FUNDAMENTA
MATHEMATICAE
184 (2004)

The equation $[B,(A-1)(A,B)]=0$ and virtual knots and links

by

Stephen Budden and **Roger Fenn** (Brighton)

Abstract. Let A, B be invertible, non-commuting elements of a ring R. Suppose that $A-1$ is also invertible and that the equation $[B,(A-1)(A,B)]=0$ called the fundamental equation is satisfied. Then this defines a representation of the algebra $\mathcal{F}=\{A,B \mid [B,(A-1)(A,B)]=0\}$. An invariant R-module can then be defined for any diagram of a (virtual) knot or link. This halves the number of previously known relations and allows us to give a complete solution in the case when R is the quaternions.

1. Introduction. In this paper (Section 2), we show that the conditions given in [BF] for a 2×2 matrix to be a linear switch can be reduced to one equation. This leads to the algebra $\mathcal{F}$ with two generators A,B and one relation given by

$$A^{-1}B^{-1}AB-BA^{-1}B^{-1}A-B^{-1}AB+A=0.$$

An alternative relation is

$$BA^{-1}B^{-1}AB-B^2A^{-1}B^{-1}A=AB-BA.$$

All possible representations of this algebra in the quaternions are given. This is a considerable advance on earlier methods of finding quaternionic representations by computer search.

This paper is organised as follows. In the next section we describe the algebraic condition and how it can be simplified. In Section 3 the application to virtual knots and links is briefly reviewed. In Section 4 the quaternion case is completely described in the sense that exact conditions are given for a pair of quaternions, A,B, to satisfy the fundamental equation. In Section 5 the problem of classifying pairs of quaternions A,B which satisfy the fundamental equation and give rise to the same invariants is considered. A sufficient condition is given in terms of the geometry of the pair of quaternions. In

2000 *Mathematics Subject Classification*: 57M25, 57M27.

Section 5 a nod is given to future work on virtual strings, also known as flat virtuals, with an indication as to why the Weyl algebra is needed.

In a further paper by the second named author this work is extended to generalized quaternions, in particular 2×2 matrices [F2].

We would like to thank Hugh Morton and Daan Krammer for helpful conversations.

2. The algebraic construction. We consider a non-commutative, associative ring R. The two commutators, $[X, Y]$ and (X, Y), are defined for suitable X, Y in R by

$$[X, Y] = XY - YX, \qquad (X, Y) = X^{-1}Y^{-1}XY.$$

Let A, B be invertible, non-commuting elements of a ring R. Suppose that $A - 1$ is also invertible. Assume that the equation

$$[B, (A - 1)(A, B)] = 0,$$

called the *fundamental equation*, is satisfied. This can be rewritten as

$$\Theta = A^{-1}B^{-1}AB - BA^{-1}B^{-1}A - B^{-1}AB + A = 0.$$

The universal object satisfying these conditions is the algebra $\mathcal{F}$. Define C, D by

$$C = A^{-1}B^{-1}A - A^{-1}B^{-1}A^2 = A^{-1}B^{-1}A(1 - A), \quad D = 1 - A^{-1}B^{-1}AB.$$

LEMMA 2.1. *With the above conditions and notations the following equations are satisfied*:

$$\left.\begin{array}{ll} 1.\ A = A^2 + BAC, & 2.\ [B, A] = BAD, \\ 3.\ [C, D] = CDA, & 4.\ D = D^2 + CDB, \\ 5.\ [A, C] = DAC, & 6.\ [D, B] = ADB, \\ 7.\ [C, B] = ADA - DAD. & \end{array}\right.$$

Moreover $C,\ D - 1$ *are invertible.*

Proof. Equations 1 and 2 are just rewritings of the defining equations, and 5 is an easy consequence. The left hand side minus the right hand side of equation 4 is $A^{-1}B^{-1}A\Theta B$. The same difference for equation 6 is ΘB and for equation 7 is $\Theta(A - 1)$. The fundamental equation can be written as

$$1 - BA^{-1}B^{-1}A = 1 - A^{-1}B^{-1}AB + B^{-1}AB - A = (1 - A)(1 - A^{-1}B^{-1}AB).$$

Since $A,\ B,\ A - 1$ are invertible this can be written as

$$\begin{aligned} &A^{-1}B^{-1}A(1 - BA^{-1}B^{-1}A)(1 - A) \\ &\qquad = A^{-1}B^{-1}A(1 - A)(1 - A^{-1}B^{-1}AB)(1 - A). \end{aligned}$$

Equation 3 can be written as

$$DC = CD(1 - A),$$

which is the above. Clearly $C, D - 1$ are invertible. ■

Define the 2×2 matrix S by

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and the 3×3 matrices

$$S \times \mathrm{id} = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{id} \times S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}.$$

LEMMA 2.2. *The matrix S is invertible and*

$$(S \times \mathrm{id})(\mathrm{id} \times S)(S \times \mathrm{id}) = (\mathrm{id} \times S)(S \times \mathrm{id})(\mathrm{id} \times S).$$

Proof. In order for the matrix S to be invertible it is sufficient that $\Delta = C^{-1}D - A^{-1}B$ is a unit (see [BF]). A calculation shows that $(1 - A)\Delta$ is $A^{-1}B(A-1)$ which is a unit. The above equation (called the *Yang–Baxter equation* in [BF]) follows from the seven equations of Lemma 1.1. ■

These conditions on S mean that S is a *linear switch* or just *switch* for short, in the sense of [BF].

3. Applications to virtual knots and links. In this section we consider virtual links. For more details see [K, FJK]. A diagram of a classical knot or link can be described by the Gauss code. However not all Gauss codes can be realised as *classical* diagrams of knots or links. Their realization may be dependent on the introduction of *virtual crossings*. These are crossing which are neither above or below in space but just indicate that the journey of the arc intersects the journey of another arc. Virtual links are represented by diagrams with ordinary crossings as for classical knots and links together with these virtual crossings. In addition to their application as a geometric realization of the combinatorics of a Gauss code, virtual links have physical, topological and homological applications. In particular, virtual links may be taken to represent a particle in space and time which dissappears and reappears. A virtual link may be represented, up to stabilisation, by a link diagram on a surface. Finally, an element of the second homology of a rack space can be represented by a labelled virtual link (see [FJK]). Since the rack spaces form classifying spaces for classical links, the study of virtual links may give information about classical knots and links.

A *diagram* for a virtual link is a 4-regular plane graph with extra structure at its nodes representing the three types of crossings in the link. A classical crossing of either sign is represented in the diagram in the usual way. A virtual crossing is represented by two crossing arcs with a small circle placed around the crossing point. The graph also lies implicitly on a two-dimensional sphere S^2. *Semi-arcs* go from one classical crossing of the graph to another ignoring virtual crossings. This is distinct from a classical link diagram where the *arcs* go from one undercrossing to another.

Two such diagrams are *equivalent* if there is a sequence of moves of the types indicated in the figures below taking one diagram to the other. They are the generalised Reidemeister moves and are local in character.

We show the classical Reidemeister moves as part (A) of Figure 1. These classical moves are part of virtual equivalence where no changes are made to the virtual crossings. Taken by themselves, the virtual crossings behave as diagrammatic permutations. Specifically, we have the flat Reidemeister moves (B) for virtual crossings as shown in Figure 1. In Figure 1 we also illustrate

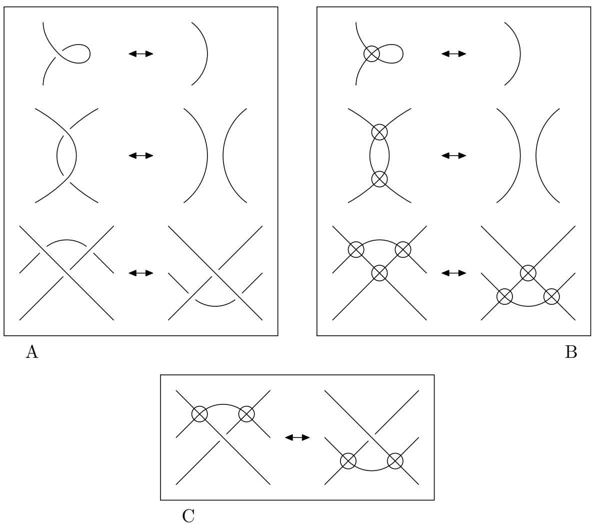


Fig. 1. Generalized Reidemeister moves for virtual knots

a basic move (C) that interrelates real and virtual crossings. In this move an arc going through a consecutive sequence of two virtual crossings can be moved across a single real crossing. In fact, it is a consequence of moves (B) and (C) for virtual crossings that an arc going through any consecutive sequence of virtual crossings can be moved anywhere in the diagram keeping the endpoints fixed and writing the places where the moved arc now crosses the diagram as new virtual crossings. This is shown schematically in Figure 2. We call the move in Figure 2 the *detour*, and note that the detour move is equivalent to having all the moves of type (B) and (C) of Figure 1. This extended move set (Reidemeister moves plus the detour move or the equivalent moves (B) and (C)) constitutes the move set for virtual knots and links.

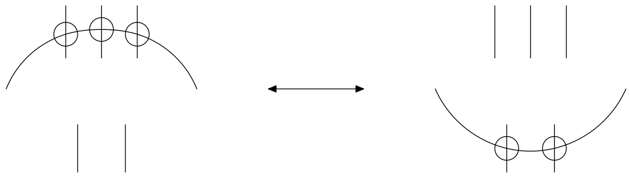


Fig. 2. The detour move

Given a switch S with entries in R, we define a *labelling* or colouring, $\mathcal{L}$, of the semi-arcs of a virtual link diagram, D, by elements of R in such a way that after a Reidemeister move converting D into D' there is a uniquely defined labelling $\mathcal{L}'$ of D' which is unchanged outside of the disturbance caused by the Reidemeister move. It follows that if D_1 and D_2 are diagrams representing the same virtual link and $D_1 \to \cdots \to D_2$ is a sequence of Reidemeister moves transforming D_1 into D_2, then any labelling $\mathcal{L}_1$ of D_1 is transferred via the sequence of Reidemeister moves to a labelling $\mathcal{L}_2$ of D_2. In particular the set of labellings of D_1 is in bijective correspondence with the set of labellings of D_2, albeit not by a uniquely defined bijection.

Let the edges of a positive real crossing in a diagram be arranged diagonally and called geographically NW, SW, NE and SE. Assume that initially the crossing is oriented and the edges oriented towards the crossing from left to right, i.e. west to east. The *input* edges, oriented towards the crossing, are in the west and the edges oriented away from the crossing, the *output* edges, are in the east. Let R be a labelling set and let a and b be labellings from R of the input edges with a labelling SW and b labelling NW. For a positive crossing, a will be the label of the undercrossing input and b the label of the overcrossing input. Suppose now that $S(a,b)^T = (c,d)^T$. Then we label the undercrossing output NE by d and we label the overcrossing output SE by c.

For a negative crossing the direction of labelling is reversed. So a labels SE, b labels NE, c labels SW and d labels NW.

Finally, for a virtual crossing the labellings carry across the strings.

Figure 3 shows the labelling for the three kind of crossings.

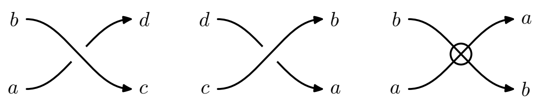


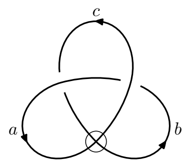
Fig. 3. Using the switch S to label the semi-arcs
$c = Aa + Bb \quad d = Ca + Db$

We shall see that a labelling defines a presentation of an R-module. Assume that the diagram contains no floating unknotted and unlinked components and has n classical crossings. Label the semi-arcs $x_1, \ldots, x_{2n}$ in some order. There will be $2n$ relations of the form $x_k = Ax_i + Bx_j$ or $x_k = Cx_i + Dx_j$. This defines a presentation of an R-module. The determinant of the presentation matrix when available gives a useful invariant. For full details see [BF].

For example let

$$S = \begin{pmatrix} 1+i & -tj \\ t^{-1}j & 1+i \end{pmatrix},$$

where i, j have their usual meanings as quaternions and t is a central variable. Consider the presentation matrix A defined by the virtual trefoil illustrated below.

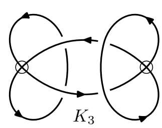


There are 4 generators for this diagram which can be reduced to two by the usual methods. Then the presentation matrix becomes

$$A = \begin{pmatrix} -t^2 + 2i & -1 + t(-j+k) + t^{-1}(j+k) \\ -1 + t(-j-k) + t^{-1}(j-k) & -t^{-1} + 2i \end{pmatrix}$$

and the Study determinant (see [As]) is $1 + 2t^2 + t^4$ up to multiplication by a unit.

This determinant could be zero for some knots, in which case we can take the gcd of the codimension 1 determinants. This happens for the Kishino knot illustrated below.



Then the gcd is $1 + (5/2)t^2 + t^4$.

4. The quaternion case. In this section we look for quaternionic solutions to the fundamental equation. We use the following notation and convention for a quaternion: if $A = \alpha + xi + yj + zk$ then $\mathbf{a} = xi + yj + zk$ so that

$$A = \alpha + \mathbf{a}.$$

Quaternion multiplication is

$$AB = \alpha\beta - \mathbf{a}\cdot\mathbf{b} + \beta\mathbf{a} + \alpha\mathbf{b} + \mathbf{a}\times\mathbf{b},$$

where $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ have their usual meanings as scalar and vector product for vectors in $\mathbb{R}^3$. Length and quaternionic conjugation are notated by

$$|A|^2 = \alpha^2 + |\mathbf{a}|^2 \quad \text{and} \quad \bar{A} = \alpha - \mathbf{a}.$$

The inverse is $A^{-1} = |A|^{-2}\bar{A}$. So, conjugation by multiplication is $B^{-1}AB = |B|^{-2}\bar{B}AB$, where

$$\bar{B}AB = \alpha(\beta^2 + |\mathbf{b}|^2) + (\beta^2 - |\mathbf{b}|^2)\mathbf{a} + 2(\mathbf{a}\cdot\mathbf{b})\mathbf{b} + 2\beta(\mathbf{a}\times\mathbf{b}).$$

The two commutators are

$$[A, B] = AB - BA = 2\mathbf{a}\times\mathbf{b},$$
$$(A, B) = A^{-1}B^{-1}AB = |A|^{-2}|B|^{-2}\bar{A}\bar{B}AB,$$

where

$$\begin{aligned}\bar{A}\bar{B}AB = {}& \alpha^2\beta^2 + \beta^2|\mathbf{a}|^2 + \alpha^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 + 2(\mathbf{a}\cdot\mathbf{b})^2 \\ & - 2(\beta(\mathbf{a}\cdot\mathbf{b}) + \alpha|\mathbf{b}|^2)\mathbf{a} + 2(\alpha(\mathbf{a}\cdot\mathbf{b}) + \beta|\mathbf{a}|^2)\mathbf{b} + 2(\alpha\beta - \mathbf{a}\cdot\mathbf{b})\mathbf{a}\times\mathbf{b}.\end{aligned}$$

From equation 2 of Lemma 2.1 we have

$$1 - A = BDB^{-1}D^{-1}.$$

So A lies on the 3-sphere $|A - 1| = 1$. The invertibility and non-commutativity condition excludes the poles 0 and 2.

The fundamental equation is

$$A^{-1}B^{-1}AB - BA^{-1}B^{-1}A = B^{-1}AB - A.$$

In terms of quaternions this is

$$\bar{A}\bar{B}AB - B\bar{A}\bar{B}A = |A|^2\bar{B}AB - |A|^2|B|^2A.$$

By the formulæ above the left hand side is

$$-4\alpha|\mathbf{b}|^2\mathbf{a} + 4\alpha(\mathbf{a}\cdot\mathbf{b})\mathbf{b} - 4(\mathbf{a}\cdot\mathbf{b})\mathbf{a}\times\mathbf{b},$$

whereas the right hand side is

$$-2(\alpha^2 + |\mathbf{a}|^2)|\mathbf{b}|^2\mathbf{a} + 2(\alpha^2 + |\mathbf{a}|^2)(\mathbf{a}\cdot\mathbf{b})\mathbf{b} + 2\beta(\alpha^2 + |\mathbf{a}|^2)\mathbf{a}\times\mathbf{b}.$$

Equating coefficients of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ gives the three equations

$$2\alpha = \alpha^2 + |\mathbf{a}|^2,$$
$$2\alpha(\mathbf{a}\cdot\mathbf{b}) = (\alpha^2 + |\mathbf{a}|^2)(\mathbf{a}\cdot\mathbf{b}),$$
$$-2(\mathbf{a}\cdot\mathbf{b}) = \beta(\alpha^2 + |\mathbf{a}|^2).$$

The first equation follows because $\mathbf{b} \neq 0$ and is a consequence of the fact that A lies on the sphere of centre 1 and radius 1. The second equation is a consequence of the first. Equations one and three imply that $A\cdot B = \alpha\beta + \mathbf{a}\cdot\mathbf{b} = 0$.

We conclude that $\mathbf{a}\cdot\mathbf{b} = -\alpha\beta$ and $\alpha^2 - 2\alpha + |\mathbf{a}|^2 = 0$. In particular A, B are perpendicular. Summing up we have

THEOREM 4.3. *Let $A = \alpha + \mathbf{a}$ and $B = \beta + \mathbf{b}$ be non-real, non-commuting quaternions. Then A, B are solutions of the fundamental equation if and only if*

$$\alpha^2 - 2\alpha + |\mathbf{a}|^2 = 0, \qquad A\cdot B = 0,$$

and B is not a multiple of $A-2$. The solutions can be given generically by six free real parameters $\mathbf{x}$, $\mathbf{y}$ *provided* $\mathbf{x}$, $\mathbf{y}$ *are not parallel.*

Proof. The first condition follows because A lies on the sphere of centre 1 and radius 1. The second condition follows because B is perpendicular to A. The only non-real quaternion to commute with A which is perpendicular to A is a multiple of $A-2$.

Inversion in the unit three-sphere turns the 3-sphere $|A-1| = 1$ into the three-plane with real part 1/2. Let $1/2+\mathbf{y} = A^{-1}$ be arbitrary on this plane. Let $B^{-1} = \mathbf{x}i+\mathbf{x}$. Then $\mathbf{x}$ and $\mathbf{y}$ define 6 real parameters with $\xi = -2\mathbf{x}\cdot\mathbf{y}$. The condition that B is not a multiple of $A-2$ reduces to $\mathbf{x}$, $\mathbf{y}$ not being parallel. ∎

5. Quaternions defining the same invariants. Let $\mathcal{S}$ denote the set of quaternion pairs $A, B \in \mathbb{H}$ satisfying the conditions of the switch algebra. Recall that A lies on the 3-sphere $|A-1| = 1$ but is not one of the poles 0, 2 and that B is perpendicular to A and does not commute with A. So A lies in a space homeomorphic to $S^2 \times \mathbb{R}$. For fixed A, B lies in the 3-dimensional vector space perpendicular to A but avoiding the 1-dimensional vector space through $A-2$. Homotopically $\mathcal{S}$ is a circle bundle over the 2-sphere. Elements of $\mathcal{S}$ can also be considered as 2×2 switch matrices S as above, since the second row entries C, D are determined by A, B, and moreover the pair C, D is in $\mathcal{S}$ and determine A, B.

We are interested in elements of $\mathcal{S}$ which determine the same knot and link invariants. Note that if

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is an element of $\mathcal{S}$ then so is

$$S(t) = \begin{pmatrix} A & tB \\ t^{-1}C & D \end{pmatrix},$$

where t is a real variable. The determinantal invariants defined in this way will all be polynomials (see [BF]). We consider a polynomial $p(t)$ "essentially" the same as $p(kt)$, where k is any non-zero real number. Call two elements of $\mathcal{S}$ *equivalent* if they give the same polynomials. We now determine sufficient conditions for two elements of $\mathcal{S}$ to be equivalent.

If A, B are in $\mathcal{S}$ then so are A, kB for any non-zero real number k. We say that B is *stretched* by k. The effect on $S(t)$ is to replace it by $S(kt)$. So stretching does not change the equivalence class.

Clearly if ψ is any automorphism or antiautomorphism of $\mathbb{H}$ then A, B is equivalent to $\psi(A), \psi(B)$. The obvious example is conjugation $X \mapsto$

$A^{-1}XA$. This fixes the 2-dimensional linear space, ϖ, spanned by 1 and A and rotates the 2-dimensional linear space, $\varpi^{\perp}$, orthogonal to ϖ through an angle 2θ, where $\alpha = |A|\cos\theta$. Equivalently, $\mathbb{R}$ is fixed and the pure quaternions $\mathbb{R}^3$ are rotated about $\mathbf{a}$ through 2θ (see [F1, p. 294]).

Given two switches, one can ask if they are obtained from each other using conjugation and stretching. Since these two operations commute, any such sequence can be written as just one conjugation and one stretch. Recall the notation $A = \alpha + \mathbf{a}$ etc.

THEOREM 5.4. *Two linear switches in $\mathcal{S}$ defined by A_i, B_i, $i = 1, 2$, are linked by conjugation and stretching precisely when $\alpha_1 = \alpha_2$ and $|\beta_1|/|\mathbf{b}_1| = |\beta_2|/|\mathbf{b}_2|$.*

Proof. Conjugation and stretching clearly implies the conditions. Conversely, suppose that the conditions are satisfied. Since C_i and D_i are functions of A_i and B_i we need only show a stretch and conjugation that takes A_1 to A_2 and B_1 to B_2.

Since $|\mathbf{a}_i|^2 = 2\alpha_i - \alpha_i^2$ it follows that $|\mathbf{a}_1| = |\mathbf{a}_2|$ and $|A_1| = |A_2|$. Hence there is some $Q \in \mathbb{H}$ such that $A_2 = Q^{-1}A_1Q$. Now let $B_1' = Q^{-1}B_1Q$. Then $\beta_1' = \beta_1$ and $|\mathbf{b}_1'| = |\mathbf{b}_1|$. Stretch by $|\mathbf{b}_2|/|\mathbf{b}_1|$. This operation leaves the A component unchanged and changes B_1' to B_2', where (using a stretch of -1 if necessary) $\beta_2' = \beta_2$ and $|\mathbf{b}_2'| = |\mathbf{b}_2|$. Let $A_2' = A_2$. Then A_2', B_2' define a switch S_2'.

We have $\mathbf{a}_2 \cdot \mathbf{b}_2' = \mathbf{a}_2 \cdot \mathbf{b}_2 = -\alpha_2\beta_2$. That is, $\mathbf{b}_2'$ and $\mathbf{b}_2$ make the same angle with $\mathbf{a}_2$. So there is a rotation using $\mathbf{a}_2$ as an axis, which takes $\mathbf{b}_2'$ to $\mathbf{b}_2$. We have shown a sequence of moves which take (A_1, B_1) to (A_2, B_2), and the proof is complete. ∎

The two conditions have a nice geometric interpretation. If S is a switch let $\varrho(S) = \alpha$ and let $\theta(S)$ be the angle between $\mathbf{a}$ and $\mathbf{b}$. So $\cos\theta = -\alpha\beta/|\mathbf{a}||\mathbf{b}|$. These are the *polar coordinates* of S. Note that $0 < \varrho < 2$ and θ is only defined in the range $0 \le \theta \le \pi$. The above theorem can now be restated as

THEOREM 5.5. *Two linear switches in $\mathcal{S}$ are linked by conjugation and stretching precisely when their polar coordinates are equal. A representative switch with polar coordinates ϱ and θ is*

$$A = \varrho + \sqrt{2\varrho - \varrho^2}\, i, \qquad B = \sqrt{2\varrho - \varrho^2} - \varrho i + \varrho \tan\theta\, j. \ \blacksquare$$

It follows, using the switch given in the above theorem, that we can define a polynomial invariant for each virtual knot or link with coefficients which are functions of ϱ and θ. However we have not calculated examples of this invariant.

Take $A = (1/2 + \mathbf{x})^{-1}$ and $B = (-2\mathbf{x} \cdot \mathbf{y} + \mathbf{y})^{-1}$. The first condition becomes $|\mathbf{x}_1| = |\mathbf{x}_2|$. The second condition states that if ϕ_i is the angle between $\mathbf{x}_i$ and $\mathbf{y}_i$ then $\phi_1 = \phi_2$.

6. Duality and Hermitian conjugation. The equations of Lemma 2.1 are invariant under the transformation

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto S^\dagger = \begin{pmatrix} D & C \\ B & A \end{pmatrix}.$$

So if $(A, B) \in \mathcal{S}$ then so is $(A, B)^\dagger = (D, C) \in \mathcal{S}$.

Lemma 6.6. *The transformation* $()^\dagger$ *is an involution on* $\mathcal{S}$. *If the underlying ring has no zero divisors then the involution is without fixed points.*

Proof. Clearly $S^{\dagger\dagger} = S$. Suppose $D = 1 - A^{-1}B^{-1}AB = A$. Then $C = A^{-1}B^{-1}A(1 - A) = A^{-1}B^{-2}AB$. If $C = B$ then $B^2 = 1$, but in a ring without zero divisors this implies that $B = \pm 1$ and so B lies in the centre. ∎

Another involution defined in the quaternion case is

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto S^* = \begin{pmatrix} \bar{A} & \bar{C} \\ \bar{B} & \bar{D} \end{pmatrix}.$$

7. Virtual strings or flat virtuals. Virtual strings (called flat virtuals by Kauffman) have been considered by Kauffman and Turaev. According to Turaev a virtual string is a chord diagram with oriented chords. Equivalently, a flat virtual knot is represented by a virtual knot diagram in the usual way but with the over and under points of the classical crossings not distinguished.

This implies that the operator S satisfies $S^2 = 1$ so on the braid level we have a representation of the symmetric group. It is not difficult to show that this implies the relation

$$B = BA^{-1} - A^{-1}B.$$

The *Weyl algebra* has two generators x, y and one relation $xy - yx = 1$. So in this case the switch algebra is a quotient of the Weyl algebra with $x = B$ and $y = B^{-1}A^{-1}$. It is in fact equal to the Weyl algebra as we shall see in the next lemma. This is important as the Weyl algebra has no non-trivial quotients (see [C, p. 363]).

Lemma 7.7. *Suppose* A, B *are invertible elements of a ring satisfying the equation*

$$B = BA^{-1} - A^{-1}B.$$

Then A, B *satisfy the fundamental equation.*

Proof. We have

$$B^{-1}AB - A^{-1}B^{-1}AB = -1$$

as can be seen by right multiplying by $B^{-1}AB$ and left multiplying by B^{-1}. If $\phi = (A-1)A^{-1}B^{-1}AB$ this implies $\phi = -1$. But the requirement of the fundamental equation is that ϕ commutes with B. ∎

We shall explore the implications of this in a future paper.

References

[As] H. Aslaksen, *Quaternionic determinants*, Math. Intel. 18 (1996), no. 3, 57–65.

[BF] A. Bartholomew and R. Fenn, *Quaternionic invariants of virtual knots and links*, submitted; http://www.maths.sussex.ac.uk/Staff/RAF/Maths/Current/Andy/.

[C] P. M. Cohn, *Algebra*, Vol. 3, Wiley, 1991.

[F] R. Fenn, *Geometry*, Springer, 2001.

[F2] —, *Generalized quaternions and knot invariants*, preprint in preparation.

[FJK] R. Fenn, M. Jordan and L. Kauffman, *The birack; an invariant of virtual knots and links*, Topology Appl., to appear.

[K] L. Kauffman, *Virtual knot theory*, European J. Combin. 20 (1999), 663–690.

[T] V. Turaev, *Virtual strings*, math.GT/0310218.

School of Mathematical Sciences
University of Sussex
Falmer, Brighton, BN1 9RH, England
E-mail: stevie_hair@yahoo.com
rogerf@sussex.ac.uk

Received 12 February 2004;
in revised form 5 September 2004