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Rectangular square-bracket operation for successor of regular cardinals

by

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Abstract. We give a uniform proof that $\lambda^+ \nrightarrow [\lambda^+;\lambda^+]^2_{\lambda^+}$ holds for every regular cardinal λ.

1. Introduction. Recall that $\lambda \nrightarrow [\lambda]^2_\kappa$ asserts the existence of a function $f : [\lambda]^2 \to \kappa$ such that $f``[X]^2 = \kappa$ for all $X \in [\lambda]^\lambda$. Recall also that $\lambda \nrightarrow [\lambda;\lambda]^2_\kappa$ asserts the existence of a function $f : [\lambda]^2 \to \kappa$ such that $f[X \circledast Y] = \kappa$ for all $X, Y \in [\lambda]^\lambda$ ([1]).

In [7], the second author introduced the method of walks on ordinals and proved that $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ holds for all infinite regular cardinals λ. This was done by defining a square-bracket operation $[\alpha\beta]$ that selects a point in the trace of the walk from β to α using the oscillation of upper traces of certain walks that start from α and from β. As for singular cardinals, it is a longstanding open problem whether $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ holds for all singular cardinals λ, but by a result of the first author [3], $\lambda^+ \nrightarrow [\lambda^+]^2_{\lambda^+}$ entails $\lambda^+ \nrightarrow [\lambda^+;\lambda^+]^2_{\lambda^+}$ for every singular cardinal λ.

In the present paper, we focus on $\lambda^+ \nrightarrow [\lambda^+;\lambda^+]^2_{\lambda^+}$ for λ regular. In [4], Shelah proved that this relation holds for all regular $\lambda > 2^{\aleph_0}$, and later in [5], he improved this to all regular $\lambda > \aleph_1$. Then, in [6], Shelah handled the case $\lambda = \aleph_1$, and finally, in [2], Moore established the missing case $\lambda = \aleph_0$. It was unknown whether there exists a uniform proof that handles all successors of regulars (or even just λ^+ for $\lambda \in \{\aleph_0, \aleph_1, \text{first inaccessible}\}$), and in particular, whether and how Moore's technique generalizes to higher cardinals. In this paper, we provide such a uniform proof. This is established by combining the analysis [2] of oscillations over the lower trace, together with the

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([1]) Here, $X \circledast Y := \{(\alpha,\beta) \in X \times Y \mid \alpha \in \beta\}$, and $[X]^2 := X \circledast X$.

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analysis [7] of the upper trace function. More specifically, we show that the ρ_1-function on λ^+ oscillates on the lower traces much the same way it does on ω_1 (regardless of the value of $\lambda^{<\lambda}$), giving us a function $o : [\lambda^+]^2 \to \omega$ whose composition with the upper trace function $\mathrm{tr} : [\lambda^+]^2 \to {}^{\omega}(\lambda^+)$ establishes $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$.

We expect that the new square-bracket operation will have applications of similar wealth as the original square-bracket operation (see, for example, the relevant chapters of [9]) and that the arrow notation $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$ captures only a small part of its properties. Judging on the basis of previous experiences, it is expected that applications will come after a deeper understanding of the relationship between the functions such as tr and o rather than on modifying the arrow notation to express more complicated statements. One example that shows this most clearly is the original proof (see, for example, [1]) that the Proper Forcing Axiom implies that $2^{\aleph_0} = \aleph_2$. That proof depends heavily on the properties of the oscillation mapping $\mathrm{osc} : (\omega^\omega)^2 \to \omega \cup \{\omega\}$ introduced in [8], properties that cannot be captured by the arrow notation such as $\mathfrak{b} \nrightarrow [\mathfrak{b}; \mathfrak{b}]^2_\omega$ nor any of its strengthenings that involve only the notion of cardinality.

2. Statement of the main result

2.1. Preliminaries. For the rest this paper, we fix an infinite regular cardinal λ, and a sequence $\overrightarrow{C} = \langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that the following two hold:

(1) $C_{\alpha+1} = \{\alpha\}$ for all $\alpha < \lambda^+$;
(2) C_α is a club subset of α of order-type $\leq \lambda$ for all limit $\alpha < \lambda^+$.

Definition 2.1. Given $\alpha < \beta < \lambda^+$, define:

- $\mathrm{tr}(\alpha, \beta) \in {}^{\omega}\lambda^+$, by recursively letting, for all $n < \omega$,

$$\mathrm{tr}(\alpha,\beta)(n) := \begin{cases} \beta, & n = 0, \\ \min(C_{\mathrm{tr}(\alpha,\beta)(n-1)} \setminus \alpha), & n > 0 \,\&\, \mathrm{tr}(\alpha,\beta)(n-1) > \alpha, \\ \alpha, & \text{otherwise}; \end{cases}$$

- $\rho_2(\alpha, \beta) := \min\{n < \omega \mid \mathrm{tr}(\alpha, \beta)(n) = \alpha\}$;
- $\rho_{1\beta} \in {}^{\beta}\lambda$, by $\rho_{1\beta}(\alpha) := \max\{\mathrm{otp}(C_{\mathrm{tr}(\alpha,\beta)(j)} \cap \alpha) \mid j < \rho_2(\alpha, \beta)\}$;
- $L(\alpha, \beta) := \{\max_{i \leq j} \sup(C_{\mathrm{tr}(\alpha,\beta)(i)} \cap \alpha) \mid j < \rho_2(\alpha, \beta)\}$;
- $\mathrm{tr}^\circ(\alpha, \beta) := \mathrm{tr}(\alpha, \beta) \restriction \rho_2(\alpha, \beta)$.

We consider $\mathrm{tr}^\circ(\alpha, \alpha)$ and $L(\alpha, \alpha)$ as the empty set.

Notation 2.2. By $A = B \oplus C$, we mean that:

- $A = B \cup C$;

- $B \neq \emptyset$, $C \neq \emptyset$;
- $\bigcup B \in \bigcap C$.

Denote $E^{\lambda^+}_{\lambda} := \{\delta < \lambda^+ \mid \mathrm{cf}(\delta) = \lambda\}$.

Fact 2.3 (Todorcevic, [9, §§2.1, 2.2, 6.2]). *If λ is a regular cardinal and $\mathrm{otp}(C_\alpha) \leq \lambda$ for every $\alpha < \lambda^+$, all of the following hold:*

(1) *for every $\alpha < \lambda^+$ and $\theta < \lambda$, we have $|\{\xi < \alpha \mid \rho_{1\alpha}(\xi) \leq \theta\}| < \lambda$;*
(2) *for every $\alpha < \beta < \lambda^+$, we have $|\{\xi < \alpha \mid \rho_{1\alpha}(\xi) \neq \rho_{1\beta}(\xi)\}| < \lambda$;*
(3) *for every $\delta \in E^{\lambda^+}_{\lambda}$ and $\beta < \lambda^+$ above δ, we have $\max(L(\delta,\beta)) < \delta$;*
(4) *for every $\alpha < \beta < \gamma < \lambda^+$, if $\alpha > \max(L(\beta,\gamma))$, then*

$$\mathrm{tr}^\circ(\alpha,\gamma) = \mathrm{tr}^\circ(\beta,\gamma)^\frown\mathrm{tr}^\circ(\alpha,\beta);$$

(5) *for every $\alpha < \beta < \gamma < \lambda^+$, if $\min(L(\alpha,\beta)) > \max(L(\beta,\gamma))$, then*

$$L(\alpha,\gamma) = L(\beta,\gamma) \oplus L(\alpha,\beta).$$

Definition 2.4. For a finite set L, and ordinal-valued functions f, g with $L \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(g)$, let

$$\mathrm{Osc}(f,g,L) := |\{\xi \in L \cap \max(L) \mid f(\xi) = g(\xi) \;\&\; f(\xi^L) > g(\xi^L)\}|,$$

where $\xi^L := \min(L \setminus \xi + 1)$.

2.2. Result. Let $\{p_l \mid l < \omega\}$ be some injective enumeration of the set of prime integers. Let $\langle S_\zeta \mid \zeta < \lambda^+\rangle$ be a partition of λ^+ into mutually disjoint sets in such a way that $S_\zeta \cap E^{\lambda^+}_{\lambda}$ is stationary for every $\zeta < \lambda^+$.

Definition 2.5. Given $\alpha < \beta < \lambda^+$, let:

- $\mathrm{osc}(\alpha,\beta) := \mathrm{Osc}(\rho_{1\alpha}, \rho_{1\beta}, L(\alpha,\beta))$;
- $o^*(\alpha,\beta) := \min\{l < \omega \mid p_l \text{ does not divide } \mathrm{osc}(\alpha,\beta)\}$;
- $c(\alpha,\beta) := \min\{\zeta < \lambda^+ \mid \mathrm{tr}(\alpha,\beta)(o^*(\alpha,\beta)) \in S_\zeta\}$.

Theorem 2.6 (Main result). *For every regular cardinal λ:*

- *o^* witnesses $\lambda^+ \nrightarrow [\lambda^+;\lambda^+]^2_\omega$;*
- *c witnesses $\lambda^+ \nrightarrow [\lambda^+;\lambda^+]^2_{\lambda^+}$.*

3. Proofs. To make the paper self-contained, we commence with a proof of Fact 2.3.

Proof of Fact 2.3. (1) Suppose not. Let $\alpha < \lambda^+$ be the least for which there exists $\theta < \lambda$ and a set $\Gamma \in [\alpha]^\lambda$ with $\rho_{1\alpha}(\gamma) \leq \theta$ for all $\gamma \in \Gamma$. In particular, $\mathrm{otp}(C_\alpha \cap \gamma) \leq \theta$ for all $\gamma \in \Gamma$. Define $o : \Gamma \to \theta + 1$ by stipulating that $o(\gamma) = \mathrm{otp}(C_\alpha \cap \gamma)$. Then there exists $\Gamma' \in [\Gamma]^\lambda$ on which o is constant. In particular, $\min(C_\alpha \setminus \gamma_1) = \min(C_\alpha \setminus \gamma_2)$ for all $\gamma_1, \gamma_2 \in \Gamma'$. Put $\alpha' := \min(C_\alpha \setminus \min(\Gamma'))$. Then $\Gamma' \in [\alpha']^\lambda$, and so by $\alpha' < \alpha$ and minimality of the latter, we may find some $\gamma' \in \Gamma'$ such that $\rho_{1\alpha'}(\gamma') > \theta$. By $\min(C_\alpha \setminus \gamma') = \alpha'$, we have $\mathrm{tr}^\circ(\gamma', \alpha) = \langle\alpha\rangle^\frown \mathrm{tr}^\circ(\gamma', \alpha')$, and hence

$$\rho_{1\alpha}(\gamma') = \max\{\mathrm{otp}(C_\alpha \cap \gamma'), \rho_{1\alpha^*}(\gamma')\} > \theta.$$

This is a contradiction.

(2) Suppose not. Let $\beta < \lambda^+$ be the least for which there exists $\alpha < \beta$ and a subset $\Gamma \subseteq \alpha$ of order-type λ with $\rho_{1\alpha}(\xi) \neq \rho_{1\beta}(\xi)$ for all $\xi \in \Gamma$. Put $\gamma := \sup(\Gamma)$, $\gamma^- := \sup(C_\beta \cap \gamma)$, and $\gamma^+ := \min(C_\beta \setminus \gamma)$. By $\mathrm{cf}(\gamma) = \lambda \geq \mathrm{otp}(C_\beta)$, we infer that $\gamma^- < \gamma \leq \alpha \leq \gamma^+ < \beta$.

Put $\theta := \mathrm{otp}(C_\beta \cap \gamma)$, and $\Gamma' := \{\xi \in \Gamma \setminus \gamma^- \mid \rho_{1\beta}(\xi) > \theta\}$. By the previous item, we know that $\mathrm{otp}(\Gamma') = \lambda$. It then follows from $\gamma^+ < \beta$ and minimality of the latter that there exists $\xi \in \Gamma'$ such that $\rho_{1\alpha}(\xi) = \rho_{1\gamma^+}(\xi)$.

By $\gamma^- \leq \xi < \gamma \leq \gamma^+$, we know that $\min(C_\beta \setminus \xi) = \min(C_\beta \setminus \gamma)$ and $\mathrm{otp}(C_\beta \cap \xi) = \mathrm{otp}(C_\beta \cap \gamma) = \theta$. That is, $\min(C_\beta \setminus \xi) = \gamma^+$, and $\rho_{1\gamma^+}(\xi) > \mathrm{otp}(C_\beta \cap \xi)$. So $\mathrm{tr}^\circ(\xi, \beta) = \langle\beta\rangle^\frown \mathrm{tr}^\circ(\xi, \gamma^+)$, and hence

$$\rho_{1\beta}(\xi) = \max\{\mathrm{otp}(C_\beta \cap \xi), \rho_{1\gamma^+}(\xi)\} = \rho_{1\gamma^+}(\xi) = \rho_{1\alpha}(\xi).$$

This is a contradiction.

(3) If $\delta \geq \max(L(\delta, \beta))$, then by Definition 2.1, there exists $i < \rho_2(\delta, \beta)$ such that $\sup(C_{\mathrm{tr}(\delta,\beta)(i)} \cap \delta) = \delta$. In particular, there exists an ordinal α with $\delta < \alpha < \beta$ such that $\sup(C_\alpha \cap \delta) = \delta$. It follows that $\mathrm{cf}(\delta) \leq \mathrm{otp}(C_\alpha \cap \delta) < \mathrm{otp}(C_\alpha) \leq \lambda$, contradicting the fact that $\delta \in E^{\lambda^+}_\lambda$.

(4) It suffices to prove that under the same hypotheses, we have $\mathrm{tr}(\beta, \gamma) = \mathrm{tr}(\alpha, \gamma) \restriction \rho_2(\beta, \gamma)$, and $\mathrm{tr}(\alpha, \gamma)(\rho_2(\beta, \gamma)) = \beta$. Clearly, $\mathrm{tr}(\alpha, \gamma)(0) = \gamma = \mathrm{tr}(\beta, \gamma)(0)$. Next, if $i < \rho_2(\beta, \gamma)$ and $\mathrm{tr}(\alpha, \gamma)(i) = \mathrm{tr}(\beta, \gamma)(i)$, then by

$$\beta > \alpha > \max(L(\beta, \gamma)) \geq \sup(C_{\mathrm{tr}(\beta,\gamma)(i)} \cap \beta),$$

we get

$$\min(C_{\mathrm{tr}^\circ(\alpha,\gamma)(i)} \setminus \alpha) = \min(C_{\mathrm{tr}^\circ(\beta,\gamma)(i)} \setminus \alpha) = \min(C_{\mathrm{tr}^\circ(\beta,\gamma)(i)} \setminus \beta),$$

and hence $\mathrm{tr}(\alpha, \gamma)(i + 1) = \mathrm{tr}(\beta, \gamma)(i + 1)$.

(5) By $\alpha \geq \min(L(\alpha, \beta)) > \max(L(\beta, \gamma))$, we deduce from the previous item that $\mathrm{tr}^\circ(\alpha, \gamma) = \mathrm{tr}^\circ(\beta, \gamma)^\frown \mathrm{tr}^\circ(\alpha, \beta)$, and hence

$$L(\alpha, \gamma) = L(\beta, \gamma) \oplus U,$$

for $U := L(\alpha, \beta) \setminus (\max(L(\beta, \gamma)) + 1)$. Recalling that $\min(L(\alpha, \beta)) > \max(L(\beta, \gamma))$, we conclude that $L(\alpha, \gamma) = L(\beta, \gamma) \oplus L(\alpha, \beta)$. ∎

Lemma 3.1. *For every subset $A \subseteq \lambda^+$, let $\hat{A}$ denote the set of all $\gamma < \lambda^+$ such that for all*

- *$\alpha \in A \setminus \gamma$,*
- *$U \in [\lambda^+ \setminus \gamma]^{<\omega}$,*
- *$L \in [\gamma]^{<\omega}$,*
- *$\theta < \lambda$,*

there exists some $\alpha' \in A$ *such that*

(1) $\alpha' > \max(U)$;
(2) $\rho_{1\alpha'}(\xi) > \theta$ *for all* $\xi \in U$;
(3) $\rho_{1\alpha'}(\xi) = \rho_{1\alpha}(\xi)$ *for all* $\xi \in L$.

If A *is cofinal in* λ^+, *then so is* $\hat{A}$.

Proof. Suppose that A is a cofinal subset of λ^+. Fix a large enough regular cardinal θ, and an elementary submodel $M \prec H_\theta$ of size λ with $\mathrm{cf}(M \cap \lambda^+) = \lambda$ such that $A, \overrightarrow{C} \in M$. Denote $\delta := M \cap \lambda^+$. As $\hat{A} \in M$ and $|M| = \lambda$, we see that $|\hat{A}| < \lambda^+$ iff $\hat{A} \subseteq M$. In particular, if $\delta \in \hat{A}$, then $\hat{A}$ is cofinal in λ^+. Thus, let us prove that $\delta \in \hat{A}$.

Suppose that $\alpha \in A \setminus \delta$, $U \in [\lambda^+ \setminus \delta]^{<\omega}$, $L \in [\delta]^{<\omega}$ and $\theta < \lambda$ are given. By $\mathrm{cf}(\delta) = \lambda$, and Fact 2.3(1), we may fix a large enough $\eta < \delta$ such that $\rho_{1\alpha}(\xi) > \theta$ whenever $\eta < \xi < \delta$. Next, put $e := \rho_{1\alpha} \restriction L$, and let

$$D := \{\nu < \lambda^+ \mid \exists \beta \in A \setminus \nu\ (\rho_{1\beta} \restriction L = e\ \&\ \rho_{1\beta}(\xi) > \theta \text{ whenever } \eta < \xi < \nu)\}.$$

Then $D \in M$, and if $\sup(D) < \lambda^+$, then $\sup(M) < \delta$. Since $\delta \in D$ (as witnessed by α), we infer that D is cofinal in λ^+. In particular, we may pick a large enough $\nu \in D$ above $\max(U)$, together with a witness $\alpha' \in A \setminus \nu$.

It follows that $\rho_{1\alpha'} \restriction L = e = \rho_{1\alpha} \restriction L$, and since $\eta < \delta \le \min(U) \le \max(U) < \nu$, we get $\rho_{1\alpha'}(\xi) > \theta$ for all $\xi \in U$. ∎

LEMMA 3.2. *Suppose* θ *is a large enough regular cardinal, and* $M \prec H_\theta$ *is an elementary submodel with* $M \cap \lambda^+ \in E^{\lambda^+}_\lambda$. *Denote* $\delta := M \cap \lambda^+$. *Suppose further that we are given* $A, B, S, \alpha, \beta, l$ *such that:*

- $A, B, \overrightarrow{C}, S \in M$;
- A, B *are cofinal subsets of* λ^+;
- S *is a stationary subset of* $E^{\lambda^+}_\lambda$;
- $\delta \in \alpha \in A$;
- $\delta \in \beta \in B$;
- $l \le \rho_2(\delta, \beta)$, *and* $\mathrm{tr}(\delta, \beta)(l) \in S$.

Then there exist $\alpha', \alpha'' \in A$, $\beta' \in B$, *and* $U \subseteq \delta$ *for which all of the following hold:*

(1) $\mathrm{tr}^\circ(\delta, \beta')(l) \in S$;
(2) $\beta' > \delta$ *and* $\rho_{1\beta'} \restriction L(\delta, \beta) = \rho_{1\beta} \restriction L(\delta, \beta)$;
(3) $\alpha' > \delta$ *and* $\rho_{1\alpha'} \restriction L(\delta, \beta) = \rho_{1\alpha} \restriction L(\delta, \beta)$;
(4) $\alpha'' > \delta$ *and* $\rho_{1\alpha''} \restriction L(\delta, \beta) = \rho_{1\alpha} \restriction L(\delta, \beta)$;
(5) $\rho_{1\alpha'}(\xi) = \rho_{1\beta'}(\xi)$ *for all* $\xi \in U$;
(6) $\rho_{1\alpha''}(\xi) > \rho_{1\beta'}(\xi)$ *for all* $\xi \in U$;
(7) $L(\delta, \beta') = L(\delta, \beta) \oplus U$.

Proof. Consider the set $\hat{A}$ as defined in Lemma 3.1. Then $\hat{A} \in M$ is a cofinal subset of λ^+, and so by Fact 2.3(2), we may pick a large enough $\gamma \in \hat{A} \cap M$ for which $\rho_{1\alpha}(\xi) = \rho_{1\beta}(\xi)$ whenever $\gamma \leq \xi < \delta$. By $\mathrm{cf}(\delta) = \lambda$, and Fact 2.3(3), we deduce that $\max(L(\delta, \beta)) \in \delta \subseteq M$, and so we may moreover require that $\gamma > \max(L(\delta, \beta))$.

Denote $\gamma^+ := \min(C_\delta \setminus \gamma + 1)$, $L := L(\delta, \beta)$, $e_\alpha := \rho_{1\alpha} \restriction L$, and $e_\beta := \rho_{1\beta} \restriction L$. Next, let T denote the set of all $\delta' \in E^{\lambda^+}_\lambda$ for which there exists $(\alpha', \beta') \in A \times B$ such that:

(a) $\mathrm{tr}(\delta', \beta')(l) \in S$;
(b) $\beta' > \delta'$ and $\rho_{1\beta'} \restriction L = e_\beta$;
(c) $\alpha' > \delta'$ and $\rho_{1\alpha'} \restriction L = e_\alpha$;
(d) $L(\delta', \beta') = L$;
(e) $\min(L(\nu, \delta')) \geq \gamma$ whenever $\gamma^+ < \nu < \delta'$;
(f) $\rho_{1\alpha'}(\xi) = \rho_{1\beta'}(\xi)$ whenever $\gamma \leq \xi < \delta'$.

As $\{l, L, e_\alpha, e_\beta, \gamma, \gamma^+, A, B, \overrightarrow{C}, S\} \subseteq M$, we get $T \in M$. Since $\delta \in T \setminus M$ as witnessed by the pair (α, β), we conclude that $|T| = \lambda^+$. Thus, let us pick some $\delta' \in T$ above δ, and a pair $(\alpha', \beta') \in A \times B$ that witnesses the fact that $\delta' \in T$. Then $\min\{\alpha', \beta'\} > \delta' > \delta$, and items (2), (3) are immediate consequences of items (b), (c), respectively.

Claim 3.2.1. *We have:*

- $L(\delta, \beta') = L(\delta, \beta) \oplus L(\delta, \delta')$;
- $\mathrm{tr}^\circ(\delta, \beta') = \mathrm{tr}^\circ(\delta', \beta')^\frown \mathrm{tr}^\circ(\delta, \delta')$.

In particular, items (1) *and* (7) *are valid.*

Proof. By item (d) and the choice of γ, we see that $\gamma > \max(L(\delta', \beta'))$. Since $\gamma^+ < \delta < \delta'$, we see from item (e) that $\min(L(\delta, \delta')) \geq \gamma > \max(L(\delta', \beta'))$. So, by $\delta < \delta' < \beta'$ and Fact 2.3(5), we infer that $L(\delta, \beta') = L(\delta', \beta') \oplus L(\delta, \delta')$. Then, by item (d), we conclude that $L(\delta, \beta') = L(\delta, \beta) \oplus L(\delta, \delta')$. Note that by Fact 2.3(3), $U := L(\delta, \delta')$ is indeed a subset of δ.

By Fact 2.3(3) and item (d), we have $\delta > \max(L(\delta, \beta)) = \max(L(\delta', \beta'))$. Then, by Fact 2.3(4), we find that $\mathrm{tr}^\circ(\delta, \beta') = \mathrm{tr}^\circ(\delta', \beta')^\frown \mathrm{tr}^\circ(\delta, \delta')$, and hence item (a) entails $\mathrm{tr}^\circ(\delta, \beta')(l) = \mathrm{tr}(\delta', \beta')(l) \in S$. ∎

As $\gamma^+ < \delta < \delta'$, we deduce from item (e) that $\xi \geq \gamma$ for all $\xi \in L(\delta, \delta')$. So, by item (f) and the preceding claim, we infer that $\rho_{1\alpha'}(\xi) = \rho_{1\beta'}(\xi)$ for all $\xi \in L(\delta, \delta') = L(\delta, \beta') \setminus L(\delta, \beta)$, thus establishing item (5).

Let $U := (L(\delta, \delta') \cup \{\delta\})$. By item (e), we have $U \in [\lambda^+ \setminus \gamma]^{<\omega}$. By $\gamma > \max(L(\delta, \beta))$, we have $L \in [\gamma]^{<\omega}$. Put $\theta := \max\{\rho_{1\beta'}(\xi) \mid \xi \in L(\delta, \delta')\}$. Recalling that γ was chosen as an element of $\hat{A}$, we infer the existence of an ordinal $\alpha'' \in A$ such that:

- $\alpha'' > \max(U) = \delta$;

- $\rho_{1\alpha''}(\xi) > \theta$ for all $\xi \in U$; in particular, item (6) holds;
- $\rho_{1\alpha''}(\xi) = \rho_{1\alpha}(\xi)$ for all $\xi \in L$; in particular, item (4) holds.

This completes the proof of Lemma 3.2. ∎

COROLLARY 3.3. *Suppose that θ is a large enough regular cardinal, and $M \prec H_\theta$ is an elementary submodel with $M \cap \lambda^+ \in E^{\lambda^+}_\lambda$. Denote $\delta := M \cap \lambda^+$. Suppose further that we are given $A, B, S, \alpha, \beta, l$ such that:*

- *$A, B, \overrightarrow{C}, S \in M$;*
- *A, B are cofinal subsets of λ^+;*
- *S is a stationary subset of $E^{\lambda^+}_\lambda$;*
- *$\delta \in \alpha \in A$;*
- *$\delta \in \beta \in B$;*
- *$l \leq \rho_2(\delta, \beta)$ and $\mathrm{tr}(\delta, \beta)(l) \in S$.*

Then there exist $\alpha^* \in A$ and $\beta^* \in B$ for which all of the following hold:

(1) *$L(\delta, \beta^*) = L(\delta, \beta) \oplus E \oplus G$ for some finite subsets E, G of δ;*
(2) *$\rho_{1\beta} \restriction L(\delta, \beta) = \rho_{1\beta^*} \restriction L(\delta, \beta)$;*
(3) *$\rho_{1\alpha} \restriction L(\delta, \beta) = \rho_{1\alpha^*} \restriction L(\delta, \beta)$;*
(4) *$\rho_{1\alpha^*}(\xi) = \rho_{1\beta^*}(\xi)$ for all $\xi \in E$;*
(5) *$\rho_{1\alpha^*}(\xi) > \rho_{1\beta^*}(\xi)$ for all $\xi \in G$;*
(6) *$\mathrm{tr}^\circ(\delta, \beta^*)(l) \in S$;*
(7) *$\min\{\alpha^*, \beta^*\} > \delta$.*

Proof. Suppose that $M, A, B, S, \delta, \alpha, \beta, l$ are as in the hypothesis. By Lemma 3.2, we may now find $(\alpha', \beta') \in A \times B$ and a finite $E \subseteq \delta$ such that:

- $L(\delta, \beta') = L(\delta, \beta) \oplus E$;
- $\beta' > \delta$ and $\rho_{1\beta'} \restriction L(\delta, \beta) = \rho_{1\beta} \restriction L(\delta, \beta)$;
- $\alpha' > \delta$ and $\rho_{1\alpha'} \restriction L(\delta, \beta) = \rho_{1\alpha} \restriction L(\delta, \beta)$;
- $\rho_{1\alpha'}(\xi) = \rho_{1\beta'}(\xi)$ for all $\xi \in E$;
- $\mathrm{tr}^\circ(\delta, \beta')(l) \in S$.

Next, appeal to Lemma 3.2 with $M, A, B, S, \delta, \alpha', \beta', l$ to find $(\alpha^*, \beta^*) \in A \times B$ and a finite $G \subseteq \delta$ such that:

- $L(\delta, \beta^*) = L(\delta, \beta') \oplus G$;
- $\beta^* > \delta$ and $\rho_{1\beta^*} \restriction L(\delta, \beta') = \rho_{1\beta'} \restriction L(\delta, \beta')$;
- $\alpha^* > \delta$ and $\rho_{1\alpha^*} \restriction L(\delta, \beta') = \rho_{1\alpha'} \restriction L(\delta, \beta')$;
- $\rho_{1\alpha^*}(\xi) > \rho_{1\beta^*}(\xi)$ for all $\xi \in G$;
- $\mathrm{tr}^\circ(\delta, \beta^*)(l) \in S$.

Then it follows that α^* and β^* have all the desired properties. ∎

THEOREM 3.4 (Main Result). *For every regular cardinal λ:*

- *o^* witnesses $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_\omega$;*
- *c witnesses $\lambda^+ \nrightarrow [\lambda^+; \lambda^+]^2_{\lambda^+}$.*

Proof. Suppose that A, B are cofinal subsets of λ^+, and $\zeta < \lambda^+$. We shall find $(\hat{\alpha}, \hat{\beta}) \in A \circledast B$ for which $c(\hat{\alpha}, \hat{\beta}) = \zeta$. The proof will also make it clear that $o^*[A \circledast B] = \omega$.

Fix a large enough regular cardinal θ, and an elementary submodel $M \prec H_\theta$ such that $A, B, \overrightarrow{C}, S_\zeta \in M$ and $M \cap \lambda^+ \in E^{\lambda^+}_\lambda \cap S_\zeta$. Denote $\delta := M \cap \lambda^+$, $\alpha := \min(A \setminus \delta+1)$, $\beta := \min(B \setminus \delta+1)$, and $l := \rho_2(\delta, \beta)$. Then, by Corollary 3.3, we may find $\alpha_0 \in A \setminus (\delta + 1)$ and $\beta_0 \in B \setminus (\delta + 1)$ such that:

- $\rho_{1\alpha_0}(\max(L(\delta, \beta_0))) > \rho_{1\beta_0}(\max(L(\delta, \beta_0)))$;
- $\mathrm{tr}^\circ(\delta, \beta_0)(l) \in S_\zeta$.

Let $n < \omega$ be large enough, so that for every $t < \omega$,

$$l \in \{\min\{\iota \mid p_\iota \text{ does not divide } k\} \mid t < k < t + n\}.$$

Next, by an iterative application of Corollary 3.3, we may find a sequence $\langle(\alpha_{m+1}, \beta_{m+1}, E_m, G_m) \mid m < \omega\rangle$ such that for all $m < \omega$, the following hold:

(1) $L(\delta, \beta_{m+1}) = L(\delta, \beta_m) \oplus E_m \oplus G_m$;
(2) $\rho_{1\beta_{m+1}} \restriction L(\delta, \beta_m) = \rho_{1\beta_m} \restriction L(\delta, \beta_m)$;
(3) $\rho_{1\alpha_{m+1}} \restriction L(\delta, \beta_m) = \rho_{1\alpha_m} \restriction L(\delta, \beta_m)$;
(4) $\rho_{1\alpha_{m+1}}(\xi) = \rho_{1\beta_{m+1}}(\xi)$ for all $\xi \in E_m$;
(5) $\rho_{1\alpha_{m+1}}(\xi) > \rho_{1\beta_{m+1}}(\xi)$ for all $\xi \in G_m$;
(6) $\mathrm{tr}^\circ(\delta, \beta_{m+1})(l) \in S_\zeta$.

By Fact 2.3(3), let us fix a large enough $\gamma \in C_\delta$ such that $\max(L(\delta, \beta_n)) < \gamma$. By Fact 2.3(2), we may further assume that

$$\gamma > \max\{\xi < \delta \mid \rho_{1\beta_m}(\xi) \neq \rho_{1\beta_{m+1}}(\xi) \text{ for some } m \leq n\}.$$

Denote $L := L(\delta, \beta_n)$, $e := \rho_{1\alpha_n} \restriction L(\delta, \beta_n)$. Consider $E := \{\alpha \in A \mid (\rho_{1\alpha} \restriction L) = e\}$. Then $E \in M$, while $\alpha_n \in E \setminus M$. In particular, $\sup(E) = \lambda^+$ and $\sup(E \cap M) = \delta$, so let us pick a large enough $\hat{\alpha} \in E \cap \delta$ above γ.

Claim 3.4.1. *For every $m \leq n$, we have:*

(a) $\rho_{1\hat{\alpha}}(\max(L(\delta, \beta_m))) > \rho_{1\beta_m}(\max(L(\delta, \beta_m)))$;
(b) $\mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\delta, \beta_m)) = \mathrm{Osc}(\rho_{1\alpha_m}, \rho_{1\beta_m}, L(\delta, \beta_m))$.

Proof. Fix $m \leq n$. Then $L(\delta, \beta_m) \subseteq L(\delta, \beta_n) = L$, so by $\hat{\alpha} \in E$, we conclude that $\rho_{1\hat{\alpha}} \restriction L(\delta, \beta_m) = \rho_{1\alpha_m} \restriction L(\delta, \beta_m)$. ■

Note that item (a) of the preceding claim implies that for every $m \leq n$ and every finite $U \subseteq \delta$ with $\min(U) > \max(L(\delta, \beta_m))$, we have

$$\mathrm{osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\delta, \beta_m) \cup U) = \mathrm{osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\delta, \beta_m)) + \mathrm{osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, U).$$

Claim 3.4.2. *For all $m \leq n$, we have:*

(a) $L(\hat{\alpha}, \beta_m) = L(\delta, \beta_m) \oplus L(\hat{\alpha}, \delta)$;
(b) $\mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \delta)) = \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\hat{\alpha}, \delta))$;

(c) $\mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_m)) = \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\delta, \beta_m))$;

(d) $\mathrm{tr}^\circ(\hat{\alpha}, \beta_m)(l) \in S_\zeta$.

Proof. Fix $m \le n$. Note that the fact that $\hat{\alpha} > \gamma \in C_\delta$ implies that $\min(L(\hat{\alpha}, \delta)) = \max(C_\delta \cap \hat{\alpha}) \ge \gamma$.

(a) follows from $\min(L(\hat{\alpha}, \delta)) \ge \gamma > \max(L(\delta, \beta_m))$ and from Fact 2.3(5) for $\hat{\alpha} < \delta \le \beta_m$.

(b) follows from $\min(L(\hat{\alpha}, \delta)) \ge \gamma > \max\{\xi < \delta \mid \rho_{1\beta_m}(\xi) \ne \rho_{1\beta_{m+1}}(\xi)\}$.

(c) follows from property (2) in the choice of $\langle(\alpha_{m+1}, \beta_{m+1}, E_m, G_m) \mid m < \omega\rangle$.

(d) By $\hat{\alpha} > \gamma > \max(L(\delta, \beta_m))$, and Fact 2.3(4) for $\hat{\alpha} < \delta \le \beta_m$, we deduce that $\mathrm{tr}^\circ(\hat{\alpha}, \beta_m) = \mathrm{tr}^\circ(\delta, \beta_m)^\frown\mathrm{tr}^\circ(\hat{\alpha}, \delta)$. In particular, $\mathrm{tr}^\circ(\hat{\alpha}, \beta_m)(l) = \mathrm{tr}(\delta, \beta_m)(l) \in S_\zeta$. ∎

Claim 3.4.3. $\mathrm{osc}(\hat{\alpha}, \beta_{m+1}) = \mathrm{osc}(\hat{\alpha}, \beta_m) + 1$ *for all* $m < n$.

Proof. Fix $m < n$. By the preceding claims, we get

$$\begin{aligned}
\mathrm{osc}(\hat{\alpha}, \beta_{m+1}) &= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \beta_{m+1}))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_{m+1}) \cup L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_{m+1})) + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_m) \cup E_m \cup G_m)\\
&\quad + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_m)) + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, E_m \cup G_m)\\
&\quad + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_m)) + \mathrm{Osc}(\rho_{1\alpha_{m+1}}, \rho_{1\beta_{m+1}}, E_m \cup G_m)\\
&\quad + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\delta, \beta_m)) + 1 + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_{m+1}}, L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\delta, \beta_m)) + 1 + \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\hat{\alpha}, \delta))\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\delta, \beta_m) \cup L(\hat{\alpha}, \delta)) + 1\\
&= \mathrm{Osc}(\rho_{1\hat{\alpha}}, \rho_{1\beta_m}, L(\hat{\alpha}, \beta_m)) + 1 = \mathrm{osc}(\hat{\alpha}, \beta_m) + 1. \ \blacksquare
\end{aligned}$$

Let $t := \mathrm{osc}(\hat{\alpha}, \beta_0)$. By our choice of n, there exists some $m^* < n$ such that $l = \min\{\iota < \omega \mid p_\iota \text{ does not divide } t + m^*\}$; thus, let $\hat{\beta} := \beta_{m^*}$ for the above m^*.

Claim 3.4.4. $\mathrm{tr}^\circ(\hat{\alpha}, \hat{\beta})(o^*(\hat{\alpha}, \hat{\beta})) \in S_\zeta$.

Proof. By the preceding claim, $\mathrm{osc}(\hat{\alpha}, \beta_m) = t + m$ for all $m < n$. In particular, $\mathrm{osc}(\hat{\alpha}, \hat{\beta}) = t + m^*$. So, $o^*(\hat{\alpha}, \hat{\beta}) = l$. It now follows from Claim 3.4.2(d) that $\mathrm{tr}^\circ(\hat{\alpha}, \hat{\beta})(o^*(\hat{\alpha}, \hat{\beta})) = \mathrm{tr}^\circ(\hat{\alpha}, \beta_{m^*})(l) \in S_\zeta$. ∎

Recalling the definition of c, we conclude that $c(\hat{\alpha}, \hat{\beta}) = \zeta$. This completes the proof of Theorem 3.4 ∎

4. Concluding remarks. In Definition 2.5, the function o^* is defined as a particular projection of the oscillation function osc. We do not know whether there are any other interesting projections for cardinals $\lambda \geq \mathfrak{c}$. In particular, we are interested in projections that directly yield an L-space at the λ^+ level. We should also point out a question appearing originally in [2], asking whether there is a variation on the oscillation mapping, or perhaps a different projection, that yields an L-space whose square is also an L-space.

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