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Chewing the Khovanov homology of tangles

by

Magnus Jacobsson (Roma)

Abstract. We present an elementary description of Khovanov's homology of tangles [K2], in the spirit of Viro's paper [V]. The formulation here is over the polynomial ring $\mathbb{Z}[c]$, unlike [K2] where the theory was presented over the integers only.

1. Introduction. In the paper [K1] Khovanov introduced a new homology theory of links, with the Jones polynomial as its graded Euler characteristic. His paper was written in a category-theoretical language which, at least to the minds of some topologists, rather obscured the simple combinatorial nature of these remarkable invariants. For this reason, Bar-Natan [BN] and Viro [V], in ensuing papers, provided what they described as the results of "chewing": the authors' more elementary understanding of the Khovanov invariants. Their chewing turned out successful, leading quickly to some new results (e.g. [L], [J]) and increasing the activity of research on Khovanov homology.

The goal of this note is to present a bit of chewing on Khovanov's follow-up paper [K2], where he extended his construction to tangles. It is in the spirit of [V] and can be regarded as a continuation of that paper by a different author. (Khovanov homology is also described in Section 2 of [J], very similarly.)

All the results in this note are due to Khovanov and can be found in his paper. This note differs from [K2] in its formulations and in that it uses $H(D)$, not $\mathcal{H}(D)$, that is, coefficients in the polynomial ring $\mathbb{Z}[c]$ rather than in $\mathbb{Z}$.

2. Khovanov homology of tangles. In this section we review Khovanov homology in its most general form, that is, with coefficients in $\mathbb{Z}[c]$ and defined for arbitrary tangle diagrams. The original definitions can be found in [K1] (for links only, but with coefficients in $\mathbb{Z}[c]$) and in [K2] (for

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tangles, but with details only with coefficients in $\mathbb{Z}$). We assume that the reader is familiar with the basic theory of tangles.

2.1. *The Frobenius algebra A.* The definition of Khovanov homology relies on a certain commutative Frobenius algebra A, generated as a free $\mathbb{Z}[c]$-module by two elements $\mathbf{1}$ and X, where $\mathbf{1}$ is the multiplicative identity and $X^2 = 0$.

There is also a comultiplication Δ, given by

$$\Delta(\mathbf{1}) = X \otimes \mathbf{1} + \mathbf{1} \otimes X + cX \otimes X, \qquad \Delta(X) = X \otimes X,$$

and a trace form $\varepsilon : A \to \mathbb{Z}[c]$ defined by

$$\varepsilon(\mathbf{1}) = -c, \qquad \varepsilon(X) = 1.$$

It is well known that a commutative Frobenius algebra gives rise to a $(1+1)$-dimensional topological quantum field theory. It associates to a disjoint planar collection of k circles the tensor product $A^{\otimes k}$. To each saddle point Morse modification on this collection of circles it associates the multiplication m or comultiplication Δ, depending on whether two circles merge under the modification or one circle splits into two. To a disappearing circle it associates the trace form, and to an appearing circle the unit map.

The basic relations in this algebra are associativity, coassociativity and the relation

$$(m \otimes \mathrm{id})(\mathrm{id} \otimes \Delta) = \Delta \circ m,$$

which can be described topologically as the isotopy relations in Figure 1.

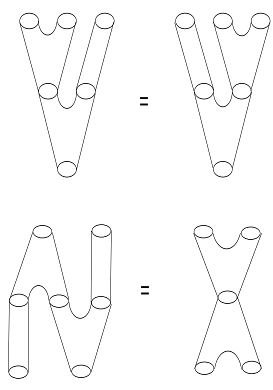

Fig. 1. Associativity/coassociativity and an additional relation

2.2. *Khovanov's rings H^n*

2.2.1. *Generators.* Khovanov homology is defined as the homology of a certain chain complex. In this subsection we review Khovanov's construction of certain rings H^n over which the chain complex will be a bimodule.

Remark. We will be concerned only with tangles with an even number of top and bottom points. We will say that a tangle (diagram) is of *type* (m, n) if it has $2m$ points at the top and $2n$ points at the bottom.

By a (*crossingless*) *matching* (*of n points*) we mean a tangle diagram M of type $(0, n)$ or $(n, 0)$, without crossings (cf. Figure 2) and without closed components.

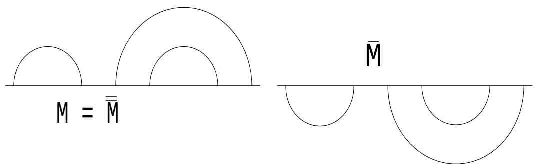

Fig. 2. Crossingless matchings of types $(3, 0)$ (left) and $(0, 3)$ (right). The bar denotes reflection in a horizontal line.

Two crossingless matchings M and M' of type $(0, n)$ and $(n, 0)$ respectively can be composed to form a diagram MM' of type $(0, 0)$. This is an unlink diagram without crossings, with the additional structure of a canonical decomposition into M and M'.

An unlink diagram has *states* in the sense of [V]. Since there are no crossings, a state is just a distribution of $\mathbf{1}$:s or X:s to the components of MM'.

Remark. In [V], $\mathbf{1}$ was denoted by a minus sign and X by a plus sign, but here we will follow Khovanov in using the symbols $\mathbf{1}$ and X instead.

Let H^n be generated as a free $\mathbb{Z}[c]$-module by all possible states of MM':s, where M is of type $(0, n)$ and M' of type $(n, 0)$. In Figure 3 such a state is displayed.

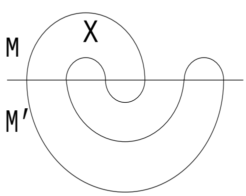

Fig. 3. An element of the ring H^3

2.2.2. *The product.* To define the multiplication on H^n, start by noting that there is an involution $M \mapsto \overline{M}$ on the set of crossingless matchings given by reflection in a horizontal line not intersecting M. This involution interchanges $(0, n)$-matchings and $(n, 0)$-matchings (cf. Figure 2).

The product ST of a state S of KL and a state T of MN will be zero if $L \neq \overline{M}$. If $L = \overline{M}$ then the product is a linear combination of states of KN, which we describe below.

Place $K\overline{M}$ above MN. Some half-circle in M can be merged with its reflection in $\overline{M}$, by a saddle point Morse move on the diagram $K\overline{M} \cup MN$, affecting only these two half-circles. This results in a pair of vertical strands connecting N to K. Continue this procedure until no half-circles are left in the space between N and K, so that N and K instead are connected by $2n$ vertical strands. The result is canonically isotopic to KN (cf. Figure 4). Each

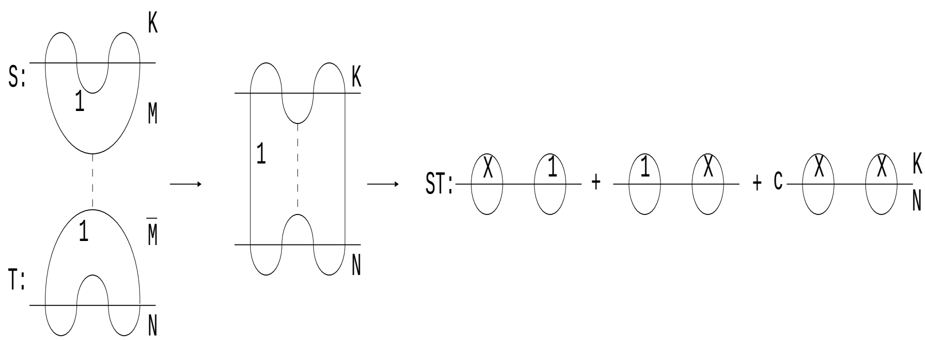

Fig. 4. A sample multiplication in H^2. The product of S and T is the sum of the three states on the right. Morse moves occur along the dashed lines.

Morse move induces a map of states coming from either the comultiplication or the multiplication of A. The full sequence of Morse moves ends in a state which is by definition the product ST.

2.2.3. *The grading.* Let S be an element of the ring H^n. Let $\sharp X(S)$ denote the number of circles marked with X:s in S and $\sharp\mathbf{1}(S)$ the number of circles marked with $\mathbf{1}$:s. Put

$$\tau(S) = \sharp X(S) - \sharp\mathbf{1}(S).$$

Then H^n becomes a graded ring $H^n = \bigoplus_j (H^n)_j$ if we put

$$j(c^k S) = -\tau(S) + 2k - n.$$

Remark. Note that H^n is a ring with 1. Namely, for each matching M consider the state of $M\overline{M}$ which has $\mathbf{1}$:s on all circles. This is clearly an idempotent, and the unit is the sum of all such idempotents in H^n.

Remark. The above j-grading is compatible with [K1], [J] and [V]. In [K2] the grading is the opposite $(-j)$. This remark also applies to the grading in the chain complex in the next section.

2.3. *The chain complex.* Let D be an oriented (m,n)-tangle diagram. Such a diagram can be turned into a link diagram by capping off its top and bottom by crossingless matchings, i.e. by composing D with an $(n,0)$-matching N from below and a $(0,m)$-matching M from above. The result is a link diagram, with a canonical decomposition into its constituent pieces as MDN.

A *state* of the tangle diagram D is a state of the link diagram MDN for some choice of matchings M,N. Recall that a *state* of a link diagram is a distribution of Kauffman markers to its crossings together with a distribution of X:s and $\mathbf{1}$:s to the components of the resolution. (A state of a tangle is also assumed to remember the decomposition MDN.)

Consider the free $\mathbb{Z}[c]$-module C generated by all states of D. Denote by $w(D)$ the writhe of the tangle diagram, by $\sigma(S)$ the sum of all signs of markers in the state S and by $\tau(S)$ the number of X:s minus the number of $\mathbf{1}$:s in the resolution of S.

We now turn C into a bigraded $\mathbb{Z}[c]$-module $C^{i,j}$, by defining the grading parameters for an element c^kS as

$$i(c^kS) = \frac{w(D)-\sigma(S)}{2}, \qquad j(c^kS) = -\frac{\sigma(S)+2\tau(S)-3w(D)}{2}+2k-n.$$

Notice that multiplication by c affects only the second grading parameter and that $\deg(c)=2$.

Given a tangle diagram D, let L be a subset of the set I of crossings of D. Let $C^{i,j}_L(D)$ be the submodule of $C^{i,j}$ generated by states S for which L is the set of crossings with negative markers.

For any finite set S, let FS be the free abelian group generated by S. For bijections $f,g:\{1,\dots,|S|\}\to S$, let $p(f,g)\in\{0,1\}$ be the parity of the permutation $f^{-1}g$ of $\{1,\dots,|S|\}$. Let $\operatorname{Enum}(S)$ be the set of all such bijections.

Definition. For S as above, we define

$$E(S) = F\operatorname{Enum}(S)/((-1)^{p(f,g)}f-g).$$

Remark. Observe that $E(S)$ is isomorphic to $\mathbb{Z}$, but not canonically.

Let $n(i)$ denote the number of negative markers in any state S with $i(S)=i$. (Note that this function is well defined.)

Definition. The (i,j)th chain group of the chain complex is

$$C^{i,j}(D) = \bigoplus_{L\subset I,\,|L|=n(i)} C^{i,j}_L(D)\otimes E(L).$$

The sum runs over all subsets L with cardinality $n(i)$.

Remark. From now on, we use the word “state” both for a state as defined above and an element $S \otimes [x] \in C^{i,j}(D)$, where S is a state and x is some sequence of crossings with negative markers. The context should prevent any confusion.

Remark. Tensoring with $E(L)$ is an invariant way of including the right incidence numbers in the complex. This can also be done by enumerating the crossings. See [V] for this approach, which necessitates a (simple) proof that the resulting invariants do not depend on the choice of enumeration.

2.4. *The bimodule structure.* The chain modules of the chain complex are in fact (H^m, H^n)-bimodules. If $S \in H^m$ is a state of $M'M$ and $T \in C(D)$ is a state of $\overline{M}DN$, then by merging M and $\overline{M}$ using the same procedure that defined the multiplication in H^n above, we get a new state of D (which is a state of the link diagram $M'DN$). This new state is the product of T with S from the left. The right module structure is defined analogously.

2.5. *The differential.* The differential in the chain complex has bidegree $(1, 0)$ and is defined in the same way as in [J]. The only difference comes from the changes in the Frobenius algebra due to c being non-zero.

The differential of a state S is built from states T which are *incident* to S in the following sense.

T is *not* incident to S unless the markers of S and T are different at exactly one crossing point a, where the marker of T is negative and the marker of S is positive. This means that the resolutions of S and T differ by a single saddle point Morse modification at a. Thus the numbers $|S|, |T|$ of components of the resolutions of S, T satisfy $|S| = |T| \pm 1$, and the resolution of T is obtained from that of S by either splitting a single circle in two or merging two circles into one.

T is *not* incident to S unless the components that their resolutions have in common are marked with the same symbols $\mathbf{1}, X$.

Thus, if T is incident to S and a is the crossing where their markers differ, then only the symbols on circles that pass a are different. It is easy to see that the requirement $j(S) = j(T)$ gives the table of incident states presented in Figure 5.

The fifth row means that T is incident to S if S has a single $\mathbf{1}$-marked circle passing a and either T has different symbols on its two circles passing a, or $T = cT'$ where T' has two X-circles passing a.

Finally, if T is incident to S in one of the ways above, then also $c^k T$ is incident to $c^k S$, for any integer k.

Remark. Observe that the states in the right column are simply obtained from those in the left by multiplication or comultiplication in A. The

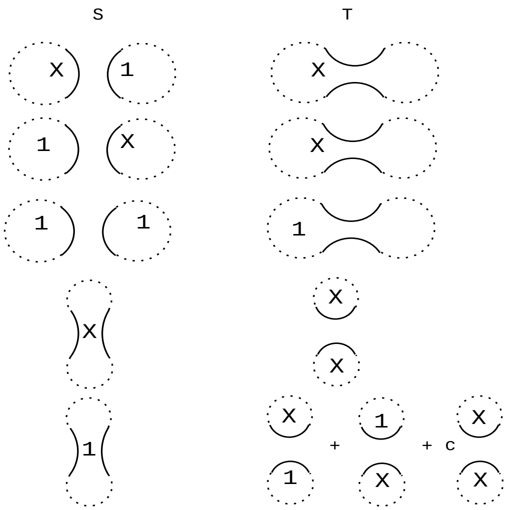

Fig. 5. Incident states

only difference from [J] occurs when the comultiplication is applied to a **1**-circle.

DEFINITION. Let S belong to $C_L^{i,j}(D)$. The *differential* of $S \otimes [x]$ is the sum

$$d(S \otimes [x]) = \sum T \otimes [xa],$$

where the T:s run over all states in $C^{i+1,j}(D)$ which are incident to S, and $a = a(T)$ is the crossing where T differs from S.

THEOREM 1 (Khovanov). *The complex of bimodules defined above is invariant up to chain homotopy equivalence under ambient isotopy of the tangle.*

3. A localization theorem. Let D and D' be tangle diagrams of types (l, m) and (m, n), respectively. Let L,M,N be crossingless matchings, and let S and S' be states of $LD\overline{M}$ and $MD'N$, respectively. Put $LD\overline{M}$ above $MD'N$. Then M and $\overline{M}$ can be merged, in the same way as when the multiplication in H^n was defined in Section 2.2. This defines a map

$$\Phi : C(D) \otimes_{\mathbb{Z}[c]} C(D') \to C(DD'),$$

which, as is easy to see, factors to give a homomorphism

$$\Phi : C(D) \otimes_{H^m} C(D') \to C(DD').$$

Khovanov proves in [K2] that this is even an isomorphism of complexes of (H^l, H^n)-bimodules:

Theorem 2 (Khovanov). *The bimodule complex $C(DD')$ of the composition of an (l,m)-tangle D and an (m,n)-tangle D' is canonically isomorphic to $C(D)\otimes_{H^m} C(D')$, via the map Φ described above.*

Remark. Khovanov proves this theorem for the coefficient ring $\mathbb{Z}$. The proof works over $\mathbb{Z}[c]$ as well.

Let D be a tangle diagram. Then D is the composition of a sequence of elementary tangles: $D = D_1 \cdots D_n$ (see Figure 6). By the above theorem $C(D)$ is canonically the tensor product of the chain complexes of $D_1, \ldots, D_n$.

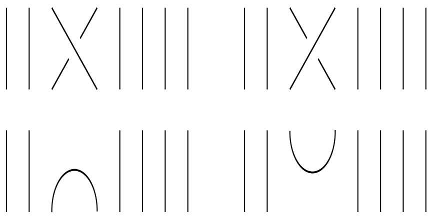

Fig. 6. Elementary (unoriented) tangles

Note that, even though the chain complexes D_i are very simple, their tensor product over the ring H^n might not be. Indeed, gluing together elementary tangles to form a link gives back the ordinary Khovanov chain complex, which in general is highly non-trivial. Thus, the localization must be used with care, so that all constructions to which one uses it respect the bimodule structure.

4. A simple example. As an example, let us compute the Khovanov chain complex $C(T)$ of the elementary $(1,1)$-tangle T in Figure 7. There

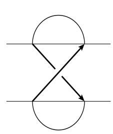

Fig. 7. The tangle T with its unique capping

is only one way to cap off this tangle, so the chain complex is isomorphic to the ordinary Khovanov complex of a trivial circle with a negative twist (up to a grading shift in j). The span of states with positive marker can be identified with the algebra A. The states with negative marker span a subspace we can identify with $A\otimes A$ (identifying e.g. the left tensor factor with the upper circle and the right tensor factor with the lower one). Hence, as a $\mathbb{Z}[c]$-module, $C(T) \cong (A\otimes A)\oplus A$. The differential in the complex is then zero on the first summand, and maps the second into the first using the comultiplication Δ.

The bimodule structures are equally easy to describe. The ring in question is H^1, which is obviously isomorphic to A. It acts on the A-summand using the (commutative) multiplication μ in A on both sides, and on the $A \otimes A$-summand by $\mu \otimes 1$ from the left and by $1 \otimes \mu$ from the right.

To illustrate the localization theorem, let us glue together two copies of T as in Figure 8. Given two states of T as in this figure, notice that the

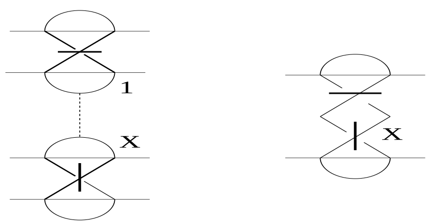

Fig. 8. Gluing two states of T into a state of TT. States are multiplied along the dashed line. The label on the topmost circle is irrelevant.

gluing map only affects the half-circles in the region between the tangles. Identifying these circles with $A \otimes A$ and the resulting circle in TT with A, we see that the gluing map Φ is given (on these circles) by the multiplication map:

$$\mu(X \otimes X) = 0, \quad \mu(X \otimes 1) = \mu(1 \otimes X) = X, \quad \mu(1 \otimes 1) = 1.$$

Over H^1, however, $X \otimes X$ is zero already, since $X \otimes X = 1X \otimes X = 1 \otimes X^2 = 0$. Similarly, $X \otimes 1 = 1 \otimes X$. With this observation it is easy to see that Φ is an isomorphism of bimodules. To show that Φ is a chain map is left to the reader. (*Hint*: Use Figure 1.)

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Istituto Nazionale di Alta Matematica (INdAM)
Città Universitaria, P.le Aldo Moro 5
00185 Roma, Italy
E-mail: jacobsso@mat.uniroma1.it

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