FUNDAMENTA
MATHEMATICAE
184 (2004)

Quantum invariants of periodic links and periodic 3-manifolds

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Abstract. We give criteria for framed links and 3-manifolds to be periodic of prime order. As applications we show that the Poincaré sphere is of periodicity 2, 3, 5 only and the Brieskorn sphere $\Sigma(2,3,7)$ is of periodicity 2, 3, 7 only.

1. Introduction. For any complex simple Lie algebra $\mathfrak{g}$ one can define a framed link invariant $J_L^{\mathfrak{g}}$ which is the Jones polynomial when $\mathfrak{g} = sl_2$. For the definition of $J_L^{\mathfrak{g}}$, see for example [Le1]. Using the Chern–Simons functional and the Feynman path integral Witten [W] introduced a 3-manifold invariant $\tau^{\mathfrak{g}}$ for any complex simple Lie algebra $\mathfrak{g}$. We will construct this 3-manifold invariant in Section 4 following Reshetikhin and Turaev [RT]. These invariants are called quantum invariants because they can be defined in terms of representations of the quantum group $\mathcal{U}_q(\mathfrak{g})$.

Murasugi [Mu] gave a congruence relation on $J_L^{sl_2}$ if a link L is p-periodic. A (framed) link L is *p-periodic* if the group $H = \mathbb{Z}/p\mathbb{Z}$ acts on S^3 smoothly, with fixed point set a circle, leaving L invariant. It is also assumed that L contains no fixed point. A framed link in S^3 is considered as embedded annuli here. Several authors have improved Murasugi's result in various directions [T, Y, P1, P2, C2, PSi]. A 3-manifold M is *p-periodic* if H acts on M smoothly with fixed point set a circle. If M is oriented then the action is required to be orientation preserving. We only consider 3-manifolds which are *oriented*, *connected* and *closed* in this note. For $\mathfrak{g} = sl_2$ Chbili [C1] and Gilmer [G1] gave independently a necessary condition similar to Murasugi's if M is p-periodic for a prime p. The drawback of their criterion is that it involves the quotient manifold. In [M1, M2] H. Murakami showed that if r is prime and ξ is a primitive rth root of unity then the 3-manifold quantum invariant $\tau_\xi^{sl_2}$ essentially takes values in $\mathbb{Z}[\xi]$. Masbaum and Roberts [MR] gave a simpler proof of this fact. Based on this result Gilmer, Kania-

2000 *Mathematics Subject Classification*: 57M27, 57R56.

Bartoszyńska and Przytycki gave a necessary condition for r-periodicity of integral homology spheres concerning only $\tau_\xi^{sl_2}(M)$.

We will generalize these congruence relations, for both periodic links and 3-manifolds, to all Lie algebras. The proof of the periodic manifold part is made possible by the integrality of quantum invariants of 3-manifolds [Le2].

Section 2 recalls some basics on Lie algebras. Section 3 deals with the link invariant. Sections 4 and 5 deal with the 3-manifold invariant. In Section 6 we show that the Poincaré sphere has only prime periodicity 2, 3 and 5, and the Brieskorn sphere $\Sigma(2,3,7)$ has only prime periodicity 2, 3 and 7. There we also discuss the periodicity of some other Brieskorn spheres.

2. Basics on Lie algebras. Let $\mathfrak{g}$ be a simple complex Lie algebra with Cartan matrix (a_{ij}), $i,j=1,\dots,l$. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set of basis roots $\alpha_1,\dots,\alpha_l$ in its dual space $\mathfrak{h}^*$. One can define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ in the following way. Multiply the ith row of (a_{ij}) by $d_i\in\{1,2,3\}$ such that (d_ia_{ij}) is a symmetric matrix. Set $(\alpha_i|\alpha_j)=d_ia_{ij}$. This bilinear form is proportional to the dual of the Killing form restricted to $\mathfrak{h}$. Let X and Y be the weight lattice and root lattice of $\mathfrak{g}$. The Weyl group W acts on X and Y naturally. The order of the group $G=X/Y$ is $\det(a_{ij})$.

Let X_+ be the set of dominant weights and $Y_+=Y\cap X_+$. According to the general theory of Lie algebras, the finite-dimensional simple representations of $\mathfrak{g}$ are parameterized by the dominant weights. Let $\mathcal{U}_q(\mathfrak{g})$ be the quantum group associated to $\mathfrak{g}$ as defined in [J] with the exception that our q is equal to q^2 of [J]. The finite-dimensional representations of type 1 for $\mathcal{U}_q(\mathfrak{g})$ are also parameterized by the dominant weights: for every $\lambda\in X_+$, there is a unique finite-dimensional simple $\mathcal{U}_q(\mathfrak{g})$-module Λ_λ of type 1 associated to it. Let $\mathcal{C}$ (resp. $\mathcal{C}^R$) be the set of all finite-dimensional simple $\mathcal{U}_q(\mathfrak{g})$-modules of type 1 associated to the elements in X_+ (resp. Y_+). Then $\mathcal{C}^R$ is a subset of $\mathcal{C}$.

If L is a (framed) link in S^3, a *coloring* (resp. *root-coloring*) of L is an assignment to each of its components of an element in $\mathcal{C}$ (resp. $\mathcal{C}^R$). Denote by $\mathcal{C}_L$ (resp. $\mathcal{C}_L^R$) the set of all colorings (resp. root-colorings) of L. If L is p-periodic then a coloring $c\in\mathcal{C}_L$ is said to be *p-periodic* if all link components in one orbit are assigned the same element in $\mathcal{C}$. Denote by $\mathcal{C}_L^p$ the set of p-periodic colorings. Set $\mathcal{C}_L^{R,p}=\mathcal{C}_L^R\cap\mathcal{C}_L^p$. Notice that a p-periodic coloring c induces a coloring c' on the quotient link. If a link L has color c, then the invariant is denoted as $J_L^{\mathfrak{g}}(c)$.

Several constants are used frequently in this note. Let $d=\max\{d_i\}$. Denote by D the least positive integer such that $D(\lambda|\mu)\in\mathbb{Z}$ for all weights λ and μ. The *Coxeter number* is $h=1+(\alpha_0|\varrho)$, where α_0 is the highest short root and ϱ is half of the sum of positive roots. The *dual Coxeter number* is

$h^\vee = 1 + \max_{\alpha>0}(\alpha|\varrho)/d$. For the exact values of these constants see, for example, [Le1].

3. Quantum invariants of periodic links. We fix p to be a prime integer throughout this paper. If a framed link $L \subset S^3$ is p-periodic then L has a diagram in $\mathbb{R}^2$ with blackboard framing such that the rotation by an angle $2\pi/p$ about a point away from the diagram leaves the link diagram invariant thanks to the positive answer to the Smith conjecture ([MB]). In what follows we do not distinguish a link and its diagrams if there is no confusion. Let $L' = L/H$ be the quotient link of L with respect to the action. The framing on L' is the blackboard framing of the quotient diagram. There exists an (n,n)-tangle T such that L' and L are the natural closures of T and T^p respectively. Here T^p is the tangle obtained by gluing p copies of T in a natural way. See Figure 3 where the arrow denotes the rotation generating the group H.

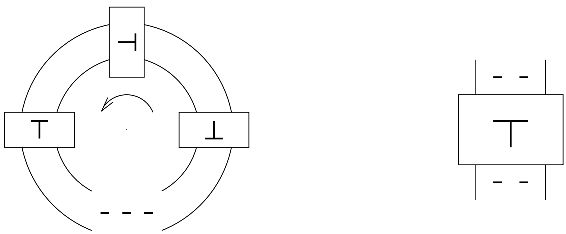

Fig. 1

Let U be the trivial knot with framing 0. The quantum dimension $\dim_q \Lambda_\lambda$ is defined to be $J^{\mathfrak{g}}_U(\Lambda_\lambda)$. Let

$$I'_{\mathfrak{g},p} = (p,\ (\dim_q \Lambda_\lambda)^p - \dim_q \Lambda_\lambda,\ \forall\lambda \in X_+)$$

be an ideal in $\mathbb{Z}[q^{\pm 1/2}]$. Note that $\dim_q \Lambda_\lambda$ is *a priori* in $\mathbb{Z}[q^{\pm 1/2D}]$, but it is actually an element in $\mathbb{Z}[q^{\pm 1/2}]$ (see the proof of Lemma 3.1). Let $I_{\mathfrak{g},p}$ be an ideal in $\mathbb{Z}[q^{\pm 1/2}]$ defined by

$$I_{\mathfrak{g},p} = \begin{cases} (p, (q^{1/2}+q^{-1/2})^p - (q^{1/2}+q^{-1/2})) = (p, (1-q^{(p-1)/2})(1-q^{(p+1)/2})) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } D \text{ is even,} \\ (p, (q+q^{-1})^p - (q+q^{-1})) = (p, (1-q^{p-1})(1-q^{p+1})) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } D \text{ is odd.} \end{cases}$$

Let

$$I_p = (p, (q+q^{-1})^p - (q+q^{-1})) = (p, (1-q^{p-1})(1-q^{p+1}))$$

be an ideal in $\mathbb{Z}[q^{\pm 1}]$. One has $I_p = I_{\mathfrak{g},p} \cap \mathbb{Z}[q^{\pm 1}]$ when D is odd. The next lemma gives the relationship between $I'_{\mathfrak{g},p}$ and $I_{\mathfrak{g},p}$.

LEMMA 3.1. $I'_{\mathfrak{g},p} \subset I_{\mathfrak{g},p}$.

Proof. By the strong integrality of the quantum link invariant (see [Le1]), $\dim_q \Lambda_\lambda = J^{\mathfrak{g}}_U(\Lambda_\lambda) \in q^{(\varrho|\lambda)}\mathbb{Z}[q^{\pm 1}]$. Since $(\varrho|\lambda)$ belongs to $\frac{1}{D}\mathbb{Z} \cap \frac{1}{2}\mathbb{Z}$, $(\varrho|\lambda)$ is an integer if D is odd. Recall that if $\overline{L}$ is the mirror image of L then $J^{\mathfrak{g}}_L$ is equal to $J^{\mathfrak{g}}_{\overline{L}}$ by substituting q with q^{-1}. Since $U = \overline{U}$ as framed links it follows that $\dim_q \Lambda_\lambda = J^{\mathfrak{g}}_U(\Lambda_\lambda)$ is in $\mathbb{Z}[q^{1/2}+q^{-1/2}]$ and is in $\mathbb{Z}[q+q^{-1}]$ when D is odd. Recall that p is prime. For any z one has

$$f^p(z) \equiv f(z) \bmod (p, z^p - z) \quad \forall f(z) \in \mathbb{Z}[z]. \ \blacksquare$$

Before we formulate the main theorem of this section let us fix some notations. Suppose that a framed link L has m components with linking matrix (l_{ij}). Let $c = (\mu_1, \ldots, \mu_m)$ be a coloring of L. Define

$$f(c,L) = \sum_{1\le i,j\le m} l_{ij}(\mu_i|\mu_j), \qquad u(c,L) = \sum_{1\le i\le m} l_{ii}(\mu_i|2\varrho),$$
$$v(c,L) = \sum_{1\le i\le m} (\mu_i|2\varrho).$$

Let

$$\mathcal{A} = \mathbb{Z}[q^{1/2}+q^{-1/2}]\cdot\mathbb{Z}[q^{\pm p/2}] \oplus p\mathbb{Z}[q^{\pm 1}],$$
$$\mathcal{B} = \mathbb{Z}[q+q^{-1}]\cdot\mathbb{Z}[q^{\pm p}] \oplus p\mathbb{Z}[q^{\pm 1}] \oplus (q^p-1)\mathbb{Z}[q^{\pm 1}].$$

A typical element in $\mathcal{A}$ is a finite sum $(\sum ab) + pc$, where $a \in \mathbb{Z}[q^{1/2} + q^{-1/2}]$, $b \in \mathbb{Z}[q^{\pm p/2}]$ and $c \in \mathbb{Z}[q^{\pm 1}]$. Elements in $\mathcal{B}$ are similarly defined. We introduce a normalization of $J^{\mathfrak{g}}_L$,

$$\hat{J}^{\mathfrak{g}}_L(c) := q^{-\frac{1}{2}(f(c,L)+u(c,L)+v(c,L))} J^{\mathfrak{g}}_L(c).$$

It is shown in [Le1] that $\hat{J}^{\mathfrak{g}}_L(c)$ does not have fractional powers and does not depend on the framing. The following theorem asserts that if a framed link L is p-periodic with quotient link L' then $\hat{J}^{\mathfrak{g}}_L$ has a special symmetry and is closely related to $\hat{J}^{\mathfrak{g}}_{L'}$.

THEOREM 3.2. *Let L be a p-periodic framed link in S^3 with quotient link L'. Their linking matrices are (l_{ij}) and (l'_{ij}). Suppose L and L' have m and m' components respectively. If p is a prime number, $c \in \mathcal{C}^p_L$ is a p-periodic coloring of L, and c' is the induced coloring on L', then*:

(a)
$$\hat{J}^{\mathfrak{g}}_L(c) \equiv q^{(pv'-v)/2}(\hat{J}^{\mathfrak{g}}_{L'}(c'))^p \bmod I'_{\mathfrak{g},p} \quad \text{in } \mathbb{Z}[q^{\pm 1/2}],$$
where $v = v(c,L)$ and $v' = v(c',L')$.

(b) *If p and D are odd then $(pv'-v)/2$ is an integer and*
$$\hat{J}^{\mathfrak{g}}_L(c) \equiv q^{(pv'-v)/2}(\hat{J}^{\mathfrak{g}}_{L'}(c'))^p \bmod I'_{\mathfrak{g},p} \cap \mathbb{Z}[q^{\pm 1}] \quad \text{in } \mathbb{Z}[q^{\pm 1}].$$

(c) *If* $(p, 2D) = 1$ *then*

$$q^{v^*} \hat{J}_L^{\mathfrak{g}}(c) \in \mathcal{A} \cap \mathcal{B},$$

where $v^* \in \mathbb{Z}$ *takes the value* $v/2$ *if* v *is even and* $(v+p)/2$ *otherwise.*

This theorem will be proved at the end of this section. The following observations are useful in applications.

Remark. Because $I'_{\mathfrak{g},p} \subset I_{\mathfrak{g},p}$ and $I_p = I_{\mathfrak{g},p} \cap \mathbb{Z}[q^{\pm 1}]$ when D is odd, statement (b) implies that for p and D odd,

$$\hat{J}_L^{\mathfrak{g}}(c) \equiv q^{(pv'-v)/2} (\hat{J}_{L'}^{\mathfrak{g}}(c'))^p \bmod I_p \quad \text{in } \mathbb{Z}[q^{\pm 1}].$$

Let ξ be a pth root of unity. Then

$$q^{2v^*} \hat{J}_L^{\mathfrak{g}}(c)|_{q=\xi} \equiv \hat{J}_L^{\mathfrak{g}}(c)|_{q=\xi^{-1}} \bmod p \quad \text{in } \mathbb{Z}[\xi]$$

if $(p, 2D) = 1$. This follows from (c).

Several authors have proved this theorem for special Lie algebras. We include some of their results in Corollary 3.3 for comparison. First we need to recall the invariants used in their papers.

A link invariant P_n can be defined by the following relations:

$$P_n(L \sqcup L') = P_n(L)\, P_n(L'),$$

$$q^{n/2} P_n(L_+) - q^{-n/2} P_n(L_-) = (q^{1/2} - q^{-1/2}) P_n(L_0),$$

where (L_0, L_-, L_+) is the standard skein triple. Notice that P_n is an invariant for non-framed links. The invariant P_n can also be defined through $\hat{J}^{\mathfrak{g}}$ as follows. Let L be a framed link whose components are colored by the fundamental representation of $U_q(sl_n)$, i.e. the one corresponding to the fundamental representation of sl_n. Denote this coloring by c_n. Let $\tilde{L}$ be any framed link obtained from L by changing the framings so that $\tilde{L} \cdot \tilde{L} = 0$, i.e. the sum of all entries of the linking matrix of $\tilde{L}$ is 0. Then $P_n(L) = J_{\tilde{L}}^{sl_n}(c_n)$. Hence

$$\hat{J}_L^{sl_n}(c_n) = q^{(e_n - u(c_n, L) - v(c_n, L))/2} P_n(L),$$

where $e_n = \sum_{1 \le i,j \le m} l_{ij} (2\varrho | \lambda_1)$. Here λ_1 is the weight corresponding to the fundamental representation. Let $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$ be the quantum integer. The Jones polynomial can be defined as

$$V_L(t) = [2]^{-1} P_2(L)|_{\sqrt{q} = -1/\sqrt{t}}.$$

Corollary 3.3. *With the same assumptions as in Theorem* 3.2 *one has*

(a) ([C2, PSi])

$$P_n(L) \equiv (P_n(L'))^p \bmod (p, [2]^p - [2]) \quad \textit{in } \mathbb{Z}[q^{\pm 1/2}]$$

and if n *is odd then*

$$P_n(L) \equiv (P_n(L'))^p \bmod (p, [3]^p - [3]) \quad \textit{in } \mathbb{Z}[q^{\pm 1/2}].$$

(b) ([Mu, Y])

$$V_L(t) \equiv (V_{L'}(t))^p \bmod (p, \eta_p(t)) \quad in\ \mathbb{Z}[t^{\pm 1/2}],$$

where $\eta_p(t) = \sum_{j=0}^{p-1}(-t)^j - t^{(p-1)/2}$.

(c) ([Y]) *If p is odd then*

$$V_L(t) - t^{2\,\mathrm{lk}} V_L(t^{-1}) \equiv 0 \bmod (p, t^p - 1) \quad in\ \mathbb{Z}[t^{\pm 1/2}],$$

where $\mathrm{lk} = \frac{1}{2}\sum_{1\le i<j\le m} l_{ij}$, *i.e.* lk *is the total linking number of L.*

Remark. The ideals $(p, [2]^p - [2])$ and $(p, [3]^p - [3])$ are used here instead of $I_{\mathfrak{g},p}\,(\supset I'_{\mathfrak{g},p})$. When $\mathfrak{g} = sl_n$ the constant D is equal to n. It follows that the ideals are actually the same. Also notice that $(t+1)\eta_p(t) \equiv q^{-p/2}([2]^p - [2])|_{\sqrt{q}=-1/\sqrt{t}} \bmod p$. One has to assume that p is odd in (c) because the Jones polynomial defined here uses $[2] = -\sqrt{t} - 1/\sqrt{t}$ whose square is 0 mod $(2, t^2 - 1)$. Yokota [Y] proved (c) when L is a knot. We skip the proof of this corollary because it follows easily from Theorem 3.2.

We begin the proof of Theorem 3.2 with the next lemma which is the $J^{\mathfrak{g}}_L$-version of Theorem 3.2. The reason we use $\hat{J}^{\mathfrak{g}}_L$ instead of $J^{\mathfrak{g}}_L$ in Theorem 3.2 is that $\hat{J}^{\mathfrak{g}}_L$ has better integrality.

Lemma 3.4. *With the same assumptions as in Theorem* 3.2 *one has*

(a)

$$J^{\mathfrak{g}}_L(c) \equiv (J^{\mathfrak{g}}_{L'}(c'))^p \bmod I''_{\mathfrak{g},p} \qquad in\ \mathbb{Z}[q^{\pm 1/2D}],$$

where $I''_{\mathfrak{g},p} = (p, (\dim_q \Lambda_\lambda)^p - \dim_q \Lambda_\lambda, \forall \lambda \in X_+)$ *is an ideal in the ring* $\mathbb{Z}[q^{\pm 1/2D}]$.

(b)

$$J^{\mathfrak{g}}_L(c) \in \mathbb{Z}[q^{1/2} + q^{-1/2}] \cdot \mathbb{Z}[q^{\pm p/2D}] \oplus p\mathbb{Z}[q^{\pm 1/2D}].$$

(c) *If $c \in \mathcal{C}^{R,p}_L$ is a p-periodic root-coloring then*

$$J^{\mathfrak{g}}_L(c) \equiv (J^{\mathfrak{g}}_{L'}(c'))^p \bmod I_p \qquad in\ \mathbb{Z}[q^{\pm 1}].$$

Proof. Let T be the (n,n)-tangle mentioned at the beginning of this section. There is a natural coloring on T by the restriction of c, which is still denoted by c. Suppose the open ends in T are colored by $V_1, \dots, V_n$. Then $J^{\mathfrak{g}}_T(c)$ is a $\mathcal{U}_q(\mathfrak{g})$-module homomorphism from $V_1 \otimes \cdots \otimes V_n$ to itself (see for example [Le1]). According to [Le1] and [Lu], $J^{\mathfrak{g}}_T(c)$ can be represented by a matrix over $\mathbb{Z}[q^{\pm 1/2D}]$ by choosing the so-called tensor product basis for $V_1 \otimes \cdots \otimes V_n$. It is known that $V_1 \otimes \cdots \otimes V_n$ can be decomposed over $\mathbb{Q}(q^{1/2D})$ into a direct sum of homogeneous parts, i.e. $V_1 \otimes \cdots \otimes V_n = \bigoplus_{\lambda \in X_+} E_\lambda$, where $E_\lambda = \bigoplus \Lambda_\lambda$. Then E_λ is isomorphic to $\Lambda_\lambda \otimes N$ as $U_q(\mathfrak{g})$-modules, where N is a finite-dimensional vector space over $\mathbb{Q}(q^{1/2D})$. It is also known that $J^{\mathfrak{g}}_T(c)$

acts on $\Lambda_\lambda \otimes N$ as $\mathrm{Id} \otimes R_\lambda$, where R_λ is a matrix over $\mathbb{Q}(q^{1/2D})$. For more details see [Le2].

If x is an eigenvalue of R_λ^j then it is also an eigenvalue of $(J_T^{\mathfrak{g}})^j$ for any positive integer j. Since $(J_T^{\mathfrak{g}})^j$ has entries in $\mathbb{Z}[q^{\pm 1/2D}]$, x must belong to O, the ring of algebraic integers over $\mathbb{Z}[q^{\pm 1/2D}]$ in some algebraic extension of $\mathbb{Q}(q^{1/2D})$. Hence $\operatorname{tr} R_\lambda^j \in O \cap \mathbb{Q}(q^{1/2D}) = \mathbb{Q}[q^{\pm 1/2D}]$. Let $K = K_{\pm 2\varrho}$ (see [Le1] for the definition of $K_{\pm 2\varrho}$). Then $\operatorname{tr} K|_{\Lambda_\lambda} = \dim_q \Lambda_\lambda$, the quantum dimension of Λ_λ, and K acts on $\Lambda_\lambda \otimes N$ as $K|_{\Lambda_\lambda} \otimes \mathrm{Id}$. So $KJ_T^{\mathfrak{g}}$ acts on $\Lambda_\lambda \otimes N$ as $K|_{\Lambda_\lambda} \otimes R_\lambda$. Therefore

$$J_{L'}^{\mathfrak{g}}(c') = \operatorname{tr}(KJ_T^{\mathfrak{g}}) = \sum_{\lambda \in X_+} \dim_q \Lambda_\lambda \operatorname{tr} R_\lambda$$

and

$$J_L^{\mathfrak{g}}(c) = \operatorname{tr}(KJ_{T^p}^{\mathfrak{g}}) = \sum_{\lambda \in X_+} \dim_q \Lambda_\lambda \operatorname{tr} R_\lambda^p$$

in $\mathbb{Z}[q^{\pm 1/2D}]$. By Lemma 3.5 below,

$$J_L^{\mathfrak{g}}(c) = \sum_{\lambda \in X_+} (\dim_q \Lambda_\lambda)(\operatorname{tr} R_\lambda)^p + pg \tag{1}$$

for some $g \in \mathbb{Z}[q^{\pm 1/2D}]$. As noticed in Lemma 3.1, $\dim_q \Lambda_\lambda$ is in $\mathbb{Z}[q^{1/2} + q^{-1/2}]$, and this proves the second statement. Now,

$$\begin{aligned}(J_{L'}^{\mathfrak{g}}(c'))^p &= \Big(\sum_{\lambda \in X_+} \dim_q \Lambda_\lambda \operatorname{tr} R_\lambda\Big)^p \equiv \sum_{\lambda \in X_+} (\dim_q \Lambda_\lambda)^p (\operatorname{tr} R_\lambda)^p \\ &\equiv \sum_{\lambda \in X_+} \dim_q \Lambda_\lambda\, (\operatorname{tr} R_\lambda)^p \equiv J_L^{\mathfrak{g}}(c) \bmod I''_{\mathfrak{g},p}.\end{aligned}$$

This proves the first statement. If c is a p-periodic root-coloring then $J_L^{\mathfrak{g}}(c)$ is in $\mathbb{Z}[q^{\pm 1}]$ by the strong integrality of the quantum link invariants ([Le1]). Then the last statement can be proved by comparing-the-terms (see the remark after Lemma 3.5). ∎

LEMMA 3.5. *Let $\mathcal{D}$ be an integrally closed domain and F be its quotient field. Suppose A is an $n \times n$ matrix over F whose eigenvalues are integral over $\mathcal{D}$. Then for any prime number p, $\operatorname{tr} A^p - (\operatorname{tr} A)^p = pg$ for some $g \in \mathcal{D}$.*

Proof. Let K be an algebraic closure of F. Denote by $K_{\mathcal{D}}$ the integral closure of $\mathcal{D}$ in K. Then $F \cap K_{\mathcal{D}} = \mathcal{D}$ since $\mathcal{D}$ is integrally closed. If $x_1, \ldots, x_n$ are the eigenvalues of A then $x_1^j, \ldots, x_n^j$ are the eigenvalues of A^j for any positive integer j. Since $x_k^j \in K_{\mathcal{D}}$ by the assumption, $\operatorname{tr} A^j = \sum_{k=1}^n x_k^j$ belongs to $K_{\mathcal{D}} \cap F = \mathcal{D}$. Then $\operatorname{tr} A^p = \sum_{k=1}^n x_k^p = (\sum_{k=1}^n x_k)^p + pg(s_1, \ldots, s_n)$, where g is an integral coefficient polynomial and s_k is the kth elementary symmetric polynomial in $x_1, \ldots, x_n$ ([L, Theorem 8.1]). Let $e_j = \sum_{k=1}^n x_k^j$.

Then the e_j's are related to the s_j's by the Newton formula ([Co])

$$e_j - s_1 e_{j-1} + s_2 e_{j-2} + \cdots + (-1)^{j-1} s_{j-1} e_1 + (-1)^j s_j j = 0,$$

where $s_k = 0$ if $k > n$. Because $e_j = \operatorname{tr} A^j \in \mathcal{D}$ one has $s_j \in F$, whence $s_j \in K_{\mathcal{D}} \cap F = \mathcal{D}$. Hence $g(s_1, \ldots, s_n) \in \mathcal{D}$ as claimed. ∎

Remark. We use a so called "comparing-the-terms" argument several times in this section. We only write down the most complicated one in detail in the proof of Theorem 3.2 below.

Proof of Theorem 3.2. Let $f = f(c, L)$, $u = u(c, L)$, $v = v(c, L)$, $f' = f(c', L')$, $u' = u(c', L')$ and $v' = v(c', L')$. Notice that f and f' are in $\frac{1}{D}\mathbb{Z}$ and the others are integers. Because f and f' are actually summations over crossings one has $f = pf'$. If the orbit of a link component of L consists of only one component then its framing must be p times the framing of its quotient component in L'. On the other hand, if an orbit consists of p link components then they must have the same framing and color. Therefore $u = pu'$. Then (a) follows from Lemma 3.4(a) and the comparing-the-terms argument.

If the orbit of a component in L consists of p components then its contribution to $(pv' - v)/2$ is zero. Otherwise the contribution is $(p - 1)/2$, which is an integer for odd p. Now (b) can be proved by using the comparing-the-terms argument again. Here one must use the fact that the generators of $I'_{\mathfrak{g},p}$ are in $\mathbb{Z}[q^{\pm 1}]$ for odd D.

By comparing-the-terms and Lemma 3.4 we can show that $q^{v^*} \hat{J}^{\mathfrak{g}}_L(c) \in \mathcal{A}$. So we only have to prove $q^{v^*} \hat{J}^{\mathfrak{g}}_L(c) \in \mathcal{B}$. The proof is again comparing-the-terms. It is clear from the definition that $p \,|\, 2(v^* - v/2)$. Define $w = 2(v^* - (f + u + v)/2)$ and $w' = w/p$. Notice that w and w' are in $\frac{1}{D}\mathbb{Z}$. By (1),

$$q^{v^*} \hat{J}^{\mathfrak{g}}_L = q^{w/2} J^{\mathfrak{g}}_L = q^{pw'/2} \Big(\sum_{\lambda \in X_+} \dim_q \Lambda_\lambda \, (\operatorname{tr} R_\lambda)^p + pg \Big) \tag{2}$$
$$= \sum_{\lambda \in X_+} \dim_q \Lambda_\lambda \, (q^{w'/2} \operatorname{tr} R_\lambda)^p + pg'.$$

We split $q^{w'/2} \operatorname{tr} R_\lambda$ and $\dim_q \Lambda_\lambda$ into integral and fractional parts as follows. Let $q^{w'/2} \operatorname{tr} R_\lambda = \alpha_\lambda + \beta_\lambda$ and $\dim_q \Lambda_\lambda = \gamma_\lambda + \delta_\lambda$, where $\alpha_\lambda \in \mathbb{Z}[q^{\pm 1}]$, $\gamma_\lambda \in \mathbb{Z}[q + q^{-1}]$, $\beta_\lambda \in \mathfrak{I}_D$ and $\delta_\lambda \in \mathfrak{I}_1 \cap \mathbb{Z}[q^{1/2} + q^{-1/2}]$, where

$$\mathfrak{I}_n = \Big\{ \sum_{\text{finite}} z_i q^i \;\Big|\; z_i \in \mathbb{Z},\, i \in \frac{1}{2n}\mathbb{Z} \setminus \mathbb{Z} \Big\} \cup \{0\}$$

for any positive integer n. Let $\alpha_\lambda = \sum a_{\lambda,i} q^i$, $\beta_\lambda = \sum b_{\lambda,j} q^j$, where $a_{\lambda,i}, b_{\lambda,j} \in \mathbb{Z}$ and $j \in \frac{1}{2D}\mathbb{Z} \setminus \mathbb{Z}$. Let $\delta_\lambda = \sum d_{\lambda,l}(q^l + q^{-l})$, where $l \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$. Then $(q^{w'/2} \operatorname{tr} R_\lambda)^p = A_\lambda + B_\lambda + pg_\lambda$, where $A_\lambda = \sum a^p_{\lambda,i} q^{ip}$, $B_\lambda = \sum b^p_{\lambda,j} q^{jp} =$

$\sum c_{\lambda,k}q^k$ and $g_\lambda \in \mathbb{Z}[q^{\pm 1/2D}]$. Because $(p, 2D) = 1$ and j is not an integer, neither is $k = jp$. Furthermore $p \,|\, 2Dk$ because $j \in \frac{1}{2D}\mathbb{Z}$. Therefore each summand on the right side of (2) becomes

$$\begin{aligned}(3)\quad \dim_q \Lambda_\lambda (q^{w'/2} \operatorname{tr} R_\lambda)^p &= (\gamma_\lambda + \delta_\lambda)(A_\lambda + B_\lambda + p\, g_\lambda)\\ &= \gamma_\lambda A_\lambda + \delta_\lambda A_\lambda + \gamma_\lambda B_\lambda + \delta_\lambda B_\lambda + p\, g_\lambda(\gamma_\lambda + \delta_\lambda).\end{aligned}$$

One has $\delta_\lambda B_\lambda = \sum_{k,l} c_{\lambda,k} d_{\lambda,l}(q^{k+l} + q^{k-l})$. Notice that $k+l \in \mathbb{Z}$ if and only if $k-l \in \mathbb{Z}$, since $2l \in \mathbb{Z}$. Moreover, if $k+l \in \mathbb{Z}$ then $2k = k+l+k-l \in \mathbb{Z}$. Hence if $k + l$ is an integer then $p \,|\, 2k$ because $p \,|\, 2Dk$ and $(p, D) = 1$. Therefore $k + l \equiv -(k - l) \bmod p$ if $k + l \in \mathbb{Z}$. Let $\delta_\lambda B_\lambda = C_\lambda + D_\lambda$, where $C_\lambda \in \mathbb{Z}[q^{\pm 1}]$ and $D_\lambda \in \mathfrak{I}_D$. Then C_λ is actually an element in $\mathbb{Z}[q + q^{-1}] \oplus (q^p - 1)\mathbb{Z}[q^{\pm 1}]$.

Using (3) one can expand (2) as follows:

$$q^{v^*} \hat{J}_L^{\mathfrak{g}} = \sum_\lambda (\gamma_\lambda A_\lambda + C_\lambda) + \sum_\lambda (D_\lambda + \delta_\lambda A_\lambda + \gamma_\lambda B_\lambda) + p \sum_\lambda (G_\lambda + G'_\lambda),$$

where $g_\lambda = G_\lambda + G'_\lambda$ and $G_\lambda \in \mathbb{Z}[q^{\pm 1}]$, $G'_\lambda \in \mathfrak{I}_D$. Notice that $\delta_\lambda A_\lambda, \gamma_\lambda B_\lambda$ and D_λ are in $\mathfrak{I}_D$. Therefore $\sum_\lambda (D_\lambda + \delta_\lambda A_\lambda + \gamma_\lambda B_\lambda + pG'_\lambda) = 0$ since $q^{w/2} J_L^{\mathfrak{g}} \in \mathbb{Z}[q^{\pm 1}]$ by the strong integrality of the quantum link invariants ([Le1]). This completes the proof. ∎

4. Quantum invariants of periodic 3-manifolds I. Let M be a connected, oriented and closed 3-manifold. Recall that for any Lie algebra $\mathfrak{g}$ and some integer $r > 1$ one can define a 3-manifold invariant $\tau_\xi^{\mathfrak{g}}(M)$, where ξ is a primitive rth root of unity. Kirby and Melvin [KM] proved that if the Lie algebra is sl_2 and r is odd, then $\tau_\xi^{sl_2}(M)$ is a product of two invariants, namely $\tau_\xi^{sl_2}(M) = \tau_\xi^G(M)\tau_\xi^{Psl_2}(M)$. The first factor is a weak invariant which is determined by the first homology group and the linking form on its torsion. The second factor is called the projective quantum invariant and will be defined later in this section. In [Le2] the second author has generalized this splitting to all quantum groups. For a p-periodic 3-manifold M one can show that the values of $\tau_\xi^{\mathfrak{g}}(M)$ and $\tau_\xi^{P\mathfrak{g}}(M)$ are restricted. We will only formulate criteria for the projective quantum invariant since the proofs for the non-projective ones are very similar. Furthermore the projective part contains "almost all" information as noted above. Another reason is that the projective invariants enjoy nice integrality ([Le2]).

First let us recall briefly the definition of projective quantum invariants of 3-manifolds. For more details see [Le2]. Let $\mathfrak{h}_{\mathbb{R}}^*$ be the $\mathbb{R}$-vector space spanned by $\alpha_1, \ldots, \alpha_l$. Then $\mathfrak{h}_{\mathbb{R}}^* \otimes \mathbb{C} = \mathfrak{h}^*$. Fix an integer $r > dh^\vee$. Let ξ be a primitive rth root of unity. The *fundamental alcove of level* k is defined as

$$C_k = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid (x|\alpha_i) \geq 0,\, (x|\alpha_0) < k,\, i = 1, \ldots, l\},$$

where $k = r - h > 0$ and α_0 is the short highest root associated to the basis roots we choose. Let

$$P_r = \{x = c_1\alpha_1 + \cdots + c_l\alpha_l \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \le c_1, \ldots, c_l < r\}.$$

Let $\overline{C}_k$ and $\overline{P}_r$ be their closures. Recall that $\mathcal{C}^R$ is the set of all finite-dimensional simple $\mathcal{U}_q(\mathfrak{g})$-modules of type 1 with highest weight in the root lattice. Let L be a framed link in S^3 with m components. A *root-coloring* of L is an assignment, to each of its components, of an element in $\mathcal{C}^R$. Denote by $\mathcal{C}^R_L$ the set of all root-colorings of L. One can extend the colors to all weights through the Weyl group action on $\mathfrak{h}^*$ ([Le2]). Then $\overline{C}^m_k \cap \mathcal{C}^R_L$ (resp. $P^m_r \cap Y$) denotes the colorings such that the color on each link component is from the set $\overline{C}_k$ (resp. P_r). Define

$$F^{P\mathfrak{g}}_L(\xi) = \sum_{c \in \overline{C}^m_k \cap \mathcal{C}^R_L} Q^{\mathfrak{g}}_L(c)|_{q=\xi},$$

where $Q^{\mathfrak{g}}_L(c) = J^{\mathfrak{g}}_L(c) J^{\mathfrak{g}}_{U^{(m)}}(c)$ and $U^{(m)}$ is the trivial link with m components. Let W be the Weyl group of $\mathfrak{g}$. By the first symmetry principle ([Le2, Le1]), if $r \ge dh^\vee$ then

$$F^{P\mathfrak{g}}_L(\xi) = \left(\frac{1}{|W|}\right)^m \sum_{c \in P^m_r \cap Y} Q^{\mathfrak{g}}_L(c)|_{q=\xi}.$$

Let $U_\pm$ be the unknot with ± 1 framing. According to [Le2],

$$F^{P\mathfrak{g}}_{U_+}(\xi) = \frac{\gamma^{P\mathfrak{g}}(\xi)}{\prod_{\alpha>0}(1 - \xi^{(\alpha|\varrho)})}, \tag{4}$$

where $\gamma^{P\mathfrak{g}}(\xi) = \sum_{\mu \in P_r \cap Y} \xi^{(|\mu+\varrho|^2 - |\varrho|^2)/2}$ is a Gauss sum.

We need one more lemma about the Gauss sum to understand $F^{P\mathfrak{g}}_{U_+}$. First let us recall some terminology. Let $\mathcal{G}$ be a finite abelian group. A *quadratic form* z on $\mathcal{G}$ is a function $z : \mathcal{G} \to \mathbb{Q}/\mathbb{Z}$ satisfying the following conditions:

- $z(ng) = n^2 z(g), \forall g \in \mathcal{G}$ and $n \in \mathbb{Z}$.
- $b_z(g, g') := z(g + g') - z(g) - z(g')$ is a symmetric bilinear form on $\mathcal{G}$.

Denote by $\operatorname{ad} b_z : \mathcal{G} \to \operatorname{Hom}_{\mathbb{Z}}(\mathcal{G}, \mathbb{Q}/\mathbb{Z})$ the adjoint representation of b_z.

Lemma 4.1 ([D]).

$$\sum_{g \in \mathcal{G}} e^{2\pi i z(g)} = \sqrt{|\ker \operatorname{ad} b_z|\, |\mathcal{G}|}\, \varepsilon,$$

where ε is 0 or an eighth root of unity.

Lemma 4.2. *If $r > dh^\vee$ and $F^{P\mathfrak{g}}_{U_\pm}(\xi) \ne 0$ then $1/F^{P\mathfrak{g}}_{U_+}(\xi) \in \mathbb{Z}[\xi, 1/r, \varepsilon]$ and*

$$\frac{F^{P\mathfrak{g}}_{U_+}(\xi)}{F^{P\mathfrak{g}}_{U_-}(\xi)} = \xi^{-((r+1)^2+2)|\varrho|^2} \omega,$$

where ε is an eighth root of unity and ω is a fourth root of unity.

Proof. If $\mu \in Y$ then $(\mu|\varrho) \in \mathbb{Z}$ and

$$|\mu+\varrho|^2 - |\varrho|^2 \equiv |\mu+(r+1)\varrho|^2 - (r+1)^2|\varrho|^2 \bmod 2r.$$

Hence

$$\gamma^{P\mathfrak{g}}(\xi) = \sum_{\mu\in P_r\cap Y} \xi^{(|\mu+(r+1)\varrho|^2-(r+1)^2|\varrho|^2)/2} = \xi^{-(r+1)^2|\varrho|^2/2} \sum_{\mu\in P_r\cap Y} \xi^{|\mu+(r+1)\varrho|^2/2}.$$

Let $\gamma' = \sum_{\mu\in P_r\cap Y} \xi^{|\mu|^2/2}$ and $\gamma'_\lambda = \sum_{\mu\in P_r\cap Y} \xi^{|\mu+\lambda|^2/2}$. It is easy to see that $\gamma' = \gamma'_\lambda$ for any $\lambda \in Y$. Therefore $\gamma^{P\mathfrak{g}}(\xi) = \xi^{-(r+1)^2|\varrho|^2/2}\gamma' \in \mathbb{Z}[\xi]$. Notice that $P_r \cap Y$ can be identified canonically with the abelian group Y/rY with r^l elements, where l is the dimension of $\mathfrak{h}$. Let $z : P_r \cap Y \to \mathbb{Q}/\mathbb{Z}$ be defined by $z(\mu) \equiv |\mu|^2/2r \bmod \mathbb{Z}$. Then $e^{2\pi i z(\mu)} = \xi^{|\mu|^2/2}$. We have

$$\frac{|\mu+r\alpha_i|^2}{2r} - \frac{|\mu|^2}{2r} = \frac{1}{2r}\left(2(\mu|r\alpha_i) - r^2(\alpha_i|\alpha_i)\right) \in \mathbb{Z},$$

where α_i is a basis root. Hence

$$z(n\mu) \equiv \frac{|n\mu|^2}{2r} = n^2\,\frac{|\mu|^2}{2r} \equiv n^2 z(\mu) \bmod \mathbb{Z}$$

and

$$b_z(\mu,\lambda) = z(\mu+\lambda) - z(\mu) - z(\lambda) \equiv \frac{|\mu+\lambda|^2}{2r} - \frac{|\mu|^2}{2r} - \frac{|\lambda|^2}{2r} = \frac{(\mu|\lambda)}{r} \bmod \mathbb{Z}.$$

Hence z is a quadratic form on $P_r\cap Y$. Then $\ker \operatorname{ad} b_z$ is a subgroup of $P_r\cap Y$, and in particular $|\ker \operatorname{ad} b_z|$ divides $|P_r\cap Y| = r^l$. By Lemma 4.1 and (4) one has $F^{P\mathfrak{g}}_{U_+}(\xi) = \sqrt{s}\,\varepsilon/f(\xi)$ for some integer s, some eighth root of unity ε and some $f(\xi) \in \mathbb{Z}[\xi]$. Here s divides r^k for some k. On the other hand, $F^{P\mathfrak{g}}_{U_+}(\xi)$ is in $\mathbb{Z}[\xi]$ by definition so $\sqrt{s}$ belongs to $\mathbb{Z}[\xi,\varepsilon]$. This proves the first statement. For the second statement one notices that $F^{P\mathfrak{g}}_{U_+}(\xi) = \overline{F^{P\mathfrak{g}}_{U_-}(\xi)}$, where the bar stands for complex conjugation. ∎

If $F^{P\mathfrak{g}}_{U_\pm}(\xi) = 0$, we define $\tau^{P\mathfrak{g}}_\xi(M) = 0$, otherwise let

$$\tau^{P\mathfrak{g}}_\xi(M) := \frac{F^{P\mathfrak{g}}_L(\xi)}{(F^{P\mathfrak{g}}_{U_+})^{\sigma_{L+}}(F^{P\mathfrak{g}}_{U_-})^{\sigma_{L-}}},$$

where M is obtained by surgery along the framed link L, and σ_{L+} (resp. σ_{L-}) is the number of positive (resp. negative) eigenvalues of the linking matrix of L. One can show that $\tau^{P\mathfrak{g}}_r$ is a 3-manifold invariant and is called the *projective quantum invariant*.

A framed link L is called *strongly p-periodic* if L is p-periodic and there is no component of L invariant under the $\mathbb{Z}/p\mathbb{Z}$ action.

The following theorem has been proved recently by Przytycki and Sokolov.

Theorem 4.3 ([PSo]). *Let p be a prime integer and M be a closed, oriented and connected 3-manifold. Then M is p-periodic iff there exists a strongly p-periodic framed link $L \subset S^3$ such that M is the result of surgery on L.*

Denote by $R^\xi_{\mathfrak{g}}$ the ring $\mathbb{Z}[\xi, 1/r, \varepsilon]$, where ε is as in Lemma 4.2. Let $I^\xi_{\mathfrak{g},p}$ be the ideal $(p, (\xi+\xi^{-1})^p - (\xi+\xi^{-1}))$ in $R^\xi_{\mathfrak{g}}$.

Theorem 4.4. *Let M be a p-periodic rational homology 3-sphere with quotient manifold M', where p is a prime number. Let $r > dh^\vee$ be an odd integer and ξ be a primitive rth root of unity. Then*

$$\tau^{P\mathfrak{g}}_\xi(M) \equiv (-\xi)^u (\tau^{P\mathfrak{g}}_\xi(M'))^p \bmod I^\xi_{\mathfrak{g},p} \quad \text{in } R^\xi_{\mathfrak{g}}$$

for some integer u if p is not a factor of $r|W|$. (W is the Weyl group.)

Remark. Theorem 4.4 is proved for $\mathfrak{g} = sl_2$ by Chbili [C1] and Gilmer [G1].

Proof of Theorem 4.4. If $I^\xi_{\mathfrak{g},p} = R^\xi_{\mathfrak{g}}$ or $F^{P\mathfrak{g}}_{U_+} = 0$ then the theorem holds trivially. (For example, if r is prime not equal to p and not a factor of p^2-1 then $I^\xi_{\mathfrak{g},p} = R^\xi_{\mathfrak{g}}$.) Now suppose otherwise; then $|W|$ is invertible in $R^\xi_{\mathfrak{g}}/I^\xi_{\mathfrak{g},p}$. By Lemma 4.2, $1/F^{P\mathfrak{g}}_{U_\pm}$ belongs to $R^\xi_{\mathfrak{g}}$ so $F^{P\mathfrak{g}}_{U_\pm}$ is invertible in $R^\xi_{\mathfrak{g}}$. By Theorem 4.3, M is the result of surgery on a strongly p-periodic framed link $L \subset S^3$. Let L' be the quotient link in S^3. Then M' is the result of surgery on L'. It is also a rational homology 3-sphere because $H_1(M) \to H_1(M')$ is a surjection under the covering projection (actually $\pi_1(M) \to \pi_1(M')$ is a surjection). By Lemma 3.4,

$$Q^{\mathfrak{g}}_L(c) = J^{\mathfrak{g}}_L(c) J^{\mathfrak{g}}_{U(m)}(c) \equiv (J^{\mathfrak{g}}_{L'}(c'))^p (J^{\mathfrak{g}}_{U(m/p)}(c'))^p = (Q^{\mathfrak{g}}_{L'}(c'))^p$$

modulo $I^\xi_{\mathfrak{g},p}$ in $R^\xi_{\mathfrak{g}}$ for any c in $\mathcal{C}^{R,p}_L$. Hence we have

$$\begin{aligned}
|W|^m F^{P\mathfrak{g}}_L(\xi) &= \sum_{c \in P^m_r \cap Y} Q^{\mathfrak{g}}_L(c)|_{q=\xi} \equiv \sum_{c \in P^m_r \cap \mathcal{R}_p} Q^{\mathfrak{g}}_L(c)|_{q=\xi} \\
&\equiv \Big(\sum_{c \in P^{m/p}_r \cap Y} Q^{\mathfrak{g}}_{L'}(c)|_{q=\xi} \Big)^p \\
&\equiv |W|^m (F^{P\mathfrak{g}}_{L'}(\xi))^p \bmod I^\xi_{\mathfrak{g},p} \quad \text{in } R^\xi_{\mathfrak{g}},
\end{aligned}$$

where $\mathcal{R}_p$ is the set of p-periodic root-colorings. Recall that we extend the set of colors to all roots, not just dominant ones. The first equivalence follows from the fact that non-periodic coloring must occur p multiple times. One can see this by letting $\mathbb{Z}/p\mathbb{Z}$ act on the set of colorings of L. Since $|W|$ and $F^{P\mathfrak{g}}_{U_\pm}(\xi)$ are invertible in $R^\xi_{\mathfrak{g}}/I^\xi_{\mathfrak{g},p}$ one has

$$\begin{aligned}
\tau^{P\mathfrak{g}}_\xi(M)(F^{P\mathfrak{g}}_{U_+})^{\sigma_{L+}}(F^{P\mathfrak{g}}_{U_-})^{\sigma_{L-}} &= F^{P\mathfrak{g}}_L(\xi) \equiv (F^{P\mathfrak{g}}_{L'}(\xi))^p \\
&= (\tau^{P\mathfrak{g}}_\xi(M')(F^{P\mathfrak{g}}_{U_+})^{\sigma_{L'+}}(F^{P\mathfrak{g}}_{U_-})^{\sigma_{L'-}})^p,
\end{aligned}$$

and hence

$$\tau_\xi^{P\mathfrak{g}}(M) \equiv (\tau_\xi^{P\mathfrak{g}}(M'))^p (F_{U_+}^{P\mathfrak{g}})^{p\sigma_{L'+}-\sigma_{L+}} (F_{U_-}^{P\mathfrak{g}})^{p\sigma_{L'-}-\sigma_{L-}} \bmod I_{\mathfrak{g},p}^\xi \quad \text{in } R_{\mathfrak{g}}^\xi.$$

Let $a = p\sigma_{L'+} - \sigma_{L+}$ and $b = p\sigma_{L'-} - \sigma_{L-}$. Then

$$a + b = p(\sigma_{L'+} + \sigma_{L'-}) - (\sigma_{L+} + \sigma_{L-}) = m - m = 0$$

because M and M' are rational homology 3-spheres. Therefore

$$\tau_\xi^{P\mathfrak{g}}(M) \equiv (\tau_\xi^{P\mathfrak{g}}(M'))^p \left(\frac{F_{U_+}^{P\mathfrak{g}}}{F_{U_-}^{P\mathfrak{g}}}\right)^a.$$

Let ω be the fourth root of unity as in Lemma 4.2. Because r is odd one has $\mathbb{Q}(\xi) \cap \mathbb{Q}(i) = \mathbb{Q}$ (see [FT]), hence $\omega = \pm 1$. Finally, $(F_{U_+}^{P\mathfrak{g}}/F_{U_-}^{P\mathfrak{g}})^a = \pm\xi^{u'} = (-\xi)^u$ since r is odd and ξ is an rth root of unity. ∎

5. Quantum invariants of periodic 3-manifolds II. Theorem 4.4 does not help us to determine whether a 3-manifold is p-periodic. It can only tell us that a 3-manifold is not a p-fold branched covering space over a particular 3-manifold branched over a circle as noted by Chbili in [C1] who showed that the Poincaré sphere is not a branched cover of S^3 of order 11 with the branched set a circle. Recently Gilmer, Kania-Bartoszyńska and Przytycki [GKP] gave a criterion for an integral homology 3-sphere M to be r-periodic, which involves only the quantum sl_2 invariant of M at an rth root of unity. The purpose of this section is to generalize their criterion to quantum invariants of all simple Lie algebras.

The proofs in [GKP] depend on the topological quantum field theory (TQFT) behind the quantum sl_2 invariants of 3-manifolds. We recall briefly some related definitions first. See [Tu, G2] for more details. A TQFT (V, Z) is a functor from the 3-dimensional cobordism category to the category $\mathrm{Mod}(K)$ of projective modules over a ring K, where V and Z are maps between objects and morphisms respectively. The objects and morphisms in the 3-dimensional cobordism category are surfaces and 3-dimensional manifolds respectively. Turaev [Tu] showed that given a modular category one can construct a TQFT. In order to remove the anomaly one also needs to consider manifolds with a certain structure $(\mathfrak{B}, \mathfrak{U})$. More explicitly, a *structure* $\mathfrak{U}(\Sigma)$ on a surface Σ is a pair (b, l), where b is a finite set of disjoint, directed line segments colored by objects from a modular category and l is a Lagrangian subspace of $H_1(\Sigma, \mathbb{Q})$. A *structure* $\mathfrak{B}(M)$ on a 3-dimensional manifold M is a pair (Ω, w), where Ω is a properly embedded ribbon graph (with coupons) in M colored by objects and morphisms from the same modular category and w is an integer. This cobordism category will be denoted as $\mathcal{C}(\mathfrak{B}, \mathfrak{U})$. Gilmer [G2] introduced a notion of *almost integral TQFT.* Let K be an integral domain which contains a Dedekind domain $\mathcal{D}$. Then the

TQFT (V, Z) is *almost* $(\mathcal{D})$*-integral* if there is an element $\mathcal{E} \in K$ such that $\mathcal{E}\langle M\rangle \in \mathcal{D}$ for any connected closed 3-manifold M, where $\langle M\rangle = Z(M)(1)$.

REMARK. $\mathcal{E}$ is required to be in $\mathcal{D}$ in [G2] but the proofs there work for the more general case.

REMARK. If the modular category comes from representations of $\mathcal{U}_q(\mathfrak{g})$ at a root of 1 then the TQFT constructed above is not almost integral. To get an almost integral TQFT one has to modify the structures on the cobordism category. The new cobordism category will be denoted by $\mathcal{C}(\mathfrak{B}^e, \mathfrak{U}^e)$. See [CL].

Let κ and η be one of the square roots of $F_{U_-}^{P\mathfrak{g}}/F_{U_+}^{P\mathfrak{g}}$ and $F_{U_-}^{P\mathfrak{g}}F_{U_+}^{P\mathfrak{g}}$ respectively such that $\kappa\eta = F_{U_-}^{P\mathfrak{g}}$. Set $K = \mathbb{C}$ and $\mathcal{D} = \mathbb{Z}[\xi, \kappa]$.

LEMMA 5.1. *Let $r > dh^\vee$ be a prime and not a factor of $|G|\,|W|$. Let ξ be a primitive rth root of unity. Then one can define an almost integral TQFT (Z, V) from $\mathcal{C}(\mathfrak{B}^e, \mathfrak{U}^e)$ to* $\mathrm{Mod}(K)$ *such that $\eta\langle M\rangle$ is in $\mathcal{D}$ for all closed, connected M. Furthermore, if M has* 0 *weight and the ribbon graph in M is empty then $\langle M\rangle = \tau_\xi^{P\mathfrak{g}}(M)$.*

Sketch of the proof. In [Le2] the second author showed that for any simple Lie algebra $\mathfrak{g}$ there exist modular tensor categories $\mathcal{C}_\xi$ using the representations of $\mathcal{U}_q(\mathfrak{g})$ associated to the root lattice. One can show that Turaev's construction of anomaly-free TQFT can be adapted to the modified cobordism category $\mathcal{C}(\mathfrak{B}^e, \mathfrak{U}^e)$. See [CL] for more details. ∎

THEOREM 5.2. *Let M be an r-periodic integral homology* 3*-sphere, where $r > dh^\vee$ is a prime number. Let ξ be a primitive rth root of unity. Then*

$$\tau_\xi^{P\mathfrak{g}}(M) \equiv \xi^v \overline{\tau_\xi^{P\mathfrak{g}}(M)} \bmod r \qquad \text{in } \mathbb{Z}[\xi]$$

for some integer v if r is not a factor of $|G|\,|W|$.

Proof. Lemma 5.1 shows that the TQFT's constructed following Turaev and the second author [Tu, Le2, CL] are almost integral and $\eta\langle M\rangle \in \mathcal{D}$. Then the proofs given in [GKP] can be used verbatim to show that the congruence relation is true in $\mathcal{D}$, i.e.,

$$x = \tau_\xi^{P\mathfrak{g}}(M) - \xi^v \overline{\tau_\xi^{P\mathfrak{g}}(M)} = rg$$

for some $g \in \mathcal{D}$. According to [Le2], x is in $\mathbb{Z}[\xi]$. So $g = x/r \in \mathbb{Q}(\xi)$, which implies that $g \in \mathbb{Z}[\xi]$ since $\mathcal{D}$ is a ring of algebraic integers. ∎

REMARK. Although one can define quantum invariants of 3-manifolds without using TQFT the proof of Theorem 5.2 depends on it. It would be interesting to find a direct proof.

6. Applications. It is known that the Poincaré sphere P, -1 surgery on the left-hand trefoil, has periodicity 2, 3 and 5. It is also known that the Brieskorn sphere $B = \Sigma(2,3,7)$, -1 surgery on the right-hand trefoil, has periodicities 2, 3 and 7. Actually these are the only prime periodicities they have. According to the second author [Le2],

$$\tau_P = \tau_\xi^{Psl_2}(P) = (1-\xi)^{-1}\sum_{n=0}^{\infty}\xi^n(1-\xi^{n+1})(1-\xi^{n+2})\cdots(1-\xi^{2n+1}),$$

$$\tau_B = \tau_\xi^{Psl_2}(B) = (1-\xi)^{-1}\sum_{n=0}^{\infty}\xi^{-n(n+2)}(1-\xi^{n+1})(1-\xi^{n+2})\cdots(1-\xi^{2n+1}).$$

Here ξ is again a primitive rth root of unity and therefore the sums are finite. Let us suppose that r is now a prime number. Every element x in the ring $\mathbb{Z}[\xi]$ can be written as

$$x = \sum_{n=0}^{r-2} a_n(x)(1-\xi)^n + x'(1-\xi)^{r-1}$$

for some $x' \in \mathbb{Z}[\xi]$ and some integers $a_n(x)$. The $a_n(x)$'s are well defined modulo r because of the following equation [O]:

$$1 + t + \cdots + t^{r-1} = \binom{r}{1} + \binom{r}{2}(t-1) + \cdots + \binom{r}{r}(t-1)^{r-1}.$$

One computes easily the following integers, well defined modulo r:

$$a_1(\tau_P) = 6, \qquad a_1(\xi^j\overline{\tau}_P) = -6-j,$$
$$a_3(\tau_P) = 464, \qquad a_3(\xi^{-12}\overline{\tau}_P) = -16,$$
$$16 + 464 = 480 = 2^5\cdot 3\cdot 5.$$

Hence P has periodicity 2, 3 and 5 only. Similarly, B has periodicity 2, 3 and 7 only:

$$a_1(\tau_B) = 6, \qquad a_1(\xi^j\overline{\tau}_B) = -6-j,$$
$$a_3(\tau_B) = 1064, \qquad a_3(\xi^{-12}\overline{\tau}_B) = -280,$$
$$1064 + 280 = 1344 = 2^6\cdot 3\cdot 7.$$

Remark. X. Zhang [Z] showed that the Poincaré sphere has only periodicity 2, 3 and 5 using the fact that the Poincaré sphere is spherical. One can also use the recently proved Orbifold Theorem to show that the Poincaré sphere is 2, 3, 5-periodic only and $\Sigma(2,3,7)$ is 2, 3, 7-periodic only as remarked in [GKP].

The calculation above suggests that the first and third Ohtsuki invariants detect very well the periodicity of homology spheres. Let us look at these two invariants for Brieskorn spheres $B_n = \Sigma(2,|n|,|2n+1|)$, where n is any odd number. It can be shown, again by the Orbifold Theorem, that B_n is p-periodic if and only if $p \mid n(2n+1)$. Following [GKP], B_n can be obtained by -1 surgery on the $(2,n)$ torus knot. Note that our B_n is the $\Sigma(2,|n|,|2n-1|)$ in [GKP] with opposite orientation. Lawrence [La] has explicit formulas for the Ohtsuki invariants of B_n. Using her formulas one can show that if $p \geq 11$ is a prime number then B_n is p-periodic only if p divides $\frac{2}{3}n^2(2n+1)(n-1)(n+1)^2$. This necessary condition is sharp when, for example, $n=7$ but not in general.

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Received 23 September 2003;
in revised form 30 November 2004