) \to \mathrm{H}_2(D(l), F) \to \mathrm{H}_1(F) \to 0$$

and the cokernel of j has rank 1, generated by a section, or equivalently, by the sum of sections. This class is clearly left invariant by $P(l)$, so we see that, as a representation of $P(l)$,

$$\mathrm{H}_2(D^\circ(l); \mathbb{Q}) = \mathrm{H}^2(D^\circ(l); \mathbb{Q}) = 1 + \mathrm{H}_2(F)$$

and so by Lemmas 4.1 and 4.5 below, inducing up to G we obtain the representation

$$(1 + 15a + 24a) + (5a + 5b + 2 \cdot 30a + 45a + 45b).$$

$\mathrm{Ind}_{P(h)}^{G} \mathrm{H}_1(C(h); \mathbb{Q})$: This is determined in Lemma 4.6 below as

$$30a + 45a + 45b.$$

$\mathrm{Ind}_{P(l)}^{G} \mathrm{H}_1(D^\circ(l); \mathbb{Q})$: We have $\mathrm{H}_1(D^\circ(l)) = \mathrm{H}_1(M^\circ(l))$ and M° is a Riemann surface of genus 0 with 4 punctures. Thus $\mathrm{H}_1(M^\circ(l))$ is generated by the loops around the punctures with the single relation that their sum bounds. Hence, as a representation of $P(l)$,

$$\mathrm{H}_1(D^\circ(l); \mathbb{Q}) = \mathrm{Ind}_{P(l,h)}^{P(l)}(1) - 1$$

and so, inducing up to G, we see that the representation in question is

$$\mathrm{Ind}_{P(l,h)}^{G}(1) - \mathrm{Ind}_{P(l)}^{G}(1),$$

which by Lemmas 4.1(iv) and 4.4 below is

$$15b + 24a + 81a.$$

Assembling these results gives the character of $\mathrm{H}_3(\mathcal{A}_2(3); \mathbb{Q}) = \mathrm{H}_3(\Gamma_2(3); \mathbb{Q})$.

Finally, for $i = 4$ we deduce that $\mathrm{H}_1(\mathfrak{T}; \mathbb{Q})$ is given by Lemma 2.11 and so by Lemmas 4.1 and 4.3 below, this is

$$(1 + 15a + 15b + 2 \cdot 24a + 81a) - (1 + 15a + 24a) - (1 + 15b + 24a) + 1 = 81a$$

(a result which is consistent with our observation in Remark 2.5).

In the next section we compute the characters that were used in the preceding arguments. ∎

4. Characters. We use the following principles throughout:

P0 If R is a real representation of G and χ and $\overline{\chi}$ are conjugate irreducible representations, then $\langle R, \chi\rangle = \langle R, \overline{\chi}\rangle$. Furthermore, if R is rational and $g, g' \in G$ are in the same "cohort" (in the sense of the Atlas), then the character R takes the same value on g and g'.

P1 If R is a transitive permutation representation of G then

(a) R contains the trivial representation with multiplicity 1.
(b) The character of R takes a value which is a nonnegative integer on each element of G.

P2 If $H \subset G$ is a subgroup and R is a representation of H and χ is a representation of G then

$$\langle \chi, \mathrm{Ind}_H^G R\rangle_G = \langle \mathrm{Res}_H^G \chi, R\rangle_H$$

(Frobenius reciprocity). This will be used here only in the case $\chi = 1$, where it simplifies to

$$\langle 1, \mathrm{Ind}_H^G R\rangle = \langle 1, R\rangle$$

and it is clear where the inner product of characters is taken.

We begin by compressing the 20×20 character table of $\mathbf{PSp}(4, \mathbf{F}_3)$ into a 15×15 table by adding rows together which correspond to pairs of conjugate representations and by retaining only one column for each cohort of elements. Note that all our representations factor through this quotient by ± 1; we will simply identify them with representations of $G = \mathbf{PSp}(4, \mathbf{F}_3)$.

The Atlas gives $\mathrm{Ind}_H^G(1)$, i.e., the permutation representation of G on the left cosets G/H, for each of the five maximal subgroups H of G. As it is short and simple (and illuminating) to do so, we derive these, using principle P3:

P3 If G acts transitively on a set X, and H is the stabilizer of any $x \in X$, then the permutation representation of G on X is $\mathrm{Ind}_H^G(1)$. Moreover, G has five maximal subgroups:

(i) The stabilizer of a line l.
(ii) The stabilizer of an isotropic plane h.
(iii) The stabilizer of a split nonsingular pair Δ.
(iv) The stabilizer of a spread of nonsingular pairs (nsp-spread).
(v) The stabilizer of a double-six.

We have seen cases (i) and (ii) in this paper, and case (iii) appeared in [21] where it was involved in $\mathrm{H}_*(\mathcal{A}_2(3)^*; \mathbb{Q})$. Cases (iv) and (v) do not enter directly, but we include them for completeness. An nsp-spread was defined in [22]; suffice it to say here that G is also the group of even automorphisms of the 27 lines on a nonsingular projective surface, and the nsp-spreads correspond to the lines, while a double-six of nsp-spreads corresponds to a double-six of lines.

LEMMA 4.1. *The representation of* $G = \mathbf{PSp}(4, \mathbf{F}_3)$ *is*:

(i) *on isotropic planes*:
$$\chi_1 + \chi_8 + \chi_{10} \qquad (1 + 15b + 24a),$$
(ii) *on lines*:
$$\chi_1 + \chi_7 + \chi_{10} \qquad (1 + 15a + 24a),$$
(iii) *on split nonsingular pairs*:
$$\chi_1 + \chi_9 + \chi_{10} \qquad (1 + 20a + 24a),$$
(iv) *on spreads of nonsingular pairs*:
$$\chi_1 + \chi_4 + \chi_9 \qquad (1 + 6a + 20),$$
(v) *on double-sixes*:
$$\chi_1 + \chi_8 + \chi_9 \qquad (1 + 15b + 20a).$$

Proof. In cases (iii) and (iv) there is a unique representation of the relevant degree satisfying P1.

There are two representations of degree 40 satisfying P1, $\chi_1 + \chi_7 + \chi_{10}$ and $\chi_1 + \chi_8 + \chi_{10}$. Calculating the characters of the two given elements of order 2 (in the next lemma) on the two representations simultaneously distinguishes between (i) and (ii), and assigns these elements to their classes in the Atlas.

As for (v), there are two representations of degree 36 satisfying P1, but the element 2A fixes 12 double-sixes, and that suffices to determine which of the two it is. ∎

LEMMA 4.2. *The following elements of* G *have the given order and behavior*:

2A $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ *order* 2, *fixes* 8 *lines and* 16 *isotropic planes*,

2B $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ *order* 2, *fixes no lines and* 4 *isotropic planes*,

3AB $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *order* 3, *fixes* 13 *lines and* 4 *isotropic planes*, 3C $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *order* 3, *fixes* 4 *lines and* 1 *isotropic plane*,

3D $\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *order* 3, *fixes* 4 *lines and* 7 *isotropic planes*,

4A $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ *order* 4, *fixes* 4 *lines and no isotropic planes*,

4B $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ *order* 4, *fixes no lines and* 2 *isotropic planes*,

5A $\begin{pmatrix} -1 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$ *order* 5, *fixes no lines and no isotropic planes*,

6AB $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ *order* 6, *fixes* 5 *lines and* 4 *isotropic planes*,

6CD $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ *order* 6, *fixes* 2 *lines and* 1 *isotropic plane*,

$$6\mathrm{E} \qquad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \begin{array}{l} \textit{order } 6, \\ \textit{fixes } 2 \textit{ lines and } 1 \textit{ isotropic plane}, \end{array}$$

$$6\mathrm{F} \qquad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{array}{l} \textit{order } 6, \\ \textit{fixes no lines and } 1 \textit{ isotropic plane}, \end{array}$$

$$9\mathrm{AB} \qquad \begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \textit{order } 9, \\ \textit{fixes } 1 \textit{ line and } 1 \textit{ isotropic plane}, \end{array}$$

$$12\mathrm{AB} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} \textit{order } 12, \\ \textit{fixes } 1 \textit{ line and no isotropic planes}, \end{array}$$

Proof. It is direct to verify that these elements have the indicated orders, and act as indicated on lines and isotropic planes. This determines their classes in the Atlas. (Note that the stabilizer of a line or an isotropic plane has order $25920/40 = 648$ so contains no element of order 5.) ∎

LEMMA 4.3. *The permutation representation of G on the flags $\{l \subset h\}$, of degree* 160, *is*

$$\chi_1 + \chi_7 + \chi_8 + 2\chi_{10} + \chi_{20} \qquad (1 + 15a + 15b + 2 \cdot 24a + 81a).$$

Proof. As 2B and 4B fix no lines, 4A fixes no isotropic planes, and 5A fixes neither, these four elements have character value 0 for the representation, since this representation is the permutation representation $\mathrm{Ind}_H^G(1)$ for the stabilizer subgroup H of $\{l \subset h\}$ (hence the character value on an element g is the number of fixed points of g acting on the set of the $\{l \subset h\}$). There is only one representation of degree 160 giving these character values and satisfying P1, the one given above. ∎

LEMMA 4.4. *The permutation representation of G on the flags $\{\{l_1, l_2\} \subset h\}$, of degree* 240, *is*

$$\chi_1 + \chi_7 + \chi_8 + \chi_9 + 2\chi_{10} + \chi_{18} + \chi_{20}$$
$$(1 + 15a + 15b + 20a + 2 \cdot 24a + 60a + 81a).$$

Proof. As 4A and 5A fix no isotropic planes, these two elements have character value 0. The element 3C fixes the single isotropic plane $(0010) \wedge (0001)$, and leaves every line therein pointwise fixed, so 3C has character 6. The element 2A fixes 16 isotropic planes, among them $(0010) \wedge (0001)$. On this plane it leaves the two pairs $\{(0010), (0001)\}$ and $\{(0011), (0012)\}$ fixed, and similarly for the other 15 fixed isotropic planes, so 2A has character 16. These character values and P1 determine the representation uniquely. ∎

Lemma 4.5. *The representation* $\mathrm{Ind}_{P(l \subset h)}^{G} \mathrm{H}_1(f)$ *of* G, *of degree* 160, *is*

$$\chi_2 + \chi_3 + 2\chi_{11} + \chi_{16} + \chi_{17} \qquad (5a + 5b + 2 \cdot 30a + 45a + 45b),$$

where f *is the exceptional fiber of* $D(l)$ *corresponding to* h.

Proof. Since, as observed above, 2B, 4A, 4B, and 5A are contained in no stabilizer $P(l \subset h)$, the character values are zero.

Recall that f is a triangle, so $\mathrm{H}_1(f) \cong \mathbb{Z}$. Thus, if an element of order 3 is in $P(l \subset h)$, its trace is 1, and if not, its trace is 0. Hence we see that every element of order 3 has character ≥ 0.

On the other hand, consider the element 2A and let $l = (0001)$, $h = (0010) \wedge (0001)$. Then the projective lines in the triangle are schematically indexed as in Figure 3.

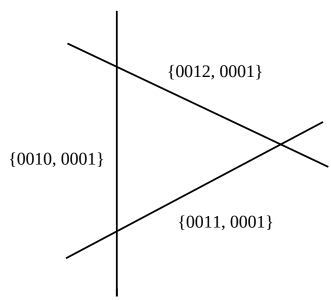

Fig. 3. Projective lines in a triangle

Note that 2A preserves the lines $(0001), (0010)$, and interchanges the lines $(0011), (0012)$. Thus it acts as multiplication by -1 on $\mathrm{H}_1(f)$. Therefore, by P2, we see that the desired representation of G does not contain the trivial representation. There is only one such representation of G consistent with the above character values, that of the lemma. ∎

Lemma 4.6. *The representation* $\mathrm{Ind}_{P(h)}^{G} \mathrm{H}_1(C(h))$ *of* G, *of degree* 120, *is*

$$\chi_{11} + \chi_{16} + \chi_{17} \qquad (30a + 45a + 45b).$$

Proof. Since 4A and 5A fix no isotropic planes, their character value is 0. The element 2A fixes 16 isotropic planes. On each $h = \{l_1, l_2, l_3, l_4\}$ it fixes two of the l_i's and interchanges the other two in pairs. Now $\mathrm{H}_1(f)$ is unchanged if we replace each $\mathbf{P}^1$ in the "tetrahedron" $C(h)$ by a segment $[0,1]$, so that we obtain a true tetrahedron (see Figure 4), and if that is

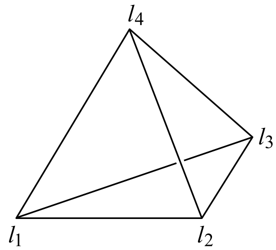

Fig. 4. Tetrahedron

done, the action of 2A on the resulting simplicial complex is:

0-cells: fix 2; interchange 2.

1-cells: fix 1; leave 1 invariant but reverse it; interchange 4 in pairs.

Hence, the trace of the action on the cell complex is $-(1-1)+2=2$, and as $\mathrm{H}_0(C(h))=\mathbb{Z}$ is acted on trivially, the trace on $\mathrm{H}_1(C(h))$ is -1, so 2A has character $16(-1)=-16$.

The element 2B fixes 4 isotropic planes, and on each such h, interchanges all four l_i in pairs. Then the action of 2B on the resulting simplicial complex is:

0-cells: interchange 4 in pairs.

1-cells: fix none; leave 2 invariant but reversed; interchange 4 in pairs.

Hence, the trace of the action on the cell complex is $-(-2)+0=2$, so again the trace on $\mathrm{H}_1(C(h))$ is -1, and 2B has character $4(-1)=-4$.

Now neither 2A nor 2B leaves any generator of $\mathrm{H}_1(C(h))$ invariant in any h that it fixes (as can easily be seen from the fact that $P(h)$ acts transitively on the triangle in h), so the representation in the lemma does not contain the trivial representation, and there is only one possibility consistent with the data.

Alternate proof. Replacing each $\mathbf{P}^1$ by a segment, we obtain a 1-complex $C'(h)$ for each h with union C' (see Section 2.2 for the notations that follow). Then $\mathrm{H}_1(C)=\mathrm{H}_1(C')$. As a representation of G,

$$-\mathrm{H}_1(C)+\mathrm{H}_0(C)=-C_1(C')+C_0(C'),$$

where C_i, $i=0,1$, is the group of i-chains, so

$$\mathrm{H}_1(C)=\mathrm{Ind}_{P(h)}^G\mathrm{H}_0(C'(h))+\mathrm{Ind}_{P(h)}^G C_1(h)-\mathrm{Ind}_{P(h)}^G C_0(h).$$

Now $\mathrm{H}_0(C'(h)) = \mathbb{Z}$ is acted on trivially by $P(h)$, so the first term on the right is simply the representation of G on $\{h\}$, which we have already determined.

For the second term: $C_1(h)$ is generated by pairs $\{l_1, l_2\}$ with $h = \pm l_1 \wedge l_2$. But this second condition holds for any distinct pair $l_1, l_2 \in h$, so that $C_1(C'(h))$ is generated by the 6 pairs of distinct $l_1, l_2 \in h$. Note that the stabilizer of any such pair is contained in the stabilizer of h. Thus again we see that the elements 4A and 5A have character 0, as they stabilize no isotropic planes.

The element 2A is in $P(h)$ for 16 isotropic planes h, and in each of them leaves two pairs l_1, l_2 invariant, preserving the orientation of one pair and reversing the orientation of the other, so 2A has character $16(-1+1) = 0$. The element 2B is in $P(h)$ for 4 values of h, and in each of them leaves two pairs invariant, reversing the orientation of both, so 2B has character $4(-2) = -8$.

Thus $\mathrm{Ind}_{P(h)}^G C_1(h)$ is a 240-dimensional representation of G, not containing the trivial representation, with character values as above; this determines the representation uniquely as

$$\chi_7 + \chi_{10} + \chi_{11} + \chi_{16} + \chi_{17} + \chi_{20}$$
$$(15a + 24a + 30a + 45a + 45b + 81a). \quad \blacksquare$$

LEMMA 4.7. *The representation* $\mathrm{Ind}_{P(h)}^G \mathrm{H}_2(D(l))$ *of* G, *of degree* 400, *is*

$$2\chi_1 + 3\chi_7 + \chi_9 + 3\chi_{10} + \chi_{11} + \chi_{16} + \chi_{17} + \chi_{18} + \chi_{20}$$
$$(2 \cdot 1 + 3 \cdot 15a + 20a + 3 \cdot 24a + 30a + 45a + 45b + 60a + 81a).$$

Proof. Let $D(l)$ be a boundary component, $\{f(h,l) : l \subset h\}$ the exceptional fibers in $D(l)$, and

$$F(l) = \bigcup_{l \subset h} f(h,l).$$

Then $\mathrm{H}_2(F(l)) \to \mathrm{H}_2(D(l))$ has 1-dimensional cokernel, represented by a section, or equivalently, by the sum of the nine sections through the 3-division points. This class is obviously acted on trivially by $P(l)$, so contributes $\mathrm{Ind}_{P(l)}^G(1)$ to the representation of the lemma.

Now for each h, $\mathrm{H}_2(f(h,l))$ is 3-dimensional (as $f(h,l)$ is a triangle) and there is one relation among the images: the sum of the fundamental classes of the $\mathbf{P}^1$'s in the triangle is the class of a general fiber, and is thus the same for all the triangles. Note that the class of a general fiber is acted on trivially by $P(l)$, so this also contributes $\mathrm{Ind}_{P(l)}^G(1)$ to the representation of the lemma. Of course the sum of the fundamental classes of the $\mathbf{P}^1$'s in $f(h,l)$ is acted on trivially by the stabilizer $P(h,l)$, so we see that, as a

Table 1. Multiplicities and character values

Conjugate pairs of irreducible representations of $\mathbf{Sp}(4, \mathbf{F}_3)$:															
	1	2	4	5	7	8	9	10	11	12	14	16	18	19	20
		3		6						13	15	17			
Dimensions:															
	1	5	6	10	15	15	20	24	30	30	40	45	60	64	81
Cohorts of conjugacy classes:															
	1	2	2	3	3	3	4	4	5	6	6	6	6	9	12
	A	A	B	AB	C	D	A	B	A	AB	CD	E	F	AB	AB
Representation on spreads of nonsingular pairs:															
mult.	1	0	1	0	0	0	1	0	0	0	0	0	0	0	0
χ	27	3	7	0	9	0	3	1	2	0	3	0	1	0	0
Representation on double-sixes:															
mult.	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0
χ	36	12	8	0	6	3	0	2	1	0	0	3	2	0	0
Representation on split nonsingular pairs:															
mult.	1	0	0	0	0	0	1	1	0	0	0	0	0	0	0
χ	45	13	5	9	6	3	1	1	0	1	4	1	2	0	1
Representation on $\{l\}$:															
mult.	1	0	0	0	1	0	0	1	0	0	0	0	0	0	0
χ	40	8	0	13	4	4	4	0	0	5	2	2	0	1	1
Representation on $\{h\}$:															
mult.	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0
χ	40	16	4	4	1	7	0	2	0	4	1	1	1	1	0
Representation on $\{l \subset h\}$:															
mult.	1	0	0	0	1	1	0	2	0	0	0	0	0	0	1
χ	160	32	0	16	4	10	0	0	0	8	2	2	0	1	0
Representation on $\{\{l_1, l_2\} \subset h\} = \mathrm{Ind}_{P(h)}^{G} \mathrm{H}_2(C(h))$:															
mult.	1	0	0	0	1	1	1	2	0	0	0	0	1	0	1
χ	240	32	8	24	6	6	0	0	0	8	2	2	2	0	0
$\mathrm{Ind}_{P(l)}^{G} \mathrm{H}_1(F(l)) = \mathrm{Ind}_{P(l \subset h)}^{G} \mathrm{H}_1(f)$:															
mult.	0	1	0	0	0	0	0	0	2	0	0	1	0	0	0
χ	160	-32	0	16	4	10	0	0	0	-8	-2	-2	0	1	0
$\mathrm{Ind}_{P(h)}^{G} \mathrm{H}_1(C(h))$:															
mult.	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
χ	120	-16	-4	12	3	3	0	2	0	-4	-1	-1	-1	0	0
$\mathrm{Ind}_{P(l \subset h)}^{G} \mathrm{H}_2(f)$:															
mult.	1	0	0	0	2	1	1	3	1	0	0	1	1	0	2
χ	480	32	0	48	12	12	0	0	0	8	2	2	0	0	0

representation of $P(l)$,

$$\mathrm{H}_2(D(l)) = \mathrm{Ind}_{P(h,l)}^{P(l)}(\mathrm{H}_2(f(h,l)) - 1) + 2 \cdot 1$$

and hence

$$\mathrm{Ind}_{P(l)}^{G}\mathrm{H}_2(D(l)) = \mathrm{Ind}_{P(h,l)}^{G}\mathrm{H}_2(f(h,l)) - \mathrm{Ind}_{P(h,l)}^{G}(1) + 2\mathrm{Ind}_{P(l)}^{G}(1).$$

We have computed the last two terms on the right, so we need only compute the first one. Classes 2B and 4B fix no lines and hence no pairs (l,h); class 4A fixes no isotropic planes and hence no pairs (l,h); class 5A fixes none of either, so these three classes all have character 0.

The element 3C fixes 4 lines, and each fixed line is contained in a fixed isotropic plane. In the corresponding exceptional fiber $f(h,l)$ of $D(l)$ it leaves all the $\mathbf{P}^1$'s in the triangle invariant, in each case, so it has character $4 \cdot 3 = 12$.

Every group element leaves the sum of the three fundamental classes of the $\mathbf{P}^1$'s in every invariant triangle fixed, so this representation contains the trivial representation once. Finally, its dimension is $160 \cdot 3 = 480$. There is only one representation satisfying these conditions:

$$\chi_1 + 2\,\chi_7 + \chi_8 + \chi_9 + 3\,\chi_{10} + \chi_{11} + \chi_{16} + \chi_{17} + \chi_{18} + \chi_{20}$$
$$(1 + 2 \cdot 15a + 15b + 20a + 3 \cdot 24a + 30a + 45a + 45b + 60a + 81a),$$

and the lemma follows. ∎

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