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The elementary-equivalence classes of clopen algebras of P-spaces

by

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Abstract. Two Boolean algebras are elementarily equivalent if and only if they satisfy the same first-order statements in the language of Boolean algebras. We prove that every Boolean algebra is elementarily equivalent to the algebra of clopen subsets of a normal P-space.

1. Introduction. Let X be a topological space and $\mathrm{Clop}(X)$ the set of all clopen subsets of X. $\mathrm{Clop}(X)$ is a Boolean algebra under the operations of set-theoretic union, intersection, and complementation. Conversely, by a famous theorem of M. Stone [5], every Boolean algebra is isomorphic to the algebra of clopen subsets of a *Boolean space*, i.e., a space that is compact, Hausdorff, and zero-dimensional. Recall that a *P-space* is a completely regular space in which every G_δ-set is open. The clopen algebras of P-spaces are σ-complete and hence are less diverse than those of Boolean spaces. However, if we require only classification up to elementary equivalence, a relation from mathematical logic weaker than isomorphism, then an analogue of Stone's result is available. Specifically, we intend to prove

THEOREM 1.1. *Every Boolean algebra is elementarily equivalent to the algebra of clopen subsets of a normal P-space.*

In [7] Theorem 1.1 plays an essential role in proofs of model-theoretic properties of certain lattice-ordered groups of continuous functions.

2. The Tarski invariants. We assume familiarity with the basic notions of Boolean algebra; for all undefined terms we refer the reader to Chapter 1 of [4]. We consider Boolean algebras as structures for the first-order language $\{+,\cdot,-,0,1\}$, where $+$ is interpreted as join, $\cdot$ as meet, $-$ as complement, and 0 and 1 as the bottom and top elements, respectively, in the algebra.

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In [6] Tarski associates with each Boolean algebra a triple of numerical invariants that completely determines its elementary-equivalence class. These triples encode information about certain ideals of the algebra. Let A be a Boolean algebra and I and J ideals of A. Let $\mathrm{sa}(I)$ be the set of all $a \in A$ such that a is atomless mod I, $\mathrm{at}(I)$ the set of all $a \in A$ such that a is atomic mod I, and $I + J$ the set of all $a \in A$ such that $a = b + c$ for some $b \in I$ and some $c \in J$. Then $\mathrm{sa}(I)$, $\mathrm{at}(I)$, and $I + J$ are all again ideals of A. Let 0^A be the bottom element of A, set $E_0 = \{0^A\}$, and for every integer $n \geq 0$ set $E_{n+1} = \mathrm{sa}(E_n) + \mathrm{at}(E_n)$.

We now define the Tarski invariants $\mathrm{inv}(A)$ of A. If A is the one-element Boolean algebra then $\mathrm{inv}(A) = (-1, 0, 0)$; if $A \neq E_n$ for every $n \geq 0$ then $\mathrm{inv}(A) = (\omega, 0, 0)$; otherwise $\mathrm{inv}(A) = (i_1, i_2, i_3)$ where (1) $A \neq E_{i_1}$ and $A = E_{i_1+1}$, (2) $i_2 = 0$ if A/E_{i_1} is atomic, otherwise $i_2 = 1$, and (3) i_3 is the number of atoms of A/E_{i_1} if that number is finite, and ω if that number is infinite. Let Inv be the set of triples consisting of $(-1, 0, 0)$, $(\omega, 0, 0)$, and all the triples (i_1, i_2, i_3) with i_1 a nonnegative integer, $i_2 \in \{0, 1\}$, i_3 a nonnegative integer or ω, and $i_2 + i_3 > 0$.

For a proof of the following result see [1] or Chapter 7 of [4].

Theorem 2.1. *Any two Boolean algebras A and B are elementarily equivalent iff* $\mathrm{inv}(A) = \mathrm{inv}(B)$. *Moreover, for any* $(i_1, i_2, i_3) \in \mathrm{Inv}$ *there is a Boolean algebra A with* $\mathrm{inv}(A) = (i_1, i_2, i_3)$.

3. P-spaces. P-spaces are a generalization of discrete spaces and were named and studied by Gillman and Henriksen in [2]. Here are two useful examples of nondiscrete P-spaces.

Example. Let S be an uncountable space in which all points are isolated except for a distinguished point s, a neighborhood of s being any set containing s whose complement is countable. Then S is a nondiscrete normal P-space and $\mathrm{Clop}(S)$ is an atomic Boolean algebra.

Example. A totally ordered set T is called an η_1-*set* if for any countable subsets A and B, with $A < B$, there is a $t \in T$ satisfying $A < t < B$. With the interval topology every η_1-set T is a normal P-space without isolated points and $\mathrm{Clop}(T)$ is an atomless Boolean algebra (see page 193 of [3]).

If $\{X_k\}_{k \in K}$ is an indexed collection of spaces, then we write $\bigoplus_{k \in K} X_k$ for their topological sum, i.e. the disjoint union of the X_k topologized so that a subset U is open in the sum if and only if its intersection with each X_k is open in X_k. Here are some basic properties of P-spaces.

Proposition 3.1. *The following hold in any P-space X:*

(i) *Every zero-set of X is open.*
(ii) *Every subspace of X is a P-space.*

(iii) *Every completely regular quotient space of X is a P-space.*
(iv) *X is zero-dimensional, i.e., has a base of clopen subsets.*

Proof. See pages 62–63 of [3]. ∎

The rest of this paper is devoted to proving Theorem 1.1. Our strategy will be to build for each triple of Tarski invariants a normal P-space whose clopen algebra has those invariants. Many of these spaces will be obtained by gluing together in a certain way copies of the two normal P-spaces mentioned at the beginning of this section.

4. Topological equivalents of algebraic notions. Let X be an arbitrary normal P-space. To simplify our analysis we characterize membership in the Tarski ideals E_n, $\mathrm{at}(E_n)$, and $\mathrm{sa}(E_n)$ of $\mathrm{Clop}(X)$ using a device reminiscent of Cantor–Bendixson derivatives. We associate with X the following descending sequence of closed subspaces: let $X_0 = X$ and for every integer $n \geq 0$ let

$$X_{n+1} = \overline{\mathrm{Is}(X_n)} \cap \overline{(X_n - \overline{\mathrm{Is}(X_n)})}$$

where the overline represents the topological closure operation in X and $\mathrm{Is}(X_n)$ is the set of isolated points of the subspace X_n.

Proposition 4.1. *Each of the following hold for any $G \in \mathrm{Clop}(X)$ and any $n \geq 0$:*

(i) *$G \in E_n$ if and only if $G \cap X_n = \emptyset$.*
(ii) *G is an atom mod E_n if and only if $|G \cap X_n| = 1$.*
(iii) *$G \in \mathrm{at}(E_n)$ if and only if $G \cap X_n \subseteq \overline{\mathrm{Is}(X_n)}$.*
(iv) *$G \in \mathrm{sa}(E_n)$ if and only if $G \cap \mathrm{Is}(X_n) = \emptyset$.*

Proof. The proof goes by induction on n. That (i) holds when $n = 0$ is clear. Suppose G is an atom mod E_0. Then G is an atom of $\mathrm{Clop}(X)$, so G must have at least one element. If G has more than one element then because X is Hausdorff and zero-dimensional G has nonempty proper clopen subsets and hence is not an atom of $\mathrm{Clop}(X)$. Thus $|G \cap X_0| = |G| = 1$. Conversely, if $|G \cap X_0| = 1$ then clearly G is an atom of $\mathrm{Clop}(X)$. Thus (ii) holds when $n = 0$. Now $G \in \mathrm{at}(E_0)$ if and only if every clopen subset of G contains an atom of $\mathrm{Clop}(X)$ and hence an isolated point of X. Since X is zero-dimensional this is equivalent to having $G \cap X_0 \subseteq \overline{\mathrm{Is}(X_0)}$. Similarly, $G \in \mathrm{sa}(E_0)$ if and only if G contains no atoms of $\mathrm{Clop}(X)$, which is the same as having $G \cap \mathrm{Is}(X) = \emptyset$. Thus (iii) and (iv) both hold when $n = 0$.

Assume the result of the proposition for $n = k \geq 0$ and let $G \in \mathrm{Clop}(X)$. We show that (i)–(iv) hold when $n = k + 1$.

Suppose $G \cap X_{k+1} = \emptyset$. Then, by the definition of X_{k+1}, we have

$$[G \cap \overline{\mathrm{Is}(X_k)}] \cap [G \cap \overline{(X_k - \overline{\mathrm{Is}(X_k)})}] = \emptyset.$$

X is normal so by Urysohn's Lemma (see page 44 of [3]) there is an $f \in C(X)$ such that

$$G \cap \overline{\mathrm{Is}(X_k)} \subseteq f^{-1}(0) \tag{1}$$

and

$$G \cap \overline{(X_k - \overline{\mathrm{Is}(X_k)})} \subseteq X - f^{-1}(0). \tag{2}$$

Note that $f^{-1}(0) \in \mathrm{Clop}(X)$ because X is a P-space, so $G \cap f^{-1}(0)$ and $G \cap (X - f^{-1}(0))$ are also in $\mathrm{Clop}(X)$. If $x \in G \cap X_k$ and $x \notin \overline{\mathrm{Is}(X_k)}$ then $x \in X - f^{-1}(0)$ by (2). It follows that $(G \cap f^{-1}(0)) \cap X_k \subseteq \overline{\mathrm{Is}(X_k)}$ and therefore $G \cap f^{-1}(0) \in \mathrm{at}(E_k)$ by the induction hypothesis. If $x \in G \cap \mathrm{Is}(X_k)$ then $x \in f^{-1}(0)$ by (1), so $(G \cap (X - f^{-1}(0)) \cap \mathrm{Is}(X_k) = \emptyset$ and thus $G \cap (X - f^{-1}(0)) \in \mathrm{sa}(E_k)$ by the induction hypothesis. Hence $G \in \mathrm{at}(E_k) + \mathrm{sa}(E_k) = E_{k+1}$.

Conversely, suppose $G \in E_{k+1}$. Then $G = H \cup F$ for some $H \in \mathrm{sa}(E_k)$ and $F \in \mathrm{at}(E_k)$. So by the induction hypothesis $H \cap \mathrm{Is}(X_k) = \emptyset$ and therefore, since H is open in X, $H \cap \overline{\mathrm{Is}(X_k)} = \emptyset$. It follows that $H \cap X_{k+1} = \emptyset$. From the induction hypothesis we also see that $F \cap X_k \subseteq \overline{\mathrm{Is}(X_k)}$. So $(F \cap X_k) \cap (X_k - \overline{\mathrm{Is}(X_k)}) = \emptyset$. Since F is open in X it follows that

$$(F \cap X_k) \cap \overline{(X_k - \overline{\mathrm{Is}(X_k)})} = \emptyset$$

and hence that $F \cap X_{k+1} = \emptyset$ as $X_{k+1} \subseteq X_k$. Thus $G \cap X_{k+1} = (H \cup F) \cap X_{k+1} = \emptyset$ and we have shown that (i) holds when $n = k+1$.

Suppose $|G \cap X_{k+1}| = 1$. Then $G \notin E_{k+1}$ by (i). Suppose $H \in \mathrm{Clop}(X)$, $H \subseteq G$, and $H \notin E_{k+1}$. Then $H \cap X_{k+1} \neq \emptyset$ by (i). Since $|G \cap X_{k+1}| = 1$ and $H \subseteq G$, G and H must contain the same member of X_{k+1}. Therefore $(G - H) \cap X_{k+1} = \emptyset$ and so $G - H \in E_{k+1}$ by (i). Hence G is an atom mod E_{k+1}. Conversely, suppose G is an atom mod E_{k+1}. Then $G \notin E_{k+1}$ so $|G \cap X_{k+1}| \geq 1$ by (i). If there were more than one member of X_{k+1} in G then it would follow from (i) and the fact that X is Hausdorff and zero-dimensional that G is not an atom mod E_{k+1}, which contradicts our supposition. Hence $|G \cap X_{k+1}| = 1$ and (ii) holds when $n = k+1$.

Suppose $G \cap X_{k+1} \subseteq \overline{\mathrm{Is}(X_{k+1})}$. Suppose $H \in \mathrm{Clop}(X)$, $H \subseteq G$, and $H \notin E_{k+1}$. Then $H \cap X_{k+1} \neq \emptyset$ by (i). Since $H \subseteq G$ and H is open in X it follows that $H \cap \mathrm{Is}(X_{k+1}) \neq \emptyset$. So there is an $F \in \mathrm{Clop}(X)$ such that $|(F \cap H) \cap X_{k+1}| = 1$. By (ii), $F \cap H$ is an atom mod E_{k+1}. Thus $G \in \mathrm{at}(E_{k+1})$. Conversely, suppose $G \in \mathrm{at}(E_{k+1})$ and $x \in G \cap X_{k+1}$. Let H be an open neighborhood of x in X. Since X is zero-dimensional we may assume that $H \in \mathrm{Clop}(X)$. By (i), $H \cap G \notin E_{k+1}$ and, since $G \in \mathrm{at}(E_{k+1})$, there must be an $F \in \mathrm{Clop}(X)$ such that $F \subseteq H \cap G$ and F is an atom mod E_{k+1}. By (ii), $F \cap X_{k+1}$ has only one element which is therefore an isolated point of X_{k+1}. Since this point is in H, it follows that $G \cap X_{k+1} \subseteq \overline{\mathrm{Is}(X_{k+1})}$. Hence (iii) holds when $n = k+1$.

Finally, $G \cap \mathrm{Is}(X_{k+1}) \neq \emptyset$ if and only if there is an $H \in \mathrm{Clop}(X)$ such that $H \subseteq G$ and $|H \cap X_{k+1}| = 1$. By (i) and (ii) this is equivalent to H being an atom mod E_{k+1}, which means that $G \notin \mathrm{sa}(E_{k+1})$. Hence (iv) holds for $n = k+1$. ∎

Remark. Inspection of the proof of Proposition 4.1 reveals that the result holds for any zero-dimensional Hausdorff space in which disjoint closed sets may be separated by disjoint clopen sets. The latter property is also possessed by zero-dimensional Lindelöf spaces (see page 247 of [3]). So, for example, Proposition 4.1 holds in any Boolean space.

Proposition 4.2. *Each of the following hold for any integers $n, m \geq 0$:*

(i) $\mathrm{inv}(\mathrm{Clop}(X)) = (n, 0, m)$ *iff* $|X_n| = m > 0$.
(ii) $\mathrm{inv}(\mathrm{Clop}(X)) = (n, 0, \omega)$ *iff* $\mathrm{Is}(X_n)$ *is infinite and* $X_n \subseteq \overline{\mathrm{Is}(X_n)}$.
(iii) $\mathrm{inv}(\mathrm{Clop}(X)) = (n, 1, m)$ *iff* $|\mathrm{Is}(X_n)| = m$ *and* $X_n \not\subseteq \overline{\mathrm{Is}(X_n)}$.
(iv) $\mathrm{inv}(\mathrm{Clop}(X)) = (n, 1, \omega)$ *iff* $\mathrm{Is}(X_n)$ *is infinite and* $X_n \not\subseteq \overline{\mathrm{Is}(X_n)}$.
(v) $\mathrm{inv}(\mathrm{Clop}(X)) = (\omega, 0, 0)$ *iff* $X_k \neq \emptyset$ *for all* $k \geq 0$.

Proof. By Proposition 4.1, $X_k \neq \emptyset$ for all $k \geq 0$ if and only if $E_k \neq \mathrm{Clop}(X)$ for all $k \geq 0$. Thus (v) holds.

Fix $n, m \geq 0$. Suppose $\mathrm{inv}(\mathrm{Clop}(X)) = (n, 0, m)$. Then $E_n \neq \mathrm{Clop}(X)$ and $\mathrm{Clop}(X)/E_n$ is atomic with m atoms. Note that $m > 0$, for otherwise $\mathrm{Clop}(X) = E_n$. Since $\mathrm{Clop}(X)/E_n$ is atomic, $X \in \mathrm{at}(E_n)$ and so $X_n = \overline{\mathrm{Is}(X_n)}$ follows from Proposition 4.1. Say $G_1, \ldots, G_m$ are representatives of the m atoms mod E_n. By Proposition 4.1, $|G_i \cap X_n| = 1$ for all i. So each G_i contains an isolated point of X_n. If $G_i \cap G_j \cap X_n \neq \emptyset$ for some $i \neq j$, then $G_i \cap G_j \notin E_n$ by Proposition 4.1, which contradicts the fact that G_i and G_j represent distinct atoms mod E_n. Thus $|\mathrm{Is}(X_n)| \geq m$. Now $\mathrm{Clop}(X)/E_n$ is atomic, so $\bigcup_{i=1}^{m} G_i$ is equal to X modulo E_n and therefore $X - \bigcup_{i=1}^{m} G_i$ contains no points of X_n by Proposition 4.1. Hence $m = |\mathrm{Is}(X_n)| = |X_n|$.

Conversely, suppose $|X_n| = m > 0$. Then $|\mathrm{Is}(X_n)| = m > 0$ and $X_n \subseteq \overline{\mathrm{Is}(X_n)}$. So $X \in \mathrm{at}(E_n)$ by Proposition 4.1 and therefore $\mathrm{Clop}(X)/E_n$ is atomic and the first Tarski invariant of $\mathrm{Clop}(X)$ is at most n. Since $|\mathrm{Is}(X_n)| = m > 0$, $X_n \neq \emptyset$ and $X \notin E_n$ by Proposition 4.1. Hence $E_n \neq \mathrm{Clop}(X)$ and the first two Tarski invariants of $\mathrm{Clop}(X)$ are n and 0. Now using the fact that X is Hausdorff and zero-dimensional we may find pairwise disjoint $G_1, \ldots, G_m \in \mathrm{Clop}(X)$ such that $|G_i \cap X_n| = 1$ for each i. It follows from Proposition 4.1 that there are at least m atoms modulo E_n. Suppose $G \in \mathrm{Clop}(X)$ is an atom mod E_n. By Proposition 4.1, G must contain exactly one member of X_n and therefore must be equal to one of the

G_i modulo E_n. Thus $\mathrm{Clop}(X)/E_n$ has exactly m atoms, the third invariant of $\mathrm{Clop}(X)$ is m, and (i) holds.

From the proof of (i) we see that having at least m atoms mod E_n is equivalent to $|\mathrm{Is}(X_n)| \geq m$. With the aid of this fact, (ii)–(iv) are easily established. ∎

5. Technical topological lemmas. Let $\{Y_k\}_{k\in K}$ be a collection of disjoint spaces, X a space disjoint from all the Y_k and of cardinality at least $|K|$, $y_k \in Y_k$ for each $k \in K$, and $\{x_k\}_{k\in K}$ a collection of distinct points in X. We call Z the *pointwise gluing* of $\{(Y_k, y_k)\}_{k\in K}$ and $(X, \{x_k\}_{k\in K})$ over K if

$$Z = \Big(\Big(\bigoplus_{k\in K} Y_k\Big) \oplus X\Big)/\sim,$$

where $\sim$ is the equivalence relation on $(\bigoplus_{k\in K} Y_k) \oplus X$ which identifies y_k with x_k for each $k \in K$. We call the y_k and x_k glue points and, for any $U \subseteq (\bigoplus_{k\in K} Y_k) \oplus X$, we write $\mathrm{gp}(U)$ for $\{k \in K : \{x_k, y_k\} \cap U \neq \emptyset\}$. Finally, we always use q to denote the canonical quotient map from $(\bigoplus_{k\in K} Y_k) \oplus X$ to Z.

LEMMA 5.1. *Let Z be the pointwise gluing of the spaces $\{(Y_k, y_k)\}_{k\in K}$ and $(X, \{x_k\}_{k\in K})$ over K. If $U \subseteq W = (\bigoplus_{k\in K} Y_k) \oplus X$ and $\mathrm{gp}(U) \cap \mathrm{gp}(W - U) = \emptyset$ then* (i) *U is open in W if and only if $q(U)$ open in Z, and* (ii) *U is closed in W if and only if $q(U)$ is closed in Z.*

Proof. Since $\mathrm{gp}(U)\cap \mathrm{gp}(W-U) = \emptyset$, we see that $q^{-1}(q(U)) = U$. Therefore U is open if and only if $q^{-1}(q(U))$ is open. But the latter is open just in case $q(U)$ is open, because q is a quotient map. Thus (i) holds.

By (i), U is closed if and only if $q(W-U)$ is open. As $\mathrm{gp}(U)\cap \mathrm{gp}(W-U) = \emptyset$ we see that $q(W-U) = q(W) - q(U)$. But $q(W) - q(U)$ is open if and only if $q(U)$ is closed because q is surjective. Hence (ii) holds. ∎

LEMMA 5.2. *Let Z be the pointwise gluing of the normal P-spaces $\{(Y_k, y_k)\}_{k\in K}$ and $(X, \{x_k\}_{k\in K})$ over K. Then Z is a normal P-space.*

Proof. Since any topological sum of P-spaces is a P-space and since any completely regular quotient of a P-space is a P-space (see page 63 of [3]), to prove the lemma it suffices to show that Z is normal. Let F and G be disjoint closed subsets of Z. Then $q^{-1}(F)$ and $q^{-1}(G)$ are disjoint closed sets in $W = (\bigoplus_{k\in K} Y_k) \oplus X$. So

$$q^{-1}(F) = \Big(\bigcup_{k\in K} F_k\Big) \cup H_F \quad \text{and} \quad q^{-1}(G) = \Big(\bigcup_{k\in K} G_k\Big) \cup H_G,$$

where F_k, G_k are disjoint closed sets in Y_k for each k, and H_F and H_G are disjoint closed sets in X. Note that

$$y_k \in F_k \quad \text{iff} \quad x_k \in H_F \tag{3}$$

and

$$y_k \in G_k \quad \text{iff} \quad x_k \in H_G \tag{4}$$

for each $k \in K$. Since X and the Y_k's are normal, there exist disjoint open $U, V \subseteq X$ and disjoint open $U_k, V_k \subseteq Y_k$ such that

$$q^{-1}(F) \subseteq \Big(\bigcup_{k\in K} U_k\Big) \cup U \quad \text{and} \quad q^{-1}(G) \subseteq \Big(\bigcup_{k\in K} V_k\Big) \cup V.$$

In order to ensure that their images under q will be open, we may need to adjust these open sets separating $q^{-1}(F)$ and $q^{-1}(G)$. First, since F_k and G_k are closed and Y_k is completely regular, we may assume that

$$y_k \in \overline{U}_k \quad \text{iff} \quad y_k \in F_k \tag{5}$$

and

$$y_k \in \overline{V}_k \quad \text{iff} \quad y_k \in G_k \tag{6}$$

for each $k \in K$. Next, let $L = \{k \in K : x_k \in U - H_F\}$ and $M = \{k \in K : x_k \in V - H_G\}$. Since $U \cap V = \emptyset$ and $H_G \subseteq U$, if $k \in L$ then $x_k \notin H_F \cup H_G$ and so $y_k \notin \overline{U}_k \cup \overline{V}_k$ by (1)–(4). Since each Y_k is completely regular we therefore may choose for each $k \in L$ an open $T_k \subseteq Y_k$ such that $y_k \in T_k$ and $T_k \cap (U_k \cup V_k) = \emptyset$. Similarly, we may choose for each $k \in M$ an open $S_k \subseteq Y_k$ with $y_k \in S_k$ and $S_k \cap (U_k \cup V_k) = \emptyset$. Finally, set

$$O_F = \Big(\bigcup_{k\in K} U_k\Big) \cup U \cup \Big(\bigcup_{k\in L} T_k\Big),$$
$$O_G = \Big(\bigcup_{k\in K} V_k\Big) \cup V \cup \Big(\bigcup_{k\in M} S_k\Big).$$

We will show that $q(O_F)$ and $q(O_G)$ separate F and G in Z.

First, clearly $F \subseteq q(O_F)$ and $G \subseteq q(O_G)$. Next we show that $q(O_F)$ and $q(O_G)$ are open in Z. Fix $k \in K$. Then from the definition of O_F we see that $x_k \in O_F$ if and only if $x_k \in U$. Now $x_k \in U$ means $x_k \in H_F$ or $x_k \in U - H_F$, so by (1) and the definition of T_k we have $x_k \in U$ if and only if either $y_k \in F_k$ or $y_k \in T_k$. Thus $x_k \in O_F$ is equivalent to $y_k \in O_F$. It follows that $\mathrm{gp}(O_F) \cap \mathrm{gp}(W - O_F) = \emptyset$ and so $q(O_F)$ is open in Z by Lemma 5.1. A similar argument shows that $\mathrm{gp}(O_G) \cap \mathrm{gp}(W - O_G) = \emptyset$ and so $q(O_G)$ is also open in Z. Finally, we show that $q(O_F) \cap q(O_G) = \emptyset$. Since $\mathrm{gp}(O_F) \cap \mathrm{gp}(W - O_F) = \mathrm{gp}(O_G) \cap \mathrm{gp}(W - O_G) = \emptyset$ we see that $q(O_F) \cap q(O_G) \neq \emptyset$ implies $O_F \cap O_G \neq \emptyset$. So to complete the proof of the lemma it suffices to show that $O_F \cap O_G = \emptyset$. Let $w \in O_F$. We show that $w \notin O_G$.

CASE (1). Suppose $w \in U_t$ for some $t \in K$. Then $w \in Y_t$ and $Y_t \cap X = \emptyset$, so $w \notin V$. Since $Y_k \cap Y_t = \emptyset$ whenever $k \neq t$, and since $U_t \cap V_t = \emptyset$, we see that $w \notin \bigcup_{k\in K} V_k$. Fix $k \in M$. If $t \neq k$ then $w \notin S_k$ because $w \in Y_t$, $S_k \subseteq Y_k$, and $Y_t \cap Y_k = \emptyset$. If $t = k$ then $w \notin V_k$ because $w \in U_t$ and $S_k \cap U_k = \emptyset$ by choice of S_k. Thus $w \notin \bigcup_{k\in M} S_k$. Hence $w \notin O_G$.

CASE (2). Suppose $w \in U$. Then $w \in X$ so $w \notin (\bigcup_{k\in K} V_k) \cup (\bigcup_{k\in M} S_k)$ because $X \cap (\bigcup_{k\in K} Y_k) = \emptyset$. Also, $U \cap V = \emptyset$ so $w \notin V$. Hence $w \notin O_G$.

CASE (3). Suppose $w \in T_l$ for some $l \in L$. Then $w \notin V$ since $T_l \subseteq Y_l$ and $Y_l \cap X = \emptyset$. Now $T_l \cap Y_k = \emptyset$ whenever $l \neq k$, and $T_l \cap V_l = \emptyset$ by choice of T_l, so we see that $w \notin \bigcup_{k\in K} V_k$. Finally, $w \notin \bigcup_{k\in M} S_k$ because $L \cap M = \emptyset$ and so $T_l \cap S_k = \emptyset$ for any $k \in M$. Hence $w \notin O_G$. ∎

LEMMA 5.3. *Let X be a normal P-space and $U \in \mathrm{Clop}(X)$. Then each of the following holds for any $n \geq 0$:*

(i) $U_n = U \cap X_n$.
(ii) $\mathrm{Is}(U_n) = U \cap \mathrm{Is}(X_n)$.
(iii) $U_n - \overline{\mathrm{Is}(U_n)} = U \cap (X_n - \overline{\mathrm{Is}(X_n)})$.

Proof. The proof goes by induction on n. Note that with the subspace topology U is a normal P-space because it is a closed subset of X. Suppose $n = 0$. Then $X_n = X$ and $U_n = U$, so (i) is obvious and (ii) holds because $U \in \mathrm{Clop}(X)$. If G is an open set in U with no isolated points of U, then G is open in X and contains no isolated points of X by (ii). Similarly, if G is an open set in X with no isolated points then $G \cap U$ is an open set in U with no isolated points in U by (ii). Thus (iii) holds when $n = 0$.

Now suppose the result holds for $n = k \geq 0$. By definition

$$U_{k+1} = \overline{\mathrm{Is}(U_k)} \cap \overline{(U_k - \overline{\mathrm{Is}(U_k)})}$$

and by the induction hypothesis

$$\overline{\mathrm{Is}(U_k)} \cap \overline{(U_k - \overline{\mathrm{Is}(U_k)})} = \overline{(U \cap \mathrm{Is}(X_k))} \cap \overline{(U \cap (X_k - \overline{\mathrm{Is}(X_k)})}.$$

Now U is clopen in X, so

$$\overline{(U \cap \mathrm{Is}(X_k))} \cap \overline{(U \cap (X_k - \overline{\mathrm{Is}(X_k)})} = (U \cap \overline{\mathrm{Is}(X_k)}) \cap (U \cap \overline{(X_k - \overline{\mathrm{Is}(X_k)})}).$$

Since the right-hand side of the last formula is equal to $U \cap X_{k+1}$, we see that (i) holds for $n = k + 1$. That (ii) holds when $n = k + 1$ follows easily from (i) and the fact that $U \in \mathrm{Clop}(X)$.

Finally,

$$U_{k+1} - \overline{\mathrm{Is}(U_{k+1})} = (U \cap X_{k+1}) - (\overline{U \cap \mathrm{Is}(X_{k+1})})$$

by (i) and (ii), and since U is clopen in X,

$$(U \cap X_{k+1}) - (\overline{U \cap \mathrm{Is}(X_{k+1})}) = (U \cap X_{k+1}) - (U \cap \overline{\mathrm{Is}(X_{k+1})}).$$

Since the right-hand side of the last formula is equal to $U \cap (X_{k+1} - \overline{\mathrm{Is}(X_{k+1})})$, (iii) holds when $n = k + 1$. ∎

LEMMA 5.4. *Let W and Z be normal P-spaces and $q : W \to Z$ a map. If $U \in \mathrm{Clop}(W)$, $q(U) \in \mathrm{Clop}(Z)$, and $q|U$ is a homeomorphism, then each of the following hold for any $n \geq 0$:*

(i) $q(U \cap W_n) = q(U) \cap Z_n$.
(ii) $q(U \cap \mathrm{Is}(W_n)) = q(U) \cap \mathrm{Is}(Z_n)$.
(iii) $q(U \cap (W_n - \overline{\mathrm{Is}(W_n)})) = q(U) \cap (Z_n - \overline{\mathrm{Is}(Z_n)})$.

Proof. The result essentially follows from Lemma 5.3. For example, $q(U \cap W_n) = q(U_n)$ by Lemma 5.3 since $U \in \mathrm{Clop}(W)$. Then $q(U_n) = q(U)_n$ because $q|U$ is a homeomorphism. Finally, $q(U) \in \mathrm{Clop}(Z)$ so $q(U)_n = q(U) \cap Z_n$ by Lemma 5.3. Similar arguments prove the other two identities. ∎

LEMMA 5.5. *Suppose $\{Y_k\}_{k\in K}$ is a collection of normal P-spaces. If $W = \bigoplus_{k\in K} Y_k$, then each of the following hold for any $n \geq 0$:*

(i) $W_n = \bigcup_{k\in K} (Y_k)_n$.
(ii) $\mathrm{Is}(W_n) = \bigcup_{k\in K} \mathrm{Is}((Y_k)_n)$.
(iii) $W_n - \overline{\mathrm{Is}(W_n)} = \bigcup_{k\in K} ((Y_k)_n - \overline{\mathrm{Is}((Y_k)_n)})$.

Proof. Since $Y_k \in \mathrm{Clop}(W)$ for all $k \in K$, we can apply Lemma 5.3. ∎

LEMMA 5.6. *Let Z be the pointwise gluing of the normal P-spaces $\{(Y_k, y_k)\}_{k\in K}$ and $(X, \{x_k\}_{k\in K})$ over K. Then $q((Y_k)_n) \subseteq Z_n$ for every $k \in K$ and every $n \geq 0$.*

Proof. Let $W = (\bigoplus_{k\in K} Y_k) \oplus X$. Note that Z is a normal P-space by Lemma 5.2. Fix $k \in K$. If $n = 0$ then the result holds because $(Y_k)_0 = Y_k$ and $Z_0 = Z$. Suppose $n > 0$ and $w \in (Y_k)_n$. Let V be a neighborhood of $q(w)$ in Z. We must show that V meets both $\mathrm{Is}(Z_{n-1})$ and $Z_{n-1} - \overline{\mathrm{Is}(Z_{n-1})}$. Now $w \in (Y_k)_n$ and $n > 0$, so we know that

$$w \in \overline{\mathrm{Is}((Y_k)_{n-1})} \cap \overline{((Y_k)_{n-1} - \overline{\mathrm{Is}((Y_k)_{n-1})})}.$$

Therefore $q^{-1}(V) \cap Y_k$ must contain infinitely many elements of $\mathrm{Is}((Y_k)_{n-1})$ and infinitely many elements of $(Y_k)_{n-1} - \overline{\mathrm{Is}((Y_k)_{n-1})}$. Pick $y \in \mathrm{Is}((Y_k)_{n-1}) \cap (q^{-1}(V) \cap Y_k)$ such that $y \neq y_k$.

Choose $U \in \mathrm{Clop}(W)$ such that $U \subseteq Y_k$, $y \in U$, and $y_k \notin U$. Then $\mathrm{gp}(U) = \emptyset$, so U satisfies the hypotheses of Lemma 5.4. Since $y \in \mathrm{Is}(W_{n-1})$ by Lemma 5.5, it follows that $q(y) \in \mathrm{Is}(Z_{n-1})$. A similar argument shows that V contains members of $Z_{n-1} - \overline{\mathrm{Is}(Z_{n-1})}$. ∎

LEMMA 5.7. *Let X be an η_1-set, γ an ordinal, $x_\gamma \in X$, and $\{x_\alpha\}_{\alpha\in\gamma}$ a strictly increasing sequence in X that is cofinal in $\{x \in X : x < x_\gamma\}$. If Z is the pointwise gluing of the normal P-spaces $\{(Y_\alpha, y_\alpha)\}_{\alpha\in\gamma}$ and $(X, \{x_\alpha\}_{\alpha\in\gamma})$ over γ, $n \geq 1$, and $(Y_\alpha)_n = \{y_\alpha\}$ for all $\alpha \in \gamma$, then $q(x_\gamma) \in \overline{\mathrm{Is}(Z_n)} - \mathrm{Is}(Z_n)$ and $Z_{n+1} = \emptyset$.*

Proof. Let $W = (\bigoplus_{\alpha\in\gamma} Y_\alpha) \oplus X$. Note that Z is a normal P-space by Lemma 5.2 and fix $\alpha < \gamma$. First we show that $q(y_\alpha) \in \overline{\mathrm{Is}(Z_n)}$.

CASE (1). Suppose there is an open set $G \subseteq X$ such that $x_\alpha \in G$ and $\mathrm{gp}(G) = \{\alpha\}$. We show that $q(y_\alpha) \in \mathrm{Is}(Z_n)$. By hypothesis $y_\alpha \in (Y_\alpha)_n$, so $q(y_\alpha) \in Z_n$ by Lemma 5.6. Since W is zero-dimensional we may find $H \in \mathrm{Clop}(W)$ such that $x_\alpha, y_\alpha \in H$ and $H \subseteq G \cup Y_\alpha$. Note that $\mathrm{gp}(H) = \{\alpha\}$ and $\mathrm{gp}(H) \cap \mathrm{gp}(W - H) = \emptyset$. So $q(H) \in \mathrm{Clop}(Z)$ by Lemma 5.1. We claim that $q(H) \cap Z_n = \{q(y_\alpha)\}$. Let $w \in H$. We already know that $q(y_\alpha) = q(x_\alpha) \in Z_n$, so suppose w is not y_α or x_α. Then there is a $U \in \mathrm{Clop}(W)$ such that $w \in U \subseteq H$ and $x_\alpha, y_\alpha \notin U$. Now X is an η_1-set and $n \geq 1$, so $X_n = \emptyset$, and $(Y_\alpha)_n = \{y_\alpha\}$. It follows that $U \cap (Y_\alpha)_n = U \cap X_n = \emptyset$ and hence that $U \cap W_n = \emptyset$, by Lemma 5.5. Since $\mathrm{gp}(U) = \emptyset$, we may apply Lemma 5.4 to conclude that $q(U) \cap Z_n = \emptyset$. Thus $q(w) \notin Z_n$, $q(H) \cap Z_n = \{q(y_\alpha)\}$, and $q(y_\alpha) \in \overline{\mathrm{Is}(Z_n)}$.

CASE (2). Suppose the hypothesis of Case (1) fails. Let V be an open neighborhood of $q(y_\alpha)$ in Z. Then $q^{-1}(V) \cap X$ is open in X, and since X is an η_1-set, there is an open interval $I \subseteq q^{-1}(V) \cap X$ such that $x_\alpha \in I$. By our supposition there is a $\beta < \gamma$ such that $\beta \neq \alpha$ and $x_\beta \in I$. Since either $x_{\alpha+1}$ or $x_{\beta+1}$ is in I, we may choose a successor ordinal $\delta < \gamma$ such that $x_\delta \in I$. Then $(x_{\delta-1}, x_{\delta+1}) \cap I$ is an open subset of X containing x_δ whose only glue point is x_δ, so $q(x_\delta) \in \mathrm{Is}(Z_n)$ by Case (1). Since $q(x_\delta) \in V$, it follows that $q(y_\alpha) \in \overline{\mathrm{Is}(Z_n)}$.

Now we show that $q(x_\gamma) \in \overline{\mathrm{Is}(Z_n)} - \mathrm{Is}(Z_n)$. Let $G \in \mathrm{Clop}(Z)$ be a neighborhood of $q(x_\gamma)$. Then $q^{-1}(G) \cap X$ is open in X, and since X is an η_1-set, there is an open interval $U \subseteq q^{-1}(G) \cap X$ containing x_γ. Since U is an open interval there must be an $x \in U$ such that $x < x_\gamma$. But $\{x_\alpha\}_{\alpha<\gamma}$ is cofinal in $\{x \in X : x < x_\gamma\}$, so $x_\alpha \in U$ for some $\alpha < \gamma$. As we know, $q(x_\alpha) \in \overline{\mathrm{Is}(Z_n)}$ and hence G must contain some member of $\mathrm{Is}(Z_n)$. Since G was arbitrary and $q(x_\alpha) \neq q(x_\gamma)$, it follows that $q(x_\gamma) \in \overline{\mathrm{Is}(Z_n)} - \mathrm{Is}(Z_n)$.

Finally, to prove that $Z_{n+1} = \emptyset$ we show that $Z_n - \overline{\mathrm{Is}(Z_n)} = \emptyset$. Let $w \in W$. If $w = y_\alpha$ for some $\alpha \in \gamma$ then $q(w) \in \overline{\mathrm{Is}(Z_n)}$. If $w \in Y_\alpha$ for some $\alpha \in \gamma$ and $w \neq y_\alpha$ then $w \notin (Y_\alpha)_n$, and so $q(w) \notin Z_n$ follows from Lemma 5.4. If $w \in X$ and there is $U \in \mathrm{Clop}(X)$ such that $\mathrm{gp}(U) = \emptyset$ then $q(w) \notin Z_n$ follows from Lemma 5.4. Otherwise, every neighborhood of w contains some x_α and therefore $q(w) \in \overline{\mathrm{Is}(Z_n)}$. Thus $Z_n - \overline{\mathrm{Is}(Z_n)} = \emptyset$. ∎

LEMMA 5.8. *Let Z be the pointwise gluing of the normal P-spaces (Y, y_0) and (X, x_0) and let $n \geq 0$. If $Y_{n+1} = \emptyset = X_{n+1}$, $y_0 \in \overline{\mathrm{Is}(Y_n)} - \mathrm{Is}(Y_n)$, and $x_0 \in X_n - \overline{\mathrm{Is}(X_n)}$, then* $\mathrm{inv}(\mathrm{Clop}(Z)) = (n+1, 0, 1)$.

Proof. Let $W = X \oplus Y$. We know that Z is a normal P-space by Lemma 5.2. Let $z_0 = q(x_0) = q(y_0)$. By Proposition 4.2, it suffices to show that $Z_{n+1} = \{z_0\}$. First we show that $z_0 \in Z_{n+1}$. Let V be any open neighborhood of z_0 in Z. Since Z is zero-dimensional we may assume that $V \in \mathrm{Clop}(Z)$. Since $y_0 \in q^{-1}(V)$ and $y_0 \in \overline{\mathrm{Is}(Y_n)} - \mathrm{Is}(Y_n)$, we may pick $y \in (q^{-1}(V) \cap Y) \cap \mathrm{Is}(Y_n)$ such that $y \neq y_0$. Then $y \in \mathrm{Is}(W_n)$ by Lemma 5.5 and there is a clopen subset $U \subseteq Y$ with $y \in U$ and $y_0 \notin U$. Since $\mathrm{gp}(U) = \emptyset$, it follows from Lemma 5.4 that $q(y) \in \mathrm{Is}(Z_n)$. Thus $V \cap \mathrm{Is}(Z_n) \neq \emptyset$ and $z_0 \in \overline{\mathrm{Is}(Z_n)}$.

Now $q^{-1}(V) \cap X \in \mathrm{Clop}(X)$ and by hypothesis $x_0 \notin \overline{\mathrm{Is}(X_n)}$, so there is an $H \in \mathrm{Clop}(X)$ with $x_0 \in H \subseteq (q^{-1}(V) \cap X) \cap (X_n - \overline{\mathrm{Is}(X_n)})$. Since $x_0 \in H$ and $x_0 \notin \mathrm{Is}(X_n)$, we may choose $x \in H$ with $x \neq x_0$. Then $x \in W_n - \overline{\mathrm{Is}(W_n)}$ by Lemma 5.5. Let U be a clopen subset of X containing x but not x_0. Then $\mathrm{gp}(U) = \emptyset$, so $q(x) \in Z_n - \overline{\mathrm{Is}(Z_n)}$ by Lemma 5.4. Thus $V \cap (Z_n - \overline{\mathrm{Is}(Z_n)}) \neq \emptyset$ and $z_0 \in \overline{Z_n - \overline{\mathrm{Is}(Z_n)}}$. Hence $z_0 \in Z_{n+1}$.

Finally, we show that z_0 is the only member of Z_{n+1}. Suppose $w \in W$ with $x_0 \neq w \neq y_0$. Then there is a clopen neighborhood U of w in W such that $x_0, y_0 \notin U$. Since $Y_{n+1} = X_{n+1} = \emptyset$, Lemma 5.5 tells us that $W_{n+1} = \emptyset$. Thus $w \notin W_{n+1}$. But $\mathrm{gp}(U) = \emptyset$, so it follows from Lemma 5.4 that $q(w) \notin Z_{n+1}$. ∎

6. Construction of P-spaces. The empty space is a normal P-space whose algebra of clopen subsets has invariants $(-1, 0, 0)$. Let X be any one-point space. Then X is a normal P-space and $|X_0| = |X| = 1$. So it follows from Proposition 4.2 that $\mathrm{inv}(\mathrm{Clop}(X)) = (0, 0, 1)$. Next let Y be the space from the first example at the start of Section 2 and let X be an η_1-set. Let y_0 be the sole nonisolated point in Y and pick some x_0 in X. Note that $\mathrm{Is}(X) = \emptyset$ and that $\mathrm{Is}(Y) = Y - \{y_0\}$. Form Z, the pointwise gluing of (Y, y_0) and (X, x_0). Then Z is a normal P-space by Lemma 5.2. Since $Y_1 = \emptyset = X_1$, $y_0 \in \overline{\mathrm{Is}(Y)} - \mathrm{Is}(Y)$, and $x_0 \in X - \overline{\mathrm{Is}(X)}$, it follows from Lemma 5.8 that $\mathrm{inv}(\mathrm{Clop}(Z)) = (1, 0, 1)$.

Let X^l and X^r be η_1-sets and pick an arbitrary point $x_l \in X^l$ and an arbitrary point $x_r \in X^r$. Choose an ordinal γ and a strictly increasing sequence $\{x^r_\alpha\}_{\alpha\in\gamma}$ in X^r such that $\{x^r_\alpha\}_{\alpha\in\gamma}$ is cofinal in the set of $x \in X^r$ with $x < x_r$. Let $\{x^l_k\}_{k\in K}$ be an enumeration of X^l. Assume that Y is a normal P-space with $\mathrm{inv}(\mathrm{Clop}(Y)) = (n, 0, 1)$ for some $n \geq 1$. Then Y_n consists of a single point by Proposition 4.2. Let $\{Y^r_\alpha\}_{\alpha\in\gamma}$ and $\{Y^l_k\}_{k\in K}$ be collections of copies of Y with y^r_α and y^l_k representing the lone elements of $(Y^r_\alpha)_n$

and $(Y^l_k)_n$, respectively. Let Z^r be the pointwise gluing of $\{(Y^r_\alpha, y^r_\alpha)\}_{\alpha\in\gamma}$ and $(X^r, \{x^r_\alpha\}_{\alpha\in\gamma})$ over γ, and Z^l the pointwise gluing of $\{(Y^l_k, y^l_k)\}_{k\in K}$ and $(X^l, \{x^l_k\}_{k\in K})$ over K. Both Z^r and Z^l are normal P-spaces by Lemma 5.2. Let $L = q(x_l) \in Z^l$ and $R = q(x_r) \in Z^r$. Finally, let Z be the pointwise gluing of (Z^l, L) and (Z^r, R). We claim that $\mathrm{inv}(\mathrm{Clop}(Z^l)) = (n, 1, 0)$ and that $\mathrm{inv}(\mathrm{Clop}(Z)) = (n + 1, 0, 1)$.

To prove that $\mathrm{inv}(\mathrm{Clop}(Z^l)) = (n, 1, 0)$ it suffices, by Proposition 4.2, to show that $\mathrm{Is}(Z^l_n) = \emptyset$ and $Z^l_n \neq \emptyset$. The latter follows from Lemma 5.6 because $y^l_k \in (Y^l_k)_n$ for all $k \in K$; in particular, note that $L \in Z^l_n$ because $x_l = x^l_k$ for some $k \in K$ and $q(x^l_k) = q(y^l_k)$. As for the former, let $G \in \mathrm{Clop}(Z^l)$ with $G \cap Z^l_n \neq \emptyset$. Then $q^{-1}(G) \cap X^l \neq \emptyset$, for otherwise $\mathrm{gp}(q^{-1}(G)) = \emptyset$ and $G \cap Z^l_n = \emptyset$ by Lemma 5.4. Now $q^{-1}(G) \cap X^l$ must be infinite because it is clopen in X^l, and X^l has no isolated points. Since $q(x^l_k) = q(y^l_k)$ for every $k \in K$, it follows that $G \cap Z^l_n$ is infinite and hence that $\mathrm{Is}(Z^l_n) = \emptyset$.

To prove that $\mathrm{inv}(\mathrm{Clop}(Z)) = (n + 1, 0, 1)$ it suffices, by Lemma 5.8, to show that $Z^l_{n+1} = \emptyset = Z^r_{n+1}$, $R \in \overline{\mathrm{Is}(Z^r_n)} - \mathrm{Is}(Z^r_n)$, and $L \in Z^l_n - \overline{\mathrm{Is}(Z^l_n)}$. We have already seen that $L \in Z^l_n$ and $\mathrm{Is}(Z^l_n) = \emptyset$, from which it follows that $Z^l_{n+1} = \emptyset$ and $L \in Z^l_n - \overline{\mathrm{Is}(Z^l_n)}$. That $R \in \overline{\mathrm{Is}(Z^r_n)} - \mathrm{Is}(Z^r_n)$ and $Z^r_{n+1} = \emptyset$ both follow from Lemma 5.7.

So by induction there are, for every integer $n \geq 0$, normal P-spaces whose clopen algebras have the invariants $(n, 0, 1)$ and also normal P-spaces whose clopen algebras have the invariants $(n, 1, 0)$. Fix $n \geq 0$ and let m be a countable cardinal greater than zero. Next we build a normal P-space whose clopen algebra has invariants $(n, 0, m)$. Let $\{Y_k\}_{k\in K}$ be a collection of normal P-spaces such that $|K| = m$ and $\mathrm{inv}(\mathrm{Clop}(Y_k)) = (n, 0, 1)$ for each $k \in K$. Let

$$X = \bigoplus_{k\in K} Y_k.$$

From Proposition 4.1 we know that $|\mathrm{Is}((Y_k)_n)| = 1$ and $(Y_k)_n = \overline{\mathrm{Is}((Y_k)_n)}$ for each $k \in K$. So $|\mathrm{Is}(X_n)| = m$ and $X_n = \overline{\mathrm{Is}(X_n)}$ both follow from Lemma 5.5. Hence $\mathrm{inv}(\mathrm{Clop}(W)) = (n, 0, m)$ by Proposition 4.2.

Next we build a normal P-space whose clopen algebra has invariants $(n, 1, m)$. Let Y be a normal P-space with $\mathrm{inv}(\mathrm{Clop}(Y)) = (n, 1, 0)$, X be a normal P-space with $\mathrm{inv}(\mathrm{Clop}(X)) = (n, 0, m)$, and $W = X \oplus Y$. Then $|\mathrm{Is}(X_n)| = m$, $X_n = \overline{\mathrm{Is}(X_n)}$, $\mathrm{Is}(Y_n) = \emptyset$, and $Y_n \neq \overline{\mathrm{Is}(Y_n)}$. It follows from Lemma 5.5 that $|\mathrm{Is}(W_n)| = m$ and $W_n \neq \overline{\mathrm{Is}(W_n)}$. Hence, by Proposition 4.2, $\mathrm{inv}(\mathrm{Clop}(W)) = (n, 1, m)$ as desired.

Finally, we build a normal P-space whose clopen algebra has the invariants $(\omega, 0, 0)$. For each integer $j \geq 0$ let Y_j be a normal P-space with

$\mathrm{inv}(\mathrm{Clop}(Y_j)) = (j, 0, 1)$. Let

$$W = \bigoplus_{j\geq 0} Y_j.$$

By Proposition 4.2, $(Y_j)_n \neq \emptyset$ whenever $n \geq 0$ and $j \geq n$. So $W_n \neq \emptyset$ for all $n \geq 0$ by Lemma 5.5. Thus $\mathrm{inv}(\mathrm{Clop}(X)) = (\omega, 0, 0)$ by Proposition 4.2 and the proof of Theorem 1.1 is complete.

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