FUNDAMENTA
MATHEMATICAE
201 (2008)

Perfect set theorems

by

Otmar Spinas (Kiel)

Abstract. We study splitting, infinitely often equal (ioe) and refining families from the descriptive point of view, i.e. we try to characterize closed, Borel or analytic such families by proving perfect set theorems. We succeed for G_δ hereditary splitting families and for analytic countably ioe families. We construct several examples of small closed ioe and refining families.

1. Introduction. In this paper we study three properties of subsets of the reals that occur in the definition (or, as in the second case, an equivalent form of it) of three well known cardinal characteristics of the continuum, namely the splitting number $\mathfrak{s}$, the uniformity of the meager ideal $\mathrm{unif}(\mathcal{M})$ and the refining number $\mathfrak{r}$. Thus we study splitting families (see Definition 2.1), infinitely often equal families (Definition 3.1) and refining families (Definition 4.1). However, we are interested in these properties from the descriptive point view, i.e. we try to characterize definable (i.e. closed, Borel or analytic) such families. Each of these three notions has its countable version. E.g. a countably splitting family is such that any countably many reals can be split simultaneously by a member of the family. In [10] we characterized analytic countably splitting families by proving a perfect set theorem for these, i.e. finding a type of closed countably splitting family that occurs as a subset of each analytic such family. The proof used a game argument, thus holds more generally, i.e. for all sets for which the splitting game is determined. Velickovic then noticed that a theorem of Solecki can be used to show that analytic countably splitting families always contain a G_δ subset which is also countably splitting. We shall show that for G_δ families the countably splitting property is the same thing as the essentially everywhere splitting property (see Definition 2.3).

2000 *Mathematics Subject Classification*: 03E05, 03E15, 03E17.

Key words and phrases: splitting family, infinitely often equal family, refining family, hereditary, perfect set, 2-colourable hypergraph.

Work partially supported by DFG grant SP 683/1-2.

But as a curious fact we will show that this equivalence fails at the $F_{\sigma\delta}$ level.

In [10] we also showed that for analytic splitting families such a neat characterization is impossible, by proving the existence of a G_δ splitting family that does not contain a closed splitting family. Here we improve a bit on this by showing that this example can be made Δ^0_2. Both these examples are not-hereditary. There is a good reason for this, as we shall show that every hereditary G_δ family A on 2^ω contains an inclusion-dense closed subfamily B. Here *inclusion-dense* means that every infinite element of A has an infinite subset in B (we identify subsets of ω with their characteristic functions in 2^ω). This easily implies that every hereditary G_δ splitting family contains a closed splitting family. Whether this or our general theorem holds for all hereditary analytic sets is an open problem.

In Section 3 we shall study analytic infinitely often equal families. It turns out that here analogous results hold with analogous proofs. So analytic countably infinitely often equal families are characterizable by a perfect set theorem, whereas analytic infinitely often equal families are not. Actually not even closed such families are well understood. A prominent related open problem is whether there exists a closed maximal almost disjoint family in ω^ω (see [4, Question 4.3]). (Such a family is infinitely often equal but not countably so.)

In Section 4 we study refining families (which is the dual notion to that of a splitting family). Here the situation seems to be even more intricate than for the previous two properties. No characterization of analytic (or even closed) refining families (countably so or not) seems to be reachable. A first candidate for such a characterization that comes to one's mind are Mathias trees. That these are no good for our purpose is shown by an example by Di Prisco and Todorcevic [1]. This is an example of a closed strongly dominating countably refining family that does not contain a Mathias tree.

We shall construct two more examples that are smaller on the scale given by the eventual dominance relation on ω^ω. The first one is a closed dominating, not strongly dominating refining family that is not even 2-refining. We have a strong conjecture that this one is minimal in the sense that no closed non-dominating subset is refining. This conjecture is related to the well-known intractable problem of characterizing 2-colourable hypergraphs. Our second example is a closed non-dominating refining family.

2. Splitting families

DEFINITION 2.1. We say that $x \in 2^\omega$ *splits* $a \in [\omega]^\omega$ iff $\exists^\infty i \in a\ \exists^\infty j \in a$ $(x(i) = 0\ \&\ x(j) = 1)$. A family $A \subseteq 2^\omega$ is called *splitting* iff every $a \in [\omega]^\omega$ is split by some $x \in A$, and A is *countably splitting* iff for every countable

$C \subseteq [\omega]^\omega$ there is a member of A that splits every $a \in C$. A splitting family A is called *everywhere splitting* iff $\forall x \in A\ \forall n\ \ A \cap [x{\restriction}n]$ is splitting. The *everywhere countably splitting* property is defined in the obvious way.

A tree $p \subseteq 2^{<\omega}$ is called a *splitting tree* if $p \neq \emptyset$ and for every $\sigma \in p$ there exists $K < \omega$ such that for every $n \geq K$ and $i < 2$ there exists $\tau \in p$ with $\sigma \subseteq \tau$, $|\tau| > n$ and $\tau(n) = i$. Note that if p is a splitting tree then $[p]$ is countably splitting and actually everywhere so.

In [10, 1.2] the following has been proved:

THEOREM 2.2. *Let $A \subseteq 2^\omega$ be analytic. Then A is countably splitting iff there exists a splitting tree p such that $[p] \subseteq A$. In particular, A contains a countably splitting closed subset.*

There exists a natural derivation process to determine whether a given splitting family contains an everywhere splitting subfamily or not. This will show that every countably splitting family contains an everywhere splitting subfamily.

DEFINITION 2.3. For any $A \subseteq 2^\omega$ let $A' = A \setminus \bigcup\{[\sigma] : \sigma \in 2^{<\omega}\ \&\ A \cap [\sigma]$ is not splitting$\}$. Recursively define $A^{(0)} = A$, $A^{(\alpha+1)} = (A^{(\alpha)})'$. For a limit ordinal λ let $A^{(\lambda)} = \bigcap_{\alpha<\lambda} A^{(\alpha)}$. Clearly there exists a least $\gamma < \omega_1$ such that $A^{(\gamma+1)} = A^{(\gamma)}$. Let us call this γ the *split-rank* of A, and denote it by $\mathrm{sprk}(A)$. Notice that $A^{(\gamma)}$ is everywhere splitting if $A^{(\gamma)} \neq \emptyset$. Moreover, $\mathrm{sprk}(A) = 0$ iff $A = \emptyset$ or A is everywhere splitting.

We say that A is *essentially everywhere splitting* (*e.e. splitting*, for short) if $A^{(\gamma)} \neq \emptyset$.

There exists an analogous notion of an everywhere unbounded, or everywhere dominating family of functions in ω^ω (e.g. in [11]). Whereas it is easy to see that every unbounded set in ω^ω contains an everywhere unbounded subset, and similarly for dominating sets, this is not the case for splitting families.

PROPOSITION 2.4. *For every $0 < \alpha < \omega_1$ there exists an F_σ splitting family $A \subseteq 2^\omega$ such that $\mathrm{sprk}(A) = \alpha$ and $A^{(\alpha)} = \emptyset$. Hence A is not e.e. splitting.*

Proof. By induction on α. For $\alpha = 1$ choose an infinite, coinfinite set $a \subseteq \omega$. Let $a^0 = a$ and $a^1 = \omega \setminus a$.

For given $\sigma \in 2^{<\omega}$ and $i < 2$ let $p_\sigma^{a,i}$ be the tree defined as follows:

$$p_\sigma^{a,i} = \{\tau \in 2^{<\omega} : \tau \subseteq \sigma \vee (\sigma \subseteq \tau\ \&\ \forall j \in |\tau| \cap a^i \setminus |\sigma|\ \ \tau(j) = 0)\}.$$

Now let $A = [p_{\langle 0\rangle}^{a,0}] \cup [p_{\langle 1\rangle}^{a,1}]$. Clearly A is closed and splitting. The only $\sigma \in 2^{<\omega}$ such that $A \cap [\sigma]$ is splitting is $\sigma = \emptyset$. Hence we have $A' = \emptyset$ and $\mathrm{sprk}(A) = 1$.

Now suppose that F is a splitting F_σ, $\mathrm{sprk}(F) = \alpha$ for some $1 < \alpha < \omega_1$ and $F^{(\alpha)} = \emptyset$. For $\sigma \in 2^{<\omega}$ let $A(\sigma, F) = \{\sigma^\frown x : x \in F\}$. Fix some infinite, coinfinite $a \subseteq \omega$. Let Q consist of all minimal (with respect to length) elements of $2^{<\omega} \setminus p^{a,0}_{\langle 0\rangle} \cup p^{a,1}_{\langle 1\rangle}$. Define

$$A = [p^{a,0}_{\langle 0\rangle}] \cup [p^{a,1}_{\langle 1\rangle}] \cup \bigcup\{A(\sigma, F) : \sigma \in Q\}.$$

Clearly for every $\sigma \in 2^{<\omega}$ we have $\mathrm{sprk}(A(\sigma, F)) = \mathrm{sprk}(F) = \alpha$, and $A(\sigma, F)$, and hence A is F_σ. Since every node of $p^{a,0}_{\langle 0\rangle} \cup p^{a,1}_{\langle 1\rangle}$ has an extension in Q, we conclude $A^{(\alpha)} = [p^{a,0}_{\langle 0\rangle}] \cup [p^{a,1}_{\langle 1\rangle}]$ and hence $\mathrm{sprk}(A) = \alpha + 1$ and $A^{(\alpha+1)} = \emptyset$.

Finally, suppose that $\alpha < \omega_1$ is a limit ordinal, that $\langle \alpha_n : n < \omega\rangle$ is an increasing sequence with $\sup_{n<\omega} \alpha_n = \alpha$, and that $F_n \subseteq 2^\omega$ are splitting F_σ's such that $\mathrm{sprk}(F_n) = \alpha_n$ and $F_n^{(\alpha_n)} = \emptyset$. Let σ_n be the sequence starting with n 0's and ending with one 1. Define $A = \bigcup_{n\in\omega} A(\sigma_n, F_n)$. Clearly A is F_σ and we have $A^{(\alpha_n)} = \bigcup_{m>n} A(\sigma_m, F_m)^{(\alpha_n)} \neq \emptyset$ for every $n < \omega$, as the α_n increase. Since the $A(\sigma_n, F_n)$ are pairwise disjoint we conclude $A^{(\alpha)} = \bigcap_{n<\omega} A^{(\alpha_n)} = \emptyset$ and hence $\mathrm{sprk}(A) = \alpha$. ∎

Question. Can we replace "F_σ" by "closed" in Proposition 2.4?

We have seen that every countably splitting family is e.e. splitting. The converse is true for G_δ sets by the next theorem. Though by Theorem 2.7, it fails for $F_{\sigma\delta}$ sets.

Theorem 2.5. *Every G_δ set $G \subseteq 2^\omega$ that is essentially everywhere splitting is countably splitting.*

Proof. Let $G = \bigcap_{n<\omega} U_n$ where each U_n is open. Let $\mathrm{sprk}(G) = \gamma$. Hence $G^{(\gamma)}$ is everywhere splitting. Let $p = \{x{\restriction}n : n < \omega \ \&\ x \in G^{(\gamma)}\}$. Now we have the following observation:

Claim. *p is a splitting tree.*

Proof of the claim. Let $\sigma \in p$. As $G^{(\gamma)} \cap [\sigma]$ is splitting we conclude that the set

$$M = \{k < \omega : \exists j < 2\ \forall \tau \in p\ (\sigma \subseteq \tau \ \&\ k < |\tau|) \Rightarrow \tau(k) = j\}$$

is finite. ∎

Now let $F \subseteq [\omega]^\omega$ be countable. It is an easy business to construct $\langle \tau_l : l < \omega\rangle$ in p with $\tau_l \subsetneq \tau_{l+1}$ such that for each n, $[\tau_l] \subseteq U_n$ for some l, and for each $a \in F$ and $i < 2, \tau_l(k) = i$ for infinitely many l and some $k \in a \cap |\tau_l| \setminus |\tau_{l-1}|$. If we let $y = \bigcup_{l<\omega} \tau_l$ then $y \in G$ and y splits every member of F. ∎

Corollary 2.6. *Every G_δ set that is essentially everywhere splitting contains a closed countably splitting subset.*

Proof. By [10, Theorem 1.2] every analytic countably splitting family contains a closed subset with the same property. ∎

Surprisingly to me, Theorem 2.5 does not hold for analytic sets. Actually it already fails at the $F_{\sigma\delta}$ level, as the following example shows. It is the ideal of nowhere dense subsets of the rationals, that occasionally showed up in the literature as kind of a marplot (see e.g. [2]). Recall that a subset of $\wp(\omega)$ is called *hereditary* if it is closed under taking subsets.

Theorem 2.7. *There exists a hereditary $F_{\sigma\delta}$ everywhere splitting family which is not countably splitting. Nevertheless this example contains a closed splitting subset.*

Proof. We replace ω by $2^{<\omega}$ and let $\mathrm{NWD}(2^{<\omega})$ be the set of all $a \subseteq 2^{<\omega}$ that are nowhere dense in $2^{<\omega}$. $\mathrm{NWD}(2^{<\omega})$ is a well-known σ-ideal that has been studied by many authors (see [2]).

Let Q be the set of all $x \in 2^\omega$ which are eventually zero. For $x \in 2^\omega$ let $a_x = \{x{\restriction}n : n < \omega\}$. Clearly every $a \subseteq 2^{<\omega}$ that splits every member of $\{a_x : x \in Q\}$ must be dense in $2^{<\omega}$. Thus $\mathrm{NWD}(2^{<\omega})$ is not countably splitting. Let R be the comparability relation on $2^{<\omega}$, thus $\{\sigma,\tau\} \in R$ iff $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. Let $C = \{a \subseteq 2^{<\omega} : a \text{ is } R\text{-homogeneous}\}$. It is easy to see that C is closed and $C \subseteq \mathrm{NWD}(2^{<\omega})$. By Ramsey's theorem, every infinite subset of $2^{<\omega}$ has an infinite subset belonging to C. As C is hereditary, C is splitting. It remains to check that $\mathrm{NWD}(2^{<\omega})$ is everywhere splitting, thus $\mathrm{sprk}(\mathrm{NWD}(2^{<\omega})) = 0$. Let σ be a finite partial function from $2^{<\omega}$ to 2, and let $x \in 2^{(2^{<\omega})}$ be arbitrary with $x^{-1}(1)$ infinite. By Ramsey's theorem there exists $y : 2^{<\omega} \setminus \mathrm{dom}(\sigma) \to 2$ such that $y^{-1}(1) \subseteq x^{-1}(1)$ is infinite and R-homogeneous and $x^{-1}(1) \setminus y^{-1}(1)$ is infinite. Then clearly we have $\sigma \cup y \in \mathrm{NWD}(2^{<\omega})$ and $\sigma \cup y$ splits x. ∎

In [10, Theorem 1.10] it has been shown that there exists an F_σ splitting family that does not contain a closed splitting family. The next result slightly improves that construction. Recall that for given $A \subseteq \wp(\omega)$ we call a collection $B \subseteq \wp(\omega)$ *inclusion-dense* in A if every infinite set in A has an infinite subset that belongs to B.

Theorem 2.8. *There exists a splitting family $A \subseteq 2^\omega$ that is both F_σ and G_δ, such that A does not contain a closed splitting family, nor does A contain an inclusion-dense closed subset.*

Proof. Let $\langle a_n : n < \omega\rangle$ be a partition of ω into infinite sets. For any $b \in [\omega]^\omega$ let $\langle b(i) : i < \omega\rangle$ be the increasing enumeration of b. Define an increasing $x \in \omega^\omega$ as follows: $x(0), x(1)$ are the first two elements of a_0. In general, $x(2n), x(2n+1)$ are the first two elements of $a_n \setminus (x(2n-1)+1)$. Now let

$$\begin{aligned} A_0 &= \{b \in [\omega]^\omega : b(0) = x(0) \ \&\ b(1) > x(1) \ \&\ \forall m \ \ |b \cap a_m| \le 1\}, \\ A_{n+1} &= \{b \in [\omega]^\omega : b \cap (x(2n+1)+1) = \{x(0), \ldots, x(2n+1)\} \\ &\qquad \&\ \forall j \ (x(2n+1) < a_n(2j) \Rightarrow b \cap \{a_n(2j), a_n(2j+1)\} \ne \emptyset) \\ &\qquad \&\ b \setminus (x(2n+1)+1) \subseteq a_n\}. \end{aligned}$$

Clearly each A_n is closed. Let $A = \bigcup_{n<\omega} A_n$. Hence A is F_σ. For each n let j_n be minimal such that $x(2n+1) < a_n(2j_n)$.

Let $\sigma_i^{n+1} \in 2^{a_n(2j_n+i)+1}$ be the characteristic function of $\{x(0), \ldots, x(2n+1), a_n(2j_n+i)\}$. Let $U_{n+1} = [\sigma_0^{n+1}] \cup [\sigma_1^{n+1}]$. Let $\sigma^0 \in 2^{x(1)+1}$ be the characteristic function of $\{x(0)\}$, and let $U_0 = [\sigma^0]$. Note that the U_n are pairwise disjoint open sets and $A_n \subseteq U_n$. For each n let $V_m^n \subseteq U_n$ be open sets such that $A_n = \bigcap_{m<\omega} V_m^n$. We conclude that

$$A = \bigcup_{m<\omega} \bigcap_{n<\omega} V_m^n = \bigcap_{n<\omega} \bigcup_{m<\omega} V_m^n,$$

and hence A is G_δ.

It is easy to see that A is splitting. Indeed, let $a \in [\omega]^\omega$. If $a \cap a_n$ is finite for all $n < \omega$, then a is split by some $b \in A_0$. Otherwise $a \cap a_n$ is infinite for some n. In this case we easily find $b \in A_{n+1}$ that splits a. Moreover, if $C \subseteq A$ is closed there must exist n such that $C \cap A_m = \emptyset$ for every $m \ge n$, as otherwise $x \in C$. But $x \notin A$. But certainly no finite union of A_n's is splitting or inclusion-dense in A. ∎

Note that the example of Theorem 2.8 as well as that of [10, Theorem 1.10] is not hereditary. The next result shows that this is necessarily so. It implies that given a hereditary G_δ splitting family $A \subseteq 2^\omega$ there exists a closed $F \subseteq A$ that is inclusion-dense in A. Letting F' be the hereditary closure of F, we see that $F' \subseteq A$ is closed and splitting.

THEOREM 2.9. *Suppose that $G \subseteq 2^\omega$ is hereditary G_δ. There exists a closed $F \subseteq G$ that is inclusion-dense in G.*

Proof. Let $G = \bigcap_{n\in\omega} U_n$ where each U_n is open. We may assume that $G \ne \emptyset$ and $U_n \supseteq U_{n+1}$ for every n. For each n choose $A_n \subseteq \omega^{<\omega}$ such that $U_n = \bigcup_{\sigma\in A_n} [\sigma]$. Let $S_n = \langle \sigma_i^n : i < N_n \rangle$ enumerate all minimal (with respect to $\subseteq$) elements of A_n. Thus $N_n \in \omega \cup \{\omega\}$. Clearly we still have $U_n = \bigcup_{i<N_n} [\sigma_i^n]$. We let $S_{-1} = \langle \emptyset \rangle$. Note that the following holds:

(1) $\forall n \ \forall i \ \forall j \ (i \ne j \Rightarrow \sigma_i^n \perp \sigma_j^n)$ (here $\perp$ denotes incomparability with respect to $\subseteq$).

Without loss of generality we may assume that $G \cap [\sigma_i^n] \ne \emptyset$ always. Otherwise delete some of the σ_i^n. Because we can replace any σ_i^n by the two sequences $\sigma_i^n{}^\frown 0$, $\sigma_i^n{}^\frown 1$, and the U_n are decreasing, we can therefore assume without loss of generality the following:

(2) $\forall n\ \forall i < N_{n+1}\ \exists j < N_n\ \ \sigma_j^n \subsetneq \sigma_i^{n+1}$.

Clearly for every $x \in 2^\omega$ we have $x \in G$ iff there exists $f \in \prod_{n\in\omega} N_n$ such that

$$x = \sigma^0_{f(0)} \cup \sigma^1_{f(1)} \cup \sigma^2_{f(2)} \cup \cdots .$$

By heredity and (1) we must have:

(3) $\forall \varrho \in S_{n-1}\ \forall \sigma \in S_n\ (\varrho \subseteq \sigma \Rightarrow \forall M \subseteq \sigma^{-1}(1)\backslash\varrho^{-1}(1)\ \exists m \geq n\ \exists \tau \in S_m$ $(\varrho \subseteq \tau\ \&\ \tau^{-1}(1) \setminus \varrho^{-1}(1) = M))$.

Inductively we define T_n as follows: Let T_0 consist of those $\sigma \in S_0$ with $|\sigma^{-1}(1)| \leq 1$. If T_n has been defined, let T_{n+1} be the set of all $\sigma \in S_{n+1}$ such that there exists $\varrho \in T_n$ with $\varrho \subsetneq \sigma$ and $|\sigma^{-1}(1) \setminus \varrho^{-1}(1)| \leq 1$. By (1) it follows that each T_n consists of pairwise incomparable elements.

Claim 1. *Each T_n is finite.*

Proof. By induction on n. By (3) there exists $\sigma \in S_0$ such that $\sigma^{-1}(1) = \emptyset$. Then $\sigma \in T_0$. By incomparability we have $\tau^{-1}(1) \subseteq |\sigma|$ and $\tau^{-1}(1) \cap \varrho^{-1}(1) = \emptyset$ for any distinct $\tau, \varrho \in T_0$. Hence T_0 is finite. Now fix $\sigma \in T_n$. By (3) and (2) there exists $\tau \in S_{n+1}$ with $\sigma \subsetneq \tau$ and $\tau^{-1}(1) \setminus \sigma^{-1}(1) = \emptyset$. We have $\tau \in T_{n+1}$. Given distinct $\mu, \nu \in T_{n+1}$ with $\sigma \subseteq \mu$ and $\sigma \subseteq \nu$, by incomparability of S_{n+1} we must have

$$\mu^{-1}(1) \setminus \sigma^{-1}(1) \subseteq |\tau| \setminus |\sigma|, \qquad (\mu^{-1}(1) \setminus \sigma^{-1}(1)) \cap (\nu^{-1}(1) \setminus \sigma^{-1}(1)) = \emptyset.$$

As T_n is finite by induction hypothesis, the same holds for T_{n+1}. ∎

Let T be the tree generated by $\bigcup_{n\in\omega} T_n$. By the observation after (2) and Claim 1, and as each element of T_{n+1} extends an element of T_n, we have:

Claim 2. $[T] \subseteq G$. ∎

The following claim will finish our proof:

Claim 3. *For every $x \in G$ with $x^{-1}(1)$ infinite there exists $y \in [T]$ such that $y^{-1}(1) \subseteq x^{-1}(1)$ and $y^{-1}(1)$ is infinite.*

Proof. Let $f \in \omega^\omega$ be such that $x = \bigcup \sigma^n_{f(n)}$. We want to construct $\langle \tau_n : n < \omega \rangle$ such that $\tau_n \in T_n$, $\tau_n \subseteq \tau_{n+1}$ and $\tau_n^{-1}(1) \subseteq x$ for all n, and such that $\tau_{n+1}^{-1}(1) \setminus \tau_n^{-1}(1)$ is non-empty for infinitely many n.

Let $k_0 < \omega$ be the minimum of $x^{-1}(1)$ and let $m_0 < \omega$ be minimal with $k_0 < |\sigma^{m_0}_{f(m_0)}|$. Let $\tau_n \in T_n$ be the constantly zero sequence for every $n < m_0$. Note that $\sigma^n_{f(n)} = \tau_n$ for every $n < m_0$. By minimality of k_0, by incomparability of S_{m_0} and by (3) there exists $\tau_{m_0} \in T_{m_0}$ such that $\tau_{m_0}(k_0) = 1$ and $\tau_{m_0-1} \subseteq \tau_{m_0}$. Suppose that we have k_i, m_i and $\tau_{m_i} \in T_{m_i}$.

Choose k_{i+1} minimal such that $k_{i+1} \geq |\tau_{m_i}|$ and $x(k_{i+1}) = 1$. Keep choosing $\tau_j \in T_j$ for $j \geq m_i + 1$ such that $\tau_{m_i} \subsetneq \tau_j \subsetneq \tau_{j+1}$ and $\tau_j^{-1}(1) \setminus \tau_{m_i}^{-1}(1) = \emptyset$ (by (3)), as long as $|\tau_j| \leq k_{i+1}$. If τ_j is the last one we let $m_{i+1} = j+1$. Hence if $\tau \in T_{m_{i+1}}$ is such that $\tau_{m_{i+1}-1} \subseteq \tau$ and $\tau^{-1}(1) \subseteq (\tau_{m_{i+1}-1})^{-1}(1)$ we have $|\tau_{m_{i+1}-1}| \leq k_{i+1} < |\tau|$.

Let $z \in 2^\omega$ be such that $\tau_{m_{i+1}-1} \subseteq z$ and $z(k_{i+1}) = 1$ and $z(l) = 0$ elsewhere. Thus we have $z^{-1}(1) \subseteq x^{-1}(1)$ and hence $z \in G$. Since $T_{m_{i+1}-1}$ consists of pairwise incomparable elements, by (2) there must exist $\tau_{m_{i+1}} \in S_{m_{i+1}}$ with $\tau_{m_{i+1}-1} \subseteq \tau_{m_{i+1}}$ and $\tau_{m_{i+1}} \subseteq z$. By the choice of m_{i+1} we cannot have $\tau_{m_{i+1}}^{-1}(1) \setminus (\tau_{m_{i+1}-1})^{-1}(1) = \emptyset$. Consequently, $|\tau_{m_{i+1}}| > k_{i+1}$ and $\tau_{m_{i+1}}(k_{i+1}) = 1$. Clearly $\tau_{m_{i+1}} \in T_{m_{i+1}}$ and $\langle \tau_n : n \in \omega \rangle$ is as desired. Now we let $y = \bigcup_{n<\omega} \tau_n$ and the Claim follows. ∎

Problem 2.10. *Is Theorem* 2.9 *true for hereditary analytic sets?*

3. Infinitely often equal families

Definition 3.1. Given $a \subseteq \omega$ we say that some family $A \subseteq \omega^\omega$ is *infinitely often equal* (*ioe* for short) *for* a if for every $x \in a^\omega$ there exists $y \in A$ such that $\exists^\infty n\ x(n) = y(n)$. We call $A \subseteq \omega^\omega$ *countably ioe for* a if for every countable $C \subseteq a^\omega$ there exists $y \in A$ such that $\forall x \in C\ \exists^\infty n\ x(n) = y(n)$. Given $a \subseteq \omega$, some tree $p \subseteq \omega^{<\omega}$ is called an *ioe tree* for a if $p \neq \emptyset$ and for every $\sigma \in p$ there exists $\tau \in p$ such that $\sigma \subseteq \tau$ and $\forall n \in a\ \tau^\frown n \in p$.

Finally, we call an ioe for a family $A \subseteq \omega^\omega$ *everywhere* ioe for a iff $\forall x \in A\ \forall n\ A \cap [x{\restriction}n]$ is ioe for a.

The following fact is easy and is left to the reader.

Fact 3.2. *Suppose that $p \subseteq \omega^{<\omega}$ is an ioe tree for a. Then $[p]$ is a countably ioe family for a.*

Theorem 3.3. *Suppose that $A \subseteq \omega^\omega$ is analytic and countably ioe for some $a \subseteq \omega$. There exists a tree $p \subseteq \omega^{<\omega}$ that is ioe for a such that $[p] \subseteq A$. In particular, A has a closed countably ioe for a subset.*

The proof of this theorem is very much analogous to that of Theorem 2.2 as outlined in Section 2.

Definition 3.4. Given $x \in \omega^\omega$ and $n < \omega$ define $F_{(x,n)} = \{y \in \omega^\omega : \forall k \geq n\ y(k) \neq x(k)\}$. Clearly $F_{(x,n)}$ is closed.

Lemma 3.5. *A is countably ioe for a iff for no countable $C \subseteq a^\omega \times \omega A \subseteq \bigcup_{(x,n)\in C} F_{(x,n)}$.*

Proof. "⇒" Suppose $C \subseteq a^\omega \times \omega$ is countable with $A \subseteq \bigcup_{(x,n)\in C} F_{(x,n)}$. Let C' be the set of all first coordinates of elements of C. Hence $C' \subseteq a^\omega$ and for no $y \in A$ do we have $\forall x \in C'\ \exists^\infty n\ x(n) = y(n)$.

"$\Leftarrow$" Let $C' \subseteq a^\omega$ be countable and let $C = C' \times \omega$. Find $y \in A \setminus \bigcup_{(x,n)\in C} F_{(x,n)}$. Then clearly $\forall x \in C'\ \exists^\infty n\ \ x(n) = y(n)$. ∎

By Solecki's theorem [8, 2.1] we obtain:

COROLLARY 3.6. *Suppose that $A \subseteq \omega^\omega$ is analytic and countably ioe for $a \subseteq \omega$. Then there exists a G_δ $B \subseteq A$ that is also countably ioe for a.*

DEFINITION 3.7. For any $A \subseteq \omega^\omega$ and $a \subseteq \omega$ let $A' = A \setminus \bigcup\{[\sigma] : \sigma \in \omega^{<\omega}\ \&\ A \cap [\sigma]$ is not ioe for $a\}$. Recursively define $A^{(0)} = A$, $A^{(\alpha+1)} = (A^{(\alpha)})'$, and for a limit ordinal λ let $A^{(\lambda)} = \bigcap_{\alpha<\lambda} A^{(\alpha)}$.

There exists a least $\gamma < \omega_1$ such that $A^{(\gamma+1)} = A^{(\gamma)}$. We call it the *ioe$_a$-rank* of A, denoted ioa$_a$-rk(A). Clearly, $A^{(\gamma)}$ is then everywhere ioe for a if $A^{(\gamma)} \neq \emptyset$. In this case we say that A is *essentially everywhere ioa for a* (*e.e. ioe*, for short).

Note that every $A \subseteq \omega^\omega$ that is countably ioe for a is e.e. ioe for a. Indeed, if we had $A^{(\gamma)} = \emptyset$ for some $\gamma < \omega_1$, then for every $x \in A$, $n < \omega$, $[x{\restriction}n]$ was removed during the derivation process at some stage $\alpha < \gamma$, as for some $y_{x\restriction n} \in a^\omega$, for no $z \in A \cap [x{\restriction}n]$ does $z(i) = y_{x\restriction n]}(i)$ hold for infinitely many $i < \omega$. Then $C = \{y_{x\restriction n} : x \in A\ \&\ n < \omega\} \subseteq a^\omega$ is countable and no $x \in A$ equals any $y \in C$ infinitely often.

LEMMA 3.8. *Suppose that $A \subseteq \omega^\omega$ is G_δ and everywhere ioe for $a \subseteq \omega$. There exists a tree $p \subseteq \omega^{<\omega}$ that is ioe for a such that $[p] \subseteq A$.*

Proof. Let $q = \{x{\restriction}n : x \in A\ \&\ n < \omega\}$. Note that q is an ioe for a tree.

Let $A = \bigcap_{n<\omega} U_n$ where each U_n is open.

We are going to construct antichains $F_n \subseteq q$, $n < \omega$, such that the following hold:

(1) If $F_n = \{\sigma_i : i < \omega\}$ then $F_{n+1} = \bigcup_{i<\omega} F_{n,i}$ where each element of $F_{n,i}$ properly extends σ_i and there exists $j > |\sigma_i|$ such that for each $k \in a$ there is $\tau \in F_{n,i}$ with $|\tau| > j$ and $\tau(j) = k$.
(2) $\forall \sigma \in F_n\ \ [\sigma] \subseteq U_n$.

The construction is straightforward, given our assumptions. If we let p be the downward closure of $\bigcup_{n<\omega} F_n$, we are done. ∎

By Corollary 3.6 and Lemma 3.8 we obtain Theorem 3.3.

Analogously to Theorem 2.7 we can show that Lemma 3.8 fails for $F_{\sigma\delta}$ sets.

THEOREM 3.9. *There exists an $F_{\sigma\delta}$ everywhere ioe family that is not countably ioe.*

Proof. Let A be the set of all $x \in (2^{<\omega})^\omega$ such that ran(x) is nowhere dense in $2^{<\omega}$. Let Q be the set of all eventually zero $y \in 2^\omega$. For each $y \in Q$ let $x_y \in (2^{<\omega})^\omega$ be a one-to-one enumeration of $\{y{\restriction}n : n < \omega\}$. Clearly

$\{x_y : y \in Q\} \subseteq A$, but no $x \in A$ equals any x_y, $y \in Q$, infinitely often. Similarly to Theorem 2.7, by applying Ramsey's theorem one shows that A is everywhere ioe. ∎

It is similarly easy to modify the example of Theorem 2.8 to obtain the following:

THEOREM 3.10. *There exists an ioe family $A \subseteq \omega^\omega$ that is both G_δ and F_σ, such that A does not contain a closed ioe subfamily.*

In [10, Problem 1.12] it was asked whether every analytic splitting family $A \subseteq 2^\omega$ is countably splitting on some infinite $a \subseteq \omega$. (A positive answer for closed A was given by [10, Corollary 1.14].) A positive solution follows from a beautiful result of Repický (see [6, Theorem 2.2]), which implies the following: *If $A \subseteq 2^\omega$ is analytic such that $A{\restriction}a := \{x{\restriction}a : x \in A\}$ is uncountable for every infinite $a \subseteq \omega$ (which is certainly the case for splitting A), then there exists some infinite $a \subseteq \omega$ such that $A{\restriction}a = 2^a$.*

In view of the similarity of Sections 2 and 3 this might lead to the conjecture that every analytic ioe family $A \subseteq \omega^\omega$ is countably ioe for some infinite $a \subseteq \omega$. However, the next result shows that there is a closed counterexample.

DEFINITION 3.11. Given some infinite $a \subseteq \omega$, let $E[a] = \{a(2n) : n < \omega\}$ and $O[a] = \{a(2n+1) : n < \omega\}$. Here $\langle a(n) : n < \omega\rangle$ denotes the increasing enumeration of a. If $a = \omega$ we omit it, and we write EE, OEO etc. in place of $E[E]$, $O[E[O]]$. Let ζ_n be the sequence of n 0's and ν_n the sequence of n 1's.

THEOREM 3.12. *There exists a closed ioe family in ω^ω that is not countably ioe for any infinite set $a \subseteq \omega$.*

Proof. We are defining a sequence $\langle p_n : n < \omega\rangle$ of (non-pruned) trees $p_n \subseteq \omega^\omega$ as follows:

1) p_0 is built as follows: p_0 contains all ζ_n's and all ν_n's. Moreover, for each $n \in \omega$, $k \in E \setminus \{0\}$ and $l \in O \setminus \{1\}$ we have $\zeta_n{}^\frown k \in p_0$ and $\nu_n{}^\frown l \in p_0$, and these are terminal nodes in p_0.
2) Given p_n, we declare $p_n \subseteq p_{n+1}$, and moreover every terminal node $\sigma \in p_n$ is extended as follows: Let $a = \{k \in \omega : \sigma{\restriction}(|\sigma|-1){}^\frown k \in p_n\}$. Then p_{n+1} will also contain all $\sigma^\frown\zeta_m$ and $\sigma^\frown\nu_m$ and moreover, for each $m < \omega$, $k \in E[a]\setminus\{0,1\}$ and $l \in O[a]\setminus\{0,1\}$, we have $\sigma^\frown\zeta_m{}^\frown k \in p_{n+1}$ and $\sigma^\frown\nu_m{}^\frown l \in p_{n+1}$.
3) We let $p = \bigcup_{n\in\omega} p_n$. We claim that $[p]$ is the desired closed set.

Let us first check that $[p]$ is ioe. Let $x \in \omega^\omega$. Without loss of generality we may assume that x is neither eventually 0 nor eventually 1, as such reals are clearly infinitely often equal to some $y \in [p]$. Recursively we shall construct terminal nodes $\sigma_n \in p_n$ such that $\sigma_n \subsetneq \sigma_{n+1}$ and $x(|\sigma_n|-1) = \sigma_n(|\sigma_n|-1)$.

Then $y = \bigcup_{n\in\omega} \sigma_n$ will be in $[p]$ and equal x infinitely often. If there are infinitely many i with $x(i) \in E \setminus \{0\}$ we choose $\sigma_0 \in p_0$ such that σ_0 is a terminal node, $\sigma_0(0) = 0$ and $\sigma_0(|\sigma_0| - 1) = x(|\sigma_0| - 1)$ (hence $x(|\sigma_0| - 1) \in E \setminus \{0\}$). Otherwise choose $\sigma_0 \in p_0$ as before except that now $\sigma_0(0) = 1$. Note that then $\sigma_0(|\sigma_0| - 1) = x(|\sigma_0| - 1) \in O \setminus \{1\}$.

Now suppose that σ_n has been constructed. We define $a = \{k < \omega : \sigma_n \restriction (|\sigma_n| - 1)^\frown k \in p_n\}$. Inductively we know $x(i) \in a$ for infinitely many i. If $x(i) \in E[a]$ for infinitely many i we choose $\sigma_{n+1} \in p_{n+1}$ such that $\sigma_{n+1}(|\sigma_n|) = 0$. Otherwise $x(i) \in O[a]$ for infinitely many i; then we choose $\sigma_{n+1} \in p_{n+1}$ as required such that $\sigma_{n+1}(|\sigma_n|) = 1$. By the definition of p_{n+1} this is certainly possible. It is clear that p does not contain any subtree that is ioe for some infinite $a \subseteq \omega$. By Theorem 3.3 we conclude that p is not countably ioe for any infinite $a \subseteq \omega$. ∎

The work of this section has been motivated by the open problem (see [4, Question 4.3]) whether there exists a closed (or analytic) maximal almost disjoint family $A \subseteq \omega^\omega$ ($x, y \in \omega^\omega$ are *almost disjoint* if $\forall^\infty n\ \ x(n) \neq y(n)$). Clearly such A is ioe but not countably ioe. If one were able to prove a perfect set theorem for closed ioe families (in the style of 3.3), one could probably solve this problem.

4. Refining families. In this section we identify $[\omega]^\omega$ with the closed subspace of ω^ω consisting of strictly increasing functions. Thus we can talk about unbounded or dominating (with respect to eventual dominance) families in $[\omega]^\omega$.

Definition 4.1. A family $A \subseteq [\omega]^\omega$ is called *refining* if for every $a \in [\omega]^\omega$ there exists $b \in A$ such that either $b \subseteq^* a$ or $b \subseteq^* \omega \setminus a$. We call A *countably refining* if for every countable $F \subseteq [\omega]^\omega$, some member of A refines every member of F.

4.1. *The Di Prisco–Todorcevic example.* The problem of characterizing analytic (or only closed) refining or countably refining families seems to be a hard one. At first glance it seems that Mathias trees might be relevant. For $s \subseteq \omega$ finite and $a \in [\omega]^\omega$ with $\max(s) < \min(a)$ let $[s, a] = \{b \in [\omega]^\omega : b \subseteq s \cup a \ \&\ b \setminus a = s\}$. (Such pairs (s, a) are the conditions of Mathias forcing. If (s, a) is viewed as a tree, then $[s, a]$ is the set of its branches.) Clearly sets of the form $[s, a]$ are closed and always countably refining. The first example of a closed countably refining family that does not contain (the branches of) a Mathias tree is due to Di Prisco and Todorcevic (see [1]). Actually they did not care about refining families, but they tried to characterize those Borel sets A in $[\omega]^\omega$ such that the shift graph they carry has infinite Borel chromatic number. Recall that the *shift graph* on A is defined by putting an edge between $a, b \in A$ iff $a = b \setminus \{\min(b)\}$ (“a is the shift of b”). Note that colouring each element of A by its minimum always defines a Borel colouring with countably many colours. In [5] it has been shown that the only possible finite Borel chromatic numbers are 1, 2, 3.

FACT 4.2. *If the shift graph on $A \subseteq [\omega]^\omega$ has infinite Borel chromatic number, then A is refining.*

Proof. Suppose that A is not refining, thus we have $a^0 \in [\omega]^\omega$ so that each $b \in A$ meets both a^0 and $a^1 := \omega \setminus a^0$ infinitely often. Now colour $b \in A$ by $(i,k) \in 2 \times \{\text{ even, odd}\}$ so that $\min(b) \in a^i$ and k is the parity of the length of the initial segment of b determined by the minimal element of b in a^{1-i}. Clearly this is a Borel graph colouring. ∎

In [3, Lemma 2.3] it has been shown that for every analytic $A \subseteq [\omega]^\omega$, A is strongly dominating iff A contains $[p]$ for some Laver tree p. Here A is *strongly dominating* iff $\forall x \in [\omega]^\omega\ \exists y \in A\ \forall^\infty k\ \ x(y(k-1)) < y(k)$, and $p \subseteq \omega^{<\omega}$ is a *Laver tree* iff every extension of its stem has infinitely many successor nodes.

The Di Prisco–Todorcevic example. We identify ω with $IP \subseteq \omega \times \omega$, the set of all increasing pairs (n,m) with $n < m$. Let E_1 be the set of all $x \in IP^\omega$ such that, letting $x(i) = (n_i, m_i)$, we have $m_i = n_{i+1}$ for every $i < \omega$. It is straightforward to check that E_1 is closed and strongly dominating (actually of the form $[p]$ for some Laver tree p) and that E_1 does not contain the set of branches of any Mathias tree. In [1] it is shown that E_1 is infinitely chromatic. Let us give a direct proof that E_1 is refining, and actually countably so. Let $r \in [IP]^\omega$ be given, thus r is a binary relation on ω. By Ramsey's theorem we can find $a \in [\omega]^\omega$ that is homogeneous for r, i.e. $[a]^2 \subseteq r$ or $[a]^2 \cap r = \emptyset$. Certainly there are $x \in E_1$ with $x(i) \in [a]^2$ for every i. These x refine r. If we have to deal with countably many $r_n \in [IP]^\omega$ we are building a descending chain of sets $a_n \in [\omega]^\omega$ such that a_n is r_n-homogeneous. Then we let $a_\omega \in [\omega]^\omega$ be an almost intersection of all a_n. Any $x \in E_1$ with $x(i) \in [a_\omega]^2$ for all i will refine all r_n.

Below we shall construct two even smaller examples of closed refining families. The first one is dominating but not strongly so, and the second one is unbounded but not dominating. Strangely, the second one is much easier to understand. We retain the first one as we have a strong conjecture that it is minimal in the sense that it does not contain any closed refining subfamily that is not dominating. This conjecture is linked with the problem of characterizing 2-colourable hypergraphs.

4.2. *A closed non-strongly dominating refining family*

DEFINITION 4.3. Let a, b be subsets of ω.

(a) If the limit
$$\lim_{n\to\infty}\frac{|a\cap b\cap n|}{|b\cap n|}$$
exists, we call it the *density* of a in b and denote it by $d(a,b)$.

(b) On the other hand, the limes superior
$$\limsup_{n\to\infty}\frac{|a\cap b\cap n|}{|b\cap n|}$$
always exists in $[0,1]$, and we call it the *sup-density* of a in b and denote it by $d_{\sup}(a,b)$.

REMARK 4.4. It is easy to see that for infinite b existence and value of $d(a,b)$ or $d_{\sup}(a,b)$ do not depend on finite changes of a or b.

Let $E[a]=\{a(2n):n<\omega\}$ and $O[a]=\{a(2n+1):n<\omega\}$. Inductively we define a tree of sets $\langle S_\sigma:\sigma\in 2^{<\omega}\rangle$ as follows: $S_\emptyset=\omega$, $S_{\sigma^\frown 0}=E[S_\sigma]$, $S_{\sigma^\frown 1}=O[S_\sigma]$. Note that $S_\sigma\cap S_\tau=\emptyset$ whenever σ and τ are incompatible, and that $S_\tau\subseteq S_\sigma$ if $\sigma\subseteq\tau$. Let $\mathcal{L}=\langle L_n:n<\omega\rangle$ be the unique family such that

(1) $L_n\subseteq\omega^{2n}$ is not empty, consisting of increasing sequences;
(2) if $\mu\in L_n$, there exists $\sigma=\sigma_\mu\in 2^n$ such that for each $i<n$, $\mu(2i)$ and $\mu(2i+1)$ are successive elements of $S_{\sigma\restriction i}$ and $\mu(2i)\in E(S_{\sigma\restriction i})$ iff $\sigma(i)=0$;
(3) all L_n are maximal such that (1) and (2) hold.

Thus $L_0=\{\emptyset\}$, $L_1=\{\langle n,n+1\rangle:n\in\omega\}$, $L_2=\{\langle 2n,2n+1,2m,2m+2\rangle : n<m<\omega\}\cup\{\langle 2n+1,2n+2,2m+1,2m+3\rangle:n<m<\omega\}$ etc.

The family $\mathcal{L}$ determines a tree $p\subseteq\omega^{<\omega}$ by letting $\sigma\in p$ iff $\sigma\subseteq\tau$ for some $\tau\in\bigcup_{n<\omega}L_n$. Clearly p is a uniform tree (see [9]) such that $\mathrm{stem}(p)=\emptyset$ and for every $\sigma\in\mathrm{split}(p)$ the successor splitnodes of σ have length $|\sigma|+2$. Hence $[p]$ is dominating, but by [3] it is not strongly dominating. Let $E_2=[p]$. Now the following is true:

THEOREM 4.5. *E_2 is refining.*

Proof. Let $a\in[\omega]^\omega$ be arbitrary. Suppose first that there exists $\sigma\in 2^{<\omega}$ such that for every $\tau\in 2^{<\omega}$ with $\sigma\subseteq\tau$ there are infinitely many $n<\omega$ such that $\{S_\tau(n),S_\tau(n+1)\}\cap a=\emptyset$. It is straightforward to construct $x\in E_2$ such that $\mathrm{ran}(x)\cap a$ is finite. Hence we may assume that for every $\sigma\in 2^{<\omega}$ there exists an extension τ of σ such that only finitely many pairs of successive elements of S_τ belong to $\omega\setminus a$. Note that this implies that for each $\sigma\in 2^{<\omega}$ there exists $k<\omega$ such that every interval of S_σ that is disjoint from a has length at most k. We conclude that the assumptions of the next lemma are satisfied by our a and $b=S_\sigma$ for a dense set of $\sigma\in 2^{<\omega}$. The other lemmas will be used to recursively construct $x\in E_2$ such that $\mathrm{ran}(x)\subseteq^* a$.

LEMMA 4.6. *Let $a, b \subseteq \omega$ be as follows:*

(i) *There are only finitely many i such that $\{b(i), b(i+1)\} \cap a = \emptyset$.*
(ii) *There exists $k_0 < \omega$ such that every interval of $E(b)$ disjoint from a has length at most k_0.*
(iii) *There exists $k_1 < \omega$ such that every interval of $O(b)$ disjoint from a has length at most k_1.*

If $k = \max\{k_0, k_1\}$, then a contains at least $k+1$ elements from every interval I of b of length $2k+1$ with $\min(I)$ large enough and therefore $d_{\sup}(a, b) \geq (k+1)/(2k+1)$. ■

Proof. Let $I = \{b(i) : j \leq i < j + 2k + 1\}$ be an interval of b of length $2k+1$ such that $\omega \setminus a$ does not contain any two successive elements of it. Let $E(I) = \{b(j+2i) : i \leq k\}$ and $O(I) = \{b(j+2i+1) : i \leq k\}$. Thus $E(I)$ has $k+1$ and $O(I)$ has k elements. By (ii) and (iii) we must have $E(I) \cap a \neq \emptyset$. If $E(I) \subseteq a$ we are done. But otherwise, as a meets any pair of successive elements of I, a contains two successive elements of I, and hence we conclude $|I \cap a| \geq k+1$. ■

LEMMA 4.7. *Suppose that $b \subseteq \omega$ and $b = c_0 \cup c_1$ is a partition. Then for every $a \subseteq \omega$,*

$$d_{\sup}(a, b) \leq d_{\sup}(a, c_0) \cdot d_{\sup}(c_0, b) + d_{\sup}(a, c_1) \cdot d_{\sup}(c_1, b).$$

Proof. Observe that for each n we have

$$\frac{|a \cap c_0 \cap n|}{|c_0 \cap n|} \cdot \frac{|c_0 \cap b \cap n|}{|b \cap n|} + \frac{|a \cap c_1 \cap n|}{|c_1 \cap n|} \cdot \frac{|c_1 \cap b \cap n|}{|b \cap n|} = \frac{|a \cap c_0 \cap n| + |a \cap c_1 \cap n|}{|b \cap n|} = \frac{|a \cap b \cap n|}{|b \cap n|}.$$

Let $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ be such that $4\delta + 2\delta^2 < \varepsilon/2$. There exists n_δ such that for all $n > n_\delta$ and $i < 2$ we have

$$\frac{|a \cap c_i \cap n|}{|c_i \cap n|} < d_{\sup}(a, c_i) + \delta \quad \text{and} \quad \frac{|c_i \cap b \cap n|}{|b \cap n|} < d_{\sup}(c_i, b) + \delta.$$

Note that this implies

$$\frac{|a \cap c_0 \cap n|}{|c_0 \cap n|} \cdot \frac{|c_0 \cap b \cap n|}{|b \cap n|} + \frac{|a \cap c_1 \cap n|}{|c_1 \cap n|} \cdot \frac{|c_1 \cap b \cap n|}{|b \cap n|} < d_{\sup}(a, c_0) \cdot d_{\sup}(c_0, b) + d_{\sup}(a, c_1) \cdot d_{\sup}(c_1, b) + \varepsilon/2.$$

Moreover, we can find arbitrarily large $n > n_\delta$ such that

$$\frac{|a \cap b \cap n|}{|b \cap n|} > d_{\sup}(a, b) - \varepsilon/2.$$

Hence

$$d_{\sup}(a, c_0) \cdot d_{\sup}(c_0, b) + d_{\sup}(a, c_1) \cdot d_{\sup}(c_1, b) > d_{\sup}(a, b) - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we have proved the desired inequality. ■

LEMMA 4.8. *Suppose b, c are infinite such that $d_{\sup}(b, c) > 1/2$. Then there are l_0, l_1 such that l_0 is even, l_1 is odd and*

$$\{c(l_0), c(l_0 + 1)\} \subseteq b \quad \textit{and} \quad \{c(l_1), c(l_1 + 1)\} \subseteq b.$$

Proof. As $\lim_{k\to\infty} \frac{k+4}{2k} = \frac{1}{2}$, by assumption we can choose k so that

$$(*) \qquad \frac{|b \cap \{c(0), \ldots, c(k-1)\}|}{k} > \frac{k+4}{2k}.$$

We may assume that b contains $\{c(l), c(l+1), c(l+2)\}$ for no l, since otherwise we are done.

Let $M = \{l < k : \{c(l), c(l+1)\} \subseteq b\}$. Certainly M is not empty, as otherwise

$$\frac{|b \cap \{c(0), \ldots, c(k-1)\}|}{k} \le \frac{k+1}{2k} < \frac{k+4}{2k},$$

contradicting $(*)$. If M contained only even numbers, then

$$\frac{|b \cap \{c(0), \ldots, c(k-1)\}|}{k} \le \frac{k/2+2}{k} = \frac{k+4}{2k},$$

which contradicts $(*)$. If M contained only odd numbers, then even

$$\frac{|b \cap \{c(0), \ldots, c(k-1)\}|}{k} \le \frac{(k+1)/2}{k} < \frac{k+4}{2k},$$

a contradiction. We conclude that M contains both even and odd numbers, and we are done. ∎

By Remark 4.4 we conclude that in Lemma 4.8 there exist infinitely many such l_0 and l_1. Now we choose $\sigma \in 2^{<\omega}$ and $k < \omega$ such that the assumptions of Lemma 4.6 hold with our a and $b = S_\sigma$. Hence $d_{\sup}(a, S_\sigma) \ge (k+1)/(2k+1)$. If $|\sigma| = n$ we can find $\mu \in L_n$ such that $\sigma = \sigma_\mu$ (see (2) of the definition of E_2). Note that $d(S_{\tau^\frown i}, S_\tau) = 1/2$ for every $\tau \in 2^{<\omega}$ and $i < 2$. Therefore, if $d_{\sup}(a, S_\tau) > 1/2$, by Lemma 4.7 we have

$$\frac{1}{2} < d_{\sup}(a, S_\tau) \le \frac{d_{\sup}(a, S_{\tau^\frown 0}) + d_{\sup}(a, S_{\tau^\frown 1})}{2}$$

and hence $d_{\sup}(a, S_{\tau^\frown i}) > 1/2$ for at least one $i < 2$. Using Lemma 4.8 it is now straightforward to construct $\langle \sigma_j : n \le j \rangle$ and $\langle \mu_j : n \le j \rangle$ such that $\sigma_n = \sigma$, $\mu_n = \mu$, $\mu_j \in L_j$, $\sigma_j = \sigma_{\mu_j}$, $d_{\sup}(a, S_{\sigma_j}) > 1/2$ and $\sigma_i \subseteq \sigma_j$, $\mu_i \subseteq \mu_j$ and $\{\mu_j(2i), \mu_j(2i+1)\} \subseteq a$ for every $n \le i < j$. Letting $x = \bigcup_{j<\omega} \mu_j$, we have $x \in E_2$ and $\operatorname{ran}(x) \subseteq^* a$. ∎

REMARK 4.9. The previous example E_2 is not even 2-refining, i.e. there are $a, b \in [\omega]^\omega$ such that no $x \in E_2$ refines a and b. To see this, define $c_i = \{n \in \omega : n = i \bmod 3\}$ for $i < 3$. Note that $x \not\subseteq^* c_i$ for any $x \in E_2$ and $i < 3$, as $x(2k+1) - x(2k)$ is a power of 2 for all k. Thus we can let $a = c_0 \cup c_1$ and $b = c_1 \cup c_2$.

Problem 4.10. *Is it true that every non-dominating closed subset of E_2 is non-refining?*

Problem 4.10 is linked to the 2-colourability problem for hypergraphs (see e.g. [7, p. 599]). Given a non-dominating subtree $q \subseteq p$ (where $[p] = E_2$), one would like to find a front $F \subseteq \mathrm{split}(q)$ (i.e. F is pairwise incomparable and for every $x \in [q]$, $x{\restriction}k \in F$ for some k) such that, letting F_σ be the set of all successor splitnodes of $\sigma \in F$ in q, the family $\mathcal{F} = \bigcup_{\sigma \in F}\{\tau \setminus \sigma : \tau \in F_\sigma\}$ considered as a hypergraph (i.e. a set of finite sets) is 2-colourable, i.e. there exists $c \subseteq \bigcup \mathcal{F}$ that has non-empty intersection with every member of $\mathcal{F}$ but does not contain any one.

4.3. *A closed refining non-dominating family.* Our third example is a closed refining family $E_3 \subseteq [\omega]^\omega$ that is non-dominating, but unbounded with respect to eventual dominance. It is easy to see that no bounded family is refining. Curiously E_3 is much easier to understand than E_2.

For the construction let $f \in \omega^\omega$ be fast enough so that $(f(k+1)-f(k))/2 > k+1$ for each k. Let

$$I_k = [f(k), f(k+1)) \quad \text{and} \quad H_k = \left\{a \subseteq I_k : |a| = \left\lfloor \frac{f(k+1)-f(k)}{2} \right\rfloor\right\}.$$

Now let E_3 be the set of all $b \in [\omega]^\omega$ of the form $b = \bigcup_{i \in \mathbb{N}} a_{k(i)}$, where $\langle k(i) : i < \omega\rangle \in \omega^\omega$ is strictly increasing and $a_{k(i)} \in H_{k(i)}$ for all $i < \omega$.

Let us see that E_3 is non-dominating, actually $\forall b \in E_3\ \exists^\infty k\ \ b(k) < f(k)$. Let $b = \bigcup_{i<\omega} a_{k(i)}$ as above. Note that $b(k(i)) \in \bigcup_{j \le i} a_{k(j)}$ by definition of $H_{k(i)}$, and hence $b(k(i)) < f(k(i))$. Moreover, E_3 is refining. Indeed, given $a \in [\omega]^\omega$, we see that for some $i < 2$, $|a^i \cap I_k| \ge \lfloor (f(k+1)-f(k))/2 \rfloor$ for infinitely many k (again we let $a^0 = a$, $a^1 = \omega \setminus a$). If $i = 0$ we easily obtain $b \in E_3$ with $b \subseteq a$, otherwise we find $b \in E_3$ disjoint from a. Finally, closedness of E_3 is easy to check.

Remark 4.11. (1) Note that each H_k is a hypergraph that is not 2-colourable. Actually, to make E_3 refining it suffices to take as H_k any such hypergraph on I_k. In order to make E_3 non-dominating we must ask that the elements of H_k are large enough with respect to $f(k+1)$.

(2) We could as well make E_3 countably refining. E.g. start with a bit faster f (say such that $(f(k+1)-f(k))/2^k > k+1$) and define

$$H_k = \left\{a \subseteq I_k : |a| = \left\lfloor \frac{f(k+1)-f(k)}{2^k} \right\rfloor\right\}.$$

E_3 is now defined as before. Given infinitely many $a_n \in [\omega]^\omega$, we first find $b_0 = \bigcup_{i\in\mathbb{N}} a^0_{k_0(i)}$ refining a_0 such that $a^0_{k_0(i)} \subseteq I_{k_0(i)}$ has size $\lfloor |I_{k_0(i)}|/2\rfloor$. Then find $\langle k_1(i) : i < \omega\rangle$, a subsequence of $\langle k_0(i) : i < \omega\rangle$, and $b_1 = \bigcup_{i\in\mathbb{N}} a^1_{k_1(i)}$ refining a_1 such that $a^1_{k_1(i)} \subseteq a^0_{k_1(i)}$ has size $\lfloor |a^0_{k_1(i)}|/2\rfloor$ etc. Finally, we have the diagonal $b_\omega = \bigcup_{n<\omega} a^n_{k_n(n)}$ which refines all a_n. Possibly $b_\omega \notin E_3$, as some $a^n_{k_n(n)}$ are too large (note $|a^n_{k_n(n)}| = |I_{k_n(n)}|/2^n$). So simply trim each $a^n_{k_n(n)}$ down to its decent size and we are done.

References

[1] C. A. Di Prisco and S. Todorcevic, *Canonical forms of shift-invariant maps on* $[\mathbb{N}]^\infty$, Discrete Math. 306 (2006), 1862–1870.

[2] I. Farah and S. Solecki, *Two $F_{\sigma\delta}$ ideals*, Proc. Amer. Math. Soc. 131 (2003), 1971–1975.

[3] M. Goldstern, M. Repický, S. Shelah and O. Spinas, *On tree ideals*, Proc. Amer. Math. Soc. 123 (1995), 1573–1581.

[4] B. Kastermans, J. Steprāns and Y. Zhang, *Analytic and coanalytic families of almost disjoint functions*, preprint.

[5] A. S. Kechris, S. Solecki and S. Todorcevic, *Borel chromatic numbers*, Adv. Math. 141 (1999), 1–44.

[6] M. Repický, *Mycielski ideal and the perfect set theorem*, Proc. Amer. Math. Soc. 132 (2004), 2141–2150.

[7] K. H. Rosen, *Handbook of Discrete and Combinatorial Mathematics*, CRC Press, 2000.

[8] S. Solecki, *Covering analytic sets by families of closed sets*, J. Symbolic Logic 59 (1994), 1022–1031.

[9] O. Spinas, *Dominating projective sets in the Baire space*, Ann. Pure Appl. Logic 68 (1994), 327–342.

[10] —, *Analytic countably splitting families*, J. Symbolic Logic 69 (2004), 101–117.

[11] S. Todorcevic, *Partition Problems in Topology*, Contemp. Math. 84, Amer. Math. Soc., 1989.

Mathematisches Seminar
Christian-Albrechts-Universität zu Kiel
24098 Kiel, Germany
E-mail: spinas@math.uni-kiel.de

Received 10 October 2007;
in revised form 28 May 2008