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Lyapunov quasi-stable trajectories

by

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Abstract. We introduce the notions of Lyapunov quasi-stability and Zhukovskiĭ quasi-stability of a trajectory in an impulsive semidynamical system defined in a metric space, which are counterparts of corresponding stabilities in the theory of dynamical systems. We initiate the study of fundamental properties of those quasi-stable trajectories, in particular, the structures of their positive limit sets. In fact, we prove that if a trajectory is asymptotically Lyapunov quasi-stable, then its limit set consists of rest points, and if a trajectory in a locally compact space is uniformly asymptotically Zhukovskiĭ quasi-stable, then its limit set is a rest point or a periodic orbit. Also, we present examples to show the differences between variant quasi-stabilities. Further, some sufficient conditions are given to guarantee the quasi-stabilities of a prescribed trajectory.

1. Introduction. The theory of impulsive differential equations now becomes an important area of investigation, since it is a lot richer than the corresponding theory of differential equations. Moreover, such equations represent a natural framework for mathematical modeling of many real world phenomena. For the elementary results in this field, we refer to the books [1, 14, 18, 21]. The research of impulsive semidynamical systems in a metric space was started by Kaul [15, 16, 17] and Rozhko [23, 24]. In particular, Kaul associated to a given impulsive semidynamical system a discrete semidynamical system defined on the range of the impulsive set under an impulsive function; thus he established many important results about the limit sets, recursive properties of orbits and stabilities of closed subsets in an impulsive semidynamical system. Later, Ciesielski [8, 9] proved several fundamental results for this theory, in fact he applied his section theory of semidynamical systems (see [7]) to obtain the continuity of some important functions associated with impulsive semidynamical systems. Recently, Bonotto, Federson and their collaborators also published a series of important papers on this subject (see [4, 5] and references therein); they developed

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a list of results on impulsive semidynamical systems, which are counterparts of basic properties of dynamical systems.

In an impulsive system, the stability of closed sets has been well studied in [4, 5, 9, 17]. However, to our knowledge, the stability of a motion or a trajectory in impulsive dynamical systems defined in metric spaces has remained untouched so far. Indeed, no notions of stability for a prescribed trajectory with an infinite number of impulses have been available. Fortunately, for an impulsive differential equation, Lakshmikantham and Liu [19] introduced the notion of quasi-stability relative to a given solution of the impulsive system. Since it deals with variant (or state-dependent) impulsive times, such a stability leads to a more complicated mechanism, and the classical notions are modified in [19] to suit the present circumstances.

Our goal in this paper is to establish the corresponding counterparts of quasi-stabilities in an impulsive semidynamical system defined in a metric space. Actually, we introduce the concepts of Lyapunov quasi-stability and Zhukovskiĭ quasi-stability, and present examples to show their differences. Several sufficient conditions are given to guarantee those quasi-stabilities of a prescribed trajectory. Next, we investigate the structure of a positive limit set for a trajectory with some quasi-stability. In particular, we prove that if a trajectory is asymptotically Lyapunov quasi-stable, then its limit set consists of rest points, and if a trajectory in a locally compact space is uniformly asymptotically Zhukovskiĭ quasi-stable, then its limit set is a rest point or a periodic orbit.

2. Definitions and notations. Throughout the paper, let $X = (X, d)$ be a metric space with metric d. For $A \subset X$, $\overline{A}$ and ∂A denote the closure and boundary of A, respectively. Let $B(x, \delta) = \{y \in X : d(x, y) < \delta\}$ be the open ball with center x and radius $\delta > 0$, and $\bar{B}(x, \delta) = \{y \in X : d(x, y) \leq \delta\}$ the closed ball. In addition, for $A \subset X$ and $r > 0$, the r-neighborhood of A is denoted by $N(A, r) = \{z \in X : d(z, A) < r\}$, where $d(z, A) = \inf\{d(z, p) : p \in A\}$. With no confusion, we also use d for the distance between a point and a set. Let $\mathbb{R}^+$ and $\mathbb{Z}^+$ be the sets of nonnegative real numbers and nonnegative integers, respectively.

A *semidynamical system* (or *semiflow*) on X is a triple $(X, \pi, \mathbb{R}^+)$, where π is a continuous mapping from $X \times \mathbb{R}^+$ onto X satisfying the following axioms:

(1) $\pi(x, 0) = x$ for each $x \in X$,
(2) $\pi(\pi(x, t), s) = \pi(x, t + s)$ for each $x \in X$ and t, $s \in \mathbb{R}^+$.

Replacing $\mathbb{R}^+$ by $\mathbb{R}$, we get the definition of a dynamical system (or flow). In this paper, we sometimes denote a semidynamical system $(X, \pi, \mathbb{R}^+)$ by (X, π). For brevity, we write $x \cdot t = \pi(x, t)$, and also let $A \cdot J = \{x \cdot t :$

$x \in A,\, t \in J\}$ for $A \subset X$ and $J \subset \mathbb{R}^+$. If either A or J is a singleton, i.e., $A = \{x\}$ or $J = \{t\}$, then we simply write $x \cdot J$ and $A \cdot t$ for $\{x\} \cdot J$ and $A \cdot \{t\}$, respectively. For any $x \in X$, the function $\pi_x : \mathbb{R}^+ \to X$ defined by $\pi_x(t) = \pi(x,t)$ is clearly continuous, and we call it the *trajectory* (or *motion*) through x. The set $x \cdot \mathbb{R}^+$ is said to be the *positive orbit* of x, and sometimes denoted by $C^+(x)$. The closure of $C^+(x)$ in X is denoted by $K^+(x)$. For the elementary properties of dynamical systems and semidynamical systems, we refer to [2, 3, 22].

Let M be a nonempty closed subset in X and $\Omega = X \backslash M$. Let $I : M \to \Omega$ be a continuous function and $I(M) = N$. If $x \in M$, we shall denote $I(x)$ by x^+ and say that x *jumps to* x^+. Moreover, I and M are said to be an *impulsive function* and an *impulsive set*, respectively. For each $x \in \Omega$, by $M^+(x)$ we mean the set $x \cdot \mathbb{R}^+ \cap M$. We can define a function $\phi : \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ (the space of extended positive reals) by

$$\phi(x) = \begin{cases} s & \text{if } x \cdot s \in M \text{ and } x \cdot t \notin M \text{ for } t \in [0,s), \\ +\infty & \text{if } M^+(x) = \emptyset. \end{cases}$$

Generally, ϕ is not continuous. However, its continuity on Ω is crucial for the analysis of dynamics. Fortunately, some easy conditions given by Ciesielski [8] guarantee the continuity of ϕ.

Assumption I. *Throughout the paper, ϕ is a continuous function on Ω.*

Following Kaul [17], we now define an impulsive semidynamical system $(\Omega, \pi, \mathbb{R}^+; M, I)$ by portraying the trajectory of each point in Ω. The *impulsive trajectory* of $x \in \Omega$ is an Ω-valued function $\tilde{\pi}_x$ defined on a subset of $\mathbb{R}^+$. If $M^+(x) = \emptyset$, then $\phi(x) = +\infty$, and we set $\tilde{\pi}_x(t) = x \cdot t$ for all $t \in \mathbb{R}^+$. If $M^+(x) \neq \emptyset$, it is easy to see that there is a positive number t_0 such that $x \cdot t_0 = x_1 \in M$ and $x \cdot t \notin M$ for $0 \le t < t_0$. Thus, we define $\tilde{\pi}_x$ on $[0, t_0]$ by

$$\tilde{\pi}_x(t) = \begin{cases} x \cdot t, & 0 \le t < t_0, \\ x_1^+, & t = t_0. \end{cases}$$

where $\phi(x) = t_0$ and $x_1^+ = I(x_1) \in \Omega$.

Since $t_0 < +\infty$, we continue the process by starting with x_1^+. Similarly, if $M^+(x_1^+) = \emptyset$, i.e., $\phi(x_1^+) = +\infty$, we define $\tilde{\pi}_x(t) = x_1^+ \cdot (t - t_0)$ for $t_0 < t < +\infty$. Otherwise, let $\phi(x_1^+) = t_1$ and $x_1^+ \cdot t_1 = x_2 \in M$; then we define $\tilde{\pi}_x(t)$ on $[t_0, t_0 + t_1]$ by

$$\tilde{\pi}_x(t) = \begin{cases} x_1^+ \cdot (t - t_0), & t_0 \le t < t_0 + t_1, \\ x_2^+, & t = t_0 + t_1, \end{cases}$$

where $x_2^+ = I(x_2)$.

Thus, continuing inductively, the process above either ends after a finite number of steps, whenever $M^+(x_n^+) = \emptyset$ for some n, or it continues indefinitely, if $M^+(x_n^+) \neq \emptyset$ for $n = 1, 2, \ldots$, and $\tilde{\pi}_x$ is defined on the interval $[0, T(x))$, where $T(x) = \sum_{i=0}^{\infty} t_i$. We call $\{t_i\}$ the *impulsive intervals* of $\tilde{\pi}_x$, and call $\{t^n = \sum_{i=0}^{n} t_i : n = 0, 1, 2, \ldots\}$ the *impulsive times* of $\tilde{\pi}_x$. Obviously, this gives rise to either a finite or an infinite number of jumps at points $\{x_n\}$ for the trajectory $\tilde{\pi}_x$. Having the trajectory $\tilde{\pi}_x$ for every point x in Ω, we let $\tilde{\pi}(x, t) = \tilde{\pi}_x(t)$ for $x \in \Omega$ and $t \in [0, T(x))$, and obtain a discontinuous system $(\Omega, \pi, \mathbb{R}^+; M, I)$, or $(\Omega, \tilde{\pi})$, with the following properties:

(i) $\tilde{\pi}(x, 0) = x$ for $x \in \Omega$,
(ii) $\tilde{\pi}(\tilde{\pi}(x, t), s) = \tilde{\pi}(x, t + s)$ for $x \in \Omega$ and t, $s \in [0, T(x))$ such that $t + s \in [0, T(x))$.

We call $(\Omega, \pi, \mathbb{R}^+; M, I)$, or $(\Omega, \tilde{\pi})$ with $\tilde{\pi}$ as defined above, the *impulsive semidynamical system* associated with (X, π). For simplicity of exposition, in the remainder of this paper we denote the trajectory $\tilde{\pi}(x, t)$ by $x * t$. Thus, (ii) reads $(x * t) * s = x * (t + s)$. Given $x \in \Omega$, if $M^+(x) = \emptyset$, the trajectory $\tilde{\pi}_x$ is continuous; otherwise, it has discontinuities at a finite or an infinite number of its *impulsive points* $\{x_n^+\}$. At any such point, however, $\tilde{\pi}_x$ is continuous from the right.

From the point of view of impulsive semidynamical systems, the trajectories of interest are those with an infinite number of discontinuities and with $[0, +\infty)$ as the interval of definition (see [15, 18]). We call them *infinite trajectories*. In this paper, we do not deal with the Zeno orbits, i.e. orbits that involve infinitely many resettings in finite time (see [14, Chap. 2]). Hence, from now on we assume $T(x) = +\infty$ for each $x \in \Omega$. Under a suitable condition about M (see [8, 15]), we can similarly define an impulsive semidynamical system $(X, \tilde{\pi})$ on X, which admits $(\Omega, \tilde{\pi})$ as a subsystem. With a mild modification, our results on $(\Omega, \tilde{\pi})$ can be applied to $(X, \tilde{\pi})$.

Now, we introduce some concepts that will be used in what follows; they were defined by Kaul in [15, 17].

DEFINITION 2.1 ([17]). A subset S of Ω is said to be *positively invariant* if $x * \mathbb{R}^+ \subset S$ for any $x \in S$, and it is said to be *invariant* if it is positively invariant and furthermore, given $x \in S$ and $t \in \mathbb{R}^+$, there exists a $y \in S$ such that $y * t = x$.

DEFINITION 2.2 ([15]). A point x in Ω is a *rest point* if $x * t = x$ for every $t \in \mathbb{R}^+$. An orbit $x * \mathbb{R}^+$ is said to be *periodic* of period τ and order k if $x * \mathbb{R}^+$ has k components and τ is the least positive number such that $x * \tau = x$.

Clearly, a periodic orbit of $(\Omega, \tilde{\pi})$ is an invariant closed set in Ω, and it is not connected as long as $k \neq 1$. A point $x \in \Omega$ is a rest point of $(\Omega, \tilde{\pi})$ if and only if it is a rest point of (X, π). If a point x is not a rest point, we call it a *regular point*.

Definition 2.3. A nonempty set S is *positively minimal* provided S is a closed, positively invariant set and whenever Z is a closed, nonempty, positively invariant subset of S, then $Z = S$.

Definition 2.4. Let $x \in \Omega$. The *omega* (or *positive*) *limit set* $\tilde{\omega}(x)$ of x in $(\Omega, \tilde{\pi})$ is defined by $\tilde{\omega}(x) = \{y \in \Omega : x * t_n \to y$ for some $t_n \to +\infty\}$.

Equivalently, $\tilde{\omega}(x) = \limsup_{n \to +\infty} \{x_n^+ \cdot [0, \phi(x_n^+))\}$. Thus, it is easy to see that $\tilde{\omega}(x)$ is closed and positively invariant. Of course, it may not be invariant.

Definition 2.5. Let $x \in \Omega$. The (positive) *prolongational limit set* $\tilde{J}(x)$ of x in $(\Omega, \tilde{\pi})$ is defined by $\tilde{J}(x) = \{y \in \Omega : x_n * t_n \to y$ for some $x_n \to x$ and $t_n \to +\infty\}$.

Let $\Omega_0 = \{x \in \Omega : \phi(x) < +\infty\}$. Then there exists a function $\varphi : \Omega_0 \to N$, defined by $\varphi(x) = x * \phi(x)$ for any $x \in \Omega_0$. Since ϕ is continuous on Ω, so is φ on Ω_0. Let $\hat{N} = \{x \in N : \tilde{\pi}_x$ is an infinite impulsive trajectory$\}$; clearly $\hat{N} \subset \Omega_0 \cap N$. Now, given any $x \in \hat{N}$ and $x_1 = x \cdot \phi(x) \in M$, we define $g : \hat{N} \to \hat{N}$ by $g(x) = \varphi(x) = I(x_1) = x_1^+$; consequently, g is continuous on $\hat{N}$. As usual, we set $g^0 =$ identity, $g^1 = g$ and define g^n inductively for $n > 1$. Thus, we obtain a discrete semidynamical system $(\hat{N}, g, \mathbb{Z}^+)$, where $g(x, n) = g^n(x) = x_n^+$. $(\hat{N}, g, \mathbb{Z}^+)$ or simply $(\hat{N}, g)$ is called the *discrete semidynamical system associated with the given impulsive system* $(\Omega, \tilde{\pi})$. For a point $x \in \hat{N}$, the dynamics of $\{g^n(x)\}$ in $(\hat{N}, g)$ is closely related to that of $\tilde{\pi}_x$ in $(\Omega, \tilde{\pi})$ (see [15, 16, 17]).

3. Quasi-stabilities. In the theory of dynamical systems, continuous dependence on initial conditions is a fundamental property (see [22, p. 327]). This is also true for $t \geq 0$ in semidynamical systems. Consider a semidynamical system (X, π). For a point x in X, given a positive number T and an $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon, T) > 0$ such that if $d(x, y) < \delta$, then $d(x \cdot t, y \cdot t) < \epsilon$ for all $t \in [0, T]$. Clearly, this is an immediate consequence of the continuity of π. In this paper, we call it the *CD property* of a semidynamical system, for brevity. If $\delta(x, \epsilon, T) > 0$ can be chosen independent of T, then the CD property implies the classical Lyapunov stability of the trajectory π_x for $t \geq 0$. Hence, Lyapunov stability of a trajectory can be interpreted as continuous dependence of trajectories uniformly in t for all $t \geq 0$. Of course, the CD property does not hold for our impulsive semidynamical system $(\Omega, \tilde{\pi})$. To remedy this, for impulsive semidynamical systems in $\mathbb{R}^n$, in [6] the authors presented a notion of quasi-continuous dependence: For every $x_0 \in \mathfrak{D} \subset \mathbb{R}^n$, there exists $J_{x_0} \subset [0, +\infty)$ such that $[0, +\infty) \setminus J_{x_0}$ is (finite or infinite) countable and for every $\epsilon > 0$, $t \in J_{x_0}$, there exists $\delta(\epsilon, x_0, t) > 0$ such that if $\|x_0 - y\| < \delta(\epsilon, x_0, t)$, $y \in \mathfrak{D}$, then $\|x_0 * t - y * t\| < \epsilon$. Similarly, if the quasi-continuous dependence holds uniformly in t for all $t \geq 0$ and not in any η-neighborhood of $[0,+\infty) \setminus J_{x_0}$, then it also leads to a quasi-stability. Such a quasi-stability was established in [18, p. 103] for impulsive differential equations.

Now, we define the quasi-stability for an impulsive semidynamical system defined in a metric space.

Definition 3.1. For an impulsive semidynamical system $(\Omega, \tilde{\pi})$, the trajectory $\tilde{\pi}_x$ (or orbit $x * \mathbb{R}^+$) of a point $x \in \Omega$ is *Lyapunov quasi-stable* provided that given any $\epsilon > 0$ and $\eta > 0$, there is a $\delta = \delta(x, \epsilon, \eta) > 0$ such that if $d(x, y) < \delta$, then $d(x * t, y * t) < \epsilon$ for all $t \geq 0$ and $|t - t^k| > \eta$, where $\{t^k : k = 0, 1, 2, \dots\}$ are impulsive times of $\tilde{\pi}_x$. The trajectory $\tilde{\pi}_x$ (or orbit $x*\mathbb{R}^+$) is *asymptotically Lyapunov quasi-stable* provided it is Lyapunov quasi-stable and for any $\eta > 0$, there exists a $\lambda > 0$ such that if $d(x, y) < \lambda$, then $d(x * t, y * t)$ $(t \notin \bigcup_{k=0}^{+\infty}[t^k - \eta, t^k + \eta])$ goes to zero as t goes to infinity.

Clearly, the notion defined above is a generalization of the classical Lyapunov stability. Since the impulsive times of an orbit differ from those of near orbits, the isochronous correspondence between orbits does not hold in an impulsive semidynamical system. So, for a Lyapunov quasi-stable orbit, the times involved and the impulsive times are at a distance at least η apart.

In the mathematical and physical literature, Zhukovskiĭ stability (see [10, 11, 20]) is also an important concept in the theory of stability, which permits a time lag. Hence, it may be more suitable for impulsive semidynamical systems. To introduce this concept, we first recall the notion of time reparametrization.

Definition 3.2. A *time reparametrization* is a homeomorphism h from $\mathbb{R}^+$ onto $\mathbb{R}^+$ with $h(0) = 0$. Further, for a $\sigma > 0$, by a *time σ-reparametrization* we mean a homeomorphism τ from $\mathbb{R}^+$ to $\mathbb{R}^+$ with $\tau(0) = 0$ such that $|\tau(t) - t| < \sigma$ for all $t \geq 0$.

Quasi-continuous Dependence. *For an impulsive semidynamical system $(\Omega, \tilde{\pi})$, let $x \in \Omega$. Given $\epsilon > 0$, $\sigma > 0$, and a positive number $\mathcal{T}$, there exists a $\delta = \delta(x, \epsilon, \sigma, \mathcal{T}) > 0$ such that if $d(x, y) < \delta$, then one can find a σ-reparametrization τ_y such that $d(x*t, y*\tau_y(t)) < \epsilon$ for all $t \in [0, \mathcal{T}]$.*

It is easy to see that quasi-continuous dependence is a generalization of the standard continuous dependence from semidynamical systems to impulsive semidynamical systems. Actually, by letting $\sigma = 0$, the quasi-continuous dependence specializes to the classical continuous dependence on initial conditions. In what follows, we sometimes abbreviate quasi-continuous dependence to the *QCD property*. In [12], it is proved that the QCD property is equivalent to the continuity of ϕ. Similarly, in the QCD property, if $\delta(x, \epsilon, \sigma, \mathcal{T}) > 0$ can be chosen independent of $\mathcal{T}$, then the QCD property implies a certain stability of a trajectory $\tilde{\pi}_x$ in the system $(\Omega, \tilde{\pi})$, which is defined as follows.

Definition 3.3. The trajectory $\tilde{\pi}_x$ (or orbit $x * \mathbb{R}^+$) of a point x in Ω is *Zhukovskiĭ quasi-stable* provided that given any $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon) > 0$ such that if $d(x, y) < \delta$, then one can find a time reparametrization τ_y such that $d(x * t, y * \tau_y(t)) < \epsilon$ for all $t \geq 0$. Moreover, if there exists a $\lambda > 0$ such that if $d(x, y) < \lambda$, then $d(x * t, p * \tau_y(t)) \to 0$ as $t \to +\infty$, then the trajectory $\tilde{\pi}_x$ (or orbit $x * \mathbb{R}^+$) is said to be *asymptotically Zhukovskiĭ quasi-stable.*

In Definition 3.3, let $\bar{t} = \tau_y(t)$ and $h(\bar{t}) = \tau_y^{-1}(\bar{t}) = t$; then it follows that $d(x * t, y * \tau_y(t)) = d(x * h(\bar{t}), y * \bar{t})$. Thus, we get an equivalent formulation: a trajectory $\tilde{\pi}_x$ is Zhukovskiĭ quasi-stable provided that given any $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon) > 0$ such that if $d(x, y) < \delta$, then one can find a time reparametrization τ_y such that $d(x * \tau_y(t), y * t) < \epsilon$ for all $t \geq 0$.

Since we are just interested in infinite trajectories, in order to avoid singularities, we only deal with infinite trajectories whose close trajectories are also infinite trajectories. Clearly, a boundary point of $\hat{N}$ in N is not Lyapunov quasi-stable or Zhukovskiĭ quasi-stable. Thus, to ensure that close trajectories of an infinite trajectory are also infinite, we need a suitable topological position of N in Ω. Fortunately, a simple and useful condition is presented in [8, 15], which is called the (TC) condition (see [8]) or that N is well placed in Ω (see [15]). Instead of introducing that condition, we apply a concrete assumption. Actually, in the remainder of this paper, we always suppose the following:

Assumption II. *For an interior point p of $\hat{N}$ in the subspace N, there exists a small ball $B(p, \delta)$ $(\delta > 0)$ in Ω such that if $q \in B(p, \delta)$, then $q \cdot t \in \hat{N}$ for a $t = t(q) > 0$ or there exists a point $z \in \hat{N}$ such that $q = z \cdot \tau$ for a $\tau = \tau(q) > 0$.*

Let $\operatorname{Int} \hat{N}$ be the interior of $\hat{N}$ in N. It is easy to see that under Assumption II, for a point $x \in \operatorname{Int}\hat{N}$, stabilities of $\tilde{\pi}_x$ defined as above are equivalent to *restricted stabilities* on $\hat{N}$, e.g., $\tilde{\pi}_x$ is Zhukovskiĭ quasi-stable if and only if given any $\epsilon > 0$, there is a $\delta = \delta(x, \epsilon) > 0$ such that if $y \in B(x, \delta) \cap \hat{N}$, then $d(x * t, y * \tau_y(t)) < \epsilon$ for all $t \geq 0$, where $\tau_y(t)$ is a time reparametrization. Similarly, one can define the restricted Lyapunov quasi-stability on $\hat{N}$. Thus, in the following, we only consider the restricted quasi-stabilities on $\hat{N}$. Of course, if $x \in \hat{N}$ is a boundary point of $\hat{N}$ in N, then $\tilde{\pi}_x$ is not Lyapunov quasi-stable or Zhukovskiĭ quasi-stable, since in every neighborhood of it there exists a trajectory with no impulses or only a finite number of impulses.

Clearly, Lyapunov quasi-stability is more restrictive, since it is an almost isochronous correspondence of orbits. However, Zhukovskiĭ quasi-stability implies that close orbits should also be close in the phase space and trace each other with a time lag. Of course, this is a kind of phase stability. In a semidynamical system, Lyapunov stability implies Zhukovskiĭ stability (see [10, 20]). However, for $(\Omega,\tilde{\pi})$, we present two examples to show that neither of the quasi-stabilities implies the other.

Example 1. We give an example with a Lyapunov quasi-stable orbit which is not Zhukovskiĭ quasi-stable. Let $X=\{(x,y)\in\mathbb{R}^2 : y>0\}$ with a dynamical system (X,π) defined by the differential equation

$$\dot{x}=\frac{x^2+1}{y+1},\qquad \dot{y}=0. \tag{3.1}$$

Let

$$M=\left\{(x,y)\in\mathbb{R}^2 : x>0,\, y=\frac{\pi}{2\arctan x}-1\right\},\qquad N=\{(0,y)\in\mathbb{R}^2 : y>0\}.$$

For $\Omega=X\setminus M$, we define $I: M\to N\subset\Omega$ by $I(x,y)=(0,y/2)\in N$ for $(x,y)\in M$. Thus, we get an impulsive semidynamical system $(\Omega,\pi,\mathbb{R}^+;M,I)$. It is easy to see that $N=\hat{N}$, i.e., each trajectory through a point in N is infinite. We assert that the trajectory $\tilde{\pi}_p$ through $p=(0,1)$ is Lyapunov quasi-stable, but not Zhukovskiĭ quasi-stable. By a simple computation, it is easy to see $\phi|_N=\pi/2$, i.e., for each point in N it takes the time $\pi/2$ to reach M. So, all the points in N go isochronously to the impulsive set M, and simultaneously jump back to N. Given an $\eta>0$, since $\pi/2-(y+1)\arctan x<\pi/2-\arctan x$ for $y>0$, there exists a $K=K(\eta)$ such that for $x\geq K$, $\pi/2-(y+1)\arctan x<\eta$. Clearly, there exists a positive integer n_0 such that for $n\geq n_0$, each p_n^+ reaches M at $p_{n+1}=(x_{n+1},2^{-n})$ with $x_{n+1}>K$. Note that the impulsive times of $\tilde{\pi}_p$ are $\{\pi i/2 : i\geq 1\}$. For large t, $|t-t^i|>\eta$ implies that $p*t$ lies in the region $\{(x,y)\in\Omega : 0\leq x\leq K\}$, where t^i is an impulsive time of $\tilde{\pi}_p$. Thus, by the QCD property of $\tilde{\pi}$, it is easy to see that $\tilde{\pi}_p$ is Lyapunov quasi-stable.

Now, let $q=(0,1+\delta)$ be a point close to p; then $q_{n+1}=(x'_{n+1}, 2^{-n}(1+\delta))$. It is not difficult to see that

$$\begin{aligned} x_{n+1}-x'_{n+1} &= \tan\frac{\pi}{2}\frac{2^n}{2^n+1}-\tan\frac{\pi}{2}\frac{2^n}{2^n+1+\delta}\\ &= \tan\frac{\pi}{2}\frac{2^n\delta}{(2^n+1)(1+\delta+2^n)}\times\left(1+\tan\frac{\pi}{2}\frac{2^n}{2^n+1}\times\tan\frac{\pi}{2}\frac{2^n}{2^n+1+\delta}\right)\\ &\sim\frac{2\delta}{\pi(1+\delta)}\times 2^n\to+\infty\qquad(n\to+\infty). \end{aligned}$$

So, we have $d(p_{n+1},q_{n+1})\to+\infty$ as $n\to+\infty$. It follows that $\tilde{\pi}_p$ is not Zhukovskiĭ quasi-stable.

Example 2. Consider the planar differential system

$$\dot{x} = 0, \quad \dot{y} = -x, \tag{3.2}$$

which defines a dynamical system in $X = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Let $M = \{(x, 0) : x > 0\}$ and $N = \{(x, 1) : x > 0\}$; then define $I : M \to \Omega$ by $I(x, 0) = (x/2, 1) \in N$ for $(x, 0) \in M$, where $\Omega = X \setminus M$. Thus, we get an impulsive semidynamical system $(\Omega, \pi, \mathbb{R}^+; M, I)$. Clearly, each trajectory through a point in $N = \hat{N}$ is infinite. Let $p = (1, 1)$. It is easy to see that $\tilde{\pi}_p$ is Zhukovskiĭ quasi-stable, since a time reparametrization τ_q (q close to p) can be obtained so that τ_q maps the impulsive times of $\tilde{\pi}_p$ to the impulsive times of $\tilde{\pi}_q$.

Now, we prove that $\tilde{\pi}_p$ is not Lyapunov quasi-stable. By a simple computation, the impulsive times of $\tilde{\pi}_p$ are $t^n = \sum_{i=0}^{n} 2^i$ ($n = 0, 1, 2, \dots$). Similarly, for a point $q = (x, 1)$ close to p, the infinite trajectory $\tilde{\pi}_q$ has impulsive times $\tau^n = \sum_{i=0}^{n} 2^i/x$ ($n = 0, 1, 2, \dots$). Thus, although $|x - 1|$ is small, $t^n - \tau^n = (2^{n+1} - 1)(x - 1)/x$ is unbounded as $n \to +\infty$. That is, the difference of the nth impulsive times for those two trajectories is unbounded. Hence, an almost isochronous correspondence of orbits is impossible, i.e., $\tilde{\pi}_p$ is not Lyapunov quasi-stable.

According to the above examples, in order to get the quasi-stabilities, it is reasonable to assume that the impulsive function I is nonexpansive and the flow π restricted on N is nonexpansive. Further, for Lyapunov quasi-stability, the difference of impulsive times should be bounded for close orbits. Therefore, we introduce the following hypotheses.

- (H_1) *For any points p, q in M we have $d(p^+, q^+) \le \lambda_1 d(p, q)$, where λ_1 is a positive constant.*
- (H_2) *For any points p, q in N we have $d(p_1, q_1) \le \lambda_2 d(p, q)$, where λ_2 is a positive constant.*
- (H_3) *For any points p, q in N we have $|\phi(p_1^+) - \phi(q_1^+)| \le \lambda_3 |\phi(p) - \phi(q)|$, where $\lambda_3 \in (0, 1)$ is a constant.*

Theorem 3.4. *Assume that H_1 and H_2 hold with $\lambda_1 \lambda_2 \le 1$. If $\hat{N}$ is compact, then for each $x \in \operatorname{Int} \hat{N}$ the infinite trajectory $\tilde{\pi}_x$ is Zhukovskiĭ quasi-stable.*

Proof. Since $\hat{N}$ is compact, ϕ is bounded on $\hat{N}$, say $\phi(x) \in [m^-, m^+]$ for all $x \in \hat{N}$, where m^- and m^+ are positive real numbers. Fix $\epsilon > 0$. Notice that π is uniformly continuous on $\hat{N} \times [0, m^+]$; it follows that there exists a positive number $\eta = \eta(\epsilon)$ such that for any p, q in $\hat{N}$ and $t_1, t_2 \in [0, m^+]$, if $\max\{d(p, q), |t_1 - t_2|\} < \eta$, then $d(p \cdot t_1, q \cdot t_2) < \epsilon$. Clearly, ϕ is also uniformly continuous on $\hat{N}$: there is a $\delta \in (0, \eta)$ such that for p, q in $\hat{N}$, if $d(p, q) < \delta$, then $|\phi(p) - \phi(q)| < \eta$.

Now, for each $x \in \operatorname{Int} \hat{N}$, if $y \in \hat{N}$ and $d(x,y) < \delta$, then $|\phi(x)-\phi(y)| < \eta$. By (H_1), (H_2) and $\lambda_1\lambda_2 \leq 1$, we obtain $d(x_1^+, y_1^+) < \delta$. Thus, $|\phi(x_1^+) - \phi(y_1^+)| < \eta$. By induction, we have $d(x_i^+, y_i^+) < \delta$ and $|\phi(x_i^+) - \phi(y_i^+)| < \eta$ for $i = 1, 2, \ldots$. Let $\{t_i : i = 0, 1, \ldots\}$ and $\{\tau_i : i = 0, 1, \ldots\}$ be the impulsive intervals of $\tilde{\pi}_x$ and $\tilde{\pi}_y$, respectively. For $n \geq 0$, we denote their impulsive times by $t^n = \sum_{i=0}^n t_i$ and $\tau^n = \sum_{i=0}^n \tau_i$. A time reparametrization $\tau_y : [0, +\infty) \to [0, +\infty)$ can be defined as follows. First, let $\tau_y(0) = 0$ and for $n \geq 0$, let $\tau_y(t^n) = \tau^n$. Next, for $n \geq 0$ and $t \in (t^{n-1}, t^n)$, define

$$\tau_y(t) = \tau^{n-1} + \frac{\tau_n}{t_n}(t - t^{n-1}),$$

where $t^{-1} = \tau^{-1} = 0$. Clearly, τ_y is a homeomorphism from $[0, +\infty)$ to $[0, +\infty)$ with $\tau_y(0) = 0$, i.e., it is a time reparametrization.

Now, we assert that $d(x * t, y * \tau_y(t)) < \epsilon$ for $t \geq 0$. In fact, if $t \in [t^{n-1}, t^n]$ $(n \geq 1)$, we have

$$d(x * t, y * \tau_y(t)) = d\left(x_n^+ \cdot (t - t^{n-1}), y_n^+ \cdot \frac{\tau_n}{t_n}(t - t^{n-1})\right).$$

Since $d(x_n^+, y_n^+) < \delta$ and

$$\left|t - t^{n-1} - \frac{\tau_n}{t_n}(t - t^{n-1})\right| \leq |t_n - \tau_n| = |\phi(x_n^+) - \phi(y_n^+)| < \eta,$$

we have $d(x * t, y * \tau_y(t)) < \epsilon$. Hence, $\tilde{\pi}_x$ is Zhukovskiĭ quasi-stable. ∎

THEOREM 3.5. *Assume that (H_1)–(H_3) are all true, and $\lambda_1\lambda_2 \leq 1$. If $\hat{N}$ is compact, then for each $x \in \operatorname{Int} \hat{N}$ the infinite trajectory $\tilde{\pi}_x$ is Lyapunov quasi-stable.*

Proof. Let $m^+ = \max\{\phi(x) : x \in \hat{N}\}$. Let $\epsilon > 0$ and $\eta > 0$ be given. Since π is uniformly continuous on $\hat{N} \times [0, m^+]$, there exists a positive number δ_1 such that for any p, q in $\hat{N}$ and t_1, $t_2 \in [0, m^+]$, if $\max\{d(p,q), |t_1 - t_2|\} < \delta_1$, then $d(p \cdot t_1, q \cdot t_2) < \epsilon$. Now, for each $x \in \operatorname{Int} \hat{N}$, there is a $\delta \in (0, \delta_1)$ such that if $y \in \hat{N}$ and $d(x,y) < \delta$, then $|\phi(x) - \phi(y)| < (1 - \lambda_3)\min\{\eta, \delta_1\}$. By induction, it follows from (H_1) and (H_2) that $d(x_i^+, y_i^+) < \delta$ for $i \geq 1$. Let $\{t_i : i = 0, 1, \ldots\}$ and $\{\tau_i : i = 0, 1, \ldots\}$ be the impulsive intervals of $\tilde{\pi}_x$ and $\tilde{\pi}_y$, respectively. For $n \geq 0$, we denote $t^n = \sum_{i=0}^n t_i$ and $\tau^n = \sum_{i=0}^n \tau_i$, which are impulsive times of $\tilde{\pi}_x$ and $\tilde{\pi}_y$. Then, by (H_3), we have

$$\begin{aligned}|t^n - \tau^n| &= \left|\sum_{i=0}^n (t_i - \tau_i)\right| \leq \sum_{i=0}^n \lambda_3^i |\phi(x) - \phi(y)| \leq \frac{1}{1 - \lambda_3}|\phi(x) - \phi(y)| \\ &< \min\{\eta, \delta_1\}.\end{aligned}$$

Thus, for each $t \in \mathbb{R}^+$, if $|t - t^n| > \eta$, i.e., $t \in [t^{k-1} + \eta, t^k - \eta]$ for some $k \geq 0$, then $t \in (\tau^{k-1}, \tau^k)$.

Now, we assert that $d(x*t, y*t) < \epsilon$ for $|t - t^n| > \eta$. In fact, if $t \in [t^{k-1} + \eta, t^k - \eta]$ for some $k \geq 0$, then $x * t = x_k^+ \cdot (t - t^{k-1})$, and also $y*t = y_k^+ \cdot (t - \tau^{k-1})$. Since $d(x_k^+, y_k^+) < \delta < \delta_1$ and $|t - t^{k-1} - (t - \tau^{k-1})| \leq |t^{k-1} - \tau^{k-1}| < \delta_1$, it follows that

$$d(x*t, y*t) = d(x_k^+ \cdot (t - t^{k-1}), y_k^+ \cdot (t - \tau^{k-1})) < \epsilon.$$

Hence, $\tilde{\pi}_x$ is Lyapunov quasi-stable. ∎

4. Limit sets. In this section, our goal is to investigate the structure of the limit sets for infinite trajectories. Let $x \in \Omega$, and suppose $\tilde{\pi}_x$ is an infinite trajectory. We prove that the limit set $\tilde{\omega}(x)$ of an asymptotically Lyapunov quasi-stable orbit $x * \mathbb{R}^+$ consists of rest points. Further, if Ω is locally compact, then the limit set of a uniformly asymptotically Zhukovskiĭ quasi-stable orbit (see Definition 4.3) is a rest point or a periodic orbit.

THEOREM 4.1. *If a trajectory $\tilde{\pi}_x$ $(x \in \Omega)$ is asymptotically Lyapunov quasi-stable and its omega limit set $\tilde{\omega}(x)$ is nonempty, then $\tilde{\omega}(x)$ consists of rest points.*

Proof. Let $q \in \tilde{\omega}(x)$. For $q \in \Omega$, we have $\phi(q) > 0$, so denote $p = q \cdot (2s)$, where $s > 0$ and $4s < \phi(q)$. Clearly, q is a rest point if and only if for each small positive s, p is a rest point. Since the boundary of Ω lies in M, analogously to the proof of Lemma 2.6 in [17] it is easy to see that $\tilde{\omega}(x)$ is positively invariant, and it follows that $p \in \tilde{\omega}(x)$. Thus, there exists a sequence $\{t_n\}$ in $\mathbb{R}^+$, $t_n \to +\infty$, such that $x * t_n \to p$. Since $\tilde{\pi}_x$ is asymptotically Lyapunov quasi-stable, there exists a $\delta > 0$ such that if $d(x, y) < \delta$, then for $t \notin \bigcup_{k=0}^{+\infty} [\tau_k - s, \tau_k + s]$, $d(x*t, y*t) \to 0$ as $t \to +\infty$, where $\{\tau_k : k = 0, 1, \dots\}$ are the impulsive times of $\tilde{\pi}_x$.

Moreover, assume that δ is so small that $B(x, \delta) \subset \Omega$. By the continuity of π, choose a $\lambda \in (0, s)$ such that for any $\tau \in [0, \lambda]$, we have $x*\tau = x\cdot\tau \in \Omega$ and $d(x, x\cdot\tau) < \delta$. Note that $x * t_n \to p$ and $\phi(p) > 2s$. From the continuity of ϕ it follows that for large n, we have $|t_n - \tau_k| \geq s$ $(k = 0, 1, \dots)$. Then $d(x * t_n, (x\cdot\tau) * t_n) \to 0$ as $t_n \to +\infty$. Letting $x_n = x * t_n$, also from the continuity of π, it follows that $x_n \cdot \tau \to p \cdot \tau$, i.e., $x * (t_n + \tau) \to p \cdot \tau$. Thus,

$$d(p, p\cdot\tau) \leq d(p, x*t_n) + d(x*t_n, x*(t_n+\tau)) + d(x*(t_n+\tau), p\cdot\tau) \to 0$$

as $t_n \to +\infty$. We have $p = p \cdot \tau$ for any $\tau \in [0, \lambda]$, which of course implies that p is a rest point (see the proof of [3, Ch. 2, Th. 2.2]); hence q is also a rest point. ∎

THEOREM 4.2. *If a trajectory $\tilde{\pi}_x$ $(x \in \Omega)$ is Zhukovskiĭ quasi-stable, then $\tilde{\omega}(x) = \tilde{J}(x)$.*

Proof. Clearly, $\tilde{\omega}(x) \subset \tilde{J}(x)$. It is sufficient to show $\tilde{J}(x) \subset \tilde{\omega}(x)$. Let $y \in \tilde{J}(x) \subset \Omega$. Then there exist a sequence $\{t_n\}_{n=1}^{\infty}$ in $\mathbb{R}^+$ and a sequence

$\{x_n\}_{n=1}^{\infty}$ in Ω such that $x_n \to x$, $t_n \to +\infty$ and $x_n * t_n \to y$. Now, given any $\epsilon > 0$, there exists a $\delta = \delta(x, \epsilon) > 0$ such that for each $p \in B(x, \delta)$, one can find a time reparametrization τ_p satisfying $d(x * \tau_p(t), p * t) < \epsilon$ for $t \geq 0$. We select a $K > 0$ such that if $n \geq K$, then $d(x, x_n) < \delta$ and $d(x_n * t_n, y) < \epsilon$. Further, it follows that $d(x * \tau_{x_n}(t), x_n * t) < \epsilon$ for $n \geq K$ and $t \geq 0$, where the time reparametrizations τ_{x_n} are defined similarly to τ_p. Thus, we obtain

$$d(x * \tau_{x_n}(t_n), y) \leq d(x * \tau_{x_n}(t_n), x_n * t_n) + d(x_n * t_n, y) < 2\epsilon \quad \text{for } n \geq K,$$

where $\tau_{x_n}(t_n) \to +\infty$ as $t_n \to +\infty$. This implies that $y \in \tilde{\omega}(x)$. ■

Definition 4.3. The trajectory $\tilde{\pi}_x$ (or orbit $x * \mathbb{R}^+$) of a point x in Ω is *uniformly asymptotically Zhukovskiĭ quasi-stable* provided that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for each $s \geq 0$ and $y \in B(x * s, \delta) \cap \Omega$, one can find a time reparametrization τ_y such that $d(x * (s + t), y * \tau_y(t)) < \epsilon$ for all $t \geq 0$, and also $d(x * (s + t), y * \tau_y(t)) \to 0$ as $t \to +\infty$.

Geometrically, the orbit $y * \mathbb{R}^+$ will stay in the tubes whose center lines are segments of $x * \mathbb{R}^+$ with different time scales, and these tubes are getting thinner and thinner as time tends to infinity.

Lemma 4.4. *If the trajectory $\tilde{\pi}_x$ of a point $x \in \Omega$ is uniformly asymptotically Zhukovskiĭ quasi-stable with $\tilde{\omega} \neq \emptyset$, then its omega limit set $\tilde{\omega}(x)$ is positively minimal.*

Proof. Otherwise, $\tilde{\omega}(x)$ has a proper closed positively invariant subset $A \subset \tilde{\omega}(x)$ with $A \neq \emptyset$. Choose $p \in \tilde{\omega}(x) \setminus A$; then $\lambda = d(p, A) > 0$. Now for a sufficiently large s, we can find $q \in A$ satisfying $d(x * s, q) < \delta$, where δ is as in Definition 4.3. Also, there exists a sequence $t_i \geq s$ such that $t_i \to +\infty$ and $x * t_i \to p$. Since A is positively invariant, so $q * \mathbb{R}^+ \subset A$. However, for large t_i we have $d(x * t_i, p) < \lambda/2$, so $d(x * t_i, q * \mathbb{R}^+) \geq d(p, A) - d(x * t_i, p) \geq \lambda/2$ for large t_i. This is a contradiction, since $d(x * s, q) < \delta$ and by Definition 4.3, $d(x * (s + t), q * \tau_q(t)) \to 0$ as $t \to +\infty$, where τ_q is a time reparametrization. Thus, $\tilde{\omega}(x)$ is positively minimal. ■

Corollary 4.5. *Assume that $\tilde{\pi}_x$ is uniformly asymptotically Zhukovskiĭ quasi-stable. If there exists a rest point p in $\tilde{\omega}(x)$, then $\tilde{\omega}(x) = \{p\}$. Also, if there is a periodic orbit γ in $\tilde{\omega}(x)$, then $\tilde{\omega}(x) = \gamma$.*

Proof. Since both a rest point and a periodic orbit are positively invariant closed sets, the results follow from Lemma 4.4. ■

For two subsets A, B in X and $\delta > 0$, A is said to be δ-*apart from* B if $d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\} \geq \delta$. From the proof of Lemma 4.4, it is easy to deduce:

Corollary 4.6. *Any nonempty positively invariant closed set A must be δ-apart from a uniformly asymptotically Zhukovskiĭ quasi-stable orbit $x * \mathbb{R}^+$ if $A \cap \tilde{\omega}(x) = \emptyset$, where δ is as in Definition* 4.3.

In order to get our next result, we recall the notion of a section in a semidynamical system, introduced by Ciesielski [7]. Consider a semidynamical system (X, π). For $x \in X$, $t \geq 0$, we define $F(x,t) = \{y : y \cdot t = x\}$ and for $A \subset X$, $J \subset \mathbb{R}^+$, let $F(A,J) = \bigcup\{F(x,t) : x \in A \text{ and } t \in J\}$. A closed set S containing a point x is called a *section* or a λ-*section* through x, with $\lambda > 0$, if there exists a closed set L in X such that

(a) $F(L, \lambda) = S$;
(b) $F(L, [0, 2\lambda])$ is a neighborhood of x;
(c) $F(L, \mu) \cap F(L, \nu) = \emptyset$ for $0 \leq \mu < \nu \leq 2\lambda$.

The set $F(L, [0, 2\lambda])$ is called a *tube* or a λ-*tube*, and the set L is called a *bar*. In [7], Ciesielski proved the following tubular theorem, which is a fundamental result in semidynamical systems with many applications.

Lemma 4.7. *If $x \in \Omega$ is a regular point, then there exists a section through x.*

Note that if S is a λ-section through x with a bar L and $0 \leq \mu \leq \lambda$, then S is also a μ-section with the bar $F(L, \lambda - \mu)$ (see [7, Lemma 1.9]). Further, we have the following result.

Lemma 4.8. *Let $F(L, [0, 2\lambda])$ be a tube with a bar L. Then, for each point p in the tube, there exists a unique $t = t(p) \in [0, 2\lambda]$ such that $p \cdot t(p)$ belongs to L. Further, if L is compact, the function $p \mapsto t(p)$ is continuous on the tube $F(L, [0, 2\lambda])$.*

Proof. Let $p \in F(L,t) \subset F(L, [0, 2\lambda])$ for a $t = t(p) \in [0, 2\lambda]$, i.e., $p \cdot t \in L$. The uniqueness of $t = t(p) \in [0, 2\lambda]$ follows immediately from condition (c) in the definition of section. Now, let L be compact and suppose a sequence $\{p_n\}$ in $F(L, [0, 2\lambda])$ tends to a point p in this tube. Since $\{p_n \cdot t(p_n)\}$ lies in the compact set L, we can assume that $p_n \cdot t(p_n)$ is convergent to $q \in L$. Also, for $t(p_n) \in [0, 2\lambda]$, we suppose that $t(p_n) \to \tau \in [0, 2\lambda]$. Hence, by the continuity of π, we have $q = p \cdot \tau$. From the uniqueness of $t(q)$, it follows that $t(p) = \tau$, i.e., $t(p_n) \to t(p)$. Thus, the function $p \mapsto t(p)$ is continuous on the tube. ∎

Finally, we also need the following fixed point theorem.

Lemma 4.9 ([13, p. 414]). *Let X be a Hausdorff topological space and $H : X \to X$ be continuous. If for each open covering $\{W_\alpha\}$ of X there is at least one $x \in X$ such that both x and $H(x)$ belong to a common W_α, then H has a fixed point.*

THEOREM 4.10. *Assume that Ω is locally compact. If an orbit $x * \mathbb{R}^+$ is uniformly asymptotically Zhukovskiĭ quasi-stable with $\tilde{\omega} \neq \emptyset$, then $\tilde{\omega}(x)$ is a rest point or a periodic orbit.*

Proof. Clearly, if $\tilde{\omega}(x)$ is a singleton, then it is a rest point. Assume that $\tilde{\omega}(x)$ is not a singleton; we shall show that $\tilde{\omega}(x)$ is a periodic orbit. Choose a point $q \in \tilde{\omega}(x)$; it is a regular point by Corollary 4.5. We take an $s \in (0, \phi(q)/2)$ with $q \cdot [0, 2s] \subset \Omega$, and let $p = q \cdot s \in \tilde{\omega}(x)$. Since $\tilde{\omega}(x)$ is positively minimal, $q * \mathbb{R}^+$ is a periodic orbit of $(\Omega, \tilde{\pi})$ if and only if $p * \mathbb{R}^+$ is a periodic orbit.

Now, let a sequence $\{t_i\}_{i=1}^{\infty} \subset \mathbb{R}^+$ be such that $t_i \to +\infty$ and $x * t_i \to p$. Pick δ as in Definition 4.3 such that $B(p, 2\delta) \subset \Omega$. Thus, there is a positive σ ($\sigma < \delta$) such that $\bar{B}(p, \sigma) \subset B(x * t_k, \delta)$ for some $t_k \in \{t_i\}_{i=1}^{\infty}$. From the local compactness of Ω, we may also suppose that $\bar{B}(p, \sigma)$ is compact. Since p is a regular point, by Lemma 4.7, there is a tube $F(L, [0, 2\lambda]) \subset \bar{B}(p, \sigma)$ with a section $S = F(L, \lambda)$ through p and a bar L. Since $F(L, [0, 2\lambda])$ is a neighborhood of p, there exists a $\rho > 0$ such that $B(p, 2\rho) \subset F(L, [0, 2\lambda])$. Also, note that L is compact, because it is closed. Then, from Lemma 4.8, it follows that for each $y \in F(L, [0, 2\lambda])$, the positive orbit of y reaches L in the tube at a unique $t = \psi(y)$, where the function $y \mapsto \psi(y)$ is continuous on $F(L, [0, 2\lambda])$. As $L \subset B(x * t_k, \delta)$, it follows from Definition 4.3 that for each $y \in L$ there is a $T(y) > 0$ such that $d(x * (t + t_k), y * \tau_y(t)) < \rho$ for $t \geq T(y)$, where $\tau_y(t)$ is a time reparametrization. We will show that, by using the compactness of L and quasi-continuous dependence of $\tilde{\pi}$, one can find a positive real number $T < +\infty$ such that

$$d(x * (t + t_k), y * \tau_y(t)) < \rho \quad \text{for all } y \in L \text{ and } t \geq T.$$

In fact, for $\rho > 0$, according to Definition 4.3 there exists a $\delta' > 0$ such that if $s \geq t_k$ and $y \in B(x * s, \delta') \cap \Omega$, then $d(x * (s + t), y * \tau_y(t)) < \rho$ for all $t \geq 0$, where τ_y is a time reparametrization. Since for each $y \in L$ we have $d(x * (t + t_k), y * \tau_y(t)) \to 0$, there exists an $s_y \geq t_k$ such that $d(x * s_y, y * (\tau_y(s_y - t_k))) < \delta'$. Thus, by the QCD property of $\tilde{\pi}$, there is a neighborhood V_y of y in L such that $d(x * s_y, y' * (\tau_{y'}(s_y - t_k))) < \delta'$ for $y' \in V_y$. Since L is compact, let $\{V_{y_1}, \ldots, V_{y_j}\}$ cover L. Define $T = \max\{s_{y_1}, \ldots, s_{y_j}\} < +\infty$; thus $d(x * (t + t_k), y * \tau_y(t)) < \rho$ for each $y \in L$ and $t \geq T$.

Fix a $t_l > t_k$ and $t_l - t_k \geq T$ with $d(x * t_l, p) < \rho$. Now, we define a Poincaré map $H : L \to L$ as follows. For $y \in L$, we have $d(x * (t + t_k), y * \tau_y(t)) < \rho$ for $t \geq T$, which implies

$$d(p, y * \tau_y(t_l - t_k)) \leq d(p, x * t_l) + d(x * t_l, y * \tau_y(t_l - t_k)) < \rho + \rho = 2\rho.$$

Clearly, this means that $y * \tau_y(t_l - t_k) \in B(p, 2\rho) \subset F(L, [0, 2\lambda])$, so we obtain $y * (\tau_y(t_l - t_k) + \psi(y)) \in L$ for some $\psi(y) \in [0, 2\lambda]$. We define $H(y) =$

$y * (\tau_y(t_l - t_k) + \psi(y))$. The continuity of H comes from the continuity of τ_y, ψ and the quasi-continuous dependence of $\tilde{\pi}$. It is easy to see that H may not be the first return map.

Next, if $\{W_\alpha\}$ is an open covering of L for its subspace topology from Ω, let $z = p \cdot \psi(p) \in W_\beta = L \cap U \in \{W_\alpha\}$, where U is an open set in Ω. Let $\mu > 0$ be so small that $B(z, 2\mu) \subset U$. By the continuity of π, we can choose an $r \in (0, \rho)$ such that $y \cdot \psi(y)$ lies in $L \cap B(z, \mu) \subset W_\beta$ for each $y \in B(p, r)$. Clearly, there exists a $T_1 > t_l$ such that $x * t_i \in B(p, r)$ for every $t_i \geq T_1$. Then, by Definition 4.3, we assert that $d(H^n(z), (x * t_i) * \psi(x * t_i)) < \mu$ for $n \geq K$ and some $t_i \geq T_2 \geq T_1$, where $H^n(z)$ is the nth iterate of z. Hence,

$$\begin{aligned} d(H^K(z), z) &\leq d(H^K(z), (x * t_m) * \psi(x * t_m)) + d((x * t_m) * \psi(x * t_m), z) \\ &< \mu + \mu = 2\mu \end{aligned}$$

for some $t_m \geq T_2$, and similarly

$$\begin{aligned} d(H^{K+1}(z), z) &\leq d(H^{K+1}(z), (x * t_n) * \psi(x * t_n)) + d((x * t_n) * \psi(x * t_n), z) \\ &< \mu + \mu = 2\mu \end{aligned}$$

for some $t_n \geq T_2$. It follows that both $H^{K+1}(z)$ and $H^K(z)$ lie in $B(z, 2\mu)$. So, $H(H^K(z))$ and $H^K(z)$ belong to W_β. By Lemma 4.9, $H : L \to L$ has a fixed point w in L. Obviously, $w * \mathbb{R}^+$ is a periodic orbit. Then, from Corollaries 4.5 and 4.6, we immediately obtain $\tilde{\omega}(p) = w * \mathbb{R}^+$, which is just $\tilde{\omega}(x)$. ■

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